PSEUDO-ABSOLUTE VALUES: FOUNDATIONS

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Abstract. In this article, we introduce pseudo-absolute values, which generalise usual absolute values. Roughly speaking, a pseudo-absolute value on a field K is a map $|\cdot|: K \to [0,+\infty]$ satisfying axioms similar to those of the usual absolute values. This notion allows to include "pathological" absolute values one can encounter trying to incorporate the analogy between Diophantine approximation and Nevanlinna theory in an Arakelov theoretic framework. It turns out that the space of all pseudo-absolute values can be endowed with a compact Hausdorff topology in a way similar to the Berkovich analytic spectrum of a Banach ring. Moreover, we introduce both local and global notions of analytic spaces over pseudo-valued fields and interpret them as analytic counterparts to Zariski-Riemann spaces.

Contents

Date: July 18, 2025.

1991 *Mathematics Subject Classification.* Primary 12J10, 13A18, 32P05; Secondary 14G40.

Key words and phrases. Valuations, valued fields, Berkovich spaces.

The author was partly supported by the collaborative research center SFB 1085 *Higher Invariants - Interactions between Arithmetic Geometry and Global Analysis* funded by the Deutsche Forschungsgemeinschaft. The main part of this research was conducted at Université Paris Cité.

Introduction

Motivations and background.

Arakelov geometry, adelic curves. Let us start by stating the following guiding principle. Given a field K, together with a distinguished set of absolute values satisfying a product formula, one can perform Diophantine geometry. The first occurrence of this principle is the case of global fields. A turning point in this philosophy is due to the seminal work of Arakelov [Ara74], which led to the development of many arithmetic analogues over number fields of classical tools in algebraic geometry. Moreover, several other instances of this principle have been studied in the literature (e.g., trivially valued fields, finitely generated extensions of $\mathbb{Q}$, infinite algebraic extensions of $\mathbb{Q}$).

In [CM19], Chen and Moriwaki introduced an Arakelov theory over an arbitrary field. The central object of the theory is called an *adelic curve*. Namely, an adelic curve is the data $S = (K, (\Omega, \nu), (|\cdot|_\omega)_{\omega\in\Omega})$, where K is a field, (Ω, ν) is a measure space and $(|\cdot|_\omega)_{\omega\in\Omega}$ is a family of absolute values on K. Moreover, an adelic curve S is called *proper* if a product formula holds (see [CM19], §3 for more details).

Adelic curves arise naturally in various number-theoretic situations, and we mention some of them.

(1) Any global field, including function fields of arbitrary characteristic, can be naturally equipped with an adelic structure whose parameter space is the set of places of the global field equipped with the discrete σ-algebra and the product formula is the usual one.
(2) Any finitely generated extension of a base field can be equipped with an adelic structure by choosing a *polarisation* of a birational model over the base field ([CM19], §3.2.4). The parameter space is again equipped with the discrete σ-algebra.
(3) A natural example of an adelic structure whose parameter space is equipped with a non-discrete is the arithmetic variant of bullet (2) above. These are adelic structures on finitely generated extensions of $\mathbb{Q}$ and come from *polarised arithmetic varieties* (*loc. cit.*, §3.2.6).
(4) More generally, any countable field can be endowed with an adelic structure ([CM21], §2.7).

The formalism of adelic curves allows one to study all these examples in uniformly.

Most tools arising in classical Arakelov geometry have a counterpart in the world of adelic curves: e.g. geometry of numbers, arithmetic intersection theory, Hilbert-Samuel formula [CM19, CM21, CM24]. However, these results hold for adelic curves such that either the parameter space is equipped with the discrete σ-algebra or the underlying field is countable. This observation is one of the motivation for this article, as we will explain in a few paragraphs.

Nevanlinna theory and M-fields. Another instance of the guiding principle is encountered in the analogy between Diophantine geometry and Nevanlinna theory. This analogy was first observed by Osgood [Osg81] and further explored by Vojta in [Voj87]. Roughly speaking, Nevanlinna theory can be seen as a generalisation of the fundamental theorem of algebra to entire functions. It builds on two fundamental theorems. Through the analogy, the first one corresponds to a suitable construction of a height function. The second one is seen as an analogue of Roth's theorem [Rot55].

In [Gub97], Gubler introduced the notion of *M-fields*, with the idea of including Nevanlinna theory in an Arakelov theoretic framework. More precisely, an *M-field K* is the data of a field K together with a measure space (M,ν) and maps defined ν-almost everywhere

$$(v \in M) \mapsto |a|_v \in \mathbb{R}_+$$

for all $a \in K$ satisfying

(i) $|a+b|_v \leq |a|_v + |b|_v$ ν-ae,
(ii) $|a \cdot b|_v = |a|_v \cdot |b|_v$ ν-ae,
(iii) $v \mapsto \log |c|_v \in L^1(M,\nu)$ and $|0|_v = 0$ ν-ae,

for all $a,b \in K$ and $c \in K^*$. This definition is notably motivated by the following example in Nevanlinna theory. Consider the field $\mathcal{M}(\mathbb{C})$ of meromorphic complex functions. Consider a real number $R > 0$ and set $M_R := \{z \in \mathbb{C} : |z|_\infty \leq R\}$ where the boundary $\{z \in \mathbb{C} : |z|_\infty = R\}$ is equipped with the Lebesgue measure and the open disc $\{z \in \mathbb{C} : |z|_\infty < R\}$ is equipped with a counting measure. For any $f \in \mathcal{M}(\mathbb{C})$, consider the map

$$(z \in M_R) \mapsto \begin{cases} |f(z)|_\infty & \text{if } |z|_\infty = R, \\ e^{-\operatorname{ord}(f,z)} & \text{if } |z|_\infty < R. \end{cases}$$

Note that the above map is well-defined everywhere except at poles of f on the circle of radius R. Then one can check that we have a M_R-field $\mathcal{M}(\mathbb{C})$.

Using M-fields, Gubler obtains the construction of a height function for fields of arithmetic nature, including the above example and thus generalising Nevanlinna's first main theorem.

Remark. Since the notions introduced above will not be further used in the following presentation, we exposed them very succinctly. For the interested reader, a more detailed exposition can be found in the introduction of [Séd24].

Goal. Let us list several objectives, questions and issues that led to the present paper.

(1) Broadly speaking, the goal is to obtain a framework in the spirit of adelic curves allowing to both include Nevanlinna theory and study possibly uncountable fields of arithmetic interest.
(2) In the case of M-fields, it seems to be a complicated problem to obtain further results, e.g. geometry of numbers. This is notably due to the fact that the "absolute values" appearing are not well-defined everywhere.
(3) In the theory of adelic curves, the countability condition on the base field is imposed by the fact that the parameter space is a measure space and various constructions for adelic vector bundles thus force this countability assumption (cf. [CM19], §4 and §6).
(4) To address the two previous points, it is aimed to work with a topological parameter space. To this purpose, the idea is to authorise "absolute values" with singularities. These are the objects we introduce in this article.
(5) Equip our topological parameter space with extra analytic data to obtain meaningful arithmetic applications using features in a similar spirit to the theory of global Berkovich spaces initiated by [Poi07, Poi10, Poi13, LP24].

This article focuses on bullet (4) above and its content is to be seen as the local aspect of my PhD thesis [Séd24]. The content of §8 and 10 will initiate the implementation of bullet (5). The global aspects, namely bullets (1)-(3) (cf. Chapters II-IV in *loc. cit.*) will be introduced in subsequent work. Since the tools we introduce next have an intrinsic interest, we decided to publish them in an independent paper.

Before moving on to the precise content of this paper, let us elaborate a bit more about the idea of working with a topological parameter space instead of a measure space. First, one can remark that the parameter spaces appearing in examples of adelic curves in [CM19] come from a topological space. Second, using (sufficiently nice) topological space allows the use of topological notions, e.g. neighbourhoods, convergence, (equi)continuity... Concerning the latter notion, it is worth mentioning the works [Voj21, DZ25]. In these articles, the authors address a generalisation of Roth theorem over a more general base than a number field. In the first mentioned article, Vojta works over the aforementioned adelic structure given by a polarised arithmetic variety. In the second paper, Dolce and Zucconi yield a class of adelic curves over which the argument of Vojta can be extended. The main technical difficulty encountered is to obtain an "equicontinuity property" relative to the adelic curve. Although the authors work over a measure parameter space, such a property suggests some underlying topological features of the parameter space. Although the current development of our topological approach does not allow the analogue of such results, we hope that the foundations presented in this paper will be helpful for the understanding of Diophantine problems over more general base fields.

Content of the article. In this article, we introduce the notion of pseudo-absolute value. Let K be a field. A *pseudo-absolute value* on K is a map $|\cdot| : K \to [0,+\infty]$ satisfying

(i) $|1| = 1$ and $|0| = 0$;
(ii) for all $a, b \in K$, $|a+b| \leq |a| + |b|$;
(iii) for all $a, b \in K$ such that $\{|a|, |b|\} \neq \{0, +\infty\}$, $|ab| = |a||b|$.

This notion includes "pathological" absolute values that one encounters in the context of Diophantine geometry or Nevanlinna theory, e.g. maps of the form $(f \in \mathbb{Q}(T)) \mapsto |f(0)|_\infty \in [0,+\infty]$ or $(f \in \mathcal{M}(\mathbb{C})) \mapsto |f(z)|_\infty$, for $z \in \mathbb{C}$. Moreover, if $|\cdot|$ is a pseudo-absolute value on a field K, then $A_{|\cdot|} := \{a \in K : |a| < +\infty\}$ is a valuation ring of K with maximal ideal is $\mathfrak{m}_{|\cdot|} := \{a \in A : |a| = 0\}$ and $|\cdot|$ induces a usual absolute value on the residue field $A_{|\cdot|}/\mathfrak{m}_{|\cdot|}$. In other words, pseudo-absolute values are the composition of a general valuation with a (possibly Archimedean) absolute value.

The first appearance of pseudo-absolute values in the literature seems to be in ([Wei51], §9). What we call pseudo-absolute values are called *absolute values*. In this seminal work, he describes pseudo-absolute values as the composition of a valuation and a residue absolute value.

In [Tem11], Temkin studies Zariski-Riemann spaces in a relative context. The notion of *semi-valuation* or *Manis valuation* is central in the valuative interpretation of these Zariski-Riemann spaces. Pseudo-absolute values can be seen as variants of semi-valuations where the residue valuation is allowed to be Archimedean. In this article, we only limit ourselves to the classical case where the algebraic morphism is just the inclusion of the generic point of an integral projective scheme. Nonetheless, more general versions should be studied using this point of view.

More recently, this notion has appeared in the context of Ben Yaacov-Hrushovski's framework of globally valued fields. They obtain several related facts to those proved in that article (cf. the independent work [BYDHS24] by Ben Yaacov-Destic-Hrushovski-Szachniewicz). More details will be given in the following of this introduction.

Using this interpretation, we obtain results on the behaviour of pseudo-absolute values with respect to extensions of the base field as well as some "Galois theoretic" results, namely, we describe the action of the Galois group on sets of extensions of pseudo-absolute values (§3-4). It is also possible to complete a field equipped with a pseudo-absolute value (§5). Note that in general, the completion is not canonical.

After that, we introduce the analogue of a normed vector space in the case where the base field is equipped with a pseudo-absolute value (§6). The generalisation of a norm on a vector space is called a *pseudo-norm* and the latter can be described as the data of a free module over the underlying valuation ring of the pseudo-absolute value together with a usual norm on the extension of scalars of this free module to the completion of the residue field. In this context, it is possible to generalise the usual algebraic constructions that are performed over normed vector spaces (e.g. subspaces, quotients, duals, tensor products, exterior products, direct sums). This step is necessary in view of developing slope theory in the global context (cf. [Séd24], Chapter 3).

The core of this article consists of exploring the analytic geometry of spaces of pseudo-absolute values. First, note that pseudo-absolute values are intrinsically related to multiplicative semi-norm on a ring: indeed, if $|\cdot|$ is a pseudo-absolute value on a field K, then $|\cdot|$ induces a multiplicative semi-norm on the valuation ring $A_{|\cdot|} = \{a \in K : |a| < +\infty\}$. It is possible to equip the space of pseudo-absolute values with a topology similar to that of the Berkovich spectrum of a Banach ring [Ber90]. It turns out that the set of all pseudo-absolute values on a field K behaves as the analytic spectrum of the Zariski-Riemann space of K, namely the set of valuation rings of K, as suggested by the following result.

Theorem A. *Let K be a field with prime subring k. Denote by M_K the set of all pseudo-absolute values on K. Then the topological space M_K is non-empty, compact and Hausdorff. Moreover, we have a specification map $j : M_K \to \mathrm{ZR}(K/k)$ which is continuous and open, where $\mathrm{ZR}(K/k)$ denotes the Zariski-Riemann space of K/k equipped with the Zariski topology.*

This will be proved in Theorem 7.1.2, except for the openness that is proven in Corollary 10.2.4. This shows that the space M_K is a choice of compactification of the set of usual absolute values over K and plays the role of an analytic spectrum, the corresponding algebraic space being the Zariski-Riemann space of K.

In §8-10, we precise the idea of this "analytic Zariski-Riemann space" interpretation of spaces of pseudo-absolute values.

In §8, we work locally, namely over a fixed pseudo-absolute value. Let v be a pseudo-absolute value on a field K determining a valuation ring A and residue field κ which is equipped with an absolute value. We denote by $\widehat{\kappa}$ the completion of κ w.r.t. this absolute value. Let K'/K be a field extension. By a *projective sub-model* of K'/A, we mean an integral projective A-scheme whose function field embeds in K'. We can attach to such data a *model local analytic space*, defined as the Berkovich analytification of the extension of scalars to $\widehat{\kappa}$ of the special fibre of $\mathcal{X}$, namely $(\mathcal{X} \otimes_A \widehat{\kappa})^{\mathrm{an}}$. We then have the following result.

Theorem B (Theorem 8.2.4 and Corollary 8.2.6)**.** *Let v be a pseudo-absolute value on K defining a valuation ring A with residue field κ. Assume that A is universally Japanese (cf. Example 1.4.2). Let K'/K be a field extension. Let $M_{K',v}$ denote the set of all pseudo-absolute values on K' extending v.*

- *We have a continuous map*
$$M_{K',v} \to \mathrm{ZR}(K'/A).$$
- *We have a commutative diagram of topological spaces*

$$\begin{tikzcd} M_{K',v} \arrow[r, "\cong"] \arrow[d] & \varprojlim_{\mathcal{X}} (\mathcal{X} \otimes_A \widehat{\kappa})^{\mathrm{an}} \arrow[d] \\ \mathrm{ZR}(K'/A) \arrow[r, "\cong"] & \varprojlim_{\mathcal{X}} \mathcal{X} \end{tikzcd},$$

where $\mathcal{X}$ runs over all projective sub-models of K'/A.

- *Assume that v is an absolute value and that K is countable. Then the subset of absolute values is dense in $M_{K',v}$.*

This result can be seen as an analytic counterpart to the usual algebraic description of Zariski-Riemann spaces, namely the bottom homeomorphism in the above diagram. It turns out that in the case where v is a usual absolute value, Theorem B can be viewed as a birational counterpart to ([Got24], Theorem 1.2).

To give a global counterpart to Theorem B. Similarly to the local case, we first introduce model analytic spaces. To do so, we have to specify an integral model of the base field, which restricts the choice of relevant pseudo-absolute values. This leads to the notion of integral structure (§9). Namely, an *integral structure* for a field K is a Prüfer Banach ring $(A, \|\cdot\|)$ with fraction field K. It turns out that the Berkovich spectrum $\mathcal{M}(A, \|\cdot\|_A)$ can be interpreted as a closed subset of M_K. This notion allows in some cases to describe explicitly some parts of spaces of pseudo-absolute values. The appearance of Prüfer domains in the theory is not surprising since they play an important role in the theory of Zariski-Riemann spaces. Namely they characterise the so-called *affine subsets* of Zariski-Riemann spaces (cf. e.g. [Olb21]).

Note that, in general, spaces of pseud-absolute values are difficult to fully describe. Integral structures can be easier to explicit and we obtain a new example of a Banach ring for which we can explicit the Berkovich spectrum (topologically and sheaf-theoretically) over which we can perform Berkovich analytic geometry (§9.4.3).

Using these integral structures, we define *model global analytic spaces* using the theory of global Berkovich spaces introduced in [LP24] (§10). These spaces will be used notably in the implementation of Nevanlinna theory in the global construction. We can now state the global analogue of Theorem B.

Theorem C (Theorem 10.2.3 and Corollary 10.2.4)**.** *Let K'/K be a field extension. Let $(A, \|\cdot\|_A)$ be an integral structure for K such that $(A, \|\cdot\|)$ is a geometric base ring (*[LP24]*, Définition 3.3.8), e.g. $(\mathbb{Z}, |\cdot|)$ or $\mathbb{F}_p$. Let V'_A denote the set of all pseudo-absolute values on K' restricting to an element of $\mathcal{M}(A, \|\cdot\|_A)$.*

- *We have a continuous map*

$$V'_A \to \mathrm{ZR}(K'/A).$$

If $(A, \|\cdot\|)$ is further assumed to be Dedekind analytic ([LP24]*, Définition 6.6.1), then the map is open.*

- *We have a commutative diagram*

$$\begin{tikzcd} V'_A \arrow[r, "\cong"] \arrow[d] & \varprojlim_{\mathcal{X}} \mathcal{X}^{\mathrm{an}} \arrow[d] \\ \mathrm{ZR}(K'/A) \arrow[r, "\cong"] & \varprojlim_{\mathcal{X}} \mathcal{X} \end{tikzcd},$$

where $\mathcal{X}$ runs over all projective, finitely presented flat sub-models of K'/A.

- *Assume that K is countable. Then the set of absolute values is dense in M_K.*

In Corollary 10.2.4, we give a list of consequences of Theorem C that shed light on the topological structure of spaces of pseudo-absolute values according to the context. In particular, we obtain a description of the whole space of pseudo-absolute values over a field as well as a description of the Archimedean part of the latter.

Future developments and relation with globally valued fields. We conclude the introduction by making more precise the link between the results presented here and those in [BYDHS24] and hinting at future developments. We have already mentioned that the authors of *loc. cit.* also work with pseudo-absolute values. In particular, they obtain Theorem A and (*loc. cit.*, Propositions 1.5 and 1.6) is a special case of Theorem C. Although our proofs are essentially the same, in this article, we aimed to give a systematic description of the "Zariski-Riemann space" structure of spaces pseudo-absolute values over an arbitrary base. This description will be used in the sequel of this paper (cf. [Séd25] or [Séd24]) and in further developments where we will study the sheaf theoretic properties of spaces of pseudo-absolute values.

In the aforementioned sequel, we will introduce *topological adelic curves*, which addresses bullet (4) of this introduction. Note that in ([BYDHS24], Theorem 7.7), the authors prove that over a countable field, GVFs and proper adelic curves are essentially equivalent, namely that a GVF structure determines a measure on the space of absolute values yielding a proper adelic curve structure. If the countability assumption is removed, their result says that a GVF structure determines a measure on the space of pseudo-absolute values yielding a proper topological adelic curve structure. The goal of developing tools in the topological adelic curve world that could also be used in the GVF world is a motivation for developing the systematic study of pseudo-absolute values. In the countable case, let us mention the recent paper [DHS24] which allows to hope for fruitful applications of such ideas.

Finally, let us mention that the development of Arakelov geometric methods over an uncountable field allows to make use of the ultraproduct construction via the GVF-topological adelic curves correspondence. Such ultraproducts could be used to define GVF structures that could formalise Diophantine approximation and Nevanlinna theory similarly ([BYDHS24], Example 11.12). In subsequent work, we will make this idea more precise and give applications in Nevanlinna theory.

Acknowledgements. We would like to thank Huayi Chen for his support and discussions during the elaboration of this paper. We also thank Keita Goto, Walter Gubler, Klaus Künnemann and Jérôme Poineau for numerous remarks and suggestions. We thank Debam Biswas for several helpful discussions. We thank Michał Szachniewicz for pointing out [Wei51] and that variants of Theorems A and C were independently obtained in [BYDHS24]. Finally, we warmly thank the anonymous referee whose questions, remarks and comments improved the exposition of the material presented here.

Conventions and notation

- All rings considered in this article are commutative with unit.
- Let A be a ring. We denote by $\mathrm{Spm}(A)$ the set of maximal ideals of A.
- By a local ring $(A, \mathfrak{m})$, we mean that A is a local ring and $\mathfrak{m}$ is its maximal ideal. In general, if A is a local ring, the maximal ideal of A is denoted by $\mathfrak{m}_A$.
- By an *algebraic function field* K/k, we mean a finitely generated field extension K/k.
- Let A be a ring and let $X \to \mathrm{Spec}(A)$ be a scheme over A. Let $A \to B$ be an A-algebra. Then we denote $X \otimes_A B := X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$.
- Let k be a field. We denote by $|\cdot|_{\mathrm{triv}}$ the trivial absolute value on k. If we have an embedding $k \hookrightarrow \mathbb{C}$, we denote by $|\cdot|_\infty$ the restriction of the usual Archimedean absolute value on $\mathbb{C}$.
- Let $(k, |\cdot|)$ be a valued field. Unless mentioned otherwise and when no confusion may arise, we will denote by $\widehat{k}$ the completion of k w.r.t. $|\cdot|$.
- Throughout this article, unless specified otherwise, all valuations are considered up to equivalence.
- We assume the *axiom of universes*, which will allow taking inverse and direct limits over collections.

1. Preliminaries

1.1. **Valuation rings and Prüfer domains.**

1.1.1. *Rank, rational rank, composite valuations and Gauss valuations.* Let V be a valuation ring. We denote its *value group* by $\Gamma_V := \mathrm{Frac}(V)^\times/V^\times$. Let V be a valuation ring with fraction field K whose underlying valuation is denoted by $v : K \to \Gamma_V \cup \{\infty\}$.

- The *rank* of V is defined as the ordinal type of the totally ordered set of prime ideals in V and is denoted by $\mathrm{rank}(V)$.
- The *rational rank* of V is defined by $\mathrm{rat.rank}(V) := \dim_{\mathbb{Q}}(\Gamma_V \otimes_{\mathbb{Z}} \mathbb{Q}) \in \mathbb{N} \cup \{\infty\}$. In general we have $\mathrm{rank}(V) \leq \mathrm{rat.rank}(V)$ ([Bou75], Chap. VI, §10.2, Proposition 3).

Let us start by recalling the construction of composite valuations, as it is reminiscent of the main object of this article.

Construction 1.1.1. Let V be a valuation ring with fraction field K, value group Γ maximal ideal $\mathfrak{m}$ and associated valuation $v : K \to \Gamma \cup \{\infty\}$. Let $\overline{v}$ be a valuation on the residue field $\kappa := V/\mathfrak{m}$. Then ([Vaq06], Proposition 1.12) implies that

$$V' := \{a \in V : \overline{v}(\overline{a}) \geq 0\}$$

is a valuation ring of K included in V. The valuation attached to V' is denoted by $v' := v \circ \overline{v}$. In that case, the residue field of V' equals the residue field of $\overline{v}$. Moreover, we have the equalities

$$\begin{aligned} \mathrm{rank}(v') &= \mathrm{rank}(v) + \mathrm{rank}(\overline{v}), \\ \mathrm{rat.rank}(v') &= \mathrm{rat.rank}(v) + \mathrm{rat.rank}(\overline{v}), \end{aligned}$$

as well as a short exact sequence of Abelian groups

$$0 \longrightarrow \overline{\Gamma} \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow 0,$$

where $\overline{\Gamma}, \Gamma'$ denote respectively the value groups of $\overline{v}, v'$. In the particular case where Γ is order isomorphic to $\mathbb{Z}^n$ for some integer $n \in \mathbb{N}_{>0}$ (in that case $\mathrm{rank}(v) = n$), then the above short exact sequence splits and Γ' is isomorphic to $\Gamma \times \overline{\Gamma}$ equipped with the lexicographic order.

Conversely, let $(V', \mathfrak{m}'), (V, \mathfrak{m})$ be valuation rings of K whose valuations, resp. residue fields, are denoted respectively by v', v and κ, κ'. Then $\mathfrak{m} \cap V'$ is a prime ideal of V' and the quotient ring $\overline{V'} := V'/(\mathfrak{m} \cap V')$ is a valuation ring of κ. Denote by $\overline{v}$ the corresponding valuation. Then we have the equality $v' = v \circ \overline{v}$.

Definition 1.1.2. With the notation of Construction 1.1.1, the valuation $v' := v \circ \overline{v}$ called the *composite valuation* with v and $\overline{v}$.

Definition 1.1.3 (Gauss valuations)**.** Let K be a field and x be transcendental over K. Let v be a valuation of K with value group Γ_1 and residue field κ. Fix an extension of totally ordered Abelian groups $\iota : \Gamma_1 \hookrightarrow \Gamma_2$. Let $a \in K$ and $\gamma \in \Gamma_2$. Let $P \in K[x]$ of degree n. Write

$$P = \sum_{i=0}^{n} c_i (x-a)^i,$$

where $c_0, ..., c_n \in K$, and set

$$v_{a,\gamma}(P) := \min_{0 \leq i \leq n} \{v(c_i) + i\gamma\}.$$

For any $P/Q \in K(X)$, where $P, Q \in K[x]$ with $Q \neq 0$, set $v_{a,\gamma}(P/Q) = v_{a,\gamma}(P) - v_{a,\gamma}(Q)$. Then $v_{a,\gamma} : K(X) \to \Gamma_1 + \mathbb{Z}\gamma$ defines a valuation of $K(X)$ extending v. From ([Kuh04], Lemma 3.10), if γ is non-torsion modulo Γ_2, then $v_{a,\gamma}$ has value group $\Gamma_1 + \mathbb{Z}\gamma$ and residue field κ. Otherwise, $v_{a,\gamma}$ has value group $\Gamma_1 + \mathbb{Z}\gamma$ and its residue field is a purely transcendental extension of transcendence degree 1 of κ. Note that in both cases, $v_{a,\gamma}$ is Abhyankar.

1.1.2. *Connection to algebraic geometry : specialisation on a scheme.* Valuation rings are of particular importance in algebraic geometry. An illustration of this fact is the following result.

Proposition 1.1.4 ([Sta23], Lemma 00PH)**.** *Let R be a Noetherian local domain which is not a field. Let $K := \mathrm{Frac}(R)$. Let L/K be a finitely generated extension. Then there exists a DVR V with fraction field L which dominates R.*

Proposition 1.1.5. *Let X be a locally Noetherian scheme and let $x, x' \in X$ be such that x is a specialisation of x', namely $x \in \overline{\{x'\}}$. Then there exist a discrete valuation ring V and a morphism $\mathrm{Spec}(V) \to X$ such that the generic point of $\mathrm{Spec}(V)$ is mapped to x' and the closed point of $\mathrm{Spec}(V)$ is mapped to x. Moreover, for any finitely generated extension $K/\kappa(x')$ we may choose V such that the extension $\mathrm{Frac}(V)/\kappa(x')$ is isomorphic to the given extension $K/\kappa(x')$.*

Proof. Let $K/\kappa(x')$ be a finitely generated extension, this induces ring morphisms $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Now Proposition 1.1.4 yields the existence of a DVR V with fraction field K which dominates $\mathcal{O}_{X,x}$. Therefore, the morphism $\mathcal{O}_{X,x} \to V$ yields the desired scheme morphism $\mathrm{Spec}(V) \to X$. □

Roughly speaking, Proposition 1.1.5 implies the fact that any specialisation of a locally Noetherian scheme can be encoded through a discrete valuation ring. In general, we have no information on the extension of residue fields of the special points. The following result gives a partial answer in this direction.

Proposition 1.1.6. *Let X be a locally Noetherian integral scheme and let $x \in X$ be a regular point. Then there exists a valuation ring V dominating $\mathcal{O}_{X,x}$ with residue field κ such that* $\mathrm{Frac}(V) = K(X)$ *and* $\kappa = \kappa(x)$.

Proof. Since x is regular, there exist $a_1, ..., a_r \in \mathcal{O}_{X,x}$ whose images in $\mathfrak{m}_x/\mathfrak{m}_x^2$ are linearly independent over $\kappa(x)$ such that the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ is $(a_1, ..., a_r)$. Moreover, for any $i = 1, ..., r$, the image of a_i is an irreducible element of the regular ring $\mathcal{O}_{X,x}/(a_1, ..., a_{i-1})$ which is a UFD, and thus is a prime element of $\mathcal{O}_{X,x}/(a_1, ..., a_{i-1})$. This yields a prime divisor $D_i \subset \mathrm{Spec}(\mathcal{O}_{X,x}/(a_1, ..., a_{i-1}))$ and thus a discrete valuation v_i of the field $\mathrm{Frac}(\mathcal{O}_{X,x}/(a_1, ..., a_{i-1}))$. We now define a valuation $v : \mathrm{Frac}(\mathcal{O}_{X,x}) \cong K(X) \to \mathbb{Z}^r_{\mathrm{lex}}$ by sending any $f \in \mathcal{O}_{X,x}$ to $(v_i(f \mod (a_1, ..., a_{i-1})))_{1 \leq i \leq r}$. Then the valuation ring V of v is a rank r valuation ring of $K(X)$ dominating $\mathcal{O}_{X,x}$ with residue field $\kappa(x)$. □

Proposition 1.1.7. *Let X be a locally Noetherian integral scheme over a field K. Assume that there exists a proper birational morphism $\pi : X' \to X$ of K-schemes such that X' is smooth. Then, for any $x \in X$, there exists a valuation ring V dominating $\mathcal{O}_{X,x}$ with residue field κ such that* $\mathrm{Frac}(R) = K(X)$ *and $\kappa/\kappa(x)$ is finite.*

Proof. We may assume that $x \in X$ is a closed point. Then, for any $x' \in \pi^{-1}(X')$, x' is a closed regular point of X' and Proposition 1.1.6 yields a valuation ring $V \subset K(X') \cong K(X)$ dominating $\mathcal{O}_{X',x'}$ with residue field $\kappa(x')$. Since the extension $\kappa(x')/\kappa(x)$ is finite, we obtain the desired property for V. □

Remark 1.1.8. In particular, Proposition 1.1.7 holds when the base field is of characteristic zero (cf. [Hir64]).

1.1.3. *Prüfer domains.* Let A be an integral domain with fraction field K. A is said to be a *Prüfer domain* if, for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, the localisation $A_{\mathfrak{p}}$ is a valuation ring. There are many characterisations of Prüfer domains (see e.g. [FHP97], Theorem 1.1.1).

Proposition 1.1.9. *Let A be a Prüfer domain with fraction field K.*

(1) *Let V be a valuation ring of K containing A, and denote by $\mathfrak{m}$ the maximal ideal. Then $\mathfrak{m} \cap A$ is a prime ideal of A and $V = A_{\mathfrak{m} \cap A}$.*
(2) *Let L/K be an algebraic extension. Then the integral closure of A in L is a Prüfer domain (*[FS01]*, Chap. III, Theorem 1.2).*
(3) *An A-module is flat if and only if it is torsion-free (*[Bou75]*, Chapitre VII, §2, Exercices 12 et 14).*
(4) *A finitely generated A-module is projective if and only if it is torsion-free (*[FS01]*, Chapter V, Theorem 2.7).*
(5) *Assume that A is Bézout. Then any projective A-module is free (*[FS01]*, Chapter VI, Theorem 1.11).*
(6) *Let $(A_i)_{i \in I}$ be direct system of Prüfer domains with injective arrows. Then $\varinjlim_{i \in I} A_i$ is a Prüfer domain (*[FS01]*, Proposition 1.8).*

Proof. By definition, $A_{\mathfrak{m}\cap A} = \{a/b : a, b \in A \text{ and } b \notin \mathfrak{m} \cap A\}$. Thus $A_{\mathfrak{m}\cap A} \subset V_{\mathfrak{m}} = V$. Since $(\mathfrak{m} \cap A) \subset \mathfrak{m}$, the inclusion $A_{\mathfrak{m}\cap A} \to V$ is a local morphism of local rings whose fraction fields are K. From the fact that $A_{\mathfrak{m}\cap A}$ is a valuation ring, we get $A_{\mathfrak{m}\cap A} = V$. □

Example 1.1.10. (1) Any field is a Prüfer domain.

(2) Any Dedekind ring is Prüfer. Indeed, these are exactly the Noetherian Prüfer domains since they are locally discrete valuation rings, i.e. Noetherian valuation rings. In particular, the ring of integers of a number field is a Prüfer domain, and so is its absolute integral closure (cf. Proposition 1.1.9 (2)).

(3) Let X be a (connected) Riemann surface. Then the ring $\mathcal{O}(X)$ of holomorphic functions on X is a Prüfer domain (it is a Dedekind domain if X is compact and the non-compact case follows from [Roy56], Proposition 1). Moreover, it is a Bézout domain.

(4) Let $C \subset U$ be a connected Stein subset of a connected non-compact Riemann surface U, namely C has a basis of Stein open subset neighbourhoods in U. Denote $A = \mathcal{O}(C)$ the ring of germs of holomorphic functions on C. It is an integral ring whose fraction field $K := \mathcal{M}(C)$ is the field of germs of meromorphic functions on C. Then A is Prüfer. Indeed, for any Stein open set $C \subset U' \subset U$, $\mathcal{O}(U')$ is a Prüfer domain and $A = \varinjlim \mathcal{O}(U')$, where U' runs over the open sets U' such that $C \subset U' \subset U$. Hence Proposition 1.1.9 (6) implies that A is Prüfer.

1.2. **Application to rings of holomorphic functions.** In this subsection, we recall useful algebraic properties of rings of holomorphic functions that are studied throughout this article. As we will be interested in non-compact Riemann surfaces and Stein subsets of such, we limit the exposition to the latter.

Proposition 1.2.1. *Let X be a non-compact Riemann surface.*

(1) *Any finitely generated ideal of $A := \mathcal{O}(X)$ is principal. Furthermore, any such ideal $I \subset A$ is prime iff it is maximal iff any generator of I has exactly one zero, namely I is the ideal of analytic functions vanishing at a point of X.*

(2) *If $\mathfrak{m}$ is a maximal ideal of A, then $\mathfrak{m}$ is principal iff $\mathfrak{m}$ is the kernel of a $\mathbb{C}$-algebra morphism $\pi : A \to \mathbb{C}$.*

Proof. Let $I = (f_1, ..., f_n)$ be a finitely generated ideal of A. Denote $d := \gcd(f_1, ..., f_n) \in A$. Then there exist $e_1, ..., e_n \in A$ such that $d = e_1 f_1 + \cdots + e_n f_n$ (cf. [Roy56], Proposition 1). Therefore $I = (d)$, i.e. I is principal. Then (1) is exactly Proposition 2 of [Roy56]. (2) is Proposition 3 of [Roy56]. □

Proposition 1.2.2. *Let X be a non-compact Riemann surface. Denote by A the ring of holomorphic functions and by K the fraction field of A. Let $|\cdot|$ be an absolute value on K such that $A \subset \{f \in K : |f| \leq 1\}$. Then $|\cdot|$ is either trivial or there exists $z \in X$ such that $|\cdot|$ is equivalent to an absolute value of the form $e^{-\operatorname{ord}(\cdot,z)}$.*

Proof. Let $|\cdot|$ be such an absolute value, it is necessarily non-Archimedean. Denote by V the valuation ring of V and by $\mathfrak{m}$ the maximal ideal. The hypothesis ensures that $A \subset V$. Then $\mathfrak{p} := \mathfrak{m} \cap A \in \operatorname{Spec}(A)$ and $A_{\mathfrak{p}}$ is a valuation ring of K and the injection $A_{\mathfrak{p}} \to V$ is local. Therefore, we have $A_{\mathfrak{p}} = V$.

Now assume that $\mathfrak{p}$ does not contain any $\mathfrak{m}_z$, for $z \in X$, and is not (0). Let us show that any element f in $\mathfrak{p}$ has an infinite number of zeros. Assume that f has a finite number of zeros $z_1, ..., z_n$. By hypothesis on $\mathfrak{p}$, there exist $f_1, ..., f_n \in \mathfrak{p}$ such that, for all $i = 1, ..., n$, $f_i \notin \mathfrak{m}_{z_i}$. Thus $\gcd(f, f_1, ..., f_n) = 1 \in \mathfrak{p}$, which gives a contradiction. Let $N = \{f \in A : f \text{ has a finite number of zeros}\}$, it is a multiplicative subset of A. From ([Gil72], §13, Exercise 21), we get that the localisation A_N is a ring whose complete integral closure is K. By the above remark, we have an inclusion $A_N \subset A_{\mathfrak{p}}$. Let S denote the complete integral closure of $A_{\mathfrak{p}}$. Then $K \subset S \subset K$ and thus $A_{\mathfrak{p}}$ is not completely integrally closed. Then ([FS01], Chapter II, Exercise 1.12) implies that $A_{\mathfrak{p}}$ is a valuation of rank greater than 1. This contradicts the fact that $A_{\mathfrak{p}}$ is the valuation ring of an absolute value on K.

Therefore either $\mathfrak{p}$ is (0) or there exists $z \in X$ such that $\mathfrak{m}_z \subset \mathfrak{p}$. In the first case, we get $V = A_{\mathfrak{p}} = K$ and $|\cdot|$ is trivial. In the second case, let $z \in X$ such that $\mathfrak{m}_z \subset \mathfrak{p}$. Then we have an inclusion $A_{\mathfrak{m}_z} \subset A_{\mathfrak{p}}$ of rank 1 valuation rings with fraction field K. Therefore $A_{\mathfrak{p}} = A_{\mathfrak{m}_z}$ and $|\cdot|$ is equivalent to the absolute value $e^{-\operatorname{ord}(\cdot, z)}$. □

Remark 1.2.3. Propositions 1.2.1 and 1.2.2 hold in the case of a compact Riemann surface due to the Noetherianity of its ring of holomorphic functions and the description of absolute values on a transcendence degree one extension of the trivial absolute value.

We now describe prime ideals of $A = \mathcal{O}(\mathbb{C})$, the ring of entire functions on $\mathbb{C}$. For any entire function $f \in A$, denote by $Z(f)$ the set of zeroes of f. For any ideal $I \subset A$, if $\bigcap_{f \in I} Z(f) \neq \varnothing$, I is called *fixed*. Otherwise, the ideal I is called *free*.

Proposition 1.2.4. *Let $\mathfrak{p}$ be a prime ideal of A.*

(1) *If $\mathfrak{p}$ is fixed, it is maximal and of the form $\mathfrak{m}_z := \{f \in A : f(z) = 0\}$ for some $z \in \mathbb{C}$.*
(2) *If $\mathfrak{p}$ is a free maximal ideal, then $A_{\mathfrak{p}}$ is a valuation ring of rank at least $2^{\aleph_1}$.*
(3) *If $\mathfrak{p}$ is free, it is contained in a unique (free) maximal ideal $\mathfrak{m}$ and*

$$\mathfrak{p}^* := \bigcap_{n>0} \mathfrak{m}^k$$

is the largest non-maximal prime ideal contained in $\mathfrak{m}$. Moreover, for any $f \in \mathfrak{p}$, f has an infinite number of zeroes.
(4) *If $\mathfrak{p}$ is free and $\mathfrak{m}$ is the unique maximal ideal containing $\mathfrak{p}$, then $A/\mathfrak{p}$ is a valuation ring whose maximal ideal $\mathfrak{m}/\mathfrak{p}$ is principal.*
(5) *Assume that $\mathfrak{p}$ is free. Then $A/\mathfrak{p}$ is a complete DVR iff $\mathfrak{p} = \mathfrak{p}^*$.*

Proof. (1), (2) and (3) follow from ([Hen52], §3, Theorems 1-5). (4) is (*loc. cit.*, Theorem 6) and (5) is (*loc. cit.*, Theorems 7 and 8). □

Remark 1.2.5. More generally, by looking at the proof of Proposition 1.2.4, one can prove that the same conclusions as the above proposition hold by replacing A by the ring of global analytic functions on a non-compact Riemann surface.

Let X be a complex analytic space. Denote by $\mathcal{O}_X$ its structure sheaf. Let $A \subset X$ be any subset. Then the space of germs of analytic functions on A is

$$\Gamma(A, \mathcal{O}_X) := \varinjlim_{U \supseteq A} \Gamma(U, \mathcal{O}_X),$$

where U runs over the open subset of X containing A. Conditions on A to study algebraic properties of $\Gamma(A, \mathcal{O}_X)$ can be found in [Fri67, All68, Siu69, Dal74].

Proposition 1.2.6. *Let $R > 0$ and let $\overline{D(R)}$ denote the closed disc of radius R in $\mathbb{C}$. Then the ring $\mathcal{O}(\overline{D(R)})$ of germs of holomorphic functions on $\overline{D(R)}$ is a principal ideal domain (hence a Dedekind domain).*

Proof. The fact that $\mathcal{O}(\overline{D(R)})$ is Noetherian is a consequence of ([Fri67], Théorème I.9), see also ([Siu69], Theorem 1). $\mathcal{O}(\overline{D(R)})$ is a unique factorisation domain by ([Dal74], Corollary to Theorem 1). Now Example 1.1.10 (4) implies that $\mathcal{O}(\overline{D(R)})$ is a Noetherian Prüfer domain, hence is Dedekind. Moreover, a Dedekind unique factorisation domain is a principal ideal domain. □

Proposition 1.2.7. *Let $R > 0$ and let $\overline{D(R)}$ denote the closed disc of radius R in $\mathbb{C}$. Then the maximal ideals of the ring $\mathcal{O}(\overline{D(R)})$ of germs of holomorphic functions on $\overline{D(R)}$ are of the form $\mathfrak{m}_z := \{f \in \mathcal{O}(\overline{D(R)}) : f(z) = 0\}$, for some $z \in \overline{D(R)}$.*

Proof. For any $R' > R$, let $D(R')$ denote the open disc of radius R' and $A_{R'} := \mathcal{O}(D(R'))$ the ring of holomorphic functions on $D(R')$. Then we have an isomorphism

$$\mathcal{O}(\overline{D(R)}) \cong \varinjlim_{R'>R} A_{R'},$$

where, for any $R < R'' < R'$, we consider the inclusion $A_{R'} \subset A_{R''}$. It follows that we have an isomorphism of schemes

$$\mathrm{Spec}(\mathcal{O}(\overline{D(R)})) \cong \varprojlim_{R'>R} \mathrm{Spec}(A_{R'}).$$

Let $\mathfrak{p} = (\mathfrak{p}_{R'})_{R'>R} \in \mathrm{Spec}(\mathcal{O}(\overline{D(R)}))$ be a non-zero prime ideal, namely, for any $R' > R$, $\mathfrak{p}_{R'} \in \mathrm{Spec}(A_{R'})$ and, for any $R < R'' < R'$, $\mathfrak{p}_{R''} \cap A_{R'} = \mathfrak{p}_{R'}$. Let us prove that, for any $R < R'$, $\mathfrak{p}_{R'}$ is fixed. Assume that $\mathfrak{p}_{R'}$ is free and non-zero for some $R < R'$. Let $f \in \mathfrak{p}_{R'} \smallsetminus \{0\}$. Then Proposition 1.2.4 (3) together with Remark 1.2.5 imply that f has an infinite number of zeroes written as a sequence $(a_n)_{n\geq 0}$. Discreteness of $(a_n)_{n\geq 0}$ yields $|a_n| \to_{n\to+\infty}= R'$. Now for any $R < R'' < R'$, $\mathfrak{p}_{R''}$ is free. Thus the restriction of f to $A_{R''}$ yields a non-zero element of $\mathfrak{p}_{R''}$, which has an infinite number of zeroes in $D(R'')$ thus accumulating at the boundary of $D(R'')$ which is included in the interior of $D(R')$. Hence we get a contradiction. Therefore, for any $R < R'$, $\mathfrak{p}_{R'}$ is fixed and corresponds to some $\mathfrak{m}_{z_{R'}} := \{f \in A_{R'} : f(z_{R'}) = 0\}$ for some $z_{R'} \in D(R')$. Since $(\mathfrak{p}_{R'})_{R'>R}$ is a projective system, we obtain that there exits $z \in \overline{D(R)}$ such that, for any $R < R'$, we have $z_{R'} = z$. Conversely, for any $z \in \overline{D(R)}$, $\mathfrak{m}_z$ is a maximal ideal of $\mathcal{O}(\overline{D(R)})$. □

1.3. **Models over a Prüfer domain.** Throughout this paragraph, we fix a Prüfer domain A with fraction field K.

Let $X \to \mathrm{Spec}(K)$ be a separated K-scheme of finite type. By a *model* of X over A, we mean a separated A-scheme $\mathcal{X} \to \mathrm{Spec}(A)$ of finite type such that the generic fibre of $\mathcal{X}$ is isomorphic to X. A model $\mathcal{X}$ of X over A is respectively called *projective*, *flat*, *coherent*, if the structural morphism $\mathcal{X} \to \mathrm{Spec}(A)$ is projective, flat, of finite presentation. Note that if $\mathcal{X}$ is a projective model of X, then X is projective.

If A is a valuation ring with residue field κ, we denote by $\mathcal{X}_s := \mathcal{X} \otimes_A \kappa$ the special fibre of $\mathcal{X}$. In general, for any $y \in \mathrm{Spec}(A)$, we denote $\mathcal{X}_y := \mathcal{X} \otimes_A A_y$ and by $\mathcal{X}_{y,s}$ the special fibre of $\mathcal{X}_y$.

Let L be an invertible $\mathcal{O}_X$-module. By a *model* $(\mathcal{X}, \mathcal{L})$ of (X, L) over A we mean the data of a model $\mathcal{X}$ of X over A together with an invertible $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ whose restriction to the generic fibre X isomorphic to L. A model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $\mathcal{L}$ is respectively called *flat*, *coherent*, if the corresponding model $\mathcal{X}$ of X over A is so. If A is a valuation ring, we denote by $\mathcal{L}_s$ the restriction of $\mathcal{L}$ to the special fibre $\mathcal{X}_s$. In general, for any $y \in \mathrm{Spec}(A)$, we denote by $\mathcal{L}_y$ the pullback of $\mathcal{L}$ to $\mathcal{X}_y$ and by $\mathcal{L}_{y,s}$ the restriction of $\mathcal{L}_y$ to the special fibre of $\mathcal{X}_y$.

Proposition 1.3.1. *Let L be a line bundle on a projective K-scheme X. Then there exists a model of (X, L) over A.*

Proof. First, assume that L is very ample. Denote by $\iota : X \hookrightarrow \mathbb{P}^n_K$ a corresponding closed immersion. Let $\mathcal{X}$ denote the schematic closure of X in $\mathbb{P}^n_A$. Then $\mathcal{X} \to \mathrm{Spec}(A)$ is a model of X over A and the pullback $\mathcal{L}$ of $\mathcal{O}_{\mathbb{P}^n_A}$ yields a model $(\mathcal{X}, \mathcal{L})$ of (X, L) over A.

In the general case, write $L = L_1 - L_2$ as a difference of very ample line bundles. The above case ensures the existence of a model $(\mathcal{X}_i, \mathcal{L}_i)$ of (X, L_i) for $i = 1, 2$. Let $\mathcal{X}$ denote the schematic closure of X in $\mathcal{X}_1 \times_{\mathrm{Spec}(A)} \mathcal{X}_2$, it is a model of X over A. Then set $\mathcal{L} := p_1^*\mathcal{L}_1 - p_2^*\mathcal{L}_2$, where $p_i : \mathcal{X} \to \mathcal{X}_i$ denote the i-th projection. □

Remark 1.3.2. The projectivity assumption in Proposition 1.3.1 is superfluous: one could only assume that the K-scheme X is proper. Indeed Nagata's compactification theorem furnishes a model $\mathcal{X}$ of X over A and one can extend L to a coherent sheaf on $\mathcal{X}$ which can be made flat by Raynaud-Gruson's flattening theorem ([RG71], Théorème I.5.2.2). We do not detail the proof since we only consider the projective case in this article and refer to ([GM19], §2.1) for the interested reader.

Proposition 1.3.3. *We assume that A is a valuation ring with residue field κ. Let $X \to \mathrm{Spec}(K)$ be a projective K-scheme and L be an invertible $\mathcal{O}_X$-module. Let $(\mathcal{X}, \mathcal{L})$ be a projective model of (X, L) over A.*

1. *There exists a flat projective model $(\mathcal{X}', \mathcal{L}')$ of (X, L) such that $\mathcal{L}' = \mathcal{L}_{|\mathcal{X}'}$ and the special fibres of $\mathcal{X}'$ and $\mathcal{X}$ coincide.*
2. *There exists a coherent projective model $(\mathcal{X}', \mathcal{L}')$ of (X, L) such that*
   (i) *$\mathcal{X}$ is a closed subscheme of $\mathcal{X}'$;*
   (ii) *the special fibres of $\mathcal{X}'$ and $\mathcal{X}$ coincide;*
   (iii) *$\mathcal{L}'_{\mathcal{X}} = \mathcal{L}$.*
3. *Assume that the restriction of $\mathcal{L}$ to every fibre of $\mathcal{X} \to \mathrm{Spec}(A)$ is ample. Then $\mathcal{L}$ is ample.*

Proof. Let $\mathcal{O}_{\mathcal{X},\mathrm{tors}}$ denote the torsion part of $\mathcal{O}_{\mathcal{X}}$ as an $\mathcal{O}_{\mathrm{Spec}(A)}$-module. Then the closed subscheme $\mathcal{X}'$ of $\mathcal{X}$ defined by the ideal sheaf $\mathcal{O}_{\mathcal{X},\mathrm{tors}}$ is a projective model of X with special fibre $\mathcal{X}' \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa) \cong \mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa)$. Moreover, the morphism $\mathcal{X}' \to \mathrm{Spec}(A)$ is flat since $\mathcal{O}_{\mathcal{X}'}$ is torsion free (cf. Proposition 1.1.9 (3)). By setting $\mathcal{L}'_{|\mathcal{X}'}$, we conclude the proof of (1).

(2) is ([CM21], Lemma 3.2.17), by noting that the proof does not use the fact that the rank of the valuation ring is less than 1.

Let us prove (3). Let $(\mathcal{X}', \mathcal{L}')$ be a model such that conditions (i)-(iii) of (2) hold. Since $\mathcal{X}' \to \mathrm{Spec}(A)$ is a proper and finitely presented and $\mathcal{L}'$ is ample along the fibres, ([Gro65], Corollaire (9.6.4)) gives the ampleness of $\mathcal{L}'$. Thus $\mathcal{L} = \mathcal{L}'_{\mathcal{X}}$ is ample. □

Proposition 1.3.4. *Let $X \to \mathrm{Spec}(K)$ be a projective K-scheme and L be an invertible $\mathcal{O}_X$-module. Let $(\mathcal{X}, \mathcal{L})$ be a projective model of (X, L) over A.*

(1) *We have an isomorphism $H^0(X, L) \cong H^0(\mathcal{X}, \mathcal{L}) \otimes_A K$.*
(2) *Assume that $(\mathcal{X}, \mathcal{L})$ is a flat model. Then the following hold.*
   (i) *$H^0(\mathcal{X}, \mathcal{L})$ is a flat A-module.*
   (ii) *Let $y \in \mathrm{Spec}(A)$, denote by $\mathcal{X}_y$ the fibre of $\mathcal{X} \to \mathrm{Spec}(A)$ over p and by $\mathcal{L}_y$ the restriction of $\mathcal{L}_y$. Then we have an injection $H^0(\mathcal{X}, \mathcal{L}) \otimes_A \kappa \hookrightarrow H^0(\mathcal{X}_y, \mathcal{L}_y)$.*
(3) *Assume that $(\mathcal{X}, \mathcal{L})$ is a coherent model. Then $H^0(\mathcal{X}, \mathcal{L})$ is a finitely generated A-module.*
(4) *Assume that $(\mathcal{X}, \mathcal{L})$ is a flat and coherent model. Then $H^0(\mathcal{X}, \mathcal{L})$ is a projective A-module of finite type. In particular, if A is Bézout, then $H^0(\mathcal{X}, \mathcal{L})$ is a free module of finite rank.*

Proof. **(1)** Since $A \to K$ is flat and $\mathcal{X} \to \mathrm{Spec}(A)$ is qcqs, (1) follows from the flat base change theorem ([GW10], Corollary 12.8).

(2.i) Since $\mathcal{X} \to \mathrm{Spec}(A)$ is flat, we obtain that $H^0(\mathcal{X}, \mathcal{L})$ is a torsion-free A-module. Now Proposition 1.1.9 (3) implies that $H^0(\mathcal{X}, \mathcal{L})$ is a flat A-module.

(2.ii) We first treat the case where A is a valuation ring with maximal ideal $\mathfrak{m}$ and residue field κ. As $f : \mathcal{X} \to \mathrm{Spec}(A)$ is a flat, we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow f^{\star}\mathfrak{m} \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_s \longrightarrow 0$$

and thus an injection $H^0(\mathcal{X}, \mathcal{L})/H^0(\mathcal{X}, \mathcal{L} \otimes f^*\mathfrak{m}) \hookrightarrow H^0(\mathcal{X}_s, \mathcal{L}_s)$. Now $\mathfrak{m}$ is a torsion-free A-module, hence is flat. By the projection formula ([GW23], Proposition 22.81), $H^0(\mathcal{X}, \mathcal{L} \otimes f^*\mathfrak{m}) \cong \mathfrak{m}H^0(\mathcal{X}, \mathcal{L})$. As $H^0(\mathcal{X}, \mathcal{L})$ is a flat A-module we have

$$H^0(\mathcal{X}, \mathcal{L}) \otimes_A \kappa \cong H^0(\mathcal{X}, \mathcal{L})/\mathfrak{m}H^0(\mathcal{X}, \mathcal{L}) \hookrightarrow H^0(\mathcal{X}_s, \mathcal{L}_s).$$

We finally treat the general case. Let $y \in \mathrm{Spec}(A)$ and denote by $\mathfrak{p}_y$ the corresponding prime ideal of A. Then $A_{\mathfrak{p}_y}$ is a valuation ring and $(\mathcal{X} \otimes_A A_{\mathfrak{p}_y}, \mathcal{L} \otimes_A A_{\mathfrak{p}_y})$ is a flat model of (X, L) over $A_{\mathfrak{p}_y}$ whose special fibre coincides with $\mathcal{X}_s$. As $A \to A_{\mathfrak{p}_y}$ is flat, the previous case combined with the flat base change theorem yields

$$H^0(\mathcal{X}, \mathcal{L}) \otimes_A \kappa \cong (H^0(\mathcal{X}, \mathcal{L}) \otimes_A A_{\mathfrak{p}_y}) \otimes_{A_{\mathfrak{p}_y}} \kappa \cong H^0(\mathcal{X} \otimes_A A_{\mathfrak{p}_y}, \mathcal{L} \otimes_A A_{\mathfrak{p}_y}) \otimes_{A_{\mathfrak{p}_y}} \kappa \hookrightarrow H^0(\mathcal{X}_y, \mathcal{L}_y),$$

which gives the conclusion.

(3) First mention that Prüfer domains and valuation domains are stably coherent, namely any polynomial algebra with finitely many indeterminates over a Prüfer domain is coherent (cf. [Gla89], Theorem 7.3.3 and Corollary 7.3.4). Since $\mathcal{X} \to \mathrm{Spec}(A)$ is projective and of finite presentation, ([Ull95], Theorem 3.5) implies that $H^0(\mathcal{X}, \mathcal{L})$ is a finitely generated A-module.

(4) Finally, (4) is a consequence of (2.i) and (3) together with Proposition 1.1.9 (4)-(5). □

1.4. **Zariski-Riemann spaces.** Let K be a field and let k be a subring of K (we do not necessarily assume that k is a domain with quotient field K). Define the *Zariski-Riemann* space of K/k, denoted by $\mathrm{ZR}(K/k)$ as the set of valuation rings of K containing k. The set $\mathrm{ZR}(K/k)$ is equipped with a topology. It is defined as follows. For any sub-k-algebra $A \subset K$ of finite type, let $E(A)$ denote the set of valuation rings on $\mathrm{ZR}(K/k)$ containing A. Then the sets $E(A)$, where A runs over the set of sub-k-algebra of finite type of K, form a basis for a topology on $\mathrm{ZR}(K/k)$, it is called the *Zariski topology.* There is a *centre map* $\mathrm{ZR}(K/k) \to \mathrm{Spec}(k)$ sending any $V \in \mathrm{ZR}(K/k)$ to $\mathfrak{m}_V \cap k$.

1.4.1. *(Sub)-models.* By a *projective sub-model* X of K/k, we mean a projective integral k-scheme X whose function field $K(X)$ embeds in K. Equivalently, a projective sub-model of K/k is a factorisation $\mathrm{Spec}(K) \to X \to \mathrm{Spec}(k)$, where the first arrow is schematically dominant and the second is projective, i.e. a $\mathrm{Spec}(K)$-modification of $\mathrm{Spec}(k)$ in the terminology of [Tem11]. We define the *domination relation* between projective sub-models of K/k as follows. Let X, Y be two projective sub-models of K/k. We say that Y *dominates* X if there exists a k-morphism of schemes $Y \to X$ compatible with the schematically dominant maps $\mathrm{Spec}(K) \to X$ and $\mathrm{Spec}(K) \to Y$. The category of projective sub-models is cofiltered: indeed given two projective sub-models $\mathrm{Spec}(K) \to X_1 \to \mathrm{Spec}(k)$, $\mathrm{Spec}(K) \to X_2 \to \mathrm{Spec}(k)$, let X be the Zariski closure of $\mathrm{Spec}(K)$ in $X_1 \times_{\mathrm{Spec}(A)} X_2$, this is a projective sub-model of K/k dominating both X_1 and X_2.

By a *projective model* of K/k, we mean a projective sub-model X of K/k such whose function field is isomorphic to K. The full subcategory of projective sub-models of K/k is cofiltered and its morphisms are birational morphisms. Let us give a criterion of existence for projective models. Since we will consider possibly non-Noetherian domains, we need the following definition. An integral domain k is called *Japanese* if its integral closure in any finite extension of its fraction field is a finite k-algebra. Moreover, we say that k is *universally Japanese* if any finite k-algebra that is an integral domain is Japanese. In the literature, universally Japanese Noetherian domains are called *Nagata* ([Sta23], Definition 032R), this notion includes most of the rings coming from algebraic geometry.

Proposition 1.4.1. *Assume that k is a universally Japanese integral domain. Then projective models of K/k exist iff $K/\mathrm{Frac}(k)$ is finitely generated.*

Proof. By definition, a projective model K/k is a projective morphism $X \to \mathrm{Spec}(k)$ with $K(X) \cong K$, thus the field extension $K/\mathrm{Frac}(k)$ is finitely generated. Conversely, there exists $x_1, ..., x_n \in K$ such that $K/\mathrm{Frac}(k)(x_1, ..., x_n)$ is algebraic. Let X denote the normalisation of $\mathbb{P}^n_k$ in $\mathrm{Spec}(K)$. Since k is universally Japanese, the morphism $X \to \mathbb{P}^n_k$ is finite and X is a projective model of K/k. □

The condition of being universally Japanese can be a bit more pathological in the non-Noetherian setting. Let us list a few examples that will cover the cases of application of interest for us.

Example 1.4.2. The following integral domains are universally Japanese (cf. [Sta23], Proposition 0335 and [Lyu25], Theorem 7).

(1) Nagata rings, e.g. fields, Dedekind domain of characteristic zero, Noetherian complete local rings and finite type ring extensions of such.
(2) A valuation ring of characteristic zero with divisible value group.
(3) The absolute integral closure of a Prüfer domain.

1.4.2. *Zariski-Riemann spaces as projective limits of sub-models.* By the valuative criterion of properness, for any $V \in \mathrm{ZR}(K/k)$, for any projective sub-model X of K/k, there exists a unique $\xi_V \in X$ such that V dominates $\mathcal{O}_{X,\xi_V}$. This defines a map $\mathrm{ZR}(K/k) \to X$ which is compatible with the domination relation. Hence we have a map $d : V \in \mathrm{ZR}(K/k) \to (d(V))_X \in \varprojlim_X X$, where X runs over the projective sub-models of K/k. Moreover, one can define a structure sheaf on $\mathrm{ZR}(K/k)$. For any open subset $U \subset \mathrm{ZR}(K/k)$, define $\mathcal{O}_{\mathrm{ZR}(K/k)}(U)$ to be the intersection of the valuation rings $V \in U$. This defines a sheaf of rings on $\mathrm{ZR}(K/k)$ such that $(\mathrm{ZR}(K/k), \mathcal{O}_{\mathrm{ZR}(K/k)})$ is a locally ringed space and the previously defined map d is a morphism of locally ringed spaces.

Theorem 1.4.3. (1) *The domination map d defined above is a homeomorphism.*
(2) *Let $V \in \mathrm{ZR}(K/k)$. Then V is the union of the local rings $\mathcal{O}_{X,d(V)_X}$, where X runs over the collection of projective sub-models of K/k.*
(3) *The map d defined above is an isomorphism of locally ringed spaces.*

Moreover, in the case where k is universally Japanese and $K/\operatorname{Frac}(k)$ is finitely generated, the conditions (1)-(3) hold for the map $d' : (V \in \mathrm{ZR}(K/k) \to (d(V))_X \in \varprojlim_X X$, where X runs over the projective models of K/k.

Proof. The first three assertions are a consequence of ([Tem11], Corollary 3.4.7). For the remaining one, it suffices to remark that the full subcategory of projective models K/k is cofinal in the category of projective sub-models of K/k. □

1.5. **Berkovich spaces.**

1.5.1. *Banach rings.* Recall that a *Banach ring* is a pair $(A, \|\cdot\|_A)$, where A is a ring and $\|\cdot\|_A$ is a (sub-multiplicative) norm on A such that A is a complete metric space. To such Banach ring, Berkovich associated its *analytic spectrum* $\mathcal{M}(A, \|\cdot\|_A)$ defined as the set of all multiplicative semi-norms on A which are bounded from above by $\|\cdot\|$. This space is equipped with the pointwise convergence topology and is a non-empty compact Hausdorff space ([Ber90], Theorem 1.2.1).

Example 1.5.1. (1) Let $(K, |\cdot|)$ be a complete valued fields. Then $(K, |\cdot|)$ is a Banach ring and the corresponding analytic spectrum is a one-point space.
(2) Let K be a field, let $|\cdot|$, be a non-trivial absolute value on K. Denote by $|\cdot|_{\mathrm{triv}}$ the trivial absolute value on K. Then $\|\cdot\|_{\mathrm{hyb}} := \max\{|\cdot|, |\cdot|_{\mathrm{triv}}\}$ is a norm on K such that $(K, \|\cdot\|_{\mathrm{hyb}})$ is a Banach ring. The norm $\|\cdot\|_{\mathrm{hyb}}$ is called the *hybrid norm* associated to $|\cdot|$. The corresponding analytic spectrum $\mathcal{M}(K, \|\cdot\|_{\mathrm{hyb}})$ is homeomorphic to $[0,1]$ via
$$(\epsilon \in [0,1]) \mapsto |\cdot|^{\epsilon} \in \mathcal{M}(K, \|\cdot\|_{\mathrm{hyb}}),$$
where $|\cdot|^0 := |\cdot|_{\mathrm{triv}}$.
(3) The construction of (2) can be generalised to the case of a ring. Let A be an arbitrary ring, let $\|\cdot\|$ be any norm on A. Denote by $\|\cdot\|_{\mathrm{triv}}$ the trivial norm on A, namely $\|0\|_{\mathrm{triv}} := 0$ and for any $a \in A \smallsetminus \{0\}$, we have $\|a\|_{\mathrm{triv}} := 1$. Define $\|\cdot\|_{\mathrm{hyb}} := \max\{\|\cdot\|, \|\cdot\|_{\mathrm{triv}}\}$. Then $\|\cdot\|_{\mathrm{hyb}}$ is a Banach norm on A. In this article, such a norm is called a *hybrid norm* and the corresponding normed ring is called a *hybrid ring*.

Definition 1.5.2. Let $(A, \|\cdot\|)$ be a Banach ring. We define the *spectral semi-norm* on A by
$$\|\cdot\|_{\mathrm{sn}} : \left| \begin{array}{ccc} A & \longrightarrow & \mathbb{R}_{\geq 0} \\ f & \longmapsto & \inf_{k \in \mathbb{N}^{\times}} \|f^k\|^{\frac{1}{k}}. \end{array} \right.$$

Theorem 1.3.1 of [Ber90] yields, for all $f \in A$, the equality $\|f\|_{\mathrm{sp}} = \max_{x\in\mathcal{M}(A)} |f(x)|$. The norm $\|\cdot\|$ is called *uniform*, and that $(A, \|\cdot\|)$is a *uniform* Banach ring, if $\|\cdot\|$ is equivalent to the spectral semi-norm. In that case, $\|\cdot\|_{\mathrm{sp}}$ is a norm and we have a homeomorphism

$$\mathcal{M}(A, \|\cdot\|) \cong \mathcal{M}(A, \|\cdot\|_{\mathrm{sp}})$$

induced by the identity on A. In practice, unless explicitly mentioned, when a ring A is equipped with a uniform norm, we always assume that the norm is the spectral norm.

Proposition-Definition 1.5.3. *Let $(A, \|\cdot|)$ be a Banach ring. For any compact subset $V \subset X := \mathcal{M}(A, \|\cdot\|)$, let*

$$S_V := \{a \in A : \forall x \in V, |a|_x \neq 0\}, \quad \mathcal{K}(V) := S_V^{-1}A.$$

$\mathcal{K}(V)$ is called the set of rational functions without poles *on V. Define a sheaf of rings $\mathcal{O}_X$ as follows. For any open subset $U \subset X$. Denote by $\mathcal{O}_X(U)$ the set of maps*

$$f : U \to \bigsqcup_{x\in U} \widehat{\kappa}(x)$$

such that

(1) *for any $x \in U$, $f(x) \in \widehat{\kappa}(x)$;*
(2) *for any $x \in U$, there exist a compact neighbourhood V of x in U and a sequence $(f_i)_{i\in\mathbb{N}}$ of non-singular rational functions on V such that $\|f_{|V} - f_i\|_V \to_{i\to+\infty} 0$.*

Then $\mathcal{O}_X$ defines a sheaf of rings on X such that $(X, \mathcal{O}_X)$ is a locally ringed space.

Proof. We refer to ([Ber90], Definition 1.5.1) and ([LP24], Définition 1.2.11 and Lemme 1.2.14). □

1.5.2. *Analytification in the sense of Berkovich: completely valued field case.* In this paragraph, we fix a completely valued field $(k, |\cdot|)$.

Let X be a k-scheme. Its *analytification in the sense of Berkovich*, denoted by X^{an}, is defined in the following way: a point $x \in X^{\mathrm{an}}$ is the data $(p, |\cdot|_x)$ where $p \in X$ and $|\cdot|_x$ is an absolute value on $\kappa(x)$ extending the absolute value on k. X^{an} can be endowed with the Zariski topology: namely the coarsest topology on X^{an} such that the first projection $j : X^{\mathrm{an}} \to X$ is continuous. There exists a finer topology on X^{an}, called the Berkovich topology: it is the initial topology on X^{an} with respect to the family defined by $j : X^{\mathrm{an}} \to X$ and the applications

$$|f|_\cdot : \left| \begin{array}{ccc} U^{\mathrm{an}} & \longrightarrow & \mathbb{R}_{\geq 0}, \\ x & \longmapsto & |f|_x, \end{array} \right.$$

where U^{an} is of the form $U^{\mathrm{an}} := j^{-1}(U)$, with U a Zariski open subset of X, and $f \in \mathcal{O}_X(U)$. Endowed with this topology, X^{an} is a locally compact topological space. There are GAGA type results: namely, X is separated, resp. proper iff X^{an} is Hausdorff, resp. compact Hausdorff. If X is a scheme of finite type, X^{an} can be endowed with a sheaf of analytic functions.

Let L be a line bundle on X. A *metric* on L is a family $\varphi := (|\cdot|_\varphi(x))_{x\in X^{\mathrm{an}}}$, where

$$\forall x \in X^{\mathrm{an}}, \quad |\cdot|_\varphi(x) : L(x) := L \otimes_{\mathcal{O}_X} \widehat{\kappa}(x) \to \mathbb{R}_{\geq 0}$$

is a norm on the $\widehat{\kappa}(x)$-vector space $L(x)$. The metric φ is called *continuous* if for all $U \subset X$ open and for all $s \in H^0(U, L)$, the map $|s|_\varphi : U^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$ is continuous with respect to the Berkovich topology.

Let φ be a continuous metric on a line bundle L. Let

$$\forall s \in H^0(X, L), \quad \|s\|_\varphi := \sup_{x \in X^{\mathrm{an}}} |s|_\varphi(x) \in \mathbb{R}_{\geq 0},$$

it defines a seminorm on $H^0(X, L)$. Moreover, if X is reduced then $\|\cdot\|_\varphi$ defines a norm on $H^0(X, L)$. In general ([CM19], Proposition 2.1.16) implies that if a section $s \in H^0(X, L)$ satisfies $\|s\|_\varphi = 0$, then there exists an integer $n \geq 1$ such that $s^{\otimes n} = 0$.

1.5.3. *Analytification in the sense of Berkovich: global case.* We now briefly recall the global counterpart of the last paragraph. The relevant class of base global analytic objects are the so-called *geometric base rings* (cf. [LP24], Définition 3.3.8 for more details). This class includes many usual examples of Banach rings studied in analytic geometry (e.g. rings of integers of number fields, hybrid fields, discretely valued Dedekind rings).

Let $(A, \|\cdot\|)$ be a Banach ring. In ([LP24], Chapitre 2), the authors define the category of analytic spaces over $(A, \|\cdot\|)$, which is denoted by $(A, \|\cdot\|)$-an.

Theorem 1.5.4 ([LP24], Corollaire 4.1.3, Lemme 6.5.1 and Proposition 6.5.3)**.** *Let $(A, \|\cdot\|)$ be a geometric base ring. Let $\mathcal{X} \to \mathrm{Spec}(A)$ be an A-scheme which is locally of finite presentation. Then the functor*

$$\Phi_{\mathcal{X}} : \left| \begin{array}{rcl} (A, \|\cdot\|)\text{−an} & \longrightarrow & \mathrm{Sets} \\ Y & \longmapsto & \mathrm{Hom}(Y, \mathcal{X}) \end{array} \right.$$

is representable. The $(A, \|\cdot\|)$-analytic space which represents $\Phi_{\mathcal{X}}$ is called the analytification *of $\mathcal{X}$ and is denoted by $\mathcal{X}^{\mathrm{an}}$. Moreover, if $\mathcal{X}$ is projective, then $\mathcal{X}^{\mathrm{an}}$ is a compact Hausdorff topological space.*

2. Pseudo-absolute values

In this section, we introduce the main object of this paper: pseudo-absolute values.

2.1. **Definitions.**

Definition 2.1.1. Let K be a field. A *pseudo-absolute value* on K is any map $|\cdot| : K \to [0, +\infty]$ such that the following conditions hold:

(i) $|1| = 1$ and $|0| = 0$;
(ii) for all $a, b \in K$, $|a + b| \leq |a| + |b|$;
(iii) for all $a, b \in K$ such that $\{|a|, |b|\} \neq \{0, +\infty\}$, $|ab| = |a||b|$.

The set of all pseudo-absolute values on K is denoted by M_K.

Proposition 2.1.2. *Let $|\cdot|$ be a pseudo-absolute value on a field K. Then $A_{|\cdot|} := \{a \in K : |a| \neq +\infty\}$ is a valuation ring of K with maximal ideal $\mathfrak{m}_{|\cdot|} := \{a \in A : |a| = 0\}$. Further, $|\cdot|$ induces a multiplicative semi-norm on $A_{|\cdot|}$ with kernel $\mathfrak{m}_{|\cdot|}$.*

Proof. From $(i), (ii)$, $A_{|\cdot|}$ and $\mathfrak{m}_{|\cdot|}$ are Abelian subgroups of K. From $(i), (iii)$, $A_{|\cdot|}$ is an integral subring of K and $\mathfrak{m}_{|\cdot|} \subset A_{|\cdot|}$ is an ideal.

We show that $A_{|\cdot|}$ is a valuation ring of K. It is enough to treat the $A_{|\cdot|} \neq K$ case. Let $x \in K \smallsetminus A_{|\cdot|}$. If $|x^{-1}| = +\infty$, (iii) yield $1 = |1| = |x \cdot x^{-1}| = +\infty$, contradicting (i). Hence $|x^{-1}| \neq +\infty$, i.e. $x^{-1} \in A_{|\cdot|}$.

We now prove that $\mathfrak{m}_{|\cdot|}$ is the maximal ideal of $A_{|\cdot|}$. Let $\overline{a} \in A_{|\cdot|}/\mathfrak{m}_{|\cdot|} \smallsetminus \{0\}$ and fix a representative $a \in A_{|\cdot|}$ of $\overline{a}$. Then $a \in A_{|\cdot|}^{\times}$ and a^{-1} is a representative of $\overline{a}$. Hence $\mathfrak{m}_{|\cdot|}$ is the maximal ideal of $A_{|\cdot|}$. The last statement is a direct consequence of $(i) - (iii)$. □

Notation 2.1.3. Let K be a field.

(1) Let $|\cdot|$ be a pseudo-absolute value on K. We call
   - $A_{|\cdot|}$ the *finiteness ring* of $|\cdot|$;
   - $\mathfrak{m}_{|\cdot|}$ the *kernel* of $|\cdot|$;
   - $\kappa_v := A_{|\cdot|}/\mathfrak{m}_{|\cdot|}$ the *residue field* of v.
   - $v_{|\cdot|} : K \to \Gamma_{v_{|\cdot|}} \cup \{\infty\}$ the *underlying valuation* of $|\cdot|$.

(2) By "let $(|\cdot|, A, \mathfrak{m}, \kappa)$ be a pseudo-absolute value", we mean that $|\cdot|$ is a pseudo-absolute value on K with finiteness ring A, kernel $\mathfrak{m}$, residue field κ.

(3) By abuse of notation, by "let v be a pseudo-absolute value" on K, we mean the pseudo-absolute value $(|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v)$.

(4) By abuse of notation, if v is a pseudo-absolute value on K, we denote by $v : K \to \Gamma_v \cup \{\infty\}$ the underlying valuation of $|\cdot|_v$.

(5) Let v be a pseudo-absolute value on K. The map $|\cdot|_v : K \to [0, +\infty]$ is uniquely determined by the *residue absolute value* on the residue field κ_v, namely the absolute value defined by $\widetilde{|\overline{x}|_v} := |x|$ for all $\overline{x} \in \kappa_v$, where x_v denotes any representative of $\overline{x}$ in A_v. By abuse of notation, we shall denote by $|\cdot|_v$ the residue absolute value. We denote by $\widehat{\kappa_v}$ the *completed residue field*, namely the completion of κ_v with respect to the residue absolute value

Remark 2.1.4. The construction of Proposition 2.1.2 can be reversed. Let A be a valuation ring of a field K with maximal ideal $\mathfrak{m}$. Let $|\cdot|$ be a multiplicative semi-norm on A. Then $|\cdot|$ can be extended to K by setting $|x| = \infty$ if $x \in K \smallsetminus A$. Then $|\cdot| : K \to [0, +\infty]$ defines a pseudo-absolute value on K with finiteness ring A and kernel $\mathfrak{m}$.

In the following, we shall implicitly use Remark 2.1.4 and we shall often describe a pseudo-absolute value by specifying the finiteness ring and the residue absolute value.

Definition 2.1.5. Let v be a pseudo-absolute value on a field K.

(1) The *rank* of v is defined as the rank of the finiteness ring A_v. It is denoted by $\mathrm{rank}(v)$.
(2) Likewise, the *rational rank* of v is defined as the rational rank of the finiteness ring A_v. It is denoted by $\mathrm{rat.\,rank}(v)$.
(3) v is respectively called *Archimedean*, *non-Archimedean*, *residually trivial* if the residue absolute value is Archimedean, non-Archimedean, trivial.

Remark 2.1.6. The notion of pseudo-absolute value is related to the notion of composite valuations (cf. Definition 1.1.2). Indeed, a non-Archimedean pseudo-absolute value on a field K is nothing else that the data of a valuation v of K with a specified decomposition $v = v' \circ \overline{v}$, where v' is a (general) valuation of K and $\overline{v}$ is a rank one valuation of the residue field of v'. Likewise, an Archimedean pseudo-absolute value on K is the "composition" of a valuation of K with an Archimedean absolute value. Roughly speaking, a pseudo-absolute value can be seen as the composition of a valuation with a "real valuation".

2.2. **Example of pseudo-absolute values.**

Example 2.2.1. (1) Any usual absolute value $|\cdot|$ on a field K defines a pseudo-absolute value with finiteness ring K and trivial underlying valuation.

(2) Let $z \in \mathbb{P}^1_{\mathbb{C}}$ and $\epsilon \in]0,1]$. Denote by $|\cdot|_\infty$ the usual absolute value on $\mathbb{C}$. Then the map

$$|\cdot|_{z,\epsilon} : \left| \begin{array}{ccc} \mathbb{C}(T) & \longrightarrow & [0,+\infty] \\ P & \longmapsto & |P(z)|_\infty^\epsilon \end{array} \right.$$

defines a pseudo-absolute value on $\mathbb{C}(T)$ denoted by $v_{z,\epsilon,\infty}$, where $P(\infty) \in \mathbb{P}^1_{\mathbb{C}}$ denotes the evaluation in 0 of $Q(T) := P(1/T)$. Its finiteness ring is $A_z := \{f \in \mathbb{C}(T) : \mathrm{ord}(f,z) \geq 0\}$, its kernel is $\mathfrak{m}_z := \{f \in \mathbb{C}(T) : \mathrm{ord}(f,z) > 0\}$, and its residue field is $\mathbb{C}$ endowed with the absolute value $|\cdot|_\infty^\epsilon$.

(3) Let $K = \mathcal{M}(U)$ be the field of meromorphic functions on a non-compact Riemann surface U and denote by $\mathcal{O}(U)$ the ring of analytic functions on U. Then $\mathcal{O}(U)$ is a Prüfer domain (cf. §1.1.3). Let $z \in U$, then the localisation $A_z := \mathcal{O}(U)_{\mathfrak{m}_z}$ of $\mathcal{O}(U)$ at the maximal ideal of functions on U vanishing in z is a valuation ring of K with residue field $\mathbb{C}$. For all $\epsilon \in]0,1]$, let $v_{z,\epsilon,\infty} \in M_K$ be the pseudo-absolute value on K with valuation ring A_z and residue absolute value $|\cdot|_\infty^\epsilon$.

(4) Let $K = \mathbb{Q}(T)$ and let $t \in [0,1]$ such that $e^{2i\pi t} \in \overline{\mathbb{Q}}$. Denote by $\mathfrak{m}_t$ the ideal of $\mathbb{Q}[T]$ generated by the minimal polynomial of $e^{2i\pi t}$. For all $\epsilon \in]0,1]$, let $v_{t,\epsilon,\infty}$ be the pseudo-absolute value on K defined by

$$\forall P \in \mathbb{Q}[T]_{\mathfrak{m}_t}, \quad |P|_{t,\epsilon,\infty} := |P(e^{2\pi i t})|_\infty,$$

where $|\cdot|_\infty$ denotes the usual absolute value on $\mathbb{C}$.

(5) Let K be a field and $(A,\mathfrak{m})$ be a valuation ring of K. Then there is a residually trivial pseudo-absolute value $v_{A,\mathrm{triv}}$ whose finiteness ring is $(A,\mathfrak{m})$ and the residue absolute value is the trivial absolute value on $A/\mathfrak{m}$. Conversely, all residually trivial pseudo-absolute values arise this way.

2.3. **Extension of pseudo-absolute values.**

Definition 2.3.1. Let K'/K be a field extension. Let $v = (|\cdot|, A, \mathfrak{m}, \kappa)$, resp. $v' = (|\cdot|', A', \mathfrak{m}', \kappa')$, be a pseudo-absolute value on K, resp. on K'. If $|x|' = |x|$ for all $x \in K$, we say that $|\cdot|'$ *extends* (alternatively is *above*) $|\cdot|$. In that case, we use the notation $v'|v$.

Proposition 2.3.2. *Let K'/K be a field extension. Let $v = (|\cdot|, A, \mathfrak{m}, \kappa)$, resp. $v' = (|\cdot|', A', \mathfrak{m}', \kappa')$, be a pseudo-absolute value on K, resp. on K'. Assume that $v'|v$. Then*

(i) *A' is an extension of A, namely there is an injective local morphism $A \to A'$;*

(ii) *the residue absolute value of $|\cdot|'$ extends the one of $|\cdot|$.*

Conversely, given any extension $A \to A'$ of valuation rings (of K and K' respectively) endowed with multiplicative semi-norms with kernel the maximal ideal satisfying (ii), the induced pseudo-absolute value on K' extends the induced pseudo-absolute value on K.

Proof. The first statement is a direct consequence of Definition 2.3.1. Let $v' = (|\cdot|', A', \mathfrak{m}', \kappa')$, resp. $v = (|\cdot|, A, \mathfrak{m}, \kappa)$, denote the induced pseudo-absolute value on K', resp. on K. Then (*ii*) yields $|x|' = |x|$ for all $x \in A$. If $x \in K \smallsetminus A$, then $|x|' = +\infty$ (otherwise $x \in A' \cap K = A$). Hence the conclusion. $\square$

Let **SVF** be the category defined as follows. An object of **SVF** is a field endowed with a pseudo-absolute value and morphisms in **SVF** are given by the extensions of pseudo-absolute values. Let **VR** be the category with objects valuation rings endowed with multiplicative semi-norms with kernel the maximal ideal and with morphisms the extensions satisfying conditions $(i), (ii)$ of Proposition 2.3.2.

Proposition 2.3.3. *The categories* **SVF** *and* **VR** *are equivalent.*

Proof. It is a consequence of Remark 2.1.4 (2) combined with Proposition 2.3.2. □

Remark 2.3.4. Proposition 2.3.3 allows to safely treat extensions of pseudo-absolute values in the context of the category **VR**

3. Algebraic extensions of pseudo-absolute values

In this section, we study extensions of pseudo-absolute values with respect to algebraic extensions of the base field. We first treat the separable case (§3.1). Then we extend the results to arbitrary finite extensions (§3.2). Finally, we introduce elementary Galois theory of pseudo-absolute values (§3.3).

3.1. **Finite separable extension.** In this subsection, we fix a finite separable extension K'/K and a pseudo-absolute value v on K. We study extensions of v to L. Let A' be the integral closure of A_v in K. ([Bou75], Chap. VI, §1.3, Théorème 3) implies that A' is the intersection of all the extensions of A_v to L.

Lemma 3.1.1. *A' is a semi-local ring.*

Proof. Let $\mathfrak{m}'$ be a maximal ideal A'. One the one hand, we have $\mathfrak{m}' \cap A_v = \mathfrak{m}$ and A' is a Prüfer domain (cf. Proposition 1.1.9 (2)). Hence $A'_{\mathfrak{m}}$ is an extension of A_v to L. On the other hand, the set of extensions of A_v to L is finite. Whence A' is semi-local. □

For any $\mathfrak{m}_w$ in the fibre of $\mathfrak{m}'_v$ of the morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A_v)$, let $A'_w := (A')_{\mathfrak{m}_w}$ the localisation in $\mathfrak{m}'_w$. From ([Bou75], Chap. VI, §8.6, Proposition 6), extensions of A_v to L are of the form A'_w, for w as above. For any such extension $A_v \to A'_w$, we have a finite extension $\kappa_v \to \kappa_w$ of residue fields.

Proposition 3.1.2. *There is a bijective correspondence between the set of pseudo-absolute values on L above v and the set of extensions of the residue absolute value of v with respect to extensions of the form $\kappa_v \to \kappa_w$, where w runs over the set of maximal ideals of A'. Furthermore, we have the equality*

$$\sum_{\mathfrak{m}_w \in \mathrm{Spm}(A')} \frac{1}{|\,\mathrm{Spm}(A')|} \sum_{i|v} \frac{[\widehat{\kappa_{w,i}} : \widehat{\kappa_v}]_s}{[\kappa_w : \kappa_v]_s} = 1, \tag{1}$$

where, for all $\mathfrak{m}_w \in \mathrm{Spm}(A')$, i runs over the set of extensions of the residue absolute value of $|\cdot|_v$ to κ_w and $\widehat{\kappa_{w,i}}$ denotes the completion of κ_w for any such absolute value.

Proof. Remark 2.1.4 gives the first assertion. We first assume that, for all $\mathfrak{m}_w \in \mathrm{Spm}(A')$, the extension $\kappa_v \to \kappa_w$ is separable. Then ([Bou75], Ch.V §8.5 Proposition 5) yields (1). In the case where $\kappa_v \to \kappa_w$ is not separable, denote by κ_v^{s} the separable closure of κ_v inside κ_w. For any $i|v$, ([CM19], Lemma 3.4.2) allows to identify the completion κ_v^{s} with the separable closure of $\widehat{\kappa_v}$ inside $\widehat{\kappa_{wi}}$. Since we have an identification of the set of extensions of v to κ_w with the set of extensions of κ_v to κ_v^{s}, (1) is obtained from the previous case. □

3.2. **Arbitrary finite extension.** In this subsection, we fix a finite extension L/K. Let K' denote the separable closure of K inside L. Then L/K' is a purely inseparable finite extension and we denote by q its degree. Therefore, for all $x \in L$, we have $x^q \in K'$.

Proposition 3.2.1. *Let $v' = (|\cdot|', A', \mathfrak{m}', \kappa')$ be a pseudo-absolute value on K'. Then $A_L := \{x \in L : x^q \in A'\}$ is the unique valuation ring of L extending A'. Its maximal ideal is $\mathfrak{m}_L := \{x \in A_L : x^q \in \mathfrak{m}'\}$. Moreover, the residue field extension $\kappa' \to A_L/\mathfrak{m}_L$ is purely inseparable and finite.*

Proof. Let $v' : K' \to \Gamma$ be the underlying valuation of A'. Then $(x \in L) \mapsto (1/q)v'(x^q)$ is the unique extension of ν to L. The corresponding valuation ring is $A_L = \{x \in L : x^q \in A'\}$ and its maximal ideal is $\mathfrak{m}_L = \{x \in L : x^q \in \mathfrak{m}'\}$. Let $\overline{a} \in A_L/\mathfrak{m}_L$ and $a \in A_L$ be a representative. Then $a^q \in A'$ and represents $\overline{a}^q \in \kappa'$. Hence the conclusion. ☐

Corollary 3.2.2. *Let v be a pseudo-absolute value K. Then the set of extensions of v to L is in bijection with the set of extensions of v on K' described in Proposition 3.1.2.*

3.3. **Galois theory of pseudo-absolute values.** Throughout this subsection, we fix a field K.

Proposition 3.3.1. *Let L/K be an algebraic extension. For all $x \in M_L$, for all $\tau \in \mathrm{Aut}(L/K)$, the map*

$$|\cdot|_{\tau(x)} : \left| \begin{array}{ccc} L & \longrightarrow & [0, +\infty] \\ a & \longmapsto & |\tau(a)|_x \end{array} \right.$$

defines a pseudo-absolute value on L denoted by $x \circ \tau$. This construction defines a right action of $\mathrm{Aut}(L/K)$ on M_L.

Proof. Let $x \in M_L$ and $\tau \in \mathrm{Aut}(L/K)$. Let us show that $|\cdot|_{\tau(x)}$ defines a pseudo-absolute value on L. Let $a, b \in L$, by linearity of τ, we have

$$|a+b|_{\tau(x)} = |\tau(a) + \tau(b)|_x \leq |a|_{\tau(x)} + |b|_{\tau(x)}$$

as well as $|0|_{\tau(x)} = 0$ and $|1|_{\tau(x)} = 1$. Assume that $\{|a|_{\tau(x)}, |b|_{\tau(x)}\} \neq \{0, +\infty\}$. Then

$$|ab|_{\tau(x)} = |\tau(a)\tau(b)|_x = |\tau(a)|_x |\tau(b)|_x = |a|_{\tau(x)} |b|_{\tau(x)}.$$

Hence $|\cdot|_{\tau(x)} \in M_L$. ☐

Lemma 3.3.2. *Let L/K be a (possibly infinite) Galois extension with Galois group G. Let B denote the integral closure of A in L. Then G acts on B and acts transitively on the set of maximal ideals of B.*

Proof. First, G acts on B since G stabilises A. Furthermore, we have inclusions $A \subset B^G \subset K = L^G$ and elements of B^G are integral over A. Hence $A = B^G$ as A is integrally closed. When L/K is finite, ([Sta23], Lemma 0BRI) gives the conclusion. In the general case, note that the set of maximal ideals of B is in bijection with the set of extensions of A to L. Hence any maximal ideal of B is mapped to $\mathfrak{m}$ via the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. We conclude by using ([Sta23], Lemma 0BRK). ☐

We shall use the following crucial result.

Proposition 3.3.3 (([Efr06], Corollary 15.2.5), ([Sta23], Lemma 0BRK)**.** *Let L/K be a normal extension. Let A' be an extension of A on L and denote by $\mathfrak{m}'$ its maximal ideal and κ' its residue field. Then the extension κ'/κ is normal and the canonical homomorphism $\{\sigma \in \mathrm{Aut}(L/K) : \sigma(\mathfrak{m}') = \mathfrak{m}'\} \to \mathrm{Aut}(\kappa'/\kappa)$ is surjective.*

Proposition 3.3.4. *Let $v = (|\cdot|, A, \mathfrak{m}, \kappa)$ be a pseudo-absolute value on K. Assume that the residue field κ is perfect. Let L/K be a Galois extension with Galois group G. Denote by $M_{L,v}$ the set of extensions of v to L. Then G acts transitively on $M_{L,v}$.*

Proof. First, remark that G acts on $M_{L,v}$ as G stabilises K. Let $w_1 = (|\cdot|_1, A_1, \mathfrak{m}_1), w_2 = (|\cdot|_2, A_2, \mathfrak{m}_2) \in M_{L,v}$ and denote by B the integral closure of A in L. Lemma 3.3.2 gives the existence of $\sigma \in G$ such that $\sigma(\mathfrak{m}_1 \cap B) = \mathfrak{m}_2 \cap B$. Hence the map

$$|\cdot|_\sigma : \left| \begin{array}{ccc} L & \longrightarrow & [0, +\infty] \\ a & \longmapsto & |\sigma(a)|_2 \end{array} \right.$$

defines an element of $M_{L,v}$ with finiteness ring A_1 and residue absolute $|\cdot|_\sigma$. Proposition 3.3.3 implies that the extension $\kappa_1 := (A_1/\mathfrak{m}_1)/\kappa$ is Galois. Hence there exists $\overline{\tau} \in \mathrm{Aut}(\kappa_1/\kappa)$ such that, for all $\overline{a} \in \kappa_1$, $|\overline{a}|_1 = |\overline{a}|_\sigma$ ([Neu99], Chapter II, Proposition 9.1). Furthermore, there exists $\tau \in G$ lifting $\overline{\tau}$ and such that $\tau(\mathfrak{m}_1) = \mathfrak{m}_1$. We then have

$$|a|_1 = |\sigma(\tau(a))|_2$$

for all $a \in K$, i.e. $v_1 = \sigma\tau(v_2)$. □

Proposition 3.3.5. *Let v be a pseudo-absolute value on K. Assume that κ_v is perfect and trivially valued if* $\mathrm{char}(\kappa_v) > 0$. *Let L/K be a Galois extension. Let $c \in A_v^s$ (the integral closure of A_v inside K^s). Let $P = T^d + a_1T^{d-1} + \cdots + a_d \in A_v[T]$ be the minimal polynomial of c over K and let L be the field of decomposition of P. Let $O_c := \{\alpha_1, ..., \alpha_d\}$ denote the orbit of c under the action of the Galois group of L/K. We fix a choice $w_0 := (|\cdot|_0, B_0, \mathfrak{m}_0, \kappa_0)$ of a pseudo-absolute value on L above v. Then*

$$\max_{1 \leq j \leq d} |\alpha_j|_0 = \max_{w|v} |c|_w = \limsup_{N \to \infty} \left| \sum_{j=1}^{d} \alpha_j^N \right|_v^{\frac{1}{N}}.$$

Proof. As L/K is Galois, Proposition 3.3.4 yields the first equality. We now prove the second equality. The $\mathrm{char}(k) > 0$ case being trivial, we assume that $\mathrm{char}(k) = 0$. Let B denote the integral closure of A_v in L. Then Lemma 3.3.2 combined with the fundamental theorem of symmetric polynomials yield, for all $j \in \{1, ..., d\}$, $\alpha_j \in B$ and $\sum_{j=1}^d \alpha_j^N \in A_v$. From §3.1, B is a Prüfer semi-local ring and denote by $\mathfrak{m}_1, ..., \mathfrak{m}_r$ its maximal ideals and respectively $(B_1, \mathfrak{m}_1, \kappa_1), ..., (B_r, \mathfrak{m}_r, \kappa_r)$ the corresponding localisations. For all $i \in \{1, ..., r\}$, for all $j \in \{1, ..., d\}$, denote by $\overline{\alpha_j}^{(i)} \in \kappa_i$ the image of α_j in κ_i. Then $\overline{\alpha_j}^{(i)}$ is a root of the minimal polynomial of $\overline{c}^{(i)}$ over κ_v, hence it is separable over κ_v. Hence, for all $i \in \{1, ..., r\}$, for all integer $N > 0$, the image of $\sum_{j=1}^d \alpha_j^N \in A_v$ in κ_i is of the form

$$\sum_{j \in J_i} n_j (\overline{\alpha_j}^{(i)})^N,$$

where $J_i \subset \{1, ..., d\}$ is of cardinality $d_i := [\kappa_i : \kappa_v]$ such that for all $j \neq j' \in J_i$, $\overline{\alpha_j}^{(i)} \neq \overline{\alpha_{j'}}^{(i)}$ and for all $j \in J$, n_j is a non-zero integer in κ_v. Fixing an extension $|\cdot|_i$ of $|\cdot|_v$ to κ_i, we have

$$\max_{w\in E_i}|c|_w=\max_{j\in J_i}|\overline{\alpha_j}^{(i)}|_i=\limsup_{N\to\infty}\left|\sum_{j\in J_i}n_j(\overline{\alpha_j}^{(i)})^N\right|_i^{\frac{1}{N}}=\left|\sum_{j\in J_i}n_j(\overline{\alpha_j}^{(i)})^N\right|_v^{\frac{1}{N}}=\limsup_{N\to\infty}\left|\sum_{j=1}^{d}\alpha_j^N\right|_v^{\frac{1}{N}},$$

where E_i denotes the set of extensions (in the sense of usual absolute values) on $|\cdot|_v$ to κ_i. The first equality comes from Proposition 3.3.4 (cf. κ_i/κ_v is Galois). The second equality comes from Lemma 3.3.6 below. Finally, the fact that, for all integer $N>0$, $\sum_{j\in J_i}n_j(\overline{\alpha_j}^{(i)})^N\in\kappa_v$ (as a symmetric functions of roots of $\overline{P}^{(i)}$, the image of P in κ_v) provides the third and last equalities. We can then conclude the proof using the description given in Proposition 3.1.2. □

Lemma 3.3.6. *Let F be a field endowed with a (usual) absolute value $|\cdot|$. We fix an extension of $|\cdot|$ to $\overline{F}$ again denoted by $|\cdot|$. Let $\{\alpha_1,...,\alpha_d\}$ be a family of pairwise distinct separable elements of F and let $\{n_1,...,n_d\}$ be a family of non-zero integers. Then*

$$\max_{j\in\{1,\dots,d\}}|\alpha_j|=\limsup_{N\to+\infty}\left|\sum_{j=1}^{d}n_j\alpha_j^N\right|^{\frac{1}{N}}.$$

Proof. The proof of ([CM19], Lemma 3.3.5) can be adapted *mutatis mutandis* replacing the α_j's by the $n_j\alpha_j$'s. □

4. Transcendental extensions of pseudo-absolute values

Throughout this section, we fix a field K. We will address the problem of extending pseudo-absolute values on K to transcendental extensions. Given study of transcendental extensions of valuations, this is a quite complicated problem (cf. e.g. [Mac36, Vaq06]). Therefore, we will mostly give specific, yet important, examples of such extensions.

4.1. Purely transcendental extension of degree 1 and usual absolute value case.

Throughout this subsection, let $K'=K(X)$, where X is transcendental over K and let v be a usual absolute value on K.

Let $v'=(|\cdot|',A',\mathfrak{m}',\kappa')$ be an extension of v to K'. Then by ([Bou75], Chap VI, §10.3, Corollaires 1-3), we have $\mathrm{rank}(v')\in\{0,1\}$.

For the $\mathrm{rank}(v')=0$ case, v' is an absolute value on K' extending v.

If $\mathrm{rank}(v)=1$, then ([Bou75], Chap VI, §10.3, Corollaires 1-3) yields $\mathrm{tr.deg}(\kappa'/K)=0$ and v' is Abhyankar, thus A' is a discrete valuation ring and there exists a closed point $x\in\mathbb{P}^1_K$ such that $A'=\{f\in K':\mathrm{ord}(f,z)\geq 0\}$. The residue absolute value of v' is an extension on v to the algebraic extension κ'/K.

From the above description, we get the following result.

Proposition 4.1.1. *Assume that K is complete with respect to the absolute value v and denote by $M_{K',v}$ the set of extensions of v to K'. Then we have a homeomorphism.*

$$M_{K',v}\cong\mathbb{P}^{1,\mathrm{an}}_K,$$

where $\mathbb{P}^{1,\mathrm{an}}_K$ denotes the Berkovich analytification of $\mathbb{P}^1_K$ (cf. §1.5.2).

Proof. From what is written above, we have a bijection $M_{K',v} \to \mathbb{P}^{1,\mathrm{an}}_{K}$. It is continuous by definition of the topologies and thus we can conclude by compactness of $M_{K',v}$ and $\mathbb{P}^{1,\mathrm{an}}_{K}$. □

4.2. **Composition with a valuation.** Let K'/K be a field extension. Let $\overline{v} = (\overline{|\cdot|}, \overline{V}, \overline{m}, \kappa)$ be a pseudo-absolute value on K. Let V' be a valuation ring with fraction field K' with residue field K, denote by v' the associated valuation of K'. For any $a \in V'$, denote by $\overline{a}$ the image of a in K. Then $V := \{a \in V' : \overline{a} \in \overline{V}\}$. The results of §1.1.1 imply that V is a valuation ring of K' with residue field κ.

Definition 4.2.1. Using the above construction together with Remark 2.1.4, we obtain a pseudo-absolute value v on K' which is denoted by $v = v' \circ \overline{v}$. v is called the *composite pseudo-absolute value* with v' and $\overline{v}$.

Lemma 4.2.2. *We use the above notation. Assume that K is a subfield of K'. Then the pseudo-absolute value v is an extension of $\overline{v}$ to K'.*

Proof. Since K'/K is an extension, we have a section $K \to V' \to K$. Thus, for any $a \in K$, we have $|a|_v = \overline{|a|}$. □

4.3. **Extension by generalisation.** Assume that K'/K is an extension of algebraic functions fields, namely K'/K is finitely generated. Let $X \to \mathrm{Spec}(K)$ be a K-variety with function field $K(X) \cong K'$, namely a model of K'/K. Let $x \in X$ be a non-singular point. Then Proposition 1.1.6 implies that there exists a valuation v of K'/K with valuation ring V dominating $\mathcal{O}_{X,x}$ and residue field isomorphic to $\kappa(x)$. Let $\overline{v_x}$ be a pseudo-absolute value on $\kappa(x)$. Using Definition 4.2.1, we obtain a pseudo-absolute value $v_x = v \circ \overline{v_x}$ on K'. Moreover, if $\kappa(x)$ is a subfield of K', Lemma 4.2.2 implies that v_x is an extension of $\overline{v_x}$ to K'. Note that this is the case when x is a regular rational point.

Definition 4.3.1. We use the above notation. The pseudo-absolute value v_x is called the *extension by generalisation* (alternatively the *extension through specialisation*) of $\overline{v}_x$ to K w.r.t. the valuation ring V.

5. Completion of pseudo-valued fields

Let K be a field and v be a pseudo-absolute value on K with residue field κ. The goal of this section is to construct a pseudo-absolute value (possibly on an extension of K) which extends v and whose residue field is $\widehat{\kappa}$, the completion of κ w.r.t. the residue absolute value. We make use of the notion of "gonflement" introduced by Bourbaki and which we recall (§5.1). Then we introduce (non-canonical) completion of a field with respect to a pseudo-absolute value (§5.2).

5.1. **Gonflement of a local ring.** We will use results from ([Bou75], Chap. IX, Appendice 2) we now recall.

Definition 5.1.1. Let $(A, \mathfrak{m})$ be a local ring with residue field κ. An A-algebra A' is called an *elementary gonflement* (*gonflement élémentaire* in French) of A if either A' is isomorphic to the A-algebra $A]X[:= A[X]_{\mathfrak{m}A[X]}$ or there exists a monic polynomial $P \in A[X]$, whose reduction in $\kappa[X]$ is irreducible, such that A' is isomorphic to the A-algebra $A[X]/(P)$.

Lemma 5.1.2. *Let A be a local ring and $\iota : A \to A'$ be an elementary gonflement of A. Then A' is local. We denote respectively by κ, κ' the residue fields of A, A'. Moreover, the following assertions hold.*

(1) *The morphism ι is flat, injective and local.*
(2) *The residue field extension κ'/κ is generated by a single element.*
(3) *The maximal ideal of A generates the maximal ideal of A'.*
(4) *If A is Noetherian, then so is A'.*
(5) *If A is a valuation ring, then so is A' and the value groups of A and A' are equal. In particular, if A is a DVR then so is A'.*

Proof. We first consider the case where ι is finite, i.e. $A' \cong A[X]/(P)$, where $P \in A[X]$ has irreducible reduction $\overline{P}$ in $\kappa[X]$. Let $\mathfrak{m}$ denote the maximal ideal of A. Since $A'/\mathfrak{m}A' \cong \kappa[X]/(\overline{P})$, mA' is a maximal ideal of A'. Let $\mathfrak{p}$ be a maximal ideal of A'. Since $\iota : A \to A'$ is finite, $\mathfrak{p} \cap A = \mathfrak{m}$ and thus $\mathfrak{m}A' \subset \mathfrak{p}$ hence $\mathfrak{p} = \mathfrak{m}A'$, i.e. A' is local. The residue field extension is generated by the class of X in κ'. We now prove (5). Assume that A is a valuation ring and denote by $v : A \to \Gamma$ the corresponding valuation. We assume that the valuation is trivial (i.e. A is not a field), otherwise the assertion is trivial itself. Since $P \in A[X]$ has irreducible image in $\kappa[X]$, it is irreducible in $A[X]$. Indeed, if we could write $P = f \cdot g$, where $f, g \in A[X]$ are non-constant polynomials, then either f or g has coefficients in $\mathfrak{m}$ and thus $P \in \mathfrak{m}[X]$, which contradicts P is monic since A is not a field. Now P is monic and irreducible in $A[X]$ and A is integrally closed. Therefore P is irreducible in $K[X]$, where K denotes the fraction field of A. Thus A' is an integral domain with quotient field $K' := K[X]/(P)$ which is a finite extension of K. Now define the map $v' : A[X] \to \Gamma$ sending $a_d X^d + \cdots a_0 \in A[X]$ to $\min_{0 \le i \le d} v(a_i)$. This defines a valuation on $K(X)$ and $A[X]$ is contained in the valuation ring $A]X[$ (cf. Definition 1.1.3). Since P is monic, $v(P) = 0$ and v' induces a map $v' : A' \to \Gamma$. It is straightforward to check that v' defines a valuation on A', which is non-negative on A'. Therefore, if V' denotes the valuation ring of K' of the valuation v', we have $A' \subset V'$. Assume that there exists $a' \in V'$ which does not belong to A'. Choose a representative $a_d X^d + \cdots a_0 \in K[X]$ of a'. By hypothesis, $v'(a') = \min_{0 \le i \le d} v(a_i) \ge 0$. Since $a' \notin A'$, there exists an index $j \in \{0, ..., d\}$ such that $v(a_j) = \min_{0 \le i \le d} v(a_i)$ and $a_j \notin A$. Thus $0 \le v(a') = v(a_j) < 0$, yielding a contradiction. Finally, A' is a valuation ring. Using (4), we obtain the final part of (5).

If ι is not finite, namely $A' \cong A]X[$, since $\mathfrak{m}A[X]$ is a prime ideal in $A[X]$, A' is local. Then (1)-(4) follow directly from the definition. (5) follows from the following claim.

Claim 5.1.3. *If A is a valuation ring, then $A]X[$ is the valuation ring of the Gauss valuation $v_{0,0}$ (cf. Definition 1.1.3).*

Proof. Let $F = P/Q$ be a non-zero element of $K(X)$, where $P, Q \in A[X]$ are coprime polynomials. Assume that both P and Q belong to $\mathfrak{m}A[X]$. Then we can write $P = aP_1$ and $Q = bQ_1$, where $P_1, Q_1 \in A[X]$ are coprime and $a, b \in \mathfrak{m}$ are non-zero. Since A is a valuation ring, either $a|b$ or $b|a$, which contradicts the fact that P and Q are coprime. Thus $A]X[$ is a valuation ring of $K(X)$.

Denote by $(V, \mathfrak{m}_V)$ the valuation ring of the Gauss valuation $v_{0,0}$ on $K(X)$. By definition of the Gauss valuation, we have $A[X] \subset V$ and $\mathfrak{m}_V \cap A[X] = \mathfrak{m}A[X]$. Thus the canonical morphism $A]X[\to V$ is local and therefore $V = A]X[$ since $A]X[$ is a valuation ring. □

□

Definition 5.1.4. Let A be a local ring. An A-algebra A' is called a *gonflement* of A if there exist a well-ordered set Λ with a greatest element ω together with an increasing family $(A'_\lambda)_{\lambda\in\Lambda}$ (w.r.t. inclusion) such that the following conditions hold:

(i) for any $\lambda \in \Lambda$, A_λ is a local ring and $A' = A'_\omega$;
(ii) let α be the least element of Λ, then A'_α is isomorphic to A;
(iii) let $\nu \in \Lambda \smallsetminus \{\alpha\}$, let $S_\nu := \{\lambda \in \Lambda : \lambda < v\}$. If S_ν has a greatest element μ, then A'_v is an elementary gonflement of A'_μ. Otherwise, then $A'_\nu = \bigcup_{\lambda\in S_\nu} A'_\lambda$.

Proposition 5.1.5. *Let A be a local ring and let $A \to A'$ be a gonflement.*

(1) *A' is a local ring and $\mathfrak{m}_{A'} = \mathfrak{m}_A A'$.*
(2) *The ring extension $A \to A'$ is faithfully flat.*
(3) *Assume that A is Noetherian. Then A' is Noetherian and* $\dim(A') = \dim(A)$*. Moreover, if A is regular, then A' is regular.*
(4) *If A is a DVR, then A' is a DVR. Moreover, the maximal ideal of A' is generated by any generator of the maximal ideal of A.*
(5) *If A is a valuation ring, then A' is a valuation ring. Moreover, the value groups of A and A' are equal.*

Proof. (1)-(3) are ([Bou75], Chap. IX, Appendice 2, Proposition 2 et Corollaire). (4) follows from (1) and (3) together with the fact that discrete valuation rings are exactly regular rings of dimension 1.

Let us prove (5). Let A be a valuation ring with value group Γ. Let $A' = (A'_\lambda)_{\lambda\in\Lambda}$ be a gonflement of A, where Λ is a well-ordered set. Denote respectively by α, ω the least and greatest elements of Λ, namely $A = A_\alpha$ and $A' = A_\omega$. Let

$$\Lambda' := \{\lambda \in \Lambda : A_\lambda \text{ is a valuation ring with value group } \Gamma\}.$$

Assume that $\Lambda \smallsetminus \Lambda' \neq \varnothing$. Since Λ is well-ordered, there exists a least element $\nu \in \Lambda'$. Let $S_\nu := \{\lambda \in \Lambda : \lambda < \nu\}$. Note that $\alpha < \nu$ and $S_\nu \subset \Lambda'$.

If S_ν has a greatest element μ, then A_ν is an elementary gonflement of A_μ and we obtain a contradiction using Lemma 5.1.2 (5).

Now assume that S_ν does not have a greatest element. Then $A_\nu = \bigcup_{\lambda\in S_\nu} A'_\lambda$. Note that, for any $\lambda \leq \lambda'$, the morphism $A_\lambda \to A_{\lambda'}$ is injective and local, thus A_ν is a direct limit of Prüfer rings with injective arrows and is thus Prüfer (Prop 1.1.9 (6)). Hence A_ν is a local Prüfer domain, i.e. a valuation ring. Since $S_\nu \subset \Lambda'$, we deduce that the value group of A_ν is Γ, providing a contradiction. □

Theorem 5.1.6 ([Bou75], Chap. IX, Appendice 2, Corollaire du Théorème 1)**.** *Let A be a local ring with residue field κ. Let κ'/κ be a field extension. Then there exists a gonflement $A \to B$ such that the field extension κ'/κ is isomorphic to the field extension κ_B/κ, where κ_B denotes the residue field of B.*

5.2. **Completion.**

Definition 5.2.1. Let K be a field and $v = (|\cdot|, A, \mathfrak{m}, \kappa)$ be a pseudo-absolute value on K. We say that K is *complete* w.r.t. v if the following conditions hold.

(i) The finiteness ring A' is Henselian.

(ii) The residue field κ' is complete w.r.t. the residue absolute value induced by v'.

Proposition 5.2.2. *Let K be a field and $v = (|\cdot|, A, \mathfrak{m}, \kappa)$ be a pseudo-absolute value on K. Assume that K is complete w.r.t. v. Then for any algebraic extension K'/K, there exists a unique extension $v'|v$ to K'.*

Proof. Let K'/K be an algebraic extension. Since A is Henselian, the underlying valuation of v extends uniquely to K', denote by A' the corresponding valuation ring of K'. Let κ' denote the residue field of A'. Then κ'/κ is an algebraic extension and, since κ is complete, the residue absolute value of v extends uniquely to κ'. □

Definition 5.2.3. Let K be a field and $v = (|\cdot|, A, \mathfrak{m}, \kappa)$ be a pseudo-absolute value on K. Let K'/K be field extension and $v' = (|\cdot|', A', \mathfrak{m}', \kappa') \in M_{K'}$ be a pseudo-absolute value on K' extending v. We say that the extension $v'|v$ is *complete* if K' is complete w.r.t. v'.

The following proposition allows to construct complete pseudo-valued fields.

Proposition 5.2.4. *Let K be a field and $v = (|\cdot|, A, \mathfrak{m}, \kappa)$ be a pseudo-absolute value on K. Denote by $\widehat{\kappa}$ the completion of κ w.r.t. the residue absolute value induced by v. Then there exist a field extension $\widehat{K}/K$ and a pseudo-absolute value $\widehat{v} \in M_{\widehat{K}}$ extending v such that $\widehat{v}|v$ is complete (Definition 5.2.3).*

Proof. Theorem 5.1.6 implies that there exists a gonflement $A \to A'$ such that A' has residue field $\widehat{\kappa}$. Moreover, Proposition 5.1.5 (5) implies that A' is a valuation ring with same value group as A. Now consider $\widehat{A} := (A')^h$ the Henselisation of A'. Then ([Sta23], Section 0BSK) implies that $\widehat{A}$ is a valuation ring extending A' with same value group as A' and A. Moreover, its residue field is $\widehat{\kappa}$. Let $\widehat{K} := \mathrm{Frac}(\widehat{A})$. Then $\widehat{A}$ is a Henselian valuation ring of $\widehat{K}$ and by considering the unique extension of the residue absolute value induced by v on κ to $\widehat{\kappa}$, we obtain a pseudo-absolute value $\widehat{v}$ on $\widehat{K}$. The extension $\widehat{v}|v$ is complete by construction. □

Remark 5.2.5. The completion construction given by Proposition 5.2.4 is unfortunately non-canonical in the following sense: it depends on the choice of a good order on the completion of the residue field ([Bou75], Chap. IX, Appendice 2, Exemple 1)). *A priori*, different choices of such orders may lead to different completions.

6. Pseudo-norms

In this section, we introduce the analogue of a normed vector space when the base field is equipped with a pseudo-absolute value. Throughout this section, we fix a field K.

6.1. Definitions.

Definition 6.1.1. Let E be a finite-dimensional vector space over K of dimension d. For any pseudo-absolute value $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in M_K$, we call *pseudo-norm* on E in v any map $\|\cdot\|_v : E \to [0, +\infty]$ such that the following conditions hold:

(i) $\|0\|_v = 0$ and there exists a basis $(e_1, ..., e_d)$ of E such that $\|e_1\|_v, \cdots, \|e_d\|_v \in \mathbb{R}_{>0}$;
(ii) for any $(\lambda, x) \in K \times E$ such that $\{|\lambda|_v, \|x\|_v\} \neq \{0, +\infty\}$, we have $\|\lambda x\|_v = |\lambda|_v \|x\|_v$;
(iii) for any $x, y \in E$, $\|x + y\|_v \leq \|x\|_v + \|y\|_v$.

Under these conditions, $(E, \|\cdot\|_v)$ is called a *pseudo-normed* vector space in v. Moreover, a basis satisfying condition (i) is called *adapted* to $\|\cdot\|_v$.

Remark 6.1.2. In [Séd24], pseudo-norms are called *local pseudo-norms*. Since in this article we only focus on local aspects, we decided to remove the qualification "local" to ease the reading.

Proposition 6.1.3. *Let $(E, \|\cdot\|_v)$ be pseudo-normed finite-dimensional K-vector space in $v \in M_K$. Let $d := \dim_K(E)$ and let $(e_1, ..., e_d)$ be a basis of E which is adapted to $\|\cdot\|_v$. Then the following assertions hold.*

(1) $\mathcal{E}_{\|\cdot\|_v} := \{x \in E : \|x\|_v < +\infty\}$ *is the restriction of scalars of* $\bigoplus_{i=1}^d K \cdot e_i$ *to* A_v. *Moreover,* $\mathcal{E}_{\|\cdot\|_v}$ *is free* A_v*-module of rank* d.

(2) $N_{\|\cdot\|_v} := \{x \in E : \|x\|_v = 0\}$ *is an* A_v*-submodule of* $\mathcal{E}_{\|\cdot\|_v}$ *and we have the equality* $N_{\|\cdot\|_v} = \mathfrak{m}_v \mathcal{E}_{\|\cdot\|_v}$.

(3) $\|\cdot\|_v$ *induces a norm on the* $\widehat{\kappa_v}$*-vector space*

$$\widehat{E}_{\|\cdot\|_v} := \mathcal{E}_{\|\cdot\|_v} \otimes_{A_v} \widehat{\kappa_v} \cong \left(\mathcal{E}_{\|\cdot\|_v}/\mathfrak{m}_v\mathcal{E}_{\|\cdot\|_v}\right) \otimes_{\kappa_v} \widehat{\kappa_v}.$$

Proof. **Proof of (1):** The defining properties of $\|\cdot\|_v$ directly imply that $\mathcal{E}_{\|\cdot\|_v}$ is an A_v-module. Let $a \in A_v \smallsetminus \{0\}$ and $x \in \mathcal{E}_{\|\cdot\|_v}$ such that $ax = 0$. Since A_v is integral, the inclusion $\mathcal{E}_{\|\cdot\|_v} \subset E$ implies that $ax = 0$ in E, thus x is torsion E and $x = 0$. Hence $\mathcal{E}_{\|\cdot\|_v}$ is a torsion-free A_v-module. Proposition 1.1.9 (5) implies that it suffices to prove that $\mathcal{E}_{\|\cdot\|_v}$ is finitely generated.

Let us show by induction on n that, for any $x \in \mathcal{E}_{\|\cdot\|_v}$, there is no linear combination $x = x_1e_1 + \cdots + x_de_d$, where $x_1, ..., x_d \in K$, such that $|\{i \in \{1, ..., d\} : x_i \in K \smallsetminus A_v\}| = n$. We first assume that $n = 1$. Let $x = x_1e_1 + \cdots + x_de_d \in \mathcal{E}_{\|\cdot\|_v}$, where $x_1, ..., x_d \in K$, such that $|\{i \in \{1, ..., d\} : x_i \in K \smallsetminus A_v\}| = 1$. By symmetry, we may assume that $x_d \notin A_v$. On the one hand, we have $\|x_de_d\|_v = +\infty$ and thus $x_de_d \notin \mathcal{E}_{\|\cdot\|_v}$. On the other hand, we have

$$x_de_d = x - x_1e_1 - \cdots - x_{d-1}e_{d-1} \in \mathcal{E}_{\|\cdot\|_v}.$$

Hence a contradiction. Assume that $n > 1$ and that the property is satisfied for $k = 1, ..., n-1$. Let $x = x_1e_1 + \cdots + x_de_d \in \mathcal{E}_{\|\cdot\|_v}$, where $x_1, ..., x_d \in K$, such that $|\{i \in \{1, ..., d\} : x_i \in K \smallsetminus A_v\}| = n$. By symmetry, we may assume that $x_1, ..., x_n \notin A_v$ and $x_{n+1}, ..., x_d \in A_v$. Since $x_n^{-1} \in A_v$, we have

$$x_n^{-1}x_1e_1 + \cdots + x_n^{-1}x_{n-1}e_{n-1} = x_n^{-1}x - e_n - x_n^{-1}x_{n+1}e_{n+1} - \cdots - x_n^{-1}x_de_d \in \mathcal{E}_{\|\cdot\|_v}.$$

The induction hypothesis yields a contradiction. Consequently, for any decomposition $x_1e_1 + \cdots + x_de_d \in \mathcal{E}_{\|\cdot\|_v}$, we have $x_1, ..., x_d \in A_v$. Hence $\mathcal{E}_{\|\cdot\|_v}$ is of finite type.

Proof of (2): The defining properties of $|\cdot\|_v$ show that $N_{\|\cdot\|_v}$ is an A_v-submodule of $\mathcal{E}_{\|\cdot\|_v}$. Moreover, we clearly have an inclusion $\mathfrak{m}_v\mathcal{E}_{\|\cdot\|_v} \subset N_{\|\cdot\|_v}$.

To prove the inverse inclusion, we show by induction on $n \geq 1$ that, for any $x \in N_{\|\cdot\|_v}$, there is no linear combination $x = x_1e_1 + \cdots + x_de_d$, where $x_1, ..., x_d \in A_v$, such that $|\{i \in \{1, ..., d\} : x_i \notin \mathfrak{m}_v\}| = n$. We first assume that $n = 1$. Let $x = x_1e_1 + \cdots + x_de_d \in \mathcal{E}_{\|\cdot\|_v}$, where $x_1, ..., x_d \in A_v$, such that $|\{i \in \{1, ..., d\} : x_i \notin \mathfrak{m}_v\}| = 1$. By symmetry, we may assume that $x_d \notin \mathfrak{m}_v$. On the one hand, we have $\|x_de_d\|_v \neq 0$, and thus $x_de_d \notin N_{\|\cdot\|_v}$. On the other hand, we have

$$x_de_d = x - x_1e_1 - \cdots - x_{d-1}e_{d-1} \in N_{\|\cdot\|_v}.$$

Hence a contradiction. Assume that $n > 1$ and that the property is satisfied for $k = 1, ..., n-1$. Let $x = x_1e_1 + \cdots + x_de_d \in N_{\|\cdot\|_v}$, where $x_1, ..., x_d \in A_v$, such that $|\{i \in \{1, ..., d\} : x_i \notin$

$\mathfrak{m}_v\}| = n$. By symmetry, we may assume that $x_1, ..., x_n \notin \mathfrak{m}_v$ and $x_{n+1}, ..., x_d \in \mathfrak{m}_v$. Since $x_n^{-1} \in A_v$, we have

$$x_n^{-1}x_1e_1 + \cdots + x_n^{-1}x_{n-1}e_{n-1} = x_n^{-1}x - e_n - x_n^{-1}x_{n+1}e_{n+1} - \cdots - x_n^{-1}x_de_d \in N_{\|\cdot\|_v}.$$

The induction hypothesis yields a contradiction. Consequently, for any decomposition $x_1e_1 + \cdots + x_de_d \in N_{\|\cdot\|_v}$, we have $x_1, ..., x_d \in \mathfrak{m}_v$.

Let $x = x_1e_1 + \cdots + x_de_d \in N_{\|\cdot\|_v}$, where $x_1, ..., x_d \in \mathfrak{m}_v$, let $\delta := \gcd(x_1, ..., x_d) \in \mathfrak{m}_v$. Then we have $x = \delta x'$, where $x' \in \mathcal{E}_{\|\cdot\|_v}$. Hence $N_{\|\cdot\|_v} \subset \mathfrak{m}_v\mathcal{E}_{\|\cdot\|_v}$. This concludes the proof of (2).

Proof of (3): By definition, $\widehat{E}_{\|\cdot\|_v}$ is the extension of scalars of $\mathcal{E}_{\|\cdot\|_v}$ via the morphism $A_v \to \widehat{\kappa_v}$. Thus we have an isomorphism of $\widehat{\kappa_v}$-modules $\widehat{E}_{\|\cdot\|_v} \cong \left(\mathcal{E}_{\|\cdot\|_v}/\mathfrak{m}_v\mathcal{E}_{\|\cdot\|_v}\right) \otimes_{\kappa_v} \widehat{\kappa_v}$. Furthermore, (1) and (2) imply that the restriction $\|\cdot\|_{v|A_v}$ induce a norm on $\widehat{E}_{\|\cdot\|v}$. □

Notation 6.1.4. Let $(E, \|\cdot\|_v)$ be pseudo-normed finite-dimensional K-vector space in $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in M_K$.

(1) In analogy with Notation 2.1.3, we call
   - $\mathcal{E}_{\|\cdot\|_v} := \{x \in E : \|x\|_v < \infty\}$ the *finiteness module* of $\|\cdot\|_v$;
   - $N_{\|\cdot\|_v} := \{x \in E : \|x\|_v = 0\}$ the *kernel* of $\|\cdot\|_v$;
   - $\widehat{E}_{\|\cdot\|_v} := \mathcal{E}_{\|\cdot\|_v} \otimes_{A_v} \widehat{\kappa_v}$ the *residue vector space* of $\|\cdot\|_v$. By abuse of notation, we denote by $\|\cdot\|_v$ the induced norm on $\widehat{E}_{\|\cdot\|_v}$, called the *residue norm* of $\|\cdot\|_v$.

(2) By "let $(\|\cdot\|, \mathcal{E}, N, \widehat{E})$ be a pseudo-norm on the K-vector space E in v", we mean that $\|\cdot\|$ is a pseudo-norm on K in v with finiteness module $\mathcal{E}$, kernel N and residue vector space $\widehat{E}$.

(3) By abuse of notation, by "let $\|\cdot\|_v$ be a pseudo-norm on E in v", we mean the pseudo-norm $(\|\cdot\|_v, \mathcal{E}_v, N_v, \widehat{E}_v)$.

In fact, in analogy with the case of pseudo-valued fields, a pseudo-normed vector space is determined by the objects defined in Notation 6.1.4. More precisely, let

- $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in M_K$;
- $\mathcal{E}_v$ be a free A_v-module of rank $d < +\infty$;
- $\|\cdot\|_v^\wedge$ be a norm on $\widehat{E}_v := \mathcal{E}_v \otimes_{A_v} \widehat{\kappa}_v$.

Define the map

$$\|\cdot\|_v : \left| \begin{array}{ccc} \mathcal{E}_v & \longrightarrow & \mathbb{R}_{\geq 0} \\ a & \longmapsto & \|\overline{a}\|_v^\wedge \end{array} \right.$$

Then $N_v := \{x \in E : \|x\|_v = 0\} = \mathfrak{m}_v\mathcal{E}_v$. By lifting to $\mathcal{E}_v$ a basis of $\widehat{E}_v$, we get a basis $(e_1, ..., e_d)$ of $\mathcal{E}_v$ such that $\|e_i\|_v > 0$ for all $i = 1, ..., d$. Then we can extend $\|\cdot\|_v$ to $E := \mathcal{E}_v \otimes_{A_v} K$ by setting $\|x\|_v = +\infty$ if $x \notin \mathcal{E}_v$.

Proposition 6.1.5. *We use the same notation as above. Then $\|\cdot\|_v$ is a pseudo-norm in v on E. Moreover, this construction is inverse to the one in Proposition 6.1.3.*

Proof. By construction of $\|\cdot\|_v$, we have $\|0\|_v = 0$. We can see $(e_1, ..., e_d)$ as a basis of E such that $0 < \|e_i\|_v < +\infty$ for all $i = 1, ..., d$. Hence $\|\cdot\|_v$ satisfies condition (i) of Definition 6.1.1.

Let $(\lambda, x) \in K \times E$ such that $\{|\lambda|_v, \|x\|_v\} \neq \{0, +\infty\}$. We distinguish three cases. First, assume that $|\lambda|_v \neq +\infty \neq \|x\|_v$. Then by definition of $\|\cdot\|_v$ on $\mathcal{E}_v$, we have $\|\lambda x\|_v = |\lambda|_v \|x\|_v$. If now $\|x\| = +\infty$, i.e. $x \in E \smallsetminus \mathcal{E}_v$, then $|\lambda|_v \neq 0$, i.e. $\lambda \in K \smallsetminus \mathfrak{m}_v$, hence $\lambda x \in E \smallsetminus \mathcal{E}_v$ and $\|\lambda x\|_v = +\infty = |\lambda|_v \|x\|_v$. Finally, if $|\lambda|_v = +\infty$, i.e. $\lambda \in K \smallsetminus A_v$, then we have $\|x\| \neq 0$, i.e $x \in E \smallsetminus \mathfrak{m}_v \mathcal{E}_v$, hence $\lambda x \in E \smallsetminus \mathcal{E}_v$ and $\|\lambda x\|_v = +\infty = |\lambda|_v \|x\|_v$. Thus $\|\cdot\|_v$ satisfies condition (ii) of Definition 6.1.1.

Since $\|\cdot\|_v^\wedge$ is a norm on $\widehat{E_v}$, for any $x, y \in \mathcal{E}_v$, we have $\|x+y\|_v \leq \|x\|_v + \|y\|_v$. Then the triangle inequality for $\|\cdot\|_v$ on E follows from the fact that $+\infty$ is the maximal element of $[0, +\infty]$.

From the construction, we directly see that it provides an inverse to the one in Proposition 6.1.3. □

Remark 6.1.6. Similarly to the case of pseudo-valued fields, we can use Proposition 6.1.5 to define properties of pseudo-normed vector spaces from the corresponding property of the residue norm. For instance, a pseudo-norm on a vector space is called *ultrametric* if so is its residue norm.

To study restrictions and quotients for pseudo-norms, we will make use of the following lemma.

Lemma 6.1.7. *Let $(E, (\|\cdot\|_v, \mathcal{E}_v, N_v, \widehat{E}_v))$ be pseudo-normed finite-dimensional K-vector space in $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in M_K$. Let $(e_1, .., e_d)$ be a basis of E such that, for all $i = 1, .., d$, we have $e_i \in \mathcal{E}_v \smallsetminus N_v$, i.e. $\|e_i\|_v \in \mathbb{R}_{>0}$. Let $(e'_1, ..., e'_d)$ be another basis of E. Then, for all $i \in \{1, ..., d\}$, there exists $\lambda_i \in K$ such that $\lambda_i e'_i \in \mathcal{E}_v \smallsetminus N_v$.*

Proof. We fix $i \in \{1, ..., d\}$. Then there exist $a_1^{(i)}, ..., a_d^{(i)} \in K$ such that $e'_i = \sum_{j=1}^d a_j^{(i)} e_j$. By symmetry, we may assume that $(a_1^{(i)})^{-1} a_j^{(i)} \in A_v$ for all $j = 1, ..., d$. Then, for all $j = 1, ..., d$, we have $(a_1^{(i)})^{-1} e'_i \in \mathcal{E}_v$. Furthermore, writing

$$(a_1^{(i)})^{-1} e'_i = e_1 + (a_1^{(i)})^{-1} a_2^{(i)} e_2 + \cdots + (a_1^{(i)})^{-1} a_d^{(i)} e_d,$$

we deduce $(a_1^{(i)})^{-1} e'_i \notin N_v = \mathfrak{m}_v \mathcal{E}_v$. □

Remark 6.1.8. Lemma 6.1.7 shows that any basis of a pseudo-normed vector space $(E, \|\cdot\|_v)$ can be scaled so that it is adapted in the sense of Definition 6.1.1. In particular, Definition 6.1.1 remains unchanged if one puts the condition

(1') For any basis $(e_1, ..., e_d)$ of E, there exists a family $(\lambda_1, ..., \lambda_d) \in K^d$ such that $\|\lambda_i e_i\|_v \in \mathbb{R}_{>0}$ for all $i = 1, ..., d$

instead of condition (1).

6.2. **Algebraic constructions for pseudo-norms.** We now extend the usual algebraic construction for normed vector spaces in our context (cf. e.g. [CM19], §1.1 for more details).

Definition 6.2.1. (1) Let $(E, \|\cdot\|_v)$ be a pseudo-normed finite-dimensional K-vector space over K in $v \in M_K$. Let $F \subset E$ be a vector subspace of E. Let $(e_1, ..., e_r)$ be a basis of F enlarged in a basis $(e_1, ..., e_r, e_{r+1}, ..., e_d)$ of E. Lemma 6.1.7 ensures that we may assume that, for all $i = 1, ..., d$, we have $e_i \in \mathcal{E}_v \smallsetminus \mathfrak{m}_v$. Then the restriction restriction of $\|\cdot\|_v$ to F is a pseudo-norm in v on F called the *restriction* of $\|\cdot\|_v$ to F. By abuse of notation, unless explicitly mentioned, we denote this pseudo-norm by

$\|\cdot\|_v$. Moreover, the finiteness module of $\|\cdot\|_v$ (the pseudo-norm on F), denoted by $\mathcal{F}_v$, has $(e_1, ..., e_r)$ as a basis.

(2) Let $(E, \|\cdot\|_v)$ be a pseudo-normed finite-dimensional K-vector space in $v \in M_K$. Let $F \subset E$ be a vector subspace of E. Let $(e_1, ..., e_r)$ be a basis of F enlarged in a basis $(e_1, ..., e_r, e_{r+1}, ..., e_d)$ of E. Lemma 6.1.7 ensures that we may assume that, for all $i = 1, ..., d$, we have $e_i \in \mathcal{E}_v \smallsetminus \mathfrak{m}_v$. Using the same notation as in (1), $\mathcal{E}_v/\mathcal{F}_v$ is a free A_v-module of rank $d - r$ and is spanned by $e_{r+1}, ..., e_d$. On the $\widehat{\kappa_v}$-quotient vector space $\widehat{E_v}/\widehat{F_v}$, we consider the quotient norm induced by the residue norm of $\|\cdot\|_v$. Then Proposition 6.1.5 yields a pseudo-norm in v on E/F, called the *quotient* pseudo-norm and denoted by $\|\cdot\|_{E/F,v}$. Moreover, we have

$$\forall \overline{x} \in E/F, \quad \|\overline{x}\|_{E/F,v} = \inf_{x \in \pi^{-1}(\{\overline{x}\})} \|x\|_v,$$

where $\pi : E \to E/F$ denotes the canonical quotient map.

(3) Let $(E, \|\cdot\|_v)$ be a pseudo-normed finite-dimensional K-vector space in $v \in M_K$. Le $\mathcal{E}_v^\vee := \mathrm{Hom}_{A_v}(\mathcal{E}_v, A_v)$, it is a free A_v-module of rank $\dim_K(E)$. We denote by $\|\cdot\|_{v,*}^\wedge$ the dual norm on $(\widehat{E}_v)^\vee$ of the residue norm $\|\cdot\|_v^\wedge$ of $\|\cdot\|_v$. Then Proposition 6.1.5 yields a pseudo-norm in v on $E^\vee \cong \mathcal{E}_v^\vee \otimes_{A_v} K$, called the *dual* pseudo-norm of $\|\cdot\|_v$ and denoted by $\|\cdot\|_{v,*}$. Furthermore, we have

$$\forall \varphi \in E^\vee, \quad \|\varphi\|_{v,*} = \sup_{x \in \mathcal{E}_v \smallsetminus \mathfrak{m}_v \mathcal{E}_v} \frac{|\varphi(x)|_v}{\|x\|_v}.$$

(4) Let $(E, \|\cdot\|_v)$ and $(E', \|\cdot\|'_v)$ be two pseudo-normed finite-dimensional K-vector spaces in $v =\in M_K$. The data of $\mathcal{E}_v \otimes_{A_v} \mathcal{E}'_v$ and respectively of the π-tensor product and the ϵ-tensor product norms on $\widehat{E}_v \otimes_{\widehat{\kappa}_v} \widehat{E'}_v$ induce pseudo-norms in v on $E \otimes_K E'$ respectively called the *π-tensor product* and the *ϵ-tensor product* pseudo-norms.

(5) Let $i \geq 1$ be an integer. $(E, \|\cdot\|_v)$ be a pseudo-normed finite-dimensional K-vector space over K in $v \in M_K$. We define the i^{th}*π-exterior power pseudo-norm* $\|\cdot\|_{v,\Lambda^i_\pi}$ of $\|\cdot\|_v$ on $\Lambda^i E$ is defined as the quotient norm of the π-tensor product norm of $\|\cdot\|_v$ on $E^{\otimes i}$. Likewise, the i^{th}*ϵ-exterior power pseudo-norm* $\|\cdot\|_{v,\Lambda^i_\epsilon}$ of $\|\cdot\|_v$ on $\Lambda^i E$ is defined the quotient norm of the ϵ-tensor product norm of $\|\cdot\|_v$ on $E^{\otimes i}$. In the $i = \dim_k(E)$ case, the $i^{th}\pi$-exterior power norm on $\Lambda^i E = \det(E)$ is called the *determinant pseudo-norm* of $\|\cdot\|_v$ and we denote it by $\|\cdot\|_{v,\det}$.

Proposition 6.2.2. *Let $(E, (\|\cdot\|_v, \mathcal{E}_v, N_v, \widehat{E}_v))$ be a pseudo-normed finite-dimensional K-vector space in $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in M_K$. Let G be a quotient of E and denote by $\|\cdot\|_{G,v}$ the quotient pseudo-norm on G. Then the dual pseudo-norm $\|\cdot\|_{G,v,*}$ on $G^\vee$ identifies with the restriction of the pseudo-norm $\|\cdot\|_{v,*}$ on $E^\vee$ to $G^\vee$.*

Proof. Let $\mathcal{G}_v$ denote the finiteness module of $\|\cdot\|_G$, it is a quotient of $\mathcal{E}_v$. Thus we have an inclusion of duals $\mathcal{G}_v^\vee \hookrightarrow \mathcal{E}_v^\vee$. The latter are respectively the finiteness modules of $\|\cdot\|_{G,v,*}$ and $\|\cdot\|_{v,*}$. To conclude the proof, it is enough to show

$$\forall \varphi \in \mathcal{G}_v^\vee \smallsetminus \mathfrak{m}_v \mathcal{G}_v^\vee, \quad \|\varphi\|_{G,*} = \|\varphi\|_{v*}.$$

This is obtained by lifting the corresponding assertion ([CM19], Proposition 1.1.20) for the residue norms. □

Proposition 6.2.3. *Let $(E, \|\cdot\|_v)$ be a pseudo-normed finite-dimensional K-vector space in $v \in M_K$ Then the inequality*

$$\|\cdot\|_{v,**} \leq \|\cdot\|_v$$

holds, where $\|\cdot\|_{v,**}$ denotes the dual pseudo-norm of $\|\cdot\|_{v,*}$ on $E^{\vee\vee} \cong E$. Moreover, if either v is Archimedean, or if v is non-Archimedean and the pseudo-norm $\|\cdot\|_v$ is ultrametric, then we have

$$\|\cdot\|_{v,**} = \|\cdot\|_v$$

Proof. Definition 6.2.1 (3) ensures that, for any $x \in \mathcal{E}_v \smallsetminus \mathfrak{m}_v\mathcal{E}_v$, for any $\varphi \in \mathcal{E}_v^\vee$, we have

$$|\varphi(x)|_v \leq \|\varphi\|_{v,*} \cdot \|x\|_v.$$

Hence, for any $x \in \mathcal{E}_v \smallsetminus \mathfrak{m}_v\mathcal{E}_v$, we have

$$\|x\|_{v,**} = \sup_{\varphi \in \mathcal{E}_v^\vee \smallsetminus \mathfrak{m}_v \mathcal{E}_v^\vee} \frac{|\varphi(x)|_v}{\|\varphi\|_{v,*}} \leq \|x\|_v.$$

Note the, for any $x \in \mathfrak{m}_v\mathcal{E}_v$, for any $\varphi \in \mathcal{E}_v^\vee$, we have $\varphi(x) \in \mathfrak{m}_v$, i.e. $|\varphi(x)|_v = 0$, and thus $\|x\|_{v,**} = \|x\|_* = 0$. This concludes the proof of the first statement.

Now assume that the pseudo-absolute value v is either Archimedean or v is non-Archimedean and the pseudo-norm $\|\cdot\|_v$ is ultrametric. Then Proposition 1.1.18 and Corollary 1.2.12 of [CM19] give

$$\forall \overline{x} \in \widehat{E}_v, \quad \|\overline{x}\|_{v,**}^\wedge = \|\overline{x}\|_v^\wedge.$$

Thus we obtain

$$\forall = x \in \mathcal{E}_v, \quad \|x\|_{v,**} = \|x\|_v.$$

To conclude the proof, it suffices to see that, for any $x \in E \smallsetminus \mathcal{E}_v$, we have $\|x\|_{v,**} = +\infty$. Let $(e_1, ..., e_r)$ a basis of E which is adapted to $\|\cdot\|_v$. Then $(e_1^\vee, ..., e_r^\vee)$ is a basis of $E^\vee$ which is adapted to $\|\cdot\|_{v,*}$. Let $x = x_1e_1 + \cdots + x_re_r \in E \smallsetminus \mathcal{E}_v$. By symmetry, we may assume that $x_1 \in K \smallsetminus A_v$. By definition,

$$\|x\|_{v,**} = \sup_{\varphi \in \mathcal{E}_v^\vee \smallsetminus \mathfrak{m}_v \mathcal{E}_v^\vee} \frac{|\varphi(x)|_v}{\|\varphi\|_{v,*}}.$$

Take $\varphi = e_1^\vee \in \mathcal{E}_v^\vee \smallsetminus \mathfrak{m}_v\mathcal{E}_v^\vee$. Then $|\varphi(x)|_v = |x_1|_v = +\infty$ and $\|\varphi\|_{v,*} \in \mathbb{R}_{>0}$. Therefore $\|x\|_{v,**} = +\infty$. □

We now generalise the Hadamard inequality to the pseudo-norm case.

Proposition 6.2.4. *Let $(E, \|\cdot\|_v)$ be a pseudo-normed finite-dimensional K-vector space in $v \in M_K$. Let $(e_1, .., e_r)$ be a basis of E which is adapted to $\|\cdot\|_v$, namely, for all $i = 1, .., r$, we have $e_i \in \mathcal{E}_v \smallsetminus N_v$, i.e. $\|e_i\|_v \in \mathbb{R}_{>0}$. Then, for any $\eta \in \det(E)$, we have the equality*

$$\|\eta\|_{v,\det} = \inf \left\{ \|x_1\|_v \cdots \|x_r\|_v : x_1, ..., x_r \in \mathcal{E}_v \text{ and } \eta = x_1 \wedge \cdots \wedge x_r \right\}.$$

Proof. We may assume that $\eta \neq 0$. Let $\eta_0 = e_1 \wedge \cdots \wedge e_r$. From the construction of tensor product and quotient pseudo-norms above, we obtain that $\|\eta_0\|_{v,\det} \in \mathbb{R}_{>0}$. Since $\det(E)$ is of dimension 1, there exists $a \in K$ such that $\eta = a\eta_0$. We first consider the $\|\eta\|_{v,\det} \in \mathbb{R}_{>0}$

case. Then there exist $x_1, ..., x_r \in \mathcal{E}_v$ such that $\eta = x_1 \wedge \cdots \wedge x_r$. From the definition of $\|\cdot\|_{v,\det}$, we directly obtain

$$\|\eta\|_{v,\det} \leq \|x_1\|_v \cdots \|x_r\|_v.$$

We assume that $a \in A_v \smallsetminus \mathfrak{m}_v$, namely $\|\eta\|_{v,\det} \in \mathbb{R}_{>0}$. Since $\|\eta\|_{v,\det}$ is computed from the residue norm on $\widehat{E}_v$, the classical Hadamard inequality implies that

$$\|\eta\|_{v,\det} = \inf\left\{\|x_1\|_v \cdots \|x_r\|_v : x_1, ..., x_r \in \mathcal{E}_v \text{ and } \eta = x_1 \wedge \cdots \wedge x_r\right\}.$$

Now assume that $a \in \mathfrak{m}_v$, then $\eta = (ae_1) \wedge \cdots \wedge e_r$ and

$$\|\eta\|_{v,\det} = \|ae_1\|_v \cdots \|e_r\| = 0,$$

hence we obtain the Hadamard equality.

Finally, we treat the $a \in K \smallsetminus A_v$ case. Now $\|\eta\|_{v,\det} = +\infty$. Moreover, there is no possible decomposition $\eta = x_1 \wedge \cdots \wedge x_r$, where $x_1, ..., x_r \in \mathcal{E}_v$ (otherwise we would have $\|\eta\|_{v,\det} < +\infty$). Thus the RHS desired equality is $\inf(\varnothing) = +\infty$. □

7. Global space of pseudo-absolute values

We now define various notions of global spaces of pseudo-absolute values on a given field. Such spaces will contain all the relevant pseudo-absolute values in the context of topological adelic curves. Throughout this section, we fix a field K.

7.1. **Definitions.**

Definition 7.1.1. Recall that we denote by M_K the set of pseudo-absolute values on K. By definition, we have an inclusion

$$M_K \subset \prod_{a \in K} [0, +\infty]$$

of sets. We equip M_K with the subspace topology induced by the product topology of the RHS above, where each $[0, +\infty]$ is endowed with the one-point compactification topology induced by the Euclidean topology on $[0, +\infty[$. This is the coarsest topology making the evaluation maps $|a|. : M_K \to [0, +\infty]$ continuous.

Theorem 7.1.2. *M_K is a non-empty, compact Hausdorff topological space.*

Proof. M_K contains the trivial absolute value on K and is thus non-empty. Moreover, by Tychonoff theorem, $\prod_{a \in K}[0, +\infty]$ is a compact Hausdorff space and thus M_K is Hausdorff and remains to prove that the space M_K is defined by closed conditions w.r.t. the pointwise convergence topology.

Let I be a directed set and let $(|\cdot|_i)_{i \in I}$ be a generalised sequence in M_K which converges to $|\cdot| : K \to [0, +\infty]$. We prove that $|\cdot|$ is a pseudo-absolute value on K. By definition of the pointwise convergence topology, for all $a \in K$, we have a generalised sequence $(|a|_i)_{i \in I}$ in $[0, +\infty]$ which converges to $|a|$. Hence $|1| = \lim_{i \in I} |1|_i = 1$ and $|0| = \lim_{i \in I} |0|_i = 0$. Let $a, b \in K$. For all $i \in I$, we have $|a + b|_i \leq |a|_i + |b|_i$, hence

$$|a + b| = \lim_{i \in I} |a + b|_i \leq \lim_{i \in I} |a|_i + \lim_{i \in I} |b|_i = |a| + |b|.$$

Finally, if $a, b \in K$ are such that $\{|a|, |b|\} \neq \{0, +\infty\}$, then there exists $i_0 \in I$ such that

$$\forall i \in I, i \geq i_0 \Rightarrow \{|a|_i, |b|_i\} \neq \{0, +\infty\} \Rightarrow |ab|_i = |a|_i |b|_i.$$

Whence

$$|ab| = \lim_{i\in I, i\geq i_0} |ab|_i = \lim_{i\in I, i\geq i_0} |a|_i|b|_i = \lim_{i\in I} |a|_i \lim_{i\in I} |b|_i = |a||b|.$$

□

Definition 7.1.3. Let L/K be a field extension. There is a restriction map $\pi_{L/K} : M_L \to M_K$. It is continuous (by definition of the topologies of M_L and M_K), surjective, and proper (cf. Proposition 7.1.2).

Notation 7.1.4. Let K be a field. We introduce the following notation.

- $M_{K,\infty}$ denotes the set of Archimedean pseudo-absolute values on K.
- $M_{K,\mathrm{um}}$ denotes the set of ultrametric pseudo-absolute values on K.
- $M_{K,\mathrm{sn}}$ denotes the set of pseudo-absolute on K values whose kernel is non-zero.
- $M_{K,\mathrm{triv}}$ denotes the set of pseudo-absolute values on K whose residual absolute value is trivial.
- $M_{K,\mathrm{disc}}$ denotes the set of discrete pseudo-absolute values, namely the set of pseudo-absolute values on K that correspond either to a discrete non-Archimedean absolute value on K or to a pseudo-absolute value whose finiteness ring is a discrete valuation ring that is not a field.

7.2. **Examples.**

7.2.1. *Number fields.* Let K be a number field and $\mathcal{O}_K$ be its ring of integers endowed with the norm

$$\|\cdot\|_\infty := \max_{\sigma\hookrightarrow\mathbb{C}} |\cdot|_\sigma,$$

where σ runs over the set of embeddings of K in $\mathbb{C}$.

Proposition 7.2.1. *M_K is homeomorphic to $\mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$, the analytic spectrum in the sense of Berkovich of the Banach ring $(\mathcal{O}_K, \|\cdot\|_\infty)$.*

Proof. Let $|\cdot| : K \to [0, +\infty]$ be a pseudo-absolute value on K which is not an absolute value. Denote by A its finiteness ring. Then A corresponds to a non-trivial valuation of K and is thus of rank 1. Therefore, by Ostrowski theorem, it corresponds to some prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$. The residue field of A being finite, the residue absolute value induced by $|\cdot|$ needs to be trivial. Therefore the restriction of $|\cdot|$ to $\mathcal{O}_K$ corresponds to the extremal point of the branch associated with $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$ in the analytic spectrum $\mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$, namely the map

$$(a \in \mathcal{O}_K) \mapsto \begin{matrix} 0 \text{ if } a \in \mathfrak{p}, \\ 1 \text{ if } a \notin \mathfrak{p}. \end{matrix}$$

Conversely, any extremal point of $\mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$ gives rise to a pseudo-absolute value on K which is not an absolute value. These constructions are inverse to each other. The remaining points of $\mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$ correspond directly to absolute values on K. Therefore we obtain a bijection $\varphi : M_K \to \mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$ which is continuous by definition of the topologies of M_K and $\mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$. We can conclude the proof as $M_K, \mathcal{M}(\mathcal{O}_K, \|\cdot\|_\infty)$ are both compact Hausdorff. □

7.2.2. *Function fields over* $\mathbb{C}$. Let us describe the Archimedean part of the space of pseudo-absolute values of a complex function field of transcendence degree 1.

Proposition 7.2.2. *Let $K = \mathbb{C}(T)$. We have a homeomorphism*

$$M_{K,\infty} \cong \mathbb{P}^1(\mathbb{C}) \times]0,1].$$

Proof. Let $v = (|\cdot|, A, \mathfrak{m}) \in M_{K,\infty}$. Then A is a non trivial valuation ring of $\mathbb{C}(T)$ and thus is of the form

$$A := \{f \in K : \mathrm{ord}(f, z) \geq 0\}$$

for some $z \in \mathbb{C} \cup \{\infty\}$. Then the residue field $A/\mathfrak{m}$ is $\mathbb{C}$ endowed with an Archimedean absolute value, necessarily of the form $|\cdot|_\infty^\epsilon$ where $\epsilon \in]0,1]$. Hence $v = v_{z,\epsilon,\infty}$, where $z \in \mathbb{P}^1(\mathbb{C})$ (using the notation of Example 2.2.1 (2)). Thus we have a bijection $\varphi : M_{k,\infty} \to \mathbb{P}^1(\mathbb{C}) \times]0,1]$. To show that φ is continuous, it suffices to prove that its composition with the maps $\pi_1 : \mathbb{P}^1(\mathbb{C}) \times]0,1] \to \mathbb{P}^1(\mathbb{C})$ and $\pi_2 : \mathbb{P}^1(\mathbb{C}) \times]0,1] \to]0,1]$ are continuous. We have

$$\pi_1 \circ \varphi : \left| \begin{array}{ccc} M_{k,\infty} & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ v_{z,\epsilon,\infty} & \longmapsto & z \end{array} \right. .$$

It is continuous by definition of the analytic topology on $\mathbb{P}^1(\mathbb{C})$. $\pi_2 \circ \varphi : |\cdot|_x \in M_{K,\infty} \mapsto \log(|2(x)|)/\log(2) \in]0,1]$ is also continuous. Furthermore, for all $f \in K$, the map

$$|\cdot|_{\varphi^{-1}(\cdot)} : \left| \begin{array}{rcl} \mathbb{P}^1(\mathbb{C}) \times]0,1] & \longrightarrow & [0,+\infty] \\ (z,\epsilon) & \longmapsto & |f(z)|^\epsilon \end{array} \right.$$

is continuous. Hence φ is a homeomorphism. ☐

Remark 7.2.3. (1) The ultrametric part of $M_{\mathbb{C}(T)}$ does not admit an easy description. This is due to the fact that the valuative structure of $\mathbb{C}(T)$ can be very wild if one does not impose any kind of continuity condition. This question will be addressed by introducing the notion of *integral structure* (cf. Proposition 9.4.1).

(2) As pointed out by the referee, studying the Archimedean part of the space of pseudo-absolute value on a function field of higher transcendence degree is a natural direction to investigate. For instance, in the case of a complex function field of two variables $K = \mathbb{C}(T_1, T_2)$, a description of the valuation rings of K that are trivial on $\mathbb{C}$ is given in [Zar39, CZ97]. A pseudo-absolute value $v \in M_{K,\infty}$ has finiteness ring whose residue field is $\mathbb{C}$. Therefore, its underlying valuation must be of type (ii)-(iv) in the terminology of ([CZ97], first Theorem of §3). This gives a set-theoretic description of $M_{K,\infty}$. Nonetheless, finding a topological description from this approach seems to be a quite challenging problem. Indeed, the pseudo-absolute belonging to $M_{K,\infty}$ correspond, roughly speaking, to branches of algebraic or analytic curves in some model of K over $\mathbb{C}$. This is a first hint that the topology of $M_{K,\infty}$ should be of "Zariski-Riemann fashion". In Corollary 10.2.4 (2.c), we will give such a topological description for general function fields over $\mathbb{C}$.

7.3. **Connection to Zariski-Riemann spaces.** Let K be a field with prime subring k. The construction of the Zariski-Riemann space $\mathrm{ZR}(K/k)$ was recalled in §1.4. Then there is a map $j : M_K \to \mathrm{ZR}(K/k)$ mapping a pseudo-absolute value $v \in M_K$ to its finiteness ring $A_v \in \mathrm{ZR}(K/k)$. The map j is called the *specification map*.

Proposition 7.3.1. *The specification map $j : M_K \to \mathrm{ZR}(K/k)$ is continuous, where $\mathrm{ZR}(K/k)$ is equipped with the Zariski topology.*

Proof. Let $a_1, ..., a_r$ be elements of K. Let $\mathcal{U} := \{V \in \mathrm{ZR}(K/k) : a_1, ..., a_r \in V\}$ be a basic open subset of $\mathrm{ZR}(K/k)$. Then we have

$$j^{-1}(\mathcal{U}) = \bigcap_{i=1}^{r} \{v \in M_K : |a_i|_v \in [0, +\infty[\},$$

which is open. $\square$

Remark 7.3.2. In the next sections, we will be able to tell more about the specification map: namely j is open and the generic fibre, i.e. the subspace of absolute values, is dense in K if K is countable.

8. Local analytic spaces

In §3, we gave a description of the behaviour of pseudo-absolute values w.r.t. algebraic extension of the base field. In view of §4, the transcendental case is much more complicated. In this section, we precise the connection with Zariski-Riemann spaces introduced in §7.3 in view of Theorem 1.4.3 to give a description of arbitrary extensions of pseudo-absolute values. More precisely, the set of extensions of a given pseudo-absolute value to an arbitrary field extension admits a natural description as a projective limit of Berkovich analytifications given by sub-models over the finiteness ring. We start this section by defining these analytic spaces that we call *(sub)-model analytic spaces* (§8.1). We then prove the announced "Zariski-Riemann type" description (Theorem 8.2.4). We conclude the section by relating local (sub)-model analytic spaces and "Zariski-Riemann type" spaces (§8.3).

Throughout this section, we fix a field K and a pseudo-absolute $v = (|\cdot|, A, \mathfrak{m}, \kappa) \in M_K$, namely $|\cdot| : K \to [0, +\infty]$ is a pseudo-absolute value with finiteness ring A, kernel $\mathfrak{m}$ and residue field κ.

8.1. Local (sub-)model analytic space.

Let $X \to \mathrm{Spec}(K)$ be a K-scheme, which is assumed to be projective for simplicity. Assume that we have a projective model $\mathcal{X}$ of X over A (cf. §1.3). The special fibre $\mathcal{X}_s := \mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa)$ is now a projective κ-scheme. Denote by $\widehat{\kappa}$ the completion of κ w.r.t. the residue absolute value on κ induced by the pseudo-absolute value v. We call $\widehat{\mathcal{X}_s} := \mathcal{X}_s \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\widehat{\kappa})$ the *completed special fibre* of the model $\mathcal{X}$. Now $\widehat{\mathcal{X}}$ is a projective $\widehat{\kappa}$-scheme and $\widehat{\kappa}$ is a completely (real-)valued field. Henceforth, we can construct the Berkovich analytic space $\widehat{\mathcal{X}_s}^{\mathrm{an}}$ attached to $\widehat{\mathcal{X}_s}$.

Definition 8.1.1. With the above notation, the space $\widehat{\mathcal{X}_s}^{\mathrm{an}}$ is called the *local model analytic space* attached to X w.r.t. the projective model $\mathcal{X}$ over v. Note that if v is a usual absolute value on K, i.e. when $A = K = \kappa$, $\mathcal{X} \cong X$ and the corresponding local analytic space is just $(X \otimes_K \widehat{K})^{\mathrm{an}}$.

In order to perform Arakelov geometry in the global setting over pseudo-absolute valued field, we will use the local model analytic spaces from Definition 8.1.1. Using the classical theory of Berkovich analytic spaces, we can adapt the usual notion of metric on a line bundle.

Definition 8.1.2. Let X be a projective scheme over $\mathrm{Spec}(K)$ and let L be an invertible $\mathcal{O}_X$-module. We call *local model pseudo-metric* in v on L the data $((\mathcal{X}, \mathcal{L}), \varphi)$ where:

(1) $(\mathcal{X},\mathcal{L})$ is a flat projective model of (X,L) over A (cf. §1.3), with special fibre $\mathcal{X}_s$ and completed special fibre $\widehat{\mathcal{X}_s}$;
(2) φ is a metric on $\widehat{\mathcal{L}_s}$, where $\widehat{\mathcal{L}_s}$ is the pullback of $\mathcal{L}$ to completed special fibre $\widehat{\mathcal{X}_s}$.

A local model pseudo-metric $((\mathcal{X},\mathcal{L}),\varphi)$ on L is called *continuous* if the metric φ is so. In the case where $\mathcal{X}$ is fixed, by "let $(\mathcal{L},\varphi)$ be a local pseudo-metric on L", we mean that $((\mathcal{X},\mathcal{L}),\varphi)$ is a local model pseudo-metric on L which is denoted by $(\mathcal{L},\varphi)$.

Let X be a projective scheme over $\mathrm{Spec}(K)$ and let L be an invertible $\mathcal{O}_X$-module. Let $((\mathcal{X},\mathcal{L}),\varphi)$ be a local model-pseudo metric in v on L. The attached local model analytic space $(\mathcal{X}\otimes_A\widehat{\kappa})^{\mathrm{an}}$ is a compact Hausdorff topological space and the metric φ defines a supremum seminorm on the $\widehat{\kappa}$-vector space of global sections $H^0(\widehat{\mathcal{X}_s},\widehat{\mathcal{L}_s})$ defined by

$$\forall \overline{s}\in H^0(\widehat{\mathcal{X}_s},\widehat{\mathcal{L}_s}),\quad \|\overline{s}\|_\varphi := \sup_{x\in X_v^{\mathrm{an}}}|\overline{s}|_\varphi(x)\in\mathbb{R}_{\geq 0}.$$

Assume that the model $(\mathcal{X},\mathcal{L})$ is flat, coherent and that the special fibre $\mathcal{X}_s$ is geometrically reduced. Then Proposition 1.3.4 implies that $H^0(\mathcal{X},\mathcal{L})$ is a free A-module of finite rank. Since $\mathcal{X}_s$ is reduced, ([CM19], Proposition 2.1.16 (1)) implies that the seminorm $\|\cdot\|_\varphi$ is a norm on $H^0(\widehat{\mathcal{X}_s},\widehat{\mathcal{L}_s})$. Proposition 1.3.4 (2.ii) implies that $H^0(\mathcal{X},\mathcal{L})\otimes_A\kappa$ is a vector subspace of $H^0(\mathcal{X}_s,\mathcal{L}_s)$. Thus $\|\cdot\|_\varphi$ induces a norm on $H^0(\mathcal{X},\mathcal{L})\otimes_A\widehat{\kappa}$ and Proposition 6.1.5 implies that we can lift $\|\cdot\|_\varphi$ to a pseudo-norm in v on $H^0(X,L)$, which is again denoted by $\|\cdot\|_\varphi$ by abuse of notation.

Now we assume that the model $(\mathcal{X},\mathcal{L})$ is flat and coherent. We denote by $(\widehat{\mathcal{X}_s})_{\mathrm{red}}$ the reduced scheme structure on $\widehat{\mathcal{X}_s}$ and by $(\widehat{\mathcal{L}_s})_{\mathrm{red}}$ the restriction of $\widehat{\mathcal{L}_s}$ to $(\widehat{\mathcal{X}_s})_{\mathrm{red}}$. Then ([CM19], Proposition 2.1.16) implies that the kernel of the supremum semi-norm on $H^0(\widehat{\mathcal{X}_s},\widehat{\mathcal{L}_s})$ coincides with the kernel of the natural map $H^0(\widehat{\mathcal{X}_s},\widehat{\mathcal{L}_s})\to H^0((\widehat{\mathcal{X}_s})_{\mathrm{red}},(\widehat{\mathcal{L}_s})_{\mathrm{red}})$. Moreover, if $\overline{s}\in H^0((\widehat{\mathcal{X}_s})_{\mathrm{red}},(\widehat{\mathcal{L}_s})_{\mathrm{red}})$ lies in the kernel of the supremum seminorm $\|\cdot\|_\varphi$, there exists some integer $n\geq 1$ such that $\overline{s}^{\otimes n}=0$.

Lemma 8.1.3. *Assume that the valuation ring A is non trivial. Let $s\in H^0(\mathcal{X},\mathcal{L})$ be a global section such that $\|s_{|\mathcal{X}_s}\|_\varphi=0$. Then $s\in\mathfrak{m}H^0(\mathcal{X},\mathcal{L})$.*

Proof. The above paragraph implies that if $\|s_{|\mathcal{X}_s}\|_{|\varphi}=0$, then there exists some integer $n\geq 1$ such that $s_{|\mathcal{X}_s}^{\otimes n}=0$. Let $\mathcal{X}=\cup_{i=1}^k U_i$ be an open covering, where $U_i=\mathrm{Spec}(A_i)$ for $i=1,...,k$, and such that $\mathcal{L}_{|U_i}\cong\mathcal{O}_{U_i}$. Then for any $i=1,...,k$, there exists $a_i\in A_i$ such that $s_{|U_i}=a_ie_i$, where e_i denotes a generator of $\mathcal{L}_{|U_i}$. Therefore, for any $i=1,...,k$, $a_i^n\in\mathfrak{m}A_i$, hence $a_i\in\mathfrak{m}A_i$. By glueing, we obtain $s\in H^0(\mathcal{X},\mathfrak{m}\mathcal{L})\cong\mathfrak{m}H^0(\mathcal{X},\mathcal{L})$. □

Therefore, we can lift the supremum seminorm $\|\cdot\|_\varphi$ on $H^0(\widehat{\mathcal{X}_s},\widehat{\mathcal{L}_s})$ to a pseudo-norm on $H^0(X,L)$. This construction is summarised in the following definition.

Definition 8.1.4. Assume that X is geometrically reduced if $A=K$. Let L be a line bundle on X. Let $(\mathcal{X},\mathcal{L})$ be a flat and coherent projective model of (X,L) over A. Let $(\mathcal{L},\varphi)$ be a local pseudo-metric on L. The pseudo-norm $\|\cdot\|_\varphi$ in v on $H^0(X,L)$ is called the *supremum pseudo-norm* attached to $((\mathcal{X},\mathcal{L}),\varphi)$. Its finiteness module is $H^0(\mathcal{X},\mathcal{L})$ and its kernel is $\mathfrak{m}H^0(\mathcal{X},\mathcal{L})$. In particular, if $A=K$ (i.e. v is a usual absolute value on K), then $\|\cdot\|_\varphi$ corresponds to a norm in the classical sense.

8.2. **Zariski-Riemann local analytic space.** We now give an analytic local analogue of Theorem 1.4.3. We assume that the finiteness ring A is universally Japanese (cf. Example 1.4.2). Let K'/K be an arbitrary field extension. Since A is universally Japanese, Proposition 1.4.1 ensures that there projective sub-models of K'/A exist.

Definition 8.2.1. The *Zariski-Riemann local analytic space* attached to K' above v is the set $M_{K',v}$ of pseudo-absolute values of K' extending v on K. In other terms, $M_{K',v}$ is the fibre of v via the projection $\pi_{K'/K} : M_{K'} \to M_K$. $M_{K',v}$ is equipped with the subspace topology induced by the topology on $M_{K'}$ defined in §7. Proposition 7.1.2 implies that $M_{K',v}$ is compact Hausdorff.

Consider the Zariski-Riemann space $\mathrm{ZR}(K'/A)$ (cf. §1.4). We have a *specification morphism* $j : M_{K',v} \to \mathrm{ZR}(K'/A)$ mapping any $v' \in M_{K',v}$ to its finiteness ring.

Lemma 8.2.2. *The specification morphism* $j : M_{K',v} \to \mathrm{ZR}(K'/A)$ *is continuous, where* $\mathrm{ZR}(K'/A)$ *is endowed with the Zariski topology.*

Proof. Let $a \in K'$ and $U := \{A' \in \mathrm{ZR}(K'/A) : a \in A'\}$ be a basic open set of $\mathrm{ZR}(K'/A)$. By definition, $j{-}1(U)$ is the set of pseudo absolute values v' on K' extending v and with finiteness ring containing a. Thus we see that $j^{-1}(U) = |a|_\cdot^{-1}([0,+\infty[)$. Since $[0,+\infty[$ is an open subset of $[0,+\infty]$, $j^{-1}(U)$ is open by definition of the topology on $M_{K',v}$. □

Let $\mathcal{X}$ be an arbitrary projective sub-model of K' over the valuation ring A. Denote by $\widehat{\mathcal{X}_s}$ the local model analytic space attached to $\mathcal{X}$. Note that in the case where v is a usual absolute value, $\mathcal{X}$ is a projective model of K'/K and $\widehat{\mathcal{X}_s} = (\mathcal{X} \otimes_K \widehat{K})^{\mathrm{an}}$.

Let $v' = (|\cdot|', A', \mathfrak{m}', \kappa') \in M_{K',v}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) \end{array}.$$

By valuative criterion of properness, there exists a unique morphism of schemes $\mathrm{Spec}(A') \to \mathcal{X}$ factorising $\mathrm{Spec}(K') \to \mathcal{X}$. Denote by p the image of the closed point of $\mathrm{Spec}(A')$ in $\mathcal{X}_s$. By considering the restriction of the residue absolute value of v' to $\kappa(p)$, we obtain an absolute value on $\kappa(p)$ extending the residue absolute value of v. Therefore we have successive valued field extensions $\kappa'/\kappa(p)/\kappa$. Denote by $\widehat{p}$ the $\widehat{\kappa(p)}$-point of $\widehat{\mathcal{X}_s}$ induced by $\mathrm{Spec}(\widehat{\kappa(p)}) \to \mathrm{Spec}(\kappa(p))$. Then we have extensions $\widehat{\kappa'}/\widehat{\kappa(p)}/\widehat{\kappa}$ of completely valued fields. Thus we obtain a point $x \in \widehat{\mathcal{X}_s}^{\mathrm{an}}$. Finally, we have obtained a map $\mathrm{red}_{\mathcal{X}} : M_{K',v} \to \widehat{\mathcal{X}_s}^{\mathrm{an}}$ such that the diagram

$$\begin{array}{ccc} M_{K',v} & \longrightarrow & \mathrm{ZR}(K'/A) \\ {\scriptstyle \mathrm{red}_{\mathcal{X}}}\downarrow & & \downarrow \\ \widehat{\mathcal{X}_s}^{\mathrm{an}} & \longrightarrow & \mathcal{X}_s \end{array}$$

commutes, where the horizontal maps are the specification morphisms and the right vertical map is the center map defined by the valuative criterion of properness.

Proposition 8.2.3. *Let* $\mathcal{X}$ *be a projective sub-model of* K'/A. *Then the map* $\mathrm{red}_{\mathcal{X}} : M_{K',v} \to \widehat{\mathcal{X}_s}^{\mathrm{an}}$ *is continuous.*

Proof. Denote by $k : \widehat{\mathcal{X}_s}^{\mathrm{an}} \to \widehat{\mathcal{X}_s}$ the specification morphism, by $j : \widehat{\mathcal{X}_s} \to \mathcal{X}_s$ the extension of scalars and by $i : \mathcal{X}_s \to \mathcal{X}$ the closed immersion. Let us first prove that the composition $f := i \circ \circ j \circ k \circ \mathrm{red}_{\mathcal{X}} : M_{K',v} \to \mathcal{X}$ is continuous. Let $U = \mathrm{Spec}(B)$ be an open affine subset of $\mathcal{X}$. By definition, we have

$$f^{-1}(U) = \{|\cdot|' \in M_{K',v} : B \subset A_{|\cdot|'}\}.$$

Let $x_1, ..., x_r$ denote generators of B as an A-algebra. Then we have

$$f^{-1}(U) := \{|\cdot|' \in M_{K',v} : \max_{i=1,\dots,r} |x_i|' < +\infty\},$$

which is an open subset of $M_{K'_v}$. Thus f is continuous, hence so is $j \circ k \circ \mathrm{red}_{\mathcal{X}}$. Since j is open, $k \circ \mathrm{red}_{\mathcal{X}}$ is continuous. Let $\mathrm{Spec}(\widehat{B})$ be an affine open subset of $\widehat{\mathcal{X}_s}$. Let $\widetilde{f} \in \widehat{B}$. By construction of $\mathrm{red}_{\mathcal{X}}$ and by definition of the topology on $M_{K',s}$, the map $|f|. : \mathrm{red}_{\mathcal{X}}^{-1}(\mathrm{Spec}(\widehat{B})^{\mathrm{an}}) \to \mathbb{R}_{\geq 0}$ is continuous. By definition of the topology of $\widehat{\mathcal{X}_s}^{\mathrm{an}}$, the map $\mathrm{red}_{\mathcal{X}}$ is continuous. □

In view of results of §1.4, we now study the compatibility of the above reduction maps w.r.t. the domination relation between sub-models.

Let $\mathcal{X}, \mathcal{Y}$ be two projective sub-models of K'/A. Assume that we have a morphism of schemes $f : \mathcal{Y} \to \mathcal{X}$. f induces a morphism $f_s : \widehat{\mathcal{Y}_s} \to \widehat{\mathcal{X}_s}$ between the special fibres and the corresponding analytification $f_s^{\mathrm{an}} : \widehat{\mathcal{Y}_s}^{\mathrm{an}} \to \widehat{\mathcal{X}_s}^{\mathrm{an}}$. By uniqueness of the maps $\mathrm{Spec}(A') \to \mathcal{X}$, $\mathrm{Spec}(A') \to \mathcal{Y}$ from the valuative criterion of properness, we have a factorisation

$$\begin{tikzcd} M_{K',v} \arrow[d, "\mathrm{red}_{\mathcal{Y}}"'] \arrow[dr, "\mathrm{red}_{\mathcal{X}}"] & \\ \widehat{\mathcal{Y}_s}^{\mathrm{an}} \arrow[r, "f_s^{\mathrm{an}}"'] & \widehat{\mathcal{X}_s}^{\mathrm{an}} \end{tikzcd}.$$

Denote by $\mathcal{M}$ the collection of all projective sub-models of K'/A. Then $\mathcal{M}$ is an inverse system of locally ringed spaces which induces an inverse system of locally ringed spaces $(\widehat{\mathcal{X}_s}^{\mathrm{an}})_{\mathcal{X} \in \mathcal{M}}$. The above construction shows that the reduction maps are compatible with this projective system. Therefore we obtain a commutative diagram

$$\begin{tikzcd} M_{K',v} \arrow[r, "j"] \arrow[d, "\mathrm{red}"'] & \mathrm{ZR}(K'/A) \arrow[d, "\cong"] \\ \varprojlim_{\mathcal{X}} \widehat{\mathcal{X}_s}^{\mathrm{an}} \arrow[r] & \varprojlim_{\mathcal{X}} \mathcal{X} \end{tikzcd}, \tag{2}$$

where the right hand side arrow is a homeomorphism by Theorem 1.4.3.

Theorem 8.2.4. *We use the above notation. The map* red $: M_{K',v} \to \varprojlim_{\mathcal{X} \in \mathcal{M}} \widehat{\mathcal{X}_s}^{\mathrm{an}}$ *is a homeomorphism. Moreover, if K'/K is finitely generated,* red *induces a homeomorphism* $M_{K',v} \to \varprojlim_{\mathcal{X} \in \mathcal{M}'} \widehat{\mathcal{X}_s}^{\mathrm{an}}$*, where $\mathcal{M}'$ denotes the collection of all projective models of K'/A.*

Proof. Note that from Proposition 8.2.3, the map red is continuous. Since $M_{K',v}$ and $\varprojlim_{\mathcal{X} \in \mathcal{M}} \widehat{\mathcal{X}_s}^{\mathrm{an}}$ are both compact Hausdorff, it suffices to prove that red is bijective.

We first prove that red : $M_{K',v} \to \varprojlim_{\mathcal{X} \in \mathcal{M}} \widehat{\mathcal{X}_s}^{\mathrm{an}}$ is injective. Let $v'_1, v'_2 \in M_{K',v}$ be such that $\mathrm{red}(v'_1) = \mathrm{red}(v'_2)$. (2) implies that the finiteness rings of v'_1 and v'_2 are equal. We denote this valuation ring by A' and by κ' its residue field. Denote respectively by $|\cdot|'_1, |\cdot|'_2$ the residue absolute values on κ' induced by v'_1, v'_2. Now Theorem 1.4.3 (2) implies that

$$A' = \bigcup_{\mathcal{X}\in\mathcal{M}} \mathcal{O}_{\mathcal{X},x_{A',\mathcal{X}}},$$

where, for any $\mathcal{X} \in \mathcal{M}$, $x_{A',\mathcal{X}}$ denotes the centre of A' on $\mathcal{X}$. Since morphisms between models are dominant morphisms of integral schemes, any morphism of models $\mathcal{X}' \to \mathcal{X}$ in $\mathcal{M}$ induces an injective morphism of local rings $\mathcal{O}_{\mathcal{X},x_{A',\mathcal{X}}} \to \mathcal{O}_{\mathcal{X}',x_{A',\mathcal{X}'}}$. Therefore ([Bou75], Chap. IX, Appendice 1, Proposition 1) implies that $A' = \varinjlim_{\mathcal{X}\in\mathcal{M}} \mathcal{O}_{\mathcal{X},x_{A',\mathcal{X}}}$ and that

$$\kappa' = \bigcup_{\mathcal{X}\in\mathcal{M}} \kappa(x_{A',\mathcal{X}}). \tag{3}$$

Now $|\cdot|'_1$ and $|\cdot|'_2$ are absolute values on κ' such that, for any $\mathcal{X} \in \mathcal{M}$, their restrictions to $\kappa(x_{V',\mathcal{X}})$ are equal. Thus (3) implies that $|\cdot|'_1$ is equal to $|\cdot|'_2$, which shows the injectivity of red.

Let us now prove the surjectivity of red. Let $x = (x_{\mathcal{X}} = (p_{\mathcal{X}}, |\cdot|_{x_{\mathcal{X}}}))_{\mathcal{X}\in\mathcal{M}}$ be an element of $\varprojlim_{\mathcal{X}\in\mathcal{M}} \widehat{\mathcal{X}_s}^{\mathrm{an}}$. Then by Theorem 1.4.3, the colimit of local rings $\varinjlim_{\mathcal{X}\in\mathcal{M}} \mathcal{O}_{\mathcal{X},p_{\mathcal{X}}}$ is a valuation ring of K' over A. Moreover, the residue field κ' of A' is the union of the $\kappa(p_{\mathcal{X}})$'s. These fields are equipped with an absolute value by construction and this data defines an absolute value on the residue field κ', thus defining a pseudo-absolute value $v' = (|\cdot|', A', \mathfrak{m}', \kappa')$ on K extending v. By construction $\mathrm{red}(v') = x$. □

Remark 8.2.5. In view of Definition 8.2.1 and Lemma 8.2.2, we see that the definition of local analytic space is birational. Roughly speaking, the "algebraic part" is a certain Zariski-Riemann space instead of a scheme in the classical case.

Corollary 8.2.6. *We use the same notation as above. Assume that v is an absolute value and that K' is countable. Then the set of absolute values extending v on K' is dense in $M_{K',v}$.*

Proof. In our case, sub-models of K'/A are integral projective K-schemes whose function field embeds in K'. The set of absolute values in $M_{K',v}$ is by definition the intersection

$$\bigcap_{a\in(K')^\times} \{|a| < +\infty\}.$$

Note that $M_{K',v}$ is a Baire space. Since K' is countable, it suffices to prove that the set $U_a := \{|a| < +\infty\}$ is dense in $M_{K',v}$ for all non-zero $a \in K'$. Let $a \in (K')^\times$. By Theorem 8.2.4, it suffices to prove that, for any projective sub-model $\mathcal{X}$ of K'/K such that $a \in \kappa(\mathcal{X})$ and for any open subset $U \subset (\mathcal{X} \otimes_K \widehat{K})^{\mathrm{an}}$, we have $U_a \cap \mathrm{red}_{\mathcal{X}}^{-1}(U) \neq \varnothing$. Let $\mathcal{X}$ be a projective sub-model of K'/K. Then the generic fibre of $(\mathcal{X} \otimes_K \widehat{K})^{\mathrm{an}} \to \mathcal{X} \otimes_K \widehat{K}$ is dense in $(\mathcal{X} \otimes_K \widehat{K})^{\mathrm{an}}$ (cf. [Ber90], Corollary 3.4.5 and Theorem 3.5.1). Thus there exists $x \in U$ whose image in $\mathcal{X}$ is the generic point. Now choosing an element v' in $M_{K',v}$ such that $\mathrm{red}_{\mathcal{X}}(v') = x$, we obtain a pseudo-absolute value $|\cdot|'$ on K' which belongs to $U_a \cap \mathrm{red}_{\mathcal{X}}^{-1}(U)$. □

8.3. **Local Zariski-Riemann analytic space associated with a scheme.** We conclude this section by saying a few words about the analytic geometry over a pseudo-valued field allowed by Theorem 8.2.4. The general picture is as follows. Let K be a field equipped with a pseudo-absolute value v. We want to associate to any K-scheme X which is locally of finite type a topological space X_v^{an} which enjoys sufficiently nice properties (e.g. locally compact Hausdorff) and which can be equipped with a sheaf of analytic functions. Morally, this space encodes all the relevant arithmetic data at v. We again assume that A is universally Japanese.

8.3.1. *Naive attempt.* Let $X \to \mathrm{Spec}(K)$ be a K-scheme, one could try mimic directly constructions of Berkovich analytic spaces by replacing absolute values and norms by pseudo-absolute values and pseudo-norms. This naive construction is the following.

Let $f : X \to \mathrm{Spec}(K)$ be a K-scheme. The *naive local analytic space* $X_v^{\mathrm{an,naive}}$ attached to X over v is defined as the set of pairs $x = (p, |\cdot|_x)$, where $p \in X$ is a scheme point and $|\cdot|_x \in M_{\kappa(p)}$ is a pseudo-absolute value on the residue field $\kappa(p)$ of p such that the restriction of $|\cdot|_x$ to K is $|\cdot|_v$. There is a *specification morphism* $j_v^{\mathrm{naive}} : X_v^{\mathrm{an,naive}} \to X$ sending any $x = (p, |\cdot|_x) \in X_v^{\mathrm{an,naive}}$ to p. For any Zariski open subset $U \subset X$, let $U_v^{\mathrm{an,naive}} := (j_v^{\mathrm{naive}})^{-1}(U)$. For such U, any regular function $f \in \mathcal{O}_X(U)$ defines a map $|f|_\cdot : U_v^{\mathrm{an,naive}} \to [0, +\infty]$ by sending any point $x \in X^{\mathrm{an}}$ to $|f(j(x))|_x$.

We now equip $X_v^{\mathrm{an,naive}}$ with the coarsest topology making j together with the maps $|f|_\cdot$, for any Zariski open subset $U \subset X$ and any $f \in \mathcal{O}_X(U)$.

Proposition 8.3.1. *Assume that X is a K-variety. Then the space $X_v^{\mathrm{an,naive}}$ is not Hausdorff.*

Proof. Let $p \in X$ be a regular closed point. Let $v_x = (|\cdot|_x, A_x, \mathfrak{m}_x, \kappa_x) \in M_{\kappa(p)}$ be a pseudo-absolute value. Proposition 1.1.6 yields the existence of a valuation ring V of $K(X)$ dominating $\mathcal{O}_{X,p}$ with residue field $\kappa(p)$. Then consider the extension by generalisation of v_x to M_K induced by V (cf. Definition 4.3.1), we denote it by $v'_x = (|\cdot|'_x, A'_x, \mathfrak{m}'_x, \kappa_x)$. Let $U \subset X$ be an open Zariski neighbourhood of p, let $f \in \mathcal{O}_X(U)$ be a regular function. By definition of v'_x and since $f \in \mathcal{O}_{X,p} \subset V$, we have

$$|f|'_x = |f(p)|_x.$$

Therefore, any open neighbourhood $V \subset X_v^{\mathrm{an}}$ of $x = (p, v_x)$ contains (η, v'_x), where η denotes the generic point of X. This implies that X_v^{an} is not Hausdorff. □

8.3.2. *Definition.* In view of Proposition 8.3.1, one has to refine the definition of the local analytic space associated with a scheme by only allowing only pseudo-absolute values on the residue field of generic points of irreducible components extending v.

Definition 8.3.2. Let X be a projective K-variety. Denote $K' := \kappa(X)$. Define the *Zariski-Riemann local analytic space* associated with X by

$$X_v^{\mathrm{an}} := \varprojlim_{\mathcal{X} \in \mathcal{M}_X} \widehat{\mathcal{X}_s}^{\mathrm{an}},$$

where $\mathcal{M}_X$ denotes the sub collection of $\mathcal{M}$ consisting of projective models of K'/A with generic fibre isomorphic to X. This is a compact Hausdorff space.

Let X be a projective K-variety. We can link the space X_v^{an} above with the space $X_v^{\mathrm{an,naive}}$ as follows. For any $\mathcal{X} \in \mathcal{M}_X$, using the valuative criterion of properness, we define a continuous map

$$\mathrm{red}_{X,\mathcal{X}} : X_v^{\mathrm{an,naive}} \to \widehat{\mathcal{X}_s}^{\mathrm{an}},$$

in a similar fashion $\mathrm{red}_{\mathcal{X}} : M_{K',v} \to \widehat{\mathcal{X}_s}^{\mathrm{an}}$ was constructed. This construction is compatible with respect to maps between elements of $\mathcal{M}_X$ and thus induces a map

$$\mathrm{red}_X : H \to X_v^{\mathrm{an}}.$$

Proposition 8.3.3. *Let X be a projective K-variety. Denote $K' := \kappa(X)$. Then X_v^{an} is homeomorphic to the quotient of $X_v^{\mathrm{an,naive}}$ by the equivalence relation $\sim$ defined by*

$$\forall x, y \in X_v^{\mathrm{an,naive}}, \quad x \sim y \Leftrightarrow \mathrm{red}_X(x) = \mathrm{red}_X(y).$$

Proof. By definition of $\sim$, red_X factorises via a continuous map $X_v^{\mathrm{an,naive}}/\sim \to X_v^{\mathrm{an}}$. Moreover, the natural map $M_{K',v} \to X_v^{\mathrm{an}}$ is a continuous surjective map between compact Hausdorff spaces, and henceforth a quotient. Thus the continuous injection $M_{K',v} \to X^{\mathrm{an}}$ induces a continuous map $i : X_v^{\mathrm{an}} \to X_v^{\mathrm{an,naive}}/\sim$. Then $\mathrm{red}_X \circ i$ is the identity on X_v^{an} and thus i is a homeomorphism. □

Remark 8.3.4. (1) If v is a usual absolute value, and X is a projective K-variety, then X_v^{an} corresponds to the usual Berkovich analytification $(X \otimes_K \widehat{K_v})^{\mathrm{an}}$.
(2) Proposition 8.3.3 illustrates a feature of spaces of pseudo-absolute values which are reminiscent of adic spaces. For such spaces, the topological structure is by essence much more algebraic and specialisation procedures can be studied to prove that, under suitable conditions, the maximal Hausdorff quotient of an adic space is a Berkovich space.
(3) In subsequent work, we will make use of these Zariski-Riemann local analytic spaces for which the description as a projective limit allows to define a sheaf of analytic functions.

9. Integral structures

In general, spaces of pseudo-absolute values might be "too big" for performing a suitably well-behaved analytic geometry. For instance, the space of (pseudo-)absolute values on $\mathbb{C}$ already contains a lot of p-adic absolute values that one would not want to include, at least at first glance. To remedy the situation, we limit the space of pseudo-absolute values in our space by imposing some *continuity condition*. This gives rise to the notion of *integral structure* which can be interpreted as affine subsets of our spaces of pseudo-absolute values. As a guideline for the reader who might be frightened by the technical content of this section, let us say that §9.1-9.3 consist of fundamental results that allow to explicit Berkovich spectra that topologically embed in spaces of pseudo-absolute values. In §7.2, we describe precisely some examples of integral structure coming from Nevanlinna theory. The main example to focus on is found in §9.4.3. We make a full description of the Berkovich spectrum associated with the ring of holomorphic functions on a closed complex disc, equipped with a hybrid norm (Example 9.2.1 (4)), as a locally ringed space. We also prove that the underlying Banach ring is a geometric base ring ([LP24], Définition 3.3.8). This gives a new example of such Banach rings that allow the employment of the machinery in *loc. cit.* and will be used to formalise Nevanlinna theory in our framework.

Throughout this section, we fix a field K.

9.1. Definition of integral structures.

Definition 9.1.1. An *integral structure* for K is the data $(A, \|\cdot\|_A)$ where:

(i) A is an integral subdomain of K with fraction field K;
(ii) A is Prüfer;
(iii) $\|\cdot\|_A$ is a Banach norm on A.

When no confusion may arise, we use the notation A for an integral structure $(A, \|\cdot\|_A)$.

Remark 9.1.2. Denote by k the prime subring of K. Let $(A, \|\cdot\|_A)$ be an integral structure for K. The latter can be interpreted through the *Zariski-Riemann space* associated with K, namely the set $\mathrm{ZR}(K/k)$ of valuation rings of K. As A is integrally closed ([Gil72], Chapter IV), it is the intersection of all its valuation overrings ([Bou75], Chapter VI, §1.3 Corollaire 2). Note that any valuation overring of A is of the form $A_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$ (cf. Proposition 1.1.9 (1)). Hence we have

$$A = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} A_{\mathfrak{p}}.$$

Let $X \subset \mathrm{ZR}(K/k)$ be the subset such that $A = \bigcap_{V \in X} V$. We have a characterisation of such subsets X in terms of affine subsets of $\mathrm{ZR}(K/k)$ (cf. [Olb21], Theorem 6.2). Let us also mention that Olberding has done extensive work in these directions [Olb08, Olb10, Olb11].

The notion of integral structure allows to define refined versions of M_K.

Definition 9.1.3. Let K be a field equipped with an integral structure $(A, \|\cdot\|_A)$. The *global space of pseudo-absolute values* relative to the integral structure A is defined as

$$V_{K,A} := \{|\cdot| \in M_K : |\cdot|_{|A} \leq \|\cdot\|_A\},$$

and is equipped with the topology induced by that of M_K.

Notation 9.1.4. Let K be a field. Let $(A, \|\cdot\|_A)$ be an integral structure for K. Denote by V the associated global space of pseudo-absolute values. We introduce the following notation.

- $V_\infty := M_{K,\infty} \cap V$ denotes the set of Archimedean pseudo-absolute values of V.
- $V_{\mathrm{um}} := M_{K,\mathrm{um}} \cap V$ denotes the set of ultrametric pseudo-absolute values of V.
- $V_{\mathrm{sn}} := M_{K,\mathrm{sn}} \cap V$ denotes the set of pseudo-absolute of V values whose kernel is non-zero.
- $V_{\mathrm{triv}} := M_{K,\mathrm{triv}} \cap V$ denotes the set of pseudo-absolute values of V whose residual absolute value is trivial.
- $V_{\mathrm{disc}} := M_{K,\mathrm{disc}} \cap V$ denotes the set of discrete pseudo-absolute values, namely the set of pseudo-absolute values of V that correspond either to a discrete non-Archimedean absolute value on K or to a pseudo-absolute value whose finiteness ring is a discrete valuation ring that is not a field.

Proposition 9.1.5. *With the notation of Definition 9.1.3, V is a non-empty compact Hausdorff topological space.*

Proof. There is a natural map $f : V_{K,A} \to \mathcal{M}(A)$, where $\mathcal{M}(A)$ denotes the Berkovich analytic spectrum of $(A, \|\cdot\|_A)$, which is given by restricting elements of $V_{K,A}$ to A. By definition of the topologies, f is continuous. Let $x \in \mathcal{M}(A)$ and denote by $\mathfrak{p}_x \in \mathrm{Spec}(A)$ its kernel. Since A is Prüfer, the localisation $A_{\mathfrak{p}_x}$ is a valuation ring of K on which x defines a multiplicative semi-norm. Therefore, every element of $\mathcal{M}(A)$ induces an element of $V_{K,A}$. This construction provides an inverse of f that is a continuous function. The conclusion follows from ([Ber90], Theorem 1.2.1). $\square$

Remark 9.1.6. The above proof shows that V is homeomorphic to $\mathcal{M}(A)$.

Proposition 9.1.7. *Let $(A, \|\cdot\|)$ be an integral structure for K and denote by V be the associated global space of pseudo-absolute values. We assume that $V_\infty \neq \varnothing$. Recall that we denote by $|\cdot|_\infty$ the usual absolute value on $\mathbb{Q}$. Let $\epsilon : V_\infty \to]0,1]$ be the function mapping $x \in V_\infty$ to the unique $\epsilon(x) \in]0,1]$ such that the residual absolute value on κ_x induces the absolute value $|\cdot|_\infty^{\epsilon(x)}$ on $\mathbb{Q}$. Then, by extending the definition of ϵ to V by setting $\epsilon(x) := 0$ if $x \in V_{\mathrm{um}}$, ϵ is a continuous function on V.*

Proof. From the assumption that $\Omega_\infty \neq \varnothing$, K has characteristic zero. By definition of ϵ, for any $x \in V_\infty$, we have $\epsilon(x) = \log|2|_x / \log 2$. It follows that for any $x \in V$, we have

$$\epsilon(x) = \frac{\max\{0, \log|2|_x\}}{\log 2}.$$

Since $\log|2|_.$ is continuous on V, we deduce the continuity of the function ϵ. $\square$

To study the topology of global spaces of pseudo-absolute values, we make use of the constructions in ([Poi10], §1.3).

Notation 9.1.8. Let $(A, \|\cdot\|)$ be an integral structure for K and denote $V := \mathcal{M}(A)$. For any $x \in V$, we define the interval

$$I_x := \{\epsilon \in \mathbb{R}_{>0} : \forall f \in A, |f|_x^\epsilon \leq |f|\}.$$

For any $\epsilon \in I_x$, we can then show that $|\cdot|_x^\epsilon$ defines an element of V that we denote by x^ϵ. In the case where I_x has 0 as a lower bound, we can extend this definition to $\epsilon = 0$ by defining x^0 as the pseudo-absolute value on K defined by

$$|\cdot|_x^0 : \left| \begin{array}{ccc} A & \longrightarrow & \mathbb{R}_{\geq 0} \\ f & \longmapsto & \left\{ \begin{array}{l} 0 \text{ if } |f|_x = 0, \\ 1 \text{ if } |f|_x \neq 1. \end{array} \right. \end{array} \right.$$

9.2. Examples of integral structures.

Example 9.2.1.
(1) Let $\|\cdot\|$ be any Banach norm on K. Then $(K, \|\cdot\|)$ is an integral structure for K. This is the case when K is either a complete valued field or a hybrid field, namely the norm $\|\cdot\|$ is of the form $\|\cdot\| = \max\{|\cdot|, |\cdot|_{\mathrm{triv}}\}$, where $|\cdot|$ denotes a non-trivial absolute value on K.
(2) Let K be a number field with ring of integers $\mathcal{O}_K$. The latter is a Prüfer domain (cf. Example 1.1.10 (2)) with fraction field K. Let $\|\cdot\|_\infty := \max_{\sigma \hookrightarrow \mathbb{C}} |\cdot|_\sigma$, where σ runs over the set of embeddings of K into $\mathbb{C}$. Then $(\mathcal{O}_K, \|\cdot\|_\infty)$ is an integral structure for K.
(3) Let U be a non-compact Riemann surface and $K := \mathcal{M}(U)$ denote its field of meromorphic functions. Then the ring of holomorphic functions $A := \mathcal{O}(U)$ is a Prüfer domain (cf. Example 1.1.10 (3)) with fraction field K. Let $C \subset U$ be a compact subset. Then define a norm on A as follows.

$$\forall f \in A, \quad \|f\|_{C,\mathrm{hyb}} := \max\{\|f\|_C, \|f\|_{\mathrm{triv}}\},$$

where $\|\cdot\|_C$ denotes the supremum norm on C and $\|\cdot\|_{\mathrm{triv}}$ denotes the trivial norm on A. Then $(A, \|\cdot\|_{C,\mathrm{hyb}})$ is a Banach ring and thus $(A, \|\cdot\|_{C,\mathrm{hyb}})$ is an integral structure for K.

(4) Let $R > 0$ and let $\overline{D(R)}$ denote the complex closed disc of radius R. We denote respectively by $A = \mathcal{O}(\overline{D(R)})$ and $K = \mathcal{M}(\overline{D(R)})$ the ring of germs of holomorphic functions and the field of germs of meromorphic functions on $\overline{D(R)}$. Then A is a principal ideal domain with field of fractions K (cf. Example 1.1.10 (4) and Proposition 1.2.6) and a Prüfer domain. Let $\|\cdot\|_R$ denote the supremum norm on $\overline{D(R)}$ and define $\|\cdot\|_{R,\mathrm{hyb}} := \max\{\|\cdot\|_R, |\cdot|_{\mathrm{triv}}\}$, where $|\|\cdot\|_{\mathrm{triv}}$ denotes the trivial norm on A. Then $(A, \|\cdot\|_{R,\mathrm{hyb}})$ is a Banach ring and $(A, \|\cdot\|_{R,\mathrm{hyb}})$ defines an integral structure for K.

9.3. **Tame global spaces of pseudo-absolute values.** The following definition is motivated by the explicit study of global spaces of pseudo-absolute values we shall consider in the context of Nevanlinna theory.

Definition 9.3.1. Let $(A, \|\cdot\|)$ be an integral structure on K and denote $V := \mathcal{M}(A)$. V is called *tame* if the following conditions are satisfied:

(i) $v_{\mathrm{triv}} \in V$;
(ii) for any $x \in V_{\mathrm{um}}$ and any $f \in A$, the inequality
$$|f|_x \leq 1$$
is satisfied;
(iii) $(A, \|\cdot\|)$ is a uniform Banach ring (cf. Definition 1.5.2).

Example 9.3.2. The spectra of Example 9.2.1 are tame spaces. This is immediate in the case of the spectrum of the ring of integers of a number field. Let us show this fact in the case of the hybrid spectrum of the ring of analytic functions on a non-compact Riemann surface. We use the notation of Example 9.2.1 (3). Let $x \in V_{\mathrm{um}} \smallsetminus V_{\mathrm{sn}}$. Then for all $f \in A \smallsetminus \{0\}$, we have $|f|_x \leq \|f\|_A = \max\{\|f\|_C, 1\}$. For all $a \in \mathbb{C}$, we have $|af|_x = |f|_x$ (since the restriction of $|\cdot|_x$ to $\mathbb{C}$ is necessarily trivial). By choosing a of sufficiently small modulus, we obtain $\|af\|_C \leq 1$ and the inequality
$$|f|_x \leq |af|_x \leq \|af\|_A = 1.$$
In §9.4.2, we will see that the norm $\|\cdot\|_A$ is uniform.

The following proposition ensures that the notion of tame space is compatible with algebraic extensions of the field of fractions, which is fundamental in what follows.

Proposition 9.3.3. *Let $(A, \|\cdot\|)$ be an integral structure on K and assume that $V := \mathcal{M}(A)$ is tame. Let L/K be an algebraic extension and let B denote the integral closure of A in L. Then B can be equipped with a norm $\|\cdot\|_B$ such that*

(1) *$(B, \|\cdot\|_B)$ is a Banach ring;*
(2) *$V_L := \mathcal{M}(B)$ is a tame space;*
(3) *the inclusion morphism $(A, \|\cdot\|) \to (B, \|\cdot\|_B)$ is an isometry.*

Moreover, V_L is "universal" in the following sense. If $v \in M_L$ is such that its restriction to A belongs to V, then $v \in V_L$.

Proof. **Case 1: the extension L/K is finite and Galois.** Let us show the existence of $\|\cdot\|_B$. First note that B is a Prüfer domain (cf. Proposition 1.1.9 (2)). Define the set
$$E := \{|\cdot| : |\cdot| \text{ is a multiplicative seminorm on } B \text{ and } |\cdot|_{|A} \in V\}.$$

Let us show that the application

$$\|\cdot\|_B : \left| \begin{array}{ccc} B & \longrightarrow & \mathbb{R}_{\geq 0} \\ b & \longmapsto & \sup_{x\in E}|b(x)| \end{array} \right.$$

defines a Banach norm on B. Let $b \in B \smallsetminus \{0\}$, we *a priori* have $\|b\|_B \in [1, +\infty]$ (since the trivial absolute value belongs to E). To show that $\|b\|_B < +\infty$, we will use the following lemma.

Lemma 9.3.4. *Let $b \in B$ and denote by $P = (T - \alpha_1)\cdots(T - \alpha_d) \in A[T]$ its minimal polynomial over K. Then there exists a constant $M > 0$ such that*

$$\forall v \in V, \quad \exists N_0 \geq 0, \quad \forall N > 0, \quad \left|\sum_{j=1}^{d} \alpha_j^N\right|_v \leq d^{N+N_0}M^N. \tag{4}$$

Proof. Write $P = a_dT^d + \cdots + a_0$ and denote $M := \max_{0\leq i\leq n}\|a_i\|_.$. We fix $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in V$. For any integer $N > 1$, define $\lambda_N := \sum_{j=1}^{d}\alpha_j^N$ and

$$\Lambda := \max_{1\leq j\leq d}|\lambda_j|_v.$$

Let $N_0 > 0$ be an integer such that $\Lambda \leq d^{N_0}$. Let us show by induction on N that (4) holds for all $N > 0$. Let N be an integer. Assume first that $N \leq d$. Then we have

$$|\lambda_N| \leq \Lambda \leq d^{N_0+N}M^N,$$

The second inequality comes from $M \geq 1$ (since $v_{\mathrm{triv}} \in V$). We now assume that $N > d$ and that (4) holds for any $N' < N$. Then, from Newton identities, we have

$$\lambda_N = \sum_{k=N-d}^{N-1}(-1)^{N-1+k}a_{N-k}\lambda_k,$$

where

$$a_j := \sum_{1\leq n_1<\cdots<n_j\leq d}\alpha_{n_1}\cdots\alpha_{n_j}$$

for any $j \in \{1, ..., d\}$. Note that for all $j \in \{1, ..., d\}$, a_j is a coefficient of P (up to sign). Hence $|a_j| \leq M$. Then the induction hypothesis yields

$$|\lambda_N|_v \leq dM \max_{N-d\leq k\leq N-1} d^{k+N_0}M^k \leq d^{N+N_0}M^N.$$

Hence the conclusion. □

We are now able to show $\|b\|_B < +\infty$. Let $\{\alpha_1, ..., \alpha_d\}$ be the Galois orbit of b. Fix an arbitrary pseudo-absolute value $v \in V$. Proposition 3.3.5 and Lemma 9.3.4 give the existence of $M > 0$ such that

$$\max_{w|v}|b|_w = \limsup_{N\to+\infty}\left|\sum_{j=1}^{d}\alpha_j^N\right|_v^{\frac{1}{N}} \leq dM.$$

Since both d and M are independent on v, we have $\|b\|_B \leq dM < +\infty$. It is immediate to see that $\|\cdot\|_B$ defines a norm on B. Since the trivial absolute value on L belongs to E, the topology on B induced by $\|\cdot\|_B$ is discrete. In particular, $(B, \|\cdot\|_B)$ is a Banach ring.

Furthermore, for any $a \in A$, we have $\|a\|_B = \|a\|$. Hence we have an isometric embedding $(A, \|\cdot\|) \to (B, \|\cdot\|_B)$. This concludes the proof of (1) and (3).

We now show that $V_L := \mathcal{M}(B)$ is a tame space. Note that V_L contains the trivial absolute value. Let $x = (|\cdot|_x, B_x, \mathfrak{m}_x) \in V_{L,\mathrm{um}}$. Let $f \in B$ whose Galois orbit is denoted by $\{\alpha_1, ..., \alpha_d\}$. Let us show that $|f|_x \leq 1$. Let $v = (|\cdot|_v, A_v, \mathfrak{m}_v) := \pi_{L/K}(x) \in V$ denote the restriction of x to K. Then Lemma 3.3.6 yields

$$|f|_x \leq \max_{x' \in \pi_{L/K}^{-1}(v)} |f|_{x'} = \limsup_{N \to +\infty} \left| \sum_{j=1}^{d} \alpha_j^N \right|_v^{\frac{1}{N}} \leq 1,$$

The last inequality comes from the fact that, for any $j \in \{1, ..., d\}$, for any $N > 0$, $\sum_{j=1}^{d} \alpha_j^N \in A$. Finally, ([Ber90], Theorem 1.3.1) implies that the norm $\|\cdot\|_B$ is uniform and V_L is tame.

Case 2: the extension L/K is finite separable. Let $L'/L/K$ be the normal closure of L/K and denote by B' the integral closure of A in L', which is equal to the integral closure of B in L'. From the finite Galois case, there exists a norm $\|\cdot\|_{B'}$ on B' such that $(B', \|\cdot\|_{B'})$ (1)-(3) hold. Since B is a subring of B', the restriction of $\|\cdot\|_{B'}$ to B, denoted by $\|\cdot\|_B$, induces the discrete topology B, hence is a Banach norm. We also have an isometric embedding $(A, \|\cdot\|) \to (B, \|\cdot\|_B)$ and $\mathcal{M}(B)$ is tame.

Case 3: the extension L/K is infinite and separable. Let $\mathcal{E}_{L/K}$ denote the set of finite sub-extensions of L/K. $\mathcal{E}_{L/K}$ is a directed set with respect to the inclusion relation and we have

$$L = \bigcup_{K' \in \mathcal{E}_{L/K}} K' \quad \text{and} \quad B = \bigcup_{K' \in \mathcal{E}_{L/K}} A_{K'},$$

where, for all $K' \in \mathcal{E}_{L/K}$, $A_{K'}$ denotes the integral closure of A in K'. For any $K' \in \mathcal{E}_{L/K}$, we denote by $\|\cdot\|_{A_{K'}}$ the norm on $A_{K'}$ constructed in the finite case. If $K''/K'/K$ are sub-extensions in $\mathcal{E}_{L/K}$, we have the compatibilities

$$\begin{tikzcd}
(A_{K''}, \|\cdot\|_{A_{K''}}) & (A_{K'}, \|\cdot\|_{A_{K'}}) \arrow[l, hook', "\iota_{K'/K}"'] \\
 & (A, \|\cdot\|) \arrow[ul] \arrow[u]
\end{tikzcd}$$

where the arrows are the isometric embeddings previously constructed. Thus we have a filtered direct system $(A_{K'}, \|\cdot\|_{A_{K'}})_{K' \in \mathcal{E}_{L/K}}$ of Banach A-algebras whose arrows are isometric embeddings. One can prove that the direct limit of this direct system exists in $\mathrm{Ban}_{A-\mathrm{alg}}^{\leq 1}$ and using the fact that all $A_{K'}$ are discrete its underlying set can be identified with B. B is Prüfer (cf. Proposition 1.1.9 (2)). Thus we have an integral structure $(B, \|\cdot\|_B)$ on L. Let us show that $V_L := \mathcal{M}(B) \cong \varprojlim_{K' \in \mathcal{E}_{L/K}} \mathcal{M}(A_{K'})$ is tame. From the description of V_L as an inverse limit, we see that V_L contains the trivial absolute value and that the norm $\|\cdot\|_B$ is uniform. Moreover, by definition of $\|\cdot\|_B$, for any $f \in B$, for any $x \in V_{L,\mathrm{um}}$, we have $|f(x)| \leq 1$.

Case 4: general case. Let K'/K be the separable closure of K in L and let q denote the degree of the purely inseparable extension L/K'. From the latter case, we have a norm

$\|\cdot\|_{A'}$ on A', the integral closure of A in K', satisfying (1)-(3). Define a map

$$\|\cdot\|_B : (b \in B) \mapsto \|b^q\|_{A'} \in \mathbb{R}_{\geq 0}.$$

Then $\|\cdot\|_B$ is a norm on B satisfying (1)-(3).

To conclude the proof of the proposition, it remains to show that V_L is "universal". Let $v \in M_L$ and denote by B_v its finiteness ring. As a valuation ring of L containing A, it is an overring of B. By construction of $\|\cdot\|_B$, we have $|b|_v \leq \|b\|_B$ for all $b \in B$, i.e. $v \in V_L$. $\square$

The following propositions ensure that the ultrametric part of global spaces of pseudo-absolute values arising from integral structures enjoys sufficiently nice properties.

Proposition 9.3.5. *Let $(A, \|\cdot\|)$ be an integral structure on K and assume that $V := \mathcal{M}(A)$ is tame. Let $x \in V_{\mathrm{um}} \smallsetminus V_{\mathrm{sn}}$, i.e. an ultrametric absolute value on K, and assume that x is non-trivial. Then, for all $\epsilon \in [0, +\infty[$, $x^\epsilon \in V$. Furthermore, the pseudo-absolute value on K is defined by*

$$|\cdot|_x^\infty : \left| \begin{array}{ccc} A & \longrightarrow & \mathbb{R}_{\geq 0} \\ f & \longmapsto & \begin{cases} 0 \text{ if } |f|_x < 1, \\ 1 \text{ if } |f|_x = 1, \end{cases} \end{array} \right.$$

belongs to V. We denote it by x^∞.

Proof. As V is tame, we have $x^0 = v_{\mathrm{triv}} \in V$. For any $\epsilon \in \mathbb{R}_{>0}$, x^ϵ is an ultrametric and nontrivial absolute value on K. For any $f \in A$, we have $|f|_x \leq 1$ and thus $|f|_x^\epsilon \leq 1$. Therefore $x^\epsilon \in V$ (since $1 \leq \|f\|$). For any $f \in A$, we have $|f|_x^\infty \leq |f|_x \leq 1 \leq \|f\|$. Hence $x^\infty \in V$. $\square$

Proposition 9.3.6. *Let $(A, |\cdot|)$ be an integral structure for K and assume that $V := \mathcal{M}(A)$ is tame. Let $x \in V_{\mathrm{um}} \smallsetminus V_{\mathrm{sn}}$ and assume that x is nontrivial. Then the map*

$$x^\cdot : \left| \begin{array}{rcl} [0, +\infty] & \longrightarrow & V \\ \epsilon & \longmapsto & x^\epsilon \end{array} \right.$$

induces a homeomorphism onto its image.

Proof. Proposition 9.3.5 implies that the map $x^\cdot$ is well-defined. Since for any $f \in A$, the map

$$\varphi_f : \left| \begin{array}{rcl} [0, +\infty] & \longrightarrow & \mathbb{R}_{\geq 0} \\ \epsilon & \longmapsto & |f|_x^\epsilon \end{array} \right.$$

is continuous, $x^\cdot$ is continuous. Thus $x^\cdot$ is a continuous bijection between compact Hausdorff spaces, and consequently a homeomorphism. $\square$

The following definition allows to get rid of pseudo-absolute values whose finiteness ring is of rank greater than 1.

Definition 9.3.7. A pseudo-absolute value $x \in M_K$ with finiteness ring A_x is called of *rank at most 1* if A_x is a valuation ring of rank ≤ 1.

Let $(A, \|\cdot\|)$ be an integral structure for K and denote $V = \mathcal{M}(A)$. We define

$$V_{\leq 1} := \{x \in V : x \text{ is of rank at most } 1\}.$$

Then $V_{\leq 1}$ is a topological subspace of V, with the subset topology.

Proposition 9.3.8. *Let $(A, \|\cdot\|)$ be an integral structure for K and assume that $V := \mathcal{M}(A)$ is tame. Let $x = (|\cdot|_x, A_x, \mathfrak{m}_x, \kappa_x) \in V_{\mathrm{sn}} \cap V_{\leq 1}$. Then the following assertions hold.*

(1) *The residue absolute value on* κ_x *is either Archimedean or trivial.*
(2) *Further assume that* $x \in V_{\mathrm{um}}$. *Then there exists* $v \in V_{\leq 1}$ *such that* $x = v^{\infty}$ *(cf. Proposition 9.3.5).*

Proof. We first show (1). Let $x = (|\cdot|_x, A_x, \mathfrak{m}_x, \kappa_x) \in V_{\mathrm{sn}} \cap V_{\leq 1}$ and assume x is ultrametric. Since V is tame, for any $a \in A$, we have $|a|_x \leq 1$. Furthermore, the canonical homomorphism $A \to \kappa_x$ is surjective. Hence, for any $\overline{a} \in \kappa_x$, we have $|\overline{a}|_x \leq 1$, i.e. κ_x is the valuation ring of the residue absolute value of x, i.e. x is residually trivial.

We now prove (2). Let $x = (|\cdot|_x, A_x, \mathfrak{m}_x) \in V_{\mathrm{sn}} \cap V_{\leq 1}$ be ultrametric. From (1), κ_x is trivially valued. Let v be a valuation (necessarily of rank 1) on K with valuation ring A_x and denote by $|\cdot|_v$ the corresponding absolute value on K. Note that, since $x, v_{\mathrm{triv}} \in V$, we have $A \subset A_x$ and, for any $a \in A$, $|a|_v \leq 1 \leq \|a\|$. Hence $v \in V_{\leq 1}$ and $x = v^{\infty}$. □

Remark 9.3.9. With the notation of Proposition 9.3.8, we have the set-theoretic description of $V_{\mathrm{um},\leq 1}$ as

$$\bigsqcup_{v \in \mathcal{P}} [0, +\infty]/\sim,$$

where:

- $\mathcal{P}$ denotes the set of equivalence classes of nontrivial ultrametric absolute values on K;
- for any $v \in \mathcal{P}$, $[0, +\infty]$ denotes the branch introduced in Proposition 9.3.6;
- $\sim$ denotes the equivalence relation which identifies the extremity 0 of each branch.

Proposition 9.3.10. *We use the notation of Remark 9.3.9. Assume that A is Dedekind. Then the bijection*

$$V_{\mathrm{um}} = V_{\mathrm{um},\leq 1} \cong \bigsqcup_{v \in \mathcal{P}} [0, +\infty]/\sim$$

is a homeomorphism.

Proof. This follows directly from the description of the Berkovich analytic spectrum of a trivially valued Dedekind ring (e.g. [LP24], Example 1.1.17). □

9.4. **Examples.** We can explicitly describe global spaces of pseudo-absolute values for various examples.

9.4.1. *Function field over* $\mathbb{C}$. Let $K = \mathbb{C}(T)$. Let $R > 0$. Denote by $\|\cdot\|_R$ the supremum norm of polynomial functions on the closed disc $\overline{D(R)}$ of radius R. Let $\|\cdot\|$ denote the hybrid norm $\|\cdot\| := \max\{\|\cdot\|_{\mathrm{triv}}, \|\cdot\|_R\}$, where $\|\cdot\|_{\mathrm{triv}}$ denotes the trivial norm on $\mathbb{C}[T]$. Then $(\mathbb{C}[T], \|\cdot\|)$ defines an integral structure for K. We describe the associated global space of pseudo-absolute values V_R.

Proposition 9.4.1. *We use the same notation as above.*

(1) *We have homeomorphisms*

$$V_{R,\infty} \cong \overline{D(R)} \times]0,1], \quad V_{R,\mathrm{um}} \cong \mathbb{P}^{1,\mathrm{triv}}_{\mathbb{C}} \cap \{|T| \leq 1\},$$

where $\mathbb{P}^{1,\mathrm{triv}}_{\mathbb{C}}$ *is the Berkovich analytic space associated with* $\mathbb{P}^1_{\mathbb{C}}$, *where* $\mathbb{C}$ *is trivially valued.*

(2) $V_{R,\infty}$ *is dense in* V_R

Proof. **(1)** Let φ be the map mapping $(z,\epsilon)\in\overline{D(R)}\times]0,1]$ to $v_{z,\epsilon,\infty}\in M_K$. First let us prove that φ has image contained in $V_{R,\infty}$. For this purpose, it suffices to show that, for any $(z,\epsilon)\in\overline{D(R)}\times]0,1]$, we have

$$\forall P\in\mathbb{C}[T],\quad |P(z)|_\infty^\epsilon\leq\|P\|.$$

Let $(z,\epsilon)\in\overline{D(R)}\times]0,1]$ and fix an arbitrary non-zero polynomial $P\in\mathbb{C}[T]$. First assume that $|P(z)|_\infty\leq 1$. Then $|P(z)|_\infty^\epsilon\leq 1=\|P\|_{\mathrm{triv}}\leq\|P\|$. Now assume that $|P(z)|_\infty>1$. Then we have the inequalities

$$1<|P(z)|_\infty^\epsilon<|P(z)|_\infty\leq\|P\|_R=\|P\|.$$

Therefore φ has image contained in $V_{R,\infty}$.

Let us now prove that φ has exactly image $V_{R,\infty}$. We first consider the case where $z\notin D(\overline{R})$ and $\epsilon=1$. Then if $P(T):=T+1\in\mathbb{C}[T]$, we have

$$\begin{cases}|P(z)|_\infty=|z|_\infty+1>1=\|P\|_{\mathrm{triv}},\\ |P(z)|_\infty=|z|_\infty+1>R+1=\|P\|_R.\end{cases}$$

Therefore $v_{z,1,\infty}\notin V_{R,\infty}$.

Now let $z\notin\overline{D(R)}$ and $\epsilon\in]0,1[$. Then we construct a polynomial $Q\in\mathbb{C}[T]$ such that $|Q(z)|_\infty^\epsilon>\|Q\|$. For any $a\in\mathbb{R}_{>0}$, denote

$$P_a(T):=\frac{T}{a}\in\mathbb{C}[T].$$

Then for any $a>0$, we have $\|P_a\|_R=R/a$ and

$$|P_a(z)|_\infty^\epsilon>\|P_a\|\Leftrightarrow\begin{cases}\frac{|z|_\infty^\epsilon}{a^\epsilon}>1=\|P_a\|_{\mathrm{triv}},\\ \frac{|z|_\infty^\epsilon}{a^\epsilon}>\frac{R}{a}=\|P_a\|_R.\end{cases}$$

Let $u:(a\in]0,|z|_\infty[)\mapsto R/a^{1-\epsilon}$. Then u is a continuous function. Note that

$$R<|z|_\infty\Leftrightarrow\lim_{a\to|z|_\infty^-}u(a)=\frac{R}{|z|_\infty^{1-\epsilon}}<|z|_\infty^\epsilon.$$

Moreover, for any $a\in]0,|z|_\infty[$, we have

$$|P_a(z)|_\infty^\epsilon>\|P\|_R\Leftrightarrow u(a)<|z|_\infty^\epsilon.$$

Since $\lim_{a\to|z|_\infty^-}u(a)<|z|_\infty^\epsilon$, by continuity, there exists $a\in]0,|z|_\infty[$ such that $u(a)<|z|_\infty^\epsilon$ and we denote $Q:=P_a$. From what precedes, we have $|Q(z)|_\infty^\epsilon>\|Q\|$, thus $v_{z,\epsilon,\infty}\notin V_{R,\infty}$.

Combining the two above paragraphs, φ defines a bijection between $\overline{D(R)}\times]0,1]$ and $V_{R,\infty}$. It is a homeomorphism by definition of the topologies of its domain and codomain.

We now prove the second homeomorphism. Let $|\cdot|\in V_{R,\mathrm{um}}$. As $|\cdot|_{|\mathbb{C}[T]}\leq 1$, we deduce that the restriction of $|\cdot|$ is the trivial absolute value. We distinguish two cases. The first one is the case where $|\cdot|$ is a usual absolute value. Then $|\cdot|$ is of the form $|\cdot|_{z,c,\mathrm{um}}:f\in K\mapsto\exp(-c\,\mathrm{ord}(f,z))\in\mathbb{R}_{>0}$, where $z\in\mathbb{C}\cup\{\infty\}$ and $c\geq 0$ (using the conventions $\mathrm{ord}(\cdot,\infty)=-\deg(\cdot)$ and $|\cdot|_{z,0,\mathrm{um}}=|\cdot|_{\mathrm{triv}}$ for all $z\in\mathbb{C}\cup\{\infty\}$). The second case is when $|\cdot|$ has a non-zero kernel. Then there exists $z\in\mathbb{C}\cup\{\infty\}$ such that $|\cdot|=|\cdot|_{z,\infty,\mathrm{um}}$, where

$$|\cdot|_{z,\infty,\mathrm{um}}:\left|\begin{array}{rcl}\mathbb{C}[T]_{(T-z)} & \longrightarrow & \mathbb{R}_{\geq 0}\\ f & \longmapsto & \begin{cases}0\text{ if }f(z)=0,\\ 1\text{ if }f(z)\neq 0.\end{cases}\end{array}\right.$$

Thus $|\cdot|$ can be identified with an element of $\mathbb{P}^{1,\mathrm{triv}}_{\mathbb{C}}$. Moreover, the condition $|\cdot|_{|\mathbb{C}[T]} \leq 1$ implies that $|\cdot|$ can be identified with an element of $\mathbb{P}^{1,\mathrm{triv}}_{\mathbb{C}} \cap \{|T| \leq 1\}$, this yields a map $\psi : V_{R,\mathrm{um}} \to \mathbb{P}^{1,\mathrm{triv}}_{\mathbb{C}} \cap \{|T| \leq 1\}$ which is continuous by definition of the topologies. Conversely, by definition, any element $x \in \mathbb{P}^{1,\mathrm{triv}}_{\mathbb{C}} \cap \{|T| \leq 1\}$ gives rise to a ultrametric pseudo-absolute value $|\cdot| \in M_{K,\mathrm{um}}$. As $x \in \{|T| \leq 1\}$, we deduce $|\cdot|_{|\mathbb{C}[T]} \leq 1 \leq \|\cdot\|$ and therefore $|\cdot| \in V_{R,\mathrm{um}}$. This construction yields an inverse to ψ. To conclude, it suffices to remark that $V_{R,\mathrm{um}}$ compact Hausdorff and that $\mathbb{A}^{1,\mathrm{triv}}_{\mathbb{C}} \cap \{|T| \leq 1\}$ is Hausdorff.

(2) We refer to the proof of Proposition 9.4.7 (2). □

Remark 9.4.2. The notion of integral structure allows to characterise the ultrametric part of our space of pseudo-absolute values. This addresses the issue mentioned in Remark 7.2.3 (1).

9.4.2. *Nevanlinna theory: open disc.* A motivation for this work is to study problems arising from Nevanlinna theory. We start by studying the Berkovich spectrum associated with the integral structure of Example 9.2.1 (3) over the ring of holomorphic functions on a closed disc. Although the description of the whole space cannot be made explicit, its study will be useful for the closed disc case (§9.4.3).

Let $0 < R' \leq +\infty$ and let $K_{R'} := \mathcal{M}(D(R'))$ be the field of meromorphic functions on $D(R') := \{z \in \mathbb{C} : |z|_\infty < R'\}$, where $D(+\infty) = \mathbb{C}$. Let $R' < R$ and denote by $\|\cdot\|_R$ the restriction to $\mathcal{O}(D(R'))$ of the hybrid norm on the disc $\overline{D(R)}$ (cf. Example 9.2.1 (3)). Then $(\mathcal{O}(D(R')), \|\cdot\|_R)$ defines an integral structure for K. Denote by $V_{R,R'}$ the corresponding global space of pseudo-absolute values. In what follows, we explicitly describe $V_{R,R',\leq 1}$.

Proposition 9.4.3. *Let $v = (|\cdot|, A, \mathfrak{m}, \kappa) \in V_{R,R',\infty}$. Then there exist $z \in \overline{D(R)}$ and $\epsilon \in]0,1]$ such that $v = v_{z,\epsilon,\infty}$ (cf. Example 2.2.1 (3)). Furthermore, we have a homeomorphism $V_{R,R',\infty} \cong \overline{D(R)} \times]0,1]$ which maps any $v_{z,\epsilon,\infty} \in V_{R,R',\infty}$ to $(z,\epsilon) \in \overline{D(R)} \times]0,1]$.*

Proof. Recall that $|\cdot|_\infty$ denotes the usual complex Archimedean value. Note that, since $v \in V_{R,R'}$, we have an inclusion $\mathcal{O}(D(R')) \subset A$. In particular, $\mathbb{C}$ is a subfield of A and thus $\mathbb{C} \subset A^\times$. We have arrows $\mathbb{C} \subset A^\times \twoheadrightarrow \kappa$ and an extension $\mathbb{C} \to \kappa$. Since v induces an Archimedean absolute value on κ, the Gelfand-Mazur theorem ([Bou75], Chapitre VI, §6, n° 4, Théorème 1) ensures that $\kappa = \mathbb{C}$ and that there exists $\epsilon \in]0,1]$ such that $|\cdot| = |\cdot|_\infty^\epsilon$ on κ.

From the inclusion $\mathbb{C} \hookrightarrow \mathcal{O}(D(R'))^\times$ we deduce that the $\mathbb{C}$-algebra morphism $\mathcal{O}(D(R')) \to \mathbb{C}$ is surjective whose kernel is the maximal ideal $\mathfrak{m}' := \mathfrak{m} \cap \mathcal{O}(D(R'))$ of $\mathcal{O}(D(R'))$. Proposition 1.2.1 implies that $\mathfrak{m}'$ is principal and that there exists $z \in D(R')$ such that $\mathfrak{m}'$ is the set of functions in $\mathcal{O}(D(R'))$ vanishing in z. Thus v corresponds to the map $(f \in \mathcal{O}(D(R'))) \mapsto |f(z)|_\infty^\epsilon \in [0,+\infty]$. The condition $v \in V_{R,R'}$ ensures that $z \in \overline{D(R)}$. Hence the conclusion of the proof of the first statement of the proposition.

For any $v \in V_{R,R',\infty}$, denote by $z(v)$ the unique $z \in \overline{D(R)}$ such that $x = v_{z(v),\epsilon(v),\infty}$. Let $\varphi : V_{R,R',\infty} \to \overline{D(R)} \times]0,1]$ be the map mapping any $v \in V_{R,R',\infty}$ to $(z(v),\epsilon(v)) \in \overline{D(R)} \times]0,1]$. Then φ is bijective. Let us show φ is a homeomorphism. To show that φ is continuous, it is enough to show that so are the induced maps $\epsilon : v \in V_{R,R',\infty} \mapsto \epsilon(v) \in]0,1]$ and $z : v \in V_{R,R',\infty} \mapsto z(v) \in \overline{D(R)}$. The first map is continuous (cf. 9.1.7). The map z is equal to the composition $V_{R,R',\infty} \to V_{R,\infty} \to \overline{D(R)}$, where $V_{R,\infty}$ is the Archimedean part described in Proposition 9.4.1. It remains to prove that φ^{-1} is continuous. For this purpose, it suffices to show that, for any $f \in \mathcal{O}(D(R'))$, the map

$$|f| \circ \varphi^{-1} : \left| \begin{array}{rcl} \overline{D(R)} \times]0,1] & \longrightarrow & \mathbb{R}_+ \\ (z, \epsilon) & \longmapsto & |f(z)|_\infty^\epsilon \end{array} \right.$$

is continuous. This fact being certainly true, concludes the proof. □

Remark 9.4.4. Let $K = \mathcal{M}(U)$ denote the field of meromorphic functions on a (connected) non-compact Riemann surface U. On can adapt the above proof to describe the Archimedean pseudo-absolute values on K: they are the pseudo-absolute values of the form $(f \in K) \mapsto |f(z)|_\infty^\epsilon \in [0, +\infty]$, for some $z \in U$ and $\epsilon \in]0, 1]$.

We now study the ultrametric pseudo-absolute values in $V_{R,R'}$. We first exhibit some of them. For any $z \in D(R')$ and $c > 0$, denote by $v_{z,c,\mathrm{um}}$ the absolute value $|\cdot|_{z,c,\mathrm{um}} : (f \in K_{R'}) \mapsto \exp(-c \operatorname{ord}(f, z)) \in \mathbb{R}_{>0}$ (it is an element $V_{R,R'}$ since for all $f \in \mathcal{O}(D(R'))$, we have $|f|_{z,c,\mathrm{um}} \leq 1$). Example 2.2.1 (4) provides, for any $z \in D(R')$, a pseudo-absolute value $v_{z,\infty,\mathrm{um}}$ defined by

$$|\cdot|_{z,\infty,\mathrm{um}} : \left| \begin{array}{rcl} A_z & \longrightarrow & \mathbb{R}_{\geq 0} \\ f & \longmapsto & \begin{cases} 0 \text{ if } f \in \mathfrak{m}_z, \\ 1 \text{ if } f \notin \mathfrak{m}_z, \end{cases} \end{array} \right.$$

where A_z and $\mathfrak{m}_z$ denote respectively the set of elements of $K_{R'}$ without pole and vanishing in z. Finally, for any $z \in D(R')$, we denote by $v_{z,0,\mathrm{um}}$ the trivial absolute value on $K_{R'}$.

Proposition 9.4.5. *$V_{R,R',\leq 1,\mathrm{um}}$ is the set of pseudo-absolute values v on K such that there exist $z \in D(R')$ and $c \in [0, +\infty]$ such that $v = v_{z,c,\mathrm{um}}$.*

Proof. Let $v \in V_{R,R',\leq 1}$. We first assume that $v \in V_{R,R'} \smallsetminus V_{R,R',\mathrm{sn}}$, i.e. v is a usual absolute value on K. Let $A \subset K$ denote the valuation ring of v. We have the inclusion $\mathcal{O}(D(R')) \subset A$ since $v \in V_{R,R'}$. Moreover, v is a valuation of rank 1, hence v is either trivial or equivalent to the valuation $\operatorname{ord}(\cdot, z)$, for some $z \in D(R')$ (cf. Proposition 1.2.2). Thus there exist $z \in D(R')$ and $c \in \mathbb{R}_+$ such that $v = v_{z,c,\mathrm{um}}$.

We now assume that $v \in V_{R,R',\mathrm{sn},\mathrm{um}}$. Since $V_{R,R'}$ is tame (Définition 9.3.1) and, for any $v \in V_{R,R',\leq 1,\mathrm{sn}}$, we have $\kappa(v) = \mathbb{C}$ and a surjection $\mathcal{O}(D(R')) \to \kappa(v)$, Proposition 9.3.8 ensures that v is of the form $v_{z,\infty,\mathrm{um}}$, for some $z \in D(R')$. □

Proposition 9.4.6. *We have homomorphisms $V_{R,R',\infty} \cong \overline{D(R')} \times]0, 1]$ and*

$$V_{R,R',\leq 1,\mathrm{um}} \cong \bigsqcup_{v \in \mathcal{P}} [0, +\infty]/\sim,$$

with the notation of Remark 9.3.9.

Proof. This is a consequence of Propositions 9.4.3 and 9.4.5. The only part needing a justification is the fact that the nontrivial ultrametric absolute values of $K_{R'}$ are discrete. This comes from the first part of the proof of Proposition 9.4.5. □

9.4.3. *Nevanlinna theory: compact disc.* We now make use of the previous example to make an explicit description of the integral structure from Example 9.2.1 (4). We also give a description of the structure sheaf of the corresponding Berkovich spectrum and show that the underlying Banach ring is a geometric base ring in the sense of [LP24], so that Berkovich analytic geometry can be performed over such a Banach ring. This gives a new explicit example of Berkovich space where the Archimedean points interact with the ultrametric ones in a less standard way.

Let $R > 0$ and denote by $K_R := \mathcal{M}(\overline{D(R)})$ the field of (germs) of meromorphic functions on the closed disc $\overline{D(R)} := \{z \in \mathbb{C} : |z|_\infty \leq R\}$. Let $A_R = \mathcal{O}(\overline{D(R)})$ denote the ring of (germs of holomorphic functions on $\overline{D(R)}$). Equip A with the hybrid norm $\|\cdot\|_{R,\mathrm{hyb}}$ (cf. Example 9.2.1 (4)). Then $(A_R, \|\cdot\|_{R,\mathrm{hyb}})$ is an integral structure for K_R. We now describe the corresponding global space of pseudo-absolute values V_R.

Firstly, $V_{R,\infty} \cong \overline{D(R)} \times]0,1]$ where $(z,\epsilon) \in \overline{D(R)} \times]0,1]$ is mapped to the pseudo-absolute value $v_{z,\epsilon,\infty}$ defined in Example 2.2.1 (3) (see also Proposition 9.4.3).

We now describe $V_{R,\mathrm{um}}$. By Propositions 1.2.6 and 1.2.7, a non-trivial ultrametric absolute value in $V_{R,\mathrm{um}}$ is given by a valuation $\mathrm{ord}(\cdot, z)$ for some $z \in \overline{D(R)}$. Hence it is of the form $|\cdot|_{z,c,\mathrm{um}} : (f \in K_R) \mapsto \exp(-c\,\mathrm{ord}(f,z)) \in \mathbb{R}_{>0}$ for some $c > 0$ and $z \in \overline{D(R)}$. The remaining elements of $V_{R,\mathrm{um}}$ are of the form

$$|\cdot|_{z,\infty,\mathrm{um}} : \left| \begin{array}{ccc} \mathcal{O}(\overline{D(R)})_{(T-z)} & \longrightarrow & \mathbb{R}_{\geq 0} \\ f & \longmapsto & \begin{cases} 0 \text{ if } f \in (T-z), \\ 1 \text{ if } f \notin (T-z), \end{cases} \end{array} \right.$$

for some $z \in \overline{D(R)}$

Proposition 9.4.7. (1) *We have homeomorphisms*

$$V_{R,\infty} \cong \overline{D(R)} \times]0,1], \quad V_{R,\mathrm{um}} \cong \bigsqcup_{z \in \overline{D(R)}} [0,+\infty]/\sim,$$

where $\sim$ denotes the equivalence relation which identifies the extremity 0 *of each branch.*

(2) $V_{R,\infty}$ *is dense in* V_R.

Proof. The homeomorphisms in (1) follow from the above paragraphs together with Proposition 9.3.10.

We now prove (2). Let $(v_n)_{n \geq 0}$ be a sequence in $V_{R,\infty}$ converging to some $v \in V_R$. (1) implies that, for any integer $n \geq 0$, $v_n = v_{z_n,\epsilon_n,\infty}$, for some $z_n \in \overline{D(R)}$ and $\epsilon_n \in]0,1]$. By compactness of $\overline{D(R)}$ and $[0,1]$, up to extracting subsequences, we may assume that there exists $(z,\epsilon) \in \overline{D(R)} \times [0,1]$ such that $\lim_{n \to +\infty} z_n = z$ and $\lim_{n\to+\infty} \epsilon_n = \epsilon$. Let us prove that $v = v_{z,\epsilon,\infty}$ if $\epsilon > 0$ and that there exists $c \in [0,\infty]$ such that $v = v_{z,c,\mathrm{um}}$ if $\epsilon = 0$.

We first assume that $\epsilon > 0$. Since $|2|_v > 1$, v is Archimedean. Moreover, $|(T-z)|_v = \lim_{n\to+\infty} |z_n - z|_\infty^{\epsilon_n} = 0$. Thus $(T-z)$ belongs to the kernel of v and v is of the form $v_{z,\epsilon',\infty}$, for some $\epsilon' \in]0,1]$. $|2|_v = 2^\epsilon = 2^{\epsilon'}$ yields $\epsilon = \epsilon'$.

We now consider the $\epsilon = 0$ case. Thus $|2|_v = 1$ and v is ultrametric. Consider the sequence $u = (\epsilon_n \log |z_n - z|_\infty)_{n \geq 0}$. Up to extracting a subsequence, we may assume that u converges to $-c \in [-\infty, 0]$. Let us prove that $v = v_{z,c,\mathrm{um}}$. By hypothesis, we have $|(T-z)|_v = e^{-c}$ (with the convention $e^{-\infty} = 0$). Moreover, for any $z' \neq z$ in $\overline{D(R)}$, we have $|(T-z')| = \lim_{n\to+\infty} |z_n - z'|_\infty^{\epsilon_n} = 1$. From the aforementioned description of $V_{R,\mathrm{um}}$, we see that $v = v_{z,c,\mathrm{um}}$. □

Remark 9.4.8. The above example shows that the subspace of absolute values need not be open, discrete or dense in the space of pseudo-absolute values. However, we will show that density holds if K is countable (Corollary 10.2.4).

Now that we have a set-theoretic description of the space V_R, we can fully describe its topology. Let us define the *center map* $z : V_R \smallsetminus_{v_{\mathrm{triv}}} \to \overline{D(R)}$ defined by mapping an element $v \in V_R \smallsetminus \{v_{\mathrm{triv}}\}$ to the unique $z(v) \in \overline{D(R)}$ such that $v = v_{z(v),\epsilon,\infty}$ for some $\epsilon \in]0,1]$or $v = v_{z(v),c,\mathrm{um}}$ for some $c \in]0,\infty]$.

Proposition 9.4.9. *Let $v \in V_R$. Define the following set $\mathcal{B}_v$.*

(1) *If $v = v_{z,\epsilon,\infty} \in V_{R,\infty}$, where $(z,\epsilon) \in \overline{D(R)} \times]0,1]$,*

$$\mathcal{B}_v := \left\{ U_{k,l}^{(z,\epsilon)} := \left(\left\{ z' \in \overline{D(R)} : |z'-z|_\infty < \frac{1}{k} \right\} \cap \overline{D(R)} \right) \times \left] \epsilon - \frac{1}{l}, \epsilon + \frac{1}{l} \right[: 0 \ll k,l \in \mathbb{N}_{>0} \right\}$$

if $\epsilon < 1$ and

$$\mathcal{B}_v := \left\{ U_{k,l}^{(z,1)} := \left(\left\{ z' \in \overline{D(R)} : |z'-z|_\infty < \frac{1}{k} \right\} \cap \overline{D(R)} \right) \times]\epsilon - \frac{1}{l}, 1] : 0 \ll k,l \in \mathbb{N}_{>0} \right\}$$

if $\epsilon = 1$.

(2) *If $v = v_{z,c,\mathrm{um}}$, for $(z,c) \in \overline{D(R)} \times [0,\infty]$,*

(a) *if $c = \infty$,*

$$\mathcal{B}_v := \left\{ U_k^{(z,\infty)} := \left\{ |T-z| < \frac{1}{k} \right\} \cap \left\{ \epsilon(\cdot) < \frac{1}{k} \right\} : k \in \mathbb{N}_{>0} \right\};$$

(b) *if $c \in \mathbb{R}_{>0}$,*

$$\mathcal{B}_v := \left\{ U_k^{(z,c)} := \left\{ e^{-c} - \frac{1}{k} < |T-z| < e^{-c} + \frac{1}{k} \right\} : k \in \mathbb{N}_{>0} \right\};$$

(c) *if $v = v_{\mathrm{triv}}$,*

$$\mathcal{B}_v := \left\{ U_{S,k}^{\mathrm{triv}} := \bigcap_{z \in S} \left\{ |T-z| > 1 - \frac{1}{k} \right\} : k \in \mathbb{N}_{>0} \text{ and } S \subset \overline{D(R)} \text{ finite} \right\}.$$

Then $\mathcal{B}_v$ defines a basis of neighbourhood of v in V_R.

Proof. **Proof of (1):** This is a direct consequence of the fact that we have a homeomorphism $V_R \cong \overline{D(R)} \times]0,1]$.

Proof of (2.a): For any $k \in \mathbb{N}_{>0}$, $U_k^{(\infty)}$ is open by definition of the topology of V_R. Note that $v \in U_k^{(\infty)}$ iff $v = v_{z,c,\mathrm{um}}$ with $c > \log k$ or $v = v_{z',\epsilon',\infty}$ with $\epsilon' < 1/k$ and $\epsilon' \log|z'-z|_\infty < -\log k$. Thus from the description in the proof of Proposition 9.4.7, it follows that $\mathcal{B}_v$ is a basis of neighbourhood of v.

Proof of (2.b): As in case (2.a), for any $k \in \mathbb{N}_{>0}$, $U_k^{(c)}$ is open and $v \in U_k^{(c)}$ iff $v = v_{z,c',\mathrm{um}}$ with $|e^{-c} - e^{c'}| < 1/k$ or $v = v_{z',\epsilon',\infty}$ with $||z'-z|_\infty^{\epsilon'} - e^{-c}| < 1/k$. Thus from the description in the proof of Proposition 9.4.7, it follows that $\mathcal{B}_v$ is a basis of neighbourhood of v.

Proof of (2.c): Let $k \in \mathbb{N}_{>0}$ and $S \subset \overline{D(R)}$ be a finite subset. Then $U_{S,k}^{\mathrm{triv}}$ is an open neighbourhood of v_{triv}. Moreover, the intersection of the $U_{S,k}^{\mathrm{triv}}$'s is v_{triv}. Indeed, a multiplicative semi-norm in the intersection must have kernel zero and thus belongs to V_R. Thus this point is v_{triv} from the description of the open neighbourhoods of v_{triv} in $V_{R,\mathrm{um}}$. □

Proposition 9.4.10. *Let $v \in V_R$. The stalks of the structure sheaf $\mathcal{O}_R$ of V_R have the following description.*

(1) *If $v = v_{z,\epsilon,\infty} \in V_{R,\infty}$, where $(z,\epsilon) \in \overline{D(R)} \times]0,1]$, then $\mathcal{O}_{R,v}$ identifies with $\mathcal{O}_{\mathbb{C},z}$, the ring of germs of holomorphic functions at z.*

(2) *If* $v = v_{z,c,\mathrm{um}}$, *for* $(z,c) \in \overline{D(R)} \times [0,\infty]$, *then* $\mathcal{O}_{R,v}$ *identifies with:*
   (a) $\mathcal{O}_{\mathbb{C},z}$, *the ring of germs of holomorphic functions at* z, *if* $c = \infty$;
   (b) $\mathcal{M}_{\mathbb{C},z}$, *the ring of germs of meromorphic functions at* z, *if* $0 < c < \infty$;
   (c) $\mathcal{M}(\overline{D(R)})$, *the field of meromorphic functions on* $\overline{D(R)}$, *if* $c = 0$.

Proof. **Proof of (1):** Let $(z,\epsilon) \in \overline{D(R)} \times]0,1]$. Let $f \in \mathcal{O}_{\mathbb{C},z}$ be a germ of holomorphic functions at v. By definition, there exists an open subset $U \subset D(R')$, where $R < R'$, such that f is holomorphic on U. Then there exist a compact neighbourhood V of v in U and a sequence $(f_i)_{i\geq 0}$ of rational functions without poles on V such that $\|f_{|V\cap\overline{D(R)}} - f_i\|_{V\cap\overline{D(R)}} \to_{i\to+\infty} 0$. Set $U_{a,b} := (U \cap \overline{D(R)}) \times]a,b]$, where $0 < a < \epsilon \leq b \leq 1$ and $V_{a,b} := (V \cap \overline{D(R)}) \times]a,b]$. Define

$$f_{a,b} : \left| \begin{array}{ccc} U_{a,b} & \longrightarrow & \displaystyle\coprod_{(z',\epsilon')\in U_{a,b}} (\mathbb{C}, |\cdot|_\infty^{\epsilon'}), \\ (z',\epsilon') & \longmapsto & f(z'). \end{array} \right.$$

Similarly, for all $i \geq 0$, set

$$f_{i,a,b} : \left| \begin{array}{ccc} U_{a,b} & \longrightarrow & \displaystyle\coprod_{(z',\epsilon')\in U_{a,b}} (\mathbb{C}, |\cdot|_\infty^{\epsilon'}), \\ (z',\epsilon') & \longmapsto & f_i(z'), \end{array} \right.$$

this is an element of $\mathcal{K}(V_{a,b})$. Using the equivalence of norms on $\mathbb{C}$, we see that

$$\|f_{a,b|V_{a,b}} - f_{i,a,b}\|_{V_{a,b}} = \sup_{(z',\epsilon')\in V_{a,b}} |f(z') - f_i(z')|_\infty^{\epsilon'} \to_{i\to+\infty} 0.$$

Thus $f_{a,b}$ defines an analytic function on a neighbourhood of $v_{z,\epsilon,\infty}$ which gives a ring homomorphism $\mathcal{O}_{\mathbb{C},z} \to \mathcal{O}_{R,v_{z,\epsilon,\infty}}$.

Conversely, let $g \in \mathcal{O}_{R,v_{z,\epsilon,\infty}}$. We may assume that g is defined by $f_{a,b} \in \mathcal{O}_R(U_{a,b})$ on a neighbourhood of the form $U_{a,b} := U \times]a,b] \subset \overline{D(R)} \times]0,1]$, where $U \subset \overline{D(R)}$ is an open subset and $0 < a < \epsilon \leq b \leq 1$. By definition, there exists a compact neighbourhood $V_{a,b}$ of $v_{z,\epsilon,\infty}$ in $U_{a,b}$ and a sequence $(f_{i,a,b})_{i\geq 0}$ of meromorphic functions on $\overline{D(R)}$ without poles on $V_{a,b}$ such that $\|f_{a,b|V_{a,b}} - f_{i,a,b}\|_{V_{a,b}} \to_{i\to+\infty} = 0$. Define $f : (z' \in U) \to f_{a,b}(z',\epsilon)\mathbb{C}$ and $f_{i,a,b} : (z' \in V) \mapsto f_{i,a,b}(z',\epsilon) \in \mathbb{C}$ for $i \geq 0$, where V denote the first projection of $V_{a,b}$. Then using again the equivalence of norms on $\mathbb{C}$, we obtain that f is on V a uniform limit of meromorphic functions without poles, hence is holomorphic. This construction defines a ring homomorphism $\mathcal{O}_{R,v_{z,\epsilon,\infty}} \to \mathcal{O}_{\mathbb{C},z}$ which is inverse to the previous one.

Proof of (2.a): Assume that $v = v_{z,\infty,\mathrm{um}}$. Consider the Berkovich spectrum $\mathcal{M}(\mathcal{O}(\overline{D(R)}), \|\cdot\|_{\mathrm{triv}})$. This is the spectrum of a trivially valued Dedekind ring which identifies as $V_{R,\mathrm{um}}$ by Proposition 9.4.7. The stalk of the structure sheaf of $\mathcal{M}(\mathcal{O}(\overline{D(R)}), \|\cdot\|_{\mathrm{triv}})$ at v identifies with the ring of power series $\mathbb{C}[\![T-z]\!]$. The inclusion $\mathcal{M}(\mathcal{O}(\overline{D(R)}), \|\cdot\|_{\mathrm{triv}}) \subset V_R$ and the construction of the structure sheaf yields an inclusion $\mathcal{O}_{R,v} \subset \mathbb{C}[\![T-z]\!]$ of rings. Assume that $f \in \mathbb{C}[\![T-z]\!]$ is a convergent power series that defines a germ of holomorphic functions at z defined on a ball of radius $r < 1$ around z. Let $k \in \mathbb{N}_{>0}$ such that $1/k < r$ and consider $U_k^{(z,\infty)} = \{|T-z| < 1/k\} \cap \{\epsilon(\cdot) < 1/k\} \subset V_R$ from Proposition 9.4.9. This is an open neighbourhood of v such that $U_k^{(z,\infty)} \cap V_{R,\mathrm{um}}$ is contained in the branch associated with z.

Moreover, $(z', \epsilon') \in \overline{D(R)} \times]0,1]$ belongs to $U_k^{(z,\infty)}$ iff $|z'-z|_\infty^\epsilon < r$. Thus we can define

$$f_k : \left| \begin{array}{ccc} U_k^{(z,\infty)} & \longrightarrow & \displaystyle\bigsqcup_{v' \in U_k^{(z,\infty)}} \widehat{\kappa(v')} \\ v' & \longmapsto & \begin{cases} f \text{ if } v' \in V_{V_k^{(z,\infty)} R,\mathrm{um}}, \\ f(z') \text{ if } v' = v_{z',\epsilon',\infty} \in V_{R,\infty}. \end{cases} \end{array} \right.$$

Then f_k defines an element of $\mathcal{O}_R(U_k^{(z,\infty)})$ and thus an element of $\mathcal{O}_{R,v}$. Moreover, $\mathcal{O}_{\mathbb{C},z} \subset \mathcal{O}_{R,v}$.

Conversely, assume that $f \in \mathbb{C}[\![T-z]\!]$ is not convergent around z. If it would correspond to a stalk in $\mathcal{O}_{R,v}$, we could find some $k \in \mathbb{N}_{>0}$ such that f is defined on $U_k^{(z,\infty)}$. But $v_{z,1/2k,\infty} \in U_k^{(z,\infty)}$ and considering the stalk at this point, f corresponds to a convergent power series at z by (1). This gives a contradiction.

Proof of (2.b): Assume that $v = v_{z,c,\mathrm{um}}$, where $c \in \mathbb{R}_{>0}$. By the same argument as in case (2.a), we have an inclusion of rings $\mathcal{O}_{R,v} \subset \mathbb{C}(\!(T-z)\!)$. Consider the open neighbourhoods $U_k^{(z,c)} = \{e^{-c} - 1/k < |T-z| < e^{-c} + 1/k\}$, where $k \in \mathbb{N}_{>0}$ from Proposition 9.4.9. Let $f \in \mathbb{C}(\!(T-z)\!)$ correspond to a germ of meromorphic functions at z. Then on a complex neighbourhood of z, f defines a meromorphic function that is holomorphic everywhere except possibly at z. Then there exists $k \in \mathbb{N}_{>0}$ such that $U_k^{(z,c)}$ contains only Archimedean points of the form $v_{z',\epsilon',\infty}$ such that f defines a germ of holomorphic functions at z'. Thus

$$f_k : \left| \begin{array}{ccc} U_k^{(z,c)} & \longrightarrow & \displaystyle\bigsqcup_{v' \in U_k^{(z,c)}} \widehat{\kappa(v')} \\ v' & \longmapsto & \begin{cases} f \text{ if } v' \in V_{R,\mathrm{um}}, \\ f(z') \text{ if } v' = v_{z',\epsilon',\infty} \in V_{R,\infty} \end{cases} \end{array} \right.$$

defines an element in $\mathcal{O}_R(U_k^{(z,c)})$ and the stalk in $\mathcal{O}_{R,v}$. Moreover, $\mathcal{M}_{\mathbb{C},z} \subset \mathcal{O}_{R,v}$.

Conversely, assume that $f \in \mathbb{C}(\!(T-z)\!)$ does not define a germ of meromorphic functions at z and corresponds to a stalk in $\mathcal{O}_{R,v}$. Then we could find an integer $k \in \mathbb{N}_{>0}$ such that f defines an analytic function on $U_k^{(z,c)}$. Then for all $(z', \epsilon') \in \overline{D(R)} \times]0,1]$ such that $|\epsilon \log |z'-z|_\infty + c| < 1/k$, $v_{z',\epsilon',\infty} \in U_k^{(z,c)}$ and considering stalks, f would define a germ in $\mathcal{O}_{\mathbb{C},z'}$. Since z' can be chosen to describe a complex open ball around z, f would define a germ of meromorphic functions at z, which gives a contradiction. This means that $\mathcal{O}_{R,v} \subset \mathcal{M}_{\mathbb{C},z}$ and therefore $\mathcal{O}_{R,v} = \mathcal{M}_{\mathbb{C},z}$.

Proof of (2.c): By the same argument as in the last two cases we have an inclusion $\mathcal{O}_{R,v} \subset \mathcal{M}(\overline{D(R)})$. Let $f \in \mathcal{M}(\overline{D(R)})$. Let S denote the finite set of poles of f. Then for any $k \in \mathbb{N}_{>0}$, define

$$f_k : \left| \begin{array}{ccc} U_{S,k}^{triv} & \longrightarrow & \displaystyle\bigsqcup_{v' \in U_{S,k}^{triv}} \widehat{\kappa(v')} \\ v' & \longmapsto & \begin{cases} f \text{ if } v' \in V_{R,\mathrm{um}}, \\ f(z') \text{ if } v' = v_{z',\epsilon',\infty} \in V_{R,\infty}. \end{cases} \end{array} \right.$$

This is an element of $\mathcal{O}_R(U_{S,k}^{\mathrm{triv}})$. This gives the inverse inclusion $\mathcal{M}(\overline{D(R)}) \subset \mathcal{O}_{R,v}$. $\square$

We conclude this section by proving that the Banach ring $(\mathcal{O}(\overline{D(R)}), \|\cdot\|_{R,\mathrm{hyb}})$ is a geometric base ring in the sense of ([LP24], Definition 3.3.8). This implies that one can perform Berkovich geometry over such a Banach ring.

Proposition 9.4.11. *Let $v \in V_R$ and consider the basis of neighbourhood $\mathcal{B}_v$ introduced in Proposition 9.4.9. Then $\mathcal{B}_v$ is a fine basis of path-connected neighbourhoods of v whose closure is spectrally convex.*

Proof. Let $v \in V_R$. The fact that $\mathcal{B}_v$ is fine is straightforward from the definition of the neighbourhoods.

Proof of path connectedness: The result is clear if $v \in V_{R,\infty}$.

Assume that $v = v_{z,\infty,\mathrm{um}}$ for some $z \in \overline{D(R)}$ and let $k \in \mathbb{N}_{>0}$. Note that $U_k^{(z,\infty)} \cap V_{R,\mathrm{um}}$ and $U_k^{(z,\infty)} \cap V_{R,\infty}$ are both path connected. Let $0 < \epsilon < 1/k$, then $(t \in [0,1]) \mapsto v_{z,\epsilon,\infty} \in U_k^{(v,\infty)}$, where $v_{z,\epsilon,\infty} := v_{z,\infty,\mathrm{um}}$ is a path between v and $v_{z,\epsilon,\infty}$. Thus $U_k^{(z,\infty)}$ is path connected.

Now assume that $v = v_{z,c,\mathrm{um}}$, where $(z,c) \in \overline{D(R)} \times \mathbb{R}_{>0}$. Let $k \in \mathbb{N}_{>0}$ such that $e^{-c} - 1/k > 0$. Again $U_k^{(z,c)} \cap V_{R,\mathrm{um}}$ and $U_k^{(z,c)} \cap V_{R,\infty}$ are both path connected. Let $v_{z',\epsilon',\infty} \in U_k^{(z,c)}$, where $(z',\epsilon') \in \overline{D(R)} \times]0,1]$. Let $\epsilon(0) := 0$ and for any $t \in]0,1]$, set

$$\epsilon(t) := \frac{\log(e^{-c} + \frac{1}{k}) + \log((e^{-c} - \frac{1}{k}))}{2(\log(t) + \log|z' - z|_\infty)}.$$

Then $(t \in [0,1]) \mapsto v_{tz'+(1-t)z,\epsilon(t),\infty}$ defines a continuous path in $U_k^{(z,c)}$ between v and $v_{z',\epsilon(1),\infty}$. Therefore we can construct a continuous path between v and $v_{z',\epsilon',\infty}$.

Finally, assume that $v = v_{\mathrm{triv}}$ and let $k \in \mathbb{N}_{>0}$ and $S \subset \overline{D(R)}$ be finite. Again $U_{S,k}^{\mathrm{triv}} \cap V_{R,\mathrm{um}}$ and $U_{S,k}^{\mathrm{triv}} \cap V_{R,\infty}$ are both path connected. We may assume that there exists a complex disc around 0 of radius $r \leq 1$ which does not intersect S. Let $z \notin S$ such that $0 < |z|_\infty \leq 1$. Then $(t \in [0,1]) \mapsto v_{tz,t|z|_\infty,\infty}$, where $v_{0,0,\infty} := v_{\mathrm{triv}}$, is a continuous path between v_{triv} and $v_{z,|z|_\infty,\infty}$ in $U_{S,k}^{\mathrm{triv}}$. Thus $U_{S,k}^{\mathrm{triv}}$ is path connected.

Proof of spectral convexity: We prove the spectral convexity only for $v = v_{z,\infty,\mathrm{um}}$, where $z \in \overline{D(R)}$, the other case are treated similarly. Let $k \in \mathbb{N}_{>0}$. By direct inspection, we have

$$V_k^{(z,\infty)} := \overline{U_k^{(z,\infty)}} = \left\{ |T - z| \leq \frac{1}{k} \right\} \cap \left\{ \epsilon(\cdot) \leq \frac{1}{k} \right\}.$$

Thus we see that $V_k^{(z,\infty)}$ is pro-rational in the sense of ([LP24], Définition 1.3.1). Thus it is spectrally convex (*loc. cit.*, Théorème 1.3.7). □

Proposition 9.4.12. *The Banach ring $(\mathcal{O}(\overline{D(R)}), \|\cdot\|_{R,\mathrm{hyb}})$ is a geometric base ring.*

Proof. The proof is in two steps: the first one is that V_R satisfies the analytic continuation principle and the second one is the proof that $(\mathcal{O}(\overline{D(R)}), \|\cdot\|_{R,\mathrm{hyb}})$ basic.

Proof of analytic continuation: Let $v \in V_R$ and U an open neighbourhood of v. Let $f \in \mathcal{O}_R(U)$ such that the stalk of f at v is non-zero in $\mathcal{O}_{R,v}$. Let us prove that there exists an open neighbourhood V in $\mathcal{B}_v$ such that f yields non-zero elements of all the stalks $\mathcal{O}_{R,v'}$, where $v' \in V$. If $v \in V_{R,\infty}$, this is a consequence of usual analytic continuation. If $v \in V_{R,\mathrm{um}}$ is of the form $v_{z,c,\mathrm{um}}$, where $(z,c) \in \overline{D(R)} \times [0,\infty]$. If $c = \infty$, then a stalk of f at v corresponds to a germ of holomorphic function at z. In the proof of Proposition 9.4.10, we described an analytic function on some $U_k^{(z,\infty)}$ giving $f_{|U_k^{(z,\infty)}}$. All the corresponding stalks are non-zero. We argue similarly in the cases $c \in \mathbb{R}_{>0}$ and $v = v_{\mathrm{triv}}$.

Proof of basicity: By Proposition 9.4.11, we have to prove that for any $v \in V_R$, the stalk $\mathcal{O}_{R,v}$ is either a strong field or a strong DVR ([LP24], Définition 1.6.15). By (*loc. cit.*, Remarque 1.6.16), the analytic continuation implies that we only have to show that $\mathcal{O}_{R,v}$ is a strong DVR, where $v = v_{z,\infty,\mathrm{um}}$ or $v = v_{z,\epsilon,\infty}$ for some $(z,\epsilon) \in \overline{D(R)} \times]0,1]$. In that case the maximal ideal $\mathfrak{m}_{R,v}$ is generated by $T - z$. Let U be a compact neighbourhood of v. Let V be the closure of an element in $\mathcal{B}_v$ such that V is contained in the interior of U. We have to show that there exists a family $K_{V,U}$ of positive constants such that for any $f = g(T-z) \in \mathfrak{m}_{R,v}$ that is defined on U and V, we have

$$\|g\|_V \leq K_{V,U}\|f\|_U.$$

Let us treat the Archimedean and ultrametric cases separately.

Assume that $v = v_{z,\epsilon,\infty}$, where $(z,c) \in \overline{D(R)} \times [0,\infty]$. Assume that $V = \overline{U_{k,l}^{(z,\epsilon)}}$, where $k,l \in \mathbb{N}_{>0}$ big enough. By Schwarz Lemma, for any $(z',\epsilon') \in U_{k,l}^{(z,\epsilon)}$, we have

$$|g(z')|_\infty^{\epsilon'} \leq \|f\|^{\epsilon'}_{\overline{D(z,1/k)} \cap \overline{D(R)}} \leq \|f\|_V \leq \|f\|_U.$$

Thus $K_{V,U} = 1$ is a suitable choice.

Assume that $v = v_{z,\infty,\mathrm{um}}$, where $z \in \overline{D(R)}$. Assume that $V = U_k^{(z,\infty)}$, where $k \in \mathbb{N}_{>0}$. Schwarz Lemma again yields

$$\sup_{v' \in V \cap V_{R,\infty}} |g(v')| \leq \sup_{v' \in V \cap V_{R,\infty}} |f(v')| \leq \sup_{v' \in U \cap V_{R,\infty}} |f(v')|.$$

Now by definition of $U_k^{(z,\infty)} \cap V_{R,\mathrm{um}}$, we have

$$\sup_{v' \in V \cap V_{R,\mathrm{um}}} |g(v')| = e^{-\frac{1}{k}\operatorname{ord}(g,z)} = e^{\frac{1}{k}} \sup_{v' \in V \cap V_{R,\mathrm{um}}} |f(v')| \leq e^{\frac{1}{k}} \sup_{v' \in U \cap V_{R,\mathrm{um}}} |f(v')|.$$

Therefore $K_{V,U} := e^{\frac{1}{k}}$ is a suitable choice. $\square$

10. Global analytic spaces

We now give alternative approaches to global analytic spaces. This is a global counterpart to §8. More precisely, we give an analogue of the Zariski-Riemann description of spaces of pseudo-absolute extending pseudo-absolute values coming from an integral structure. As an application, we describe the full space of pseudo-absolute values on a field as an analytic Zariski-Riemann space over the prime ring. We also obtain the density of the set of absolute values in the countable case. Throughout this section, we fix a field K.

10.1. Model global analytic space.

In this subsection, we assume that we have an integral structure $(A, \|\cdot\|)$ for K (Definition 9.1.1). We denote by $V := \mathcal{M}(A, \|\cdot\|)$ the corresponding global space of pseudo-absolute value. We will make heavy use of the theory developed by Lemanissier-Poineau in [LP24].

We assume that $(A, \|\cdot\|)$ is a geometric base ring (cf. §1.5.3) and that A is universally Japanese (cf. Example 1.4.2). Let $X \to \operatorname{Spec}(K)$ be a projective K-scheme and let $\mathcal{X} \to \operatorname{Spec}(A)$ be a projective model of X over A. Assume that $\mathcal{X}$ is a coherent model, namely

$\mathcal{X} \to \mathrm{Spec}(A)$ is finitely presented. Then one can consider the A-analytic space associated with $\mathcal{X}$, denoted by $\mathcal{X}^{\mathrm{an}}_{(A,\|\cdot\|)}$, or $\mathcal{X}^{\mathrm{an}}$ when no confusion may arise (cf. §1.5.3).

Definition 10.1.1. With the above notation, the A-analytic space $\mathcal{X}^{\mathrm{an}}$ is called the *global model analytic space* attached to the model $\mathcal{X}$. It is a compact Hausdorff topological space ([LP24], Lemme 6.5.1 and Proposition 6.5.3).

From now on, we fix a coherent model $\mathcal{X}$ of X over A and denote by X^{an} the associated global analytic space. The global analytic space $\mathcal{X}$ comes with two maps of locally ringed spaces $p : \mathcal{X}^{\mathrm{an}} \to V := \mathcal{M}(A)$ and $j : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$. In this thesis, we will only use the topological properties of analytic spaces.

Let $v = (|\cdot|_v, A_v, \mathfrak{m}_v, \kappa_v) \in V$. Then $\mathcal{X}_v := \mathcal{X} \otimes_A A_v$ is a coherent model of $\mathcal{X}$, which is flat if $\mathcal{X}$ is flat itself. Thus $(\mathcal{X}_v \otimes_A \widehat{\kappa_v})^{\mathrm{an}}$ is a local model analytic space in v of X in the sense of Definition 8.1.1.

Proposition 10.1.2 ([LP24], Proposition 4.5.3)**.** *We have a homeomorphism*

$$(\mathcal{X}_v \otimes_A \widehat{\kappa_v})^{\mathrm{an}} \cong p^{-1}(v).$$

10.2. **General global analytic space.** We now try to introduce a general approach to global analytic spaces. Let K'/K be a field extension. Intuitively, the global analytic space attached to any sub-model X of K'/K should be a fibration over M_K whose fibres correspond to the local analytic spaces attached to X. For the same reasons that were explained in Proposition 8.3.1, an approach mimicking directly Berkovich spaces is not suitable if one wants to obtain Hausdorff spaces.

10.2.1. *Global analytic space associated with a sub-model.* As in the previous subsection, we fix an integral structure $(A, \|\cdot\|)$ for K and we assume that $(A, \|\cdot\|)$ is a geometric base ring and that A is universally Japanese.

Definition 10.2.1. Let K'/K be an algebraic function field and X be a projective sub-model of K'/K. The *global analytic space* attached to X is defined to be

$$X^{\mathrm{an}}_{(A,\|\cdot\|)} := \varprojlim_{\mathcal{X}} \mathcal{X}^{\mathrm{an}}_{(A,\|\cdot\|)},$$

where $\mathcal{X}$ runs over the projective sub-models of K'/A that are flat and coherent and whose generic fibre is isomorphic to X. This is a compact Hausdorff space.

10.2.2. *Zariski-Riemann global analytic space.* We keep the same notation as in the above paragraph. Denote by $V'_{(A,\|\cdot\|)}$, or V'_A if no confusion may arise, the subset of $M_{K'}$ consisting of all pseudo-absolute values on K' extending a pseudo-absolute belonging to $\mathcal{M}(A, \|\cdot\|)$. There is a specification map $V'_A \to \mathrm{ZR}(K'/A)$ which is continuous by a similar argument to Lemma 8.2.2.

Theorem 1.4.3 (1) gives a homeomorphism

$$\mathrm{ZR}(K'/A) \cong \varprojlim_{\mathcal{X}} \mathcal{X}, \tag{5}$$

where $\mathcal{X}$ runs over the projective models of K'/A. Let us prove that the isomorphism (5) has a counterpart in our context.

Lemma 10.2.2. *Let $\mathcal{X}$ be a projective sub-model of K'/A. Then there exists a flat and coherent projective model $\mathcal{X}'$ of K'/A dominating $\mathcal{X}$.*

Proof. Consider the sheaf of ideals $(\mathcal{O}_{\mathcal{X}})_{\mathrm{tor}}$ of $\mathcal{O}_{\mathcal{X}}$ as an $\mathcal{O}_{\mathrm{Spec}(A)}$-module. Then let $\mathcal{X}'$ denote the closed subscheme of $\mathcal{X}$ defined by $(\mathcal{O}_{\mathcal{X}})_{\mathrm{tor}}$. It is a projective sub-model of K'/A. Moreover, since $\mathcal{O}_{\mathcal{X}}/(\mathcal{O}_{\mathcal{X}})_{\mathrm{tor}}$ is a torsion-free $\mathcal{O}_{\mathrm{Spec}(A)}$, hence $\mathcal{X}' \to \mathrm{Spec}(A)$ is flat. By the flattening theorem of Raynaud-Gruson ([RG71], Corollaire 3.4.7), $\mathcal{X}'$ is also coherent. □

By Lemma 10.2.2, we see that any two projective sub-models of K'/A are dominated by a flat and coherent projective sub-model of K'/A. Therefore we have a homeomorphism

$$\mathrm{ZR}(K'/A) \cong \varprojlim_{\mathcal{X}} \mathcal{X},$$

where $\mathcal{X}$ runs over the flat and coherent projective models of K'/A. We will now prove an analytic analogue of this homeomorphism.

Let $\mathcal{X}$ be a flat and coherent projective model of K'/A and set $\mathcal{X}^{\mathrm{an}} := \mathcal{X}^{\mathrm{an}}_{(A,\|\cdot\|)}$. Let $v' = (|\cdot|', A', \mathfrak{m}', \kappa') \in V'_A$. Denote by v the restriction of v' to K and let $\mathcal{X}_v := \mathcal{X} \otimes_A A_v$. Then the construction of the map $\mathrm{red}_{\mathcal{X}_v} : M_{K',v} \to \widehat{\mathcal{X}_v}^{\mathrm{an}} := (\mathcal{X}_v \otimes_{A_v} \widehat{\kappa_v})^{\mathrm{an}}$ in §8.2 yields a point $x \in \widehat{\mathcal{X}_v}^{\mathrm{an}}$. Let $\pi : \mathcal{X}^{\mathrm{an}} \to \mathcal{M}(A, \|\cdot\|)$ denote the structural morphism. By Proposition 10.1.2, we have a homeomorphism $\mathcal{X}^{\mathrm{an}}_v \cong p^{-1}(v)$. Thus we obtain a map $\mathrm{red}_{\mathcal{X}} : V'_A \to \mathcal{X}^{\mathrm{an}}$. Moreover the arguments in §8.2 adapt *mutatis mutandis* to show that the construction of the maps $\mathrm{red}_{\mathcal{X}}$ is compatible with the domination relation between projective models of K'/A, so that we obtain a commutative diagram

$$\begin{tikzcd} V'_A \arrow[r, "j"] \arrow[d, "\mathrm{red}"'] & \mathrm{ZR}(K'/A) \arrow[d, "\cong"] \\ \varprojlim_{\mathcal{X}\in\mathcal{M}_{K'/A}} \mathcal{X}^{\mathrm{an}} \arrow[r] & \varprojlim_{\mathcal{X}\in\mathcal{M}_{K'/A}} \mathcal{X} \end{tikzcd},$$

where $\mathcal{M}_{K'/A}$ denotes the collection of flat and coherent projective models of K'/A.

Theorem 10.2.3. *We use the above notation. The map* $\mathrm{red} : V'_A \to \varprojlim_{\mathcal{X}\in\mathcal{M}} \mathcal{X}^{\mathrm{an}}$ *is a homeomorphism. Moreover, if K'/K is finitely generated, one can only consider the projective models of K'/A in the projective limit above.*

Proof. First note that the arguments in the proof of Proposition 8.2.3 adapt directly in our setting to obtain the continuity of the map $\mathrm{red}_{\mathcal{X}}$, for any $\mathcal{X} \in \mathcal{M}$, and thus the continuity of red. By compactness of V'_A and $\varprojlim_{\mathcal{X}} \mathcal{X}^{\mathrm{an}}$, it suffices to prove that red is bijective.

For the injectivity of red, the elements in the proof of Theorem 8.2.4 adapt *mutatis mutandis* (using Theorem 1.4.3 (2) and ([Bou75], Chap. IX, Appendice 1, Proposition 1)).

The surjectivity of red is given by Theorem 8.2.4 and Proposition 10.1.2. □

We list several consequences of Theorem 10.2.3.

Corollary 10.2.4. *We use the same notation as above.*

(1) *We have a homeomorphism*

$$V'_{(A,\|\cdot\|)} \cong \varprojlim_{X} X^{\mathrm{an}}_{(A,\|\cdot\|)},$$

where X runs over the projective sub-models of K'/K.

(2) *Assume that K is of characteristic zero. The following assertion hold.*

(a) *M_K is homeomorphic to $\varprojlim_{\mathcal{X}\in\mathcal{M}_{K/\mathbb{Z}}} \mathcal{X}^{\mathrm{an}}$, where $\mathbb{Z}$ is equipped with the Banach norm $|\cdot|_\infty$.*

(b) *$M_{K,\infty}$ is homeomorphic to $(\varprojlim_{X\in\mathcal{M}_{K/\mathbb{Q}}} X(\mathbb{C})/\sim)\times]0,1]$, where $\sim$ denotes the complex conjugation.*

(c) *Assume that K is a finitely generated extension of $\mathbb{C}$. Then $M_{K,\infty}$ is homeomorphic to $(\varprojlim_X X(\mathbb{C}))\times]0,1]$, where X runs over the projective models of $K/\mathbb{C}$.*

(3) *Assume that K is of characteristic $p>0$. Then M_K is homeomorphic to $\varprojlim_{\mathcal{X}\in\mathcal{M}_{K/\mathbb{F}_p}} \mathcal{X}^{\mathrm{an}}$, where $\mathbb{F}_p$ is equipped with the trivial norm.*

(4) *Assume that $(A,\|\cdot\|)$ is Dedekind analytic (*[LP24]*, Définition 6.6.1), e.g. if A is the prime ring of K'. Then the specification morphism $j : V'_{(A,\|\cdot\|)} \to \mathrm{ZR}(K'/A)$ is open.*

(5) *Assume that K is countable. Then the set of absolute values on K is dense in M_K.*

Proof. (1) is clear from Theorem 10.2.3 and (2.a), (3) follow directly from the cases $(\mathbb{Z},\|\cdot\|)$ and $(\mathbb{F}_p,\|\cdot\|_{\mathrm{triv}})$.

To prove (2.b), we set $(A,\|\cdot\|):=(\mathbb{Q},\max\{|\cdot|_{\mathrm{triv}},|\cdot|_\infty\})$. Using the description of hybrid analytifications (cf. e.g. [Poi25]), Theorems 8.2.4 and 10.2.3 yield homeomorphisms

$$M_{K,\infty} = V'_{(A,\|\cdot\|)} \smallsetminus M_{K,|\cdot|_{\mathrm{triv}}} \cong \varprojlim_{X\in\mathcal{M}_{K/\mathbb{Q}}} (X^{\mathrm{an}}_{(A,\|\cdot\|)} \smallsetminus X^{\mathrm{an}}_{(\mathbb{Q},|\cdot|_{\mathrm{triv}})}) \cong \left(\varprojlim_{X\in\mathcal{M}_{K/\mathbb{Q}}} X(\mathbb{C})/\sim\right)\times]0,1].$$

(2.c) is proved similarly.

(4) follows from Theorem 10.2.3 combined with the fact that for any $\mathcal{X}\in\mathcal{M}_{K/A}$, the map $\mathcal{X}^{\mathrm{an}}\to\mathcal{M}(A,\|\cdot\|)$ is open ([LP24], Propositions 6.4.1 and 6.6.10).

The proof of (5) goes along the same lines as the one of Corollary 8.2.6. Note that the positive characteristic case is Corollary 8.2.6. In the characteristic zero case, since K is countable and M_K is a Baire space, it suffices to prove that, for all $a\in K$, $U_a:=\{|a|<+\infty\}$ is dense in M_K. By Theorem 10.2.3, it suffices to prove that, for any projective sub-model $\mathcal{X}$ in $\mathcal{M}_{K/\mathbb{Z}}$ such that $a\in\kappa(\mathcal{X})$ and for any open subset $U\subset\mathcal{X}^{\mathrm{an}}$, we have $U_a\cap\mathrm{red}_{\mathcal{X}}^{-1}(U)\neq\varnothing$. This is a consequence of the following claim.

Claim 10.2.5. *Let $\mathcal{X}$ be a projective sub-model in $\mathcal{M}_{K/\mathbb{Z}}$. Then for any Zariski-open subset $V\subset\mathcal{X}$, V^{an} is dense in $\mathcal{X}^{\mathrm{an}}$.*

Proof. Let $U\subset\mathcal{X}^{\mathrm{an}}$ be an open subset. Since $\mathcal{X}^{\mathrm{an}}\to\mathcal{M}(\mathbb{Z},|\cdot|_\infty)$ is open, there exists $x=(p,|\cdot|_x)\in U$ such that $|\cdot|_{x|\mathbb{Q}}$ is a non-trivial absolute value on $\mathbb{Q}$. Denote by $\mathcal{X}^{\mathrm{an}}_{\mathbb{Q},x}$ the Berkovich analytification of $\mathcal{X}\otimes_{\mathbb{Z}}\mathbb{Q}_x$. Then, as mentioned in the proof of Corollary 8.2.6, the preimage of $V\otimes_{\mathbb{Z}}\mathbb{Q}$ in $\mathcal{X}^{\mathrm{an}}_{\mathbb{Q},x}$ is dense in $\mathcal{X}^{\mathrm{an}}_{\mathbb{Q},x}$. Thus $V^{\mathrm{an}}\cap U\neq\varnothing$. □

□

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