Geometric Measure Theory in Padova

List of Talks

Giovanni Alberti, *Frobenius theorem for non-smooth sets and currents*

Let V be a distribution of k-planes in $\mathbb{R}^d$, and let S be a k-dimensional surface which is tangent to V. Then Frobenius theorem states that V must be involutive at every point of S. In this talk I will give an overview of some recent (and not so recent) research with A. Massaccesi (University of Padova), Evgeni Stepanov (Steklov Institute, Saint Petersburg) and Andrea Merlo (University of Pisa), where we investigated the extension of this statement to weaker notions of surfaces, such as rectifiable sets and currents. It turns out that if S is a k-rectifiable set then the validity of the statement is strictly related to the regularity of the boundary of S, understood in terms of currents (in particular the statement holds if the boundary has finite mass, and may not hold otherwise). If S is normal current but is not rectifiable, then the key is a certain geometric property of the boundary of S. These questions are strictly related to the problem of decomposing normal currents into integral/rectifiable ones.

Katrin Fässler, *Singular integrals on regular curves in the Heisenberg group*

A result by A. P. Calderon from 1977 says that the Cauchy transform is bounded on L^2 on Lipschitz graphs in the plane with sufficiently small Lipschitz constant. The assumption on the slope of the graphs could later be removed by R. R. Coifman, A. McIntosh, and Y. Meyer, and the result was extended by G. David to a more general class of singular integral operators (SIOs) induced by smooth -1-homogeneous odd kernels on 1-regular curves in $\mathbb{R}^n$. These works mark the beginning of an active line of research at the interface of geometric measure theory and harmonic analysis. In the talk I will discuss an extension of David's theorem to a non-Euclidean metric space, the Heisenberg group. This is joint work with Tuomas Orponen.

Jonathan Fraser, *Box dimensions of projections and dimension interpolation*

Marstrand's celebrated projection theorem from 1954 says that for a Borel set F in the plane, the Hausdorff dimension of the projection of F onto almost every line is the minimum of 1 and the Hausdorff dimension of F. The box dimensions of projections was studied some years later by Falconer, Howroyd and Jarvenpaa. Here the situation is more complicated: the box dimension of the projection assumes the same value almost surely, but the value is given by a "dimension profile" and is more awkward to work with.

In this talk I will discuss recent joint work with Stuart Burrell, Kenneth Falconer, and Pablo Shmerkin, where we apply the concept of "dimension interpolation" to study the box dimensions of projections.

Jonas Hirsch, *Nonclassical minimizing surfaces with smooth boundary*

We construct a Riemannian metric g on $\mathbb{R}^4$ (arbitrarily close to the euclidean one) and a smooth simple closed curve $\Gamma \subset \mathbb{R}^4$ such that the unique area minimizing surface spanned by Γ has infinite topology. Furthermore the metric is almost Kähler and the area minimizing surface is calibrated.

This example suggests that a conjecture by B. White is sharp. It states that the Federer-Fleming solution has finite topology if the boundary curve $\Gamma \subset \mathbb{R}^n$ is real analytic. If White's conjecture were true, then for real analytic boundary curves the Federer-Fleming solution T would coincide with the Douglas-Rado solution for some genus g.

In codimension one this holds true already if the boundary curve Γ is sufficient regular ($C^{k,\alpha}$ for $k+\alpha > 2$) as a consequence of De Giorgi's interior regularity theorem and Hardt-Simon's boundary regularity result.

In contrast by our example the situation seems to change dramatically if we go to higher co-dimension. In my talk I would like to present the construction of our example and its link to the known boundary regularity result in higher codimension.

Joint work with C. De Lellis and G. De Philippis.

Luca Lussardi, *Varifold minimizers of the multiphase Canham-Helfrich functional*

This seminar concerns the minimization of the Canham-Helfrich functional in presence of multiple phases. The problem is inspired by the modelization of heterogeneous biological membranes, which may feature variable bending rigidities and spontaneous curvatures. With respect to previous contributions, no symmetry of the minimizers is here assumed. Correspondingly, the problem is reformulated in a Geometric Measure Theory setting, precisely the framework of oriented curvature varifolds with boundary. I will consider both single and multiphase minimizers under area and enclosed volume constrains. Additionally, I will briefly discuss regularity of minimizers and lower and upper diameter bounds.

Andrea Merlo, *Geometry and rectifiability of 1-codimensional measures in the Heisenberg groups*

Characterisation of rectifiable measures in Euclidean spaces through the existence of the density has been a longstanding problem for Geometric Measure Theory until the complete answer by D. Preiss in 1987. The question of how in more general metric spaces existence of density can affect any kind of gain in terms of regularity of the measure is a completely open problem. In this talk I will discuss how the mere existence of the 1-codimensional density for a measure in the Heisenberg groups endowed with the Koranyi metric implies that almost everywhere the tangents to the measure are flat. If time permits I will also give an idea of how this regularity of the tangents implies also the rectifiability of the measure in the sense of Franchi, Serapioni and Serra Cassano.

Tuomas Orponen, *Progress on the visibility problem*

Let e be a unit vector in the plane, and let $l(e)$ be a closed half-line parallel to e. The "visible part" $Vis(K,e)$ of a planar compact set K in direction e is the set of points x in K with the property that $x + l(e)$ only meets K at x. It may seem intuitive that the visible parts of any compact set should be at most 1-dimensional, but the existence of 2-dimensional graphs shows that this is not completely clear. However, the visibility problem asks to show that if K is a fixed compact set, then $Vis(K,e)$ is at most 1-dimensional for almost every e. This remains open, but I will discuss how to prove that $Vis(K,e)$ is at most 1.99-dimensional for almost every e."

Bozhidar Velichkov, *Regularity of the two-phase free boundaries*

We consider the two-phase functional J_{TP} defined, for every open set $D \subset \mathbb{R}^d$ and every function $u : D \to \mathbb{R}$, as

$$J_{\text{TP}}(u,D) := \int_D |\nabla u|^2\,dx + \lambda_+^2|\Omega_u^+ \cap D| + \lambda_-^2|\Omega_u^- \cap D|, \tag{TP}$$

where the constants $\lambda_+ > 0$ and $\lambda_- > 0$ are fixed, and the two phases are given by

$$\Omega_u^+ = \{u > 0\} \qquad \text{and} \qquad \Omega_u^- = \{u < 0\}.$$

We say that a function $u : D \to \mathbb{R}$ is a *local minimizer of* J_{TP} *in* D, if

$$J_{\text{TP}}\big(u,\Omega\big) \le J_{\text{TP}}\big(v,\Omega\big),$$

for all open sets Ω and functions $v : D \to \mathbb{R}$ such that $\overline{\Omega} \subset D$ and $v = u$ on $D \setminus \Omega$.

In this talk, we will present some new results on the regularity of the free boundary $\partial\Omega_u^+ \cup \partial\Omega_u^- \cap D$ of local minimizers u of the two-phase functional J_{TP}. Precisely, we will show that:

– in dimension $d = 2$, the free boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ are $C^{1,\alpha}$-regular curves (this result was proved in the recent works Spolaor-Velichkov [CPAM, 2019] and Spolaor-Trey-Velichkov [Comm.PDE, 2019]);

– in dimension $d > 2$, the free boundaries $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ are $C^{1,\alpha}$-regular, up to a (possibly empty) one-phase singular set of lower dimension (De Philippis-Spolaor-V.).

In particular, these results complete (in any dimension) the analysis of the two-phase free boundaries started by Alt, Caffarelli and Friedman in 1984.

Roger Züst, *Rough integration on paths and surfaces, an overview*

The goal of this talk is to present some classical and recent results on rough integration as well as applications to geometry. We start with the results of Young and Chen that lead to the introduction of rough paths by Lyons in the 90s in order to solve certain stochastic differential equations. In recent years there has been a growing interest in developing a similar integration theory in higher-dimensions. Among other things we want to present (different versions of) functions of bounded fractional variation and extensions of Young's integral to higher dimensions. Two of the applications we discuss are concerned with the rigidity in the Weyl problem about isometric immersions of spheres in space and Gromov's Hölder equivalence problem in the Heisenberg group.