# Butterflies, CATs, and billiards.

Anton Zorich
Mathematical Institute of Jussieu
Institut Universitaire de France

Singapore

11 May 2017

# Linear dynamics

## Fibonacci sequence

The *Fibonacci sequence* is the sequence of integer numbers $\{f_0, f_1, f_2, \ldots\}$ having $f_0 = 0$ and $f_1 = 1$ as the first two entries, and having any other entry equal to the sum of the two preceding ones:

$$f_{n-1} + f_n =: f_{n+1} \tag{Fi}$$

## Fibonacci sequence

The *Fibonacci sequence* is the sequence of integer numbers $\{f_0, f_1, f_2, \ldots\}$ having $f_0 = 0$ and $f_1 = 1$ as the first two entries, and having any other entry equal to the sum of the two preceding ones:

$$f_{n-1} + f_n =: f_{n+1} \tag{Fi}$$

Making a simple computation

$$\begin{array}{ccccccccccc}
0 & + & 1 & = & 1 & & & & & & \\
 & & 1 & + & 1 & = & 2 & & & & \\
 & & & & 1 & + & 2 & = & 3 & & \\
 & & & & & & 2 & + & 3 & = & 5 \\
 & & & & & & & & \ldots & \ldots & \ldots
\end{array}$$

we get the first several terms of the sequence $\{f_0, f_1, f_2, f_3, f_4, \ldots\} =$

$$= \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 \ldots, \}$$

## Brothers and sisters of Fibonacci sequence

Let us study ALL sequences satisfying recurrence relation (Fi) and not only the Fibonacci sequence. Are there any geometric progressions among such sequences? For a geometric progression, $a_k = \lambda^k \cdot a_0$ equation (Fi) becomes

$$\lambda^{n-1} \cdot a_0 + \lambda^n \cdot a_0 = \lambda^{n+1} \cdot a_0$$

Simplifying, we see that a geometric progression is a solution of (Fi) if and only if $\lambda$ is a root of the quadratic equation

$$1 + \lambda = \lambda^2 \,.$$

There are two distinct roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \qquad \lambda_2 = -\frac{1}{\lambda_1} = \frac{1-\sqrt{5}}{2} \approx -0.618$$

and, thus, there are two basic geometric sequences which we will denote by $\{u_n\} = \{1, \lambda_1, \lambda_1^2, \lambda_1^3, \dots\}$ and by $\{w_n\} = \{1, \lambda_2, \lambda_2^2, \lambda_2^3, \dots\}$.

## Back to Fibonacci

1) The sum of any two solutions of (Fi) is also a solution of (Fi).
2) Multiplying any solution of (Fi) by a constant factor we still get a solution of (Fi).
3) The first two terms of any solution of (Fi) uniquely determine all subsequent ones.

Let us represent the vector $\vec{a}_{init} = (a_0, a_1) = (0, 1)$ of the initial data defining the Fibonacci sequence as a linear combination of the vectors $\vec{u}_{init} = (1, \lambda_1)$ and $\vec{w}_{init} = (1, \lambda_2)$ defining solutions of (Fi) represented by the geometric progressions $\{u_n\} = \{\lambda_1^n\}$ and $\{w_n\} = \{\lambda_2^n\}$. Here $c_1$ and $c_2$ are unknown.

$$\vec{a}_{init} = (0, 1) = c_1 \cdot (1, \lambda_1) + c_2 \cdot (1, \lambda_2) = c_1 \cdot \vec{u}_{init} + c_2 \cdot \vec{w}_{init} \,.$$

Solving the system of linear equations

$$\begin{cases} \frac{1+\sqrt{5}}{2}\, c_1 + \frac{1-\sqrt{5}}{2}\, c_2 & = 1 \\ c_1 + c_2 & = 0 \end{cases}$$

we find the solutions $c_1 = \sqrt{5}/5, \quad c_2 = -\sqrt{5}/5$.

## Fibonacci sequence: the answer

According to observations $1$ and $2$, the sequence $a_n$ defined as

$$a_n := c_1 u_n + c_2 w_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

satisfies the equation (Fi).

By construction the initial vector $\vec{a}_{init} = (a_0, a_1) = (0, 1)$ of the latter sequence coincides with the initial vector of the Fibonacci sequence. Hence, by observation 3, the sequence $\{a_n\}$ coincides with the Fibonacci one. We have found a formula for the term number $n$ of the Fibonacci sequence:

$$f_n := \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since $|\lambda_2| = \left|(1-\sqrt{5})/2\right| \approx 0.618 < 1$, we conclude that for $n \gg 1$ we get $\lambda_2^n \to 0$ and hence

$$f_n \approx \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n .$$

## Fibonacci sequence: the answer

According to observations $1$ and $2$, the sequence $a_n$ defined as

$$a_n := c_1 u_n + c_2 w_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

satisfies the equation (Fi).

By construction the initial vector $\vec{a}_{init} = (a_0, a_1) = (0, 1)$ of the latter sequence coincides with the initial vector of the Fibonacci sequence. Hence, by observation 3, the sequence $\{a_n\}$ coincides with the Fibonacci one. We have found a formula for the term number $n$ of the Fibonacci sequence:

$$f_n := \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since $|\lambda_2| = \left|(1-\sqrt{5})/2\right| \approx 0.618 < 1$, we conclude that for $n \gg 1$ we get $\lambda_2^n \to 0$ and hence

$$f_n \approx \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n .$$

## How to impress your friends

Computing,

$$\log_{10} a_n \approx \log_{10}(\sqrt{5}/5) + n \log_{10}((1+\sqrt{5})/2) \approx 0.209n - 0.349$$

we observe that $a_n$ contains approximatively $[0.209n + 0.65]$ decimal digits, where $[x]$ denotes the integer part. For example, for $n = 1000$ this predicts $[209 + 0.65] = 209$ digits.

## How to impress your friends

Computing,

$$\log_{10} a_n \approx \log_{10}(\sqrt{5}/5) + n \log_{10}((1+\sqrt{5})/2) \approx 0.209n - 0.349$$

we observe that $a_n$ contains approximatively $[0.209n + 0.65]$ decimal digits, where $[x]$ denotes the integer part. For example, for $n = 1000$ this predicts $[209 + 0.65] = 209$ digits.

The exact number
$a_{1000} = 434665576869374564356885276750406258025646605173717804024817290895365554179490518904038798400792551692959$
$225930803226347752096896232398733224711616429964409065331879382989696499285160037044761377951668492288 75$ indeed contains $209$ digits as estimated!

# Butterfly effect

## The little sister of the Fibonacci sequence

Writing the characteristic equation

$$1 + \lambda = \lambda^2$$

associated to the recurrence relation

$$a_{n-1} + a_n = a_{n+1} \tag{Fi}$$

we have found that the geometric progression $\{w_n\} = \{1, \lambda_2, \lambda_2^2, \ldots\}$ with $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$ satisfies the recurrence relation (Fi).

Since $|\lambda_2| \approx 0.618 < 1$ this sequence tends rapidly to $0$.

Let us try to compute $w_n$ using a computer. Here is a program for “Maple” to compute $w_{100}$.

## The little sister of the Fibonacci sequence

Writing the characteristic equation

$$1 + \lambda = \lambda^2$$

associated to the recurrence relation

$$a_{n-1} + a_n = a_{n+1} \qquad \text{(Fi)}$$

we have found that the geometric progression $\{w_n\} = \{1, \lambda_2, \lambda_2^2, \ldots\}$ with $\lambda_2 = \dfrac{1 - \sqrt{5}}{2}$ satisfies the recurrence relation (Fi).

Since $|\lambda_2| \approx 0.618 < 1$ this sequence tends rapidly to $0$.

Let us try to compute $w_n$ using a computer. Here is a program for “Maple” to compute $w_{100}$.

## Program for computing $w_{100}$ in "Maple"

```
Fibonacci := proc(u0,u1,n::nonnegint)
local old, new, buffer, i;
    if n = 0 then return u0
    elif n = 1 then return u1
    else
        old := u0;
        new := u1;
        for i from 2 to n do
            buffer := new;
            new := new + old;
            old := buffer
        end do
    end if;
    return new
end proc;
```

## Wrong computation

The program gives the value $w_{100} \approx 10^{11}$ while in reality $w_{100} \approx 10^{-21}$.

You can use your preferred computer and your preferred software to compute $w_{100}$ or $w_{1000}$ by recurrence to realize that the results are disastrously wrong (provided you make floating point calculations and NOT the algebraic ones).

## Wrong computation

The program gives the value $w_{100} \approx 10^{11}$ while in reality $w_{100} \approx 10^{-21}$.

You can use your preferred computer and your preferred software to compute $w_{100}$ or $w_{1000}$ by recurrence to realize that the results are disastrously wrong (provided you make floating point calculations and NOT the algebraic ones).

## Wrong computation

The program gives the value $w_{100} \approx 10^{11}$ while in reality $w_{100} \approx 10^{-21}$.

You can use your preferred computer and your preferred software to compute $w_{100}$ or $w_{1000}$ by recurrence to realize that the results are disastrously wrong (provided you make floating point calculations and NOT the algebraic ones).

To understand what is going on, let us study the recurrence relation (Fi) geometrically. Consider the following transformation $F$ of the plane $\mathbb{R}^2$:

$$F : (x, y) \mapsto (y, x + y) \, .$$

If we take two consecutive elements $(a_{n-1}, a_n)$ of any sequence satisfying (Fi) and apply $F$ to the planar vector $(x, y) = (a_{n-1}, a_n)$ we get

$$F(a_{n-1}, a_n) = (a_n, a_{n-1} + a_n) = (a_n, a_{n+1}) \, .$$

Thus, to produce any sequence satisfying (Fi) we can apply $F$ many times iteratively starting from the initial vector $\vec{a}_{init} = (a_0, a_1)$:

$$(a_n, a_{n+1}) = \underbrace{F(F(\ldots F}_{n}(a_0, a_1))\ldots) \, .$$

## Very special vectors for the map $F$

The geometric progressions $\{u_n\}, \{w_n\} = \{1, \lambda, \lambda^2, \dots\}$ with $\lambda = \lambda_1, \lambda_2$ are special for the map $F$. Every next vector is collinear to the previous one:

$$(u_n, u_{n+1}) = (\lambda^n, \lambda^{n+1}) = F(\lambda^{n-1}, \lambda^n) = \lambda\cdot(\lambda^{n-1}, \lambda^n) = \lambda\cdot(u_{n-1}, u_n)\,.$$

For the geometric progression with $\lambda_1 \approx 1.6$ every next vector is obtained by the dilatation of the previous one by the factor $\lambda_1$. For the geometric progression with $\lambda_2 \approx -0.6$ every next vector is obtained by the contraction of the previous one by the factor $\lambda_1 = -1/\lambda_2$ followed by flipping it to the symmetric one.

Consider now two basic vectors $\vec{u} = (1, \lambda_1)$ and $\vec{w} = (1, \lambda_2)$. Any other vector $\vec{v} = c_1\vec{u} + c_2\vec{w}$ of the plane $\mathbb{R}^2$ is mapped to the vector

$$\vec{v} \mapsto F(\vec{v}) = c_1F(\vec{u}) + c_2F(\vec{w}) = c_1\lambda_1\vec{u} + c_2\lambda_2\vec{w}$$

It is the same story as:

1) The sum of any two solutions of (Fi) is also a solution of (Fi).

2) Multiplying any solution of (Fi) by a constant factor we still get a solution of (Fi).

## Very special vectors for the map $F$

The geometric progressions $\{u_n\}, \{w_n\} = \{1, \lambda, \lambda^2, \dots\}$ with $\lambda = \lambda_1, \lambda_2$ are special for the map $F$. Every next vector is collinear to the previous one:

$$(u_n, u_{n+1}) = (\lambda^n, \lambda^{n+1}) = F(\lambda^{n-1}, \lambda^n) = \lambda\cdot(\lambda^{n-1}, \lambda^n) = \lambda\cdot(u_{n-1}, u_n)\,.$$

For the geometric progression with $\lambda_1 \approx 1.6$ every next vector is obtained by the dilatation of the previous one by the factor $\lambda_1$. For the geometric progression with $\lambda_2 \approx -0.6$ every next vector is obtained by the contraction of the previous one by the factor $\lambda_1 = -1/\lambda_2$ followed by flipping it to the symmetric one.

Consider now two basic vectors $\vec{u} = (1, \lambda_1)$ and $\vec{w} = (1, \lambda_2)$. Any other vector $\vec{v} = c_1\vec{u} + c_2\vec{w}$ of the plane $\mathbb{R}^2$ is mapped to the vector

$$\vec{v} \mapsto F(\vec{v}) = c_1 F(\vec{u}) + c_2 F(\vec{w}) = c_1\lambda_1\vec{u} + c_2\lambda_2\vec{w}$$

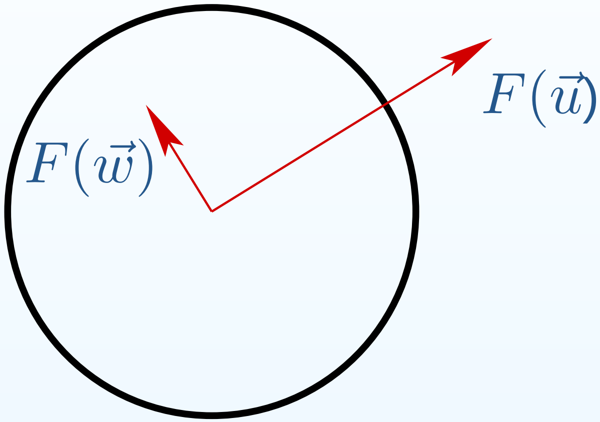
It is the same story as:

1) The sum of any two solutions of (Fi) is also a solution of (Fi).

2) Multiplying any solution of (Fi) by a constant factor we still get a solution of (Fi).

# Iterations of a linear transformation

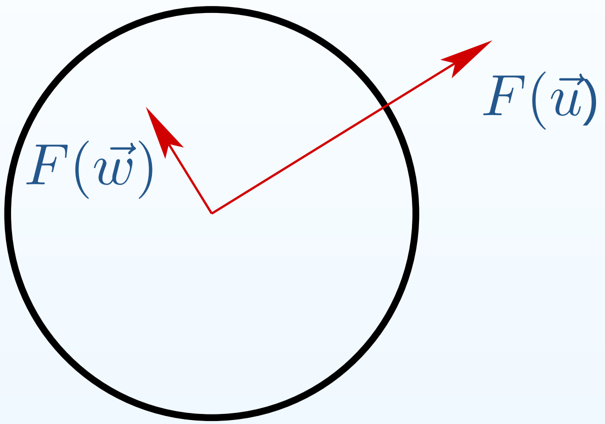
Let us follow several consecutive images of a unit circle under iterations of the map $F$.

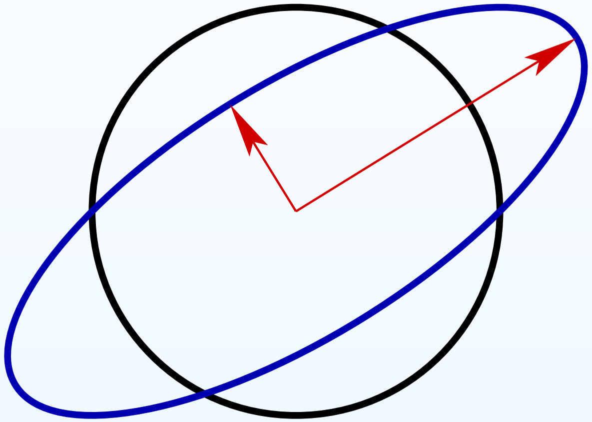



The circle is transformed to an ellipse where the directions of the axes are exactly the directions of the special vectors $\vec{u}$ and $\vec{w}$, and the lengths of the demi-axes are exactly $\lambda_1$ and $|\lambda_2| = 1/\lambda_1$. When we pass to iterates $F \circ F \circ \cdots \circ F$ of the linear transformation $F$, the directions of the demi-axes stay unchanged while their lengths change drastically. For a sufficiently long $n$ an ellipse is basically smashed to an interval aligned along $\vec{u}$. Morally, a large iterate of the transformation $F$ projects the entire plane to the line spanned by $\vec{u}$ and then expands it by an enormous dilatation.

## Iterations of a linear transformation

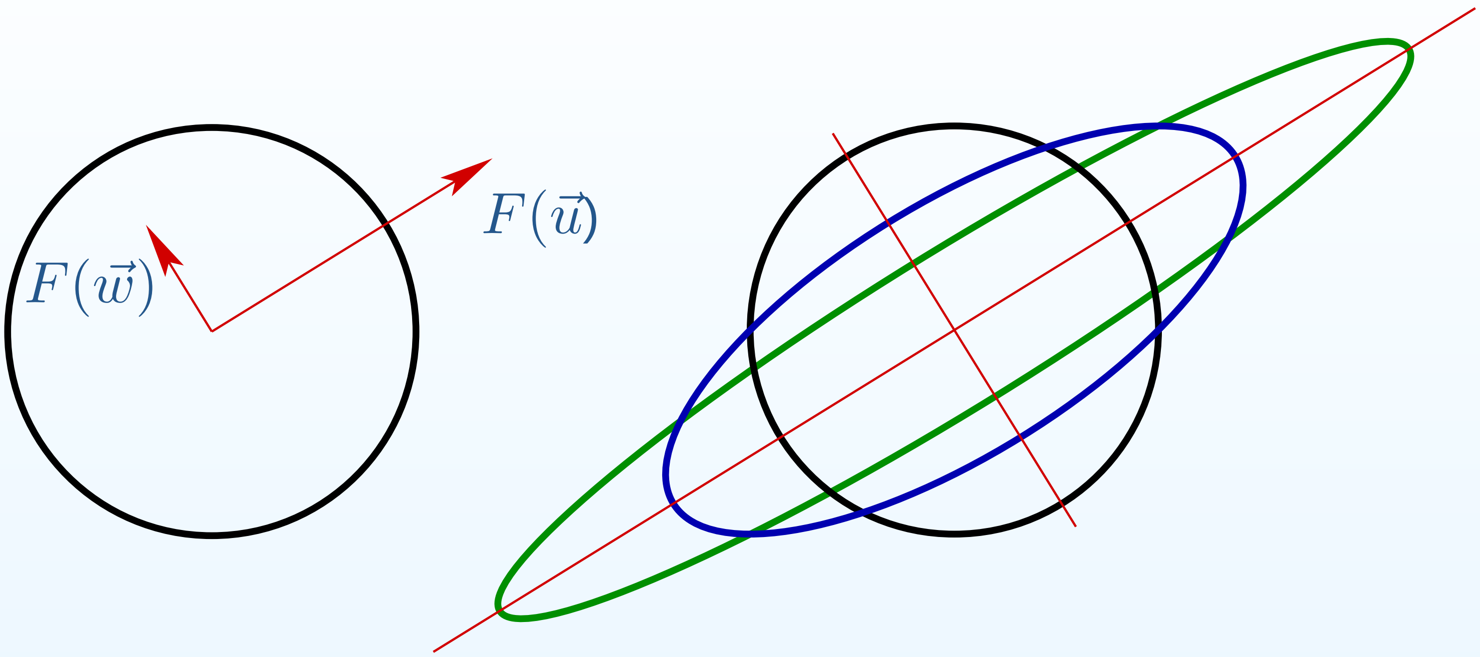
Let us follow several consecutive images of a unit circle under iterations of the map $F$.



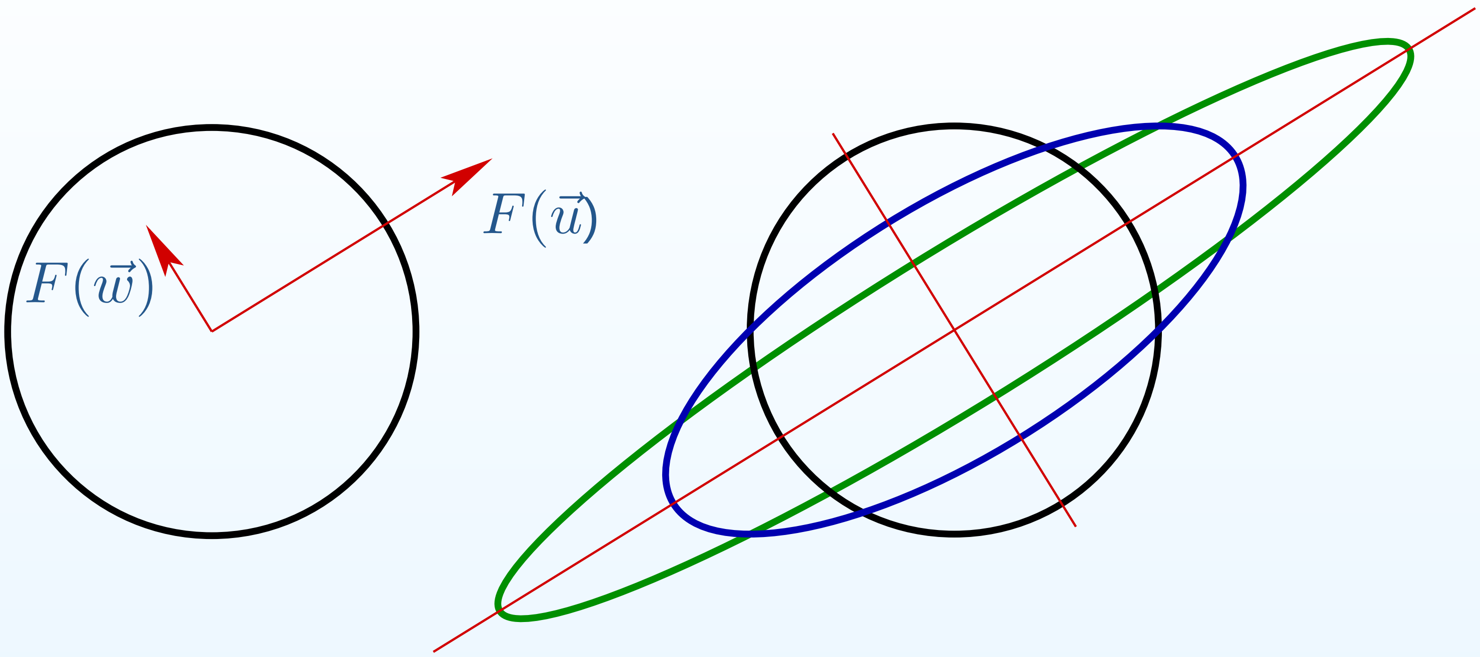
The circle is transformed to an ellipse where the directions of the axes are exactly the directions of the special vectors $\vec{u}$ and $\vec{w}$, and the lengths of the demi-axes are exactly $\lambda_1$ and $|\lambda_2| = 1/\lambda_1$. When we pass to iterates $F \circ F \circ \cdots \circ F$ of the linear transformation $F$, the directions of the demi-axes stay unchanged while their lengths change drastically. For a sufficiently long $n$ an ellipse is basically smashed to an interval aligned along $\vec{u}$. Morally, a large iterate of the transformation $F$ projects the entire plane to the line spanned by $\vec{u}$ and then expands it by an enormous dilatation.

## Iterations of a linear transformation




The circle is transformed to an ellipse where the directions of the axes are exactly the directions of the special vectors $\vec{u}$ and $\vec{w}$, and the lengths of the demi-axes are exactly $\lambda_1$ and $|\lambda_2| = 1/\lambda_1$. When we pass to iterates $F \circ F \circ \cdots \circ F$ of the linear transformation $F$, the directions of the demi-axes stay unchanged while their lengths change drastically. For a sufficiently long $n$ an ellipse is basically smashed to an interval aligned along $\vec{u}$. Morally, a large iterate of the transformation $F$ projects the entire plane to the line spanned by $\vec{u}$ and then expands it by an enormous dilatation.

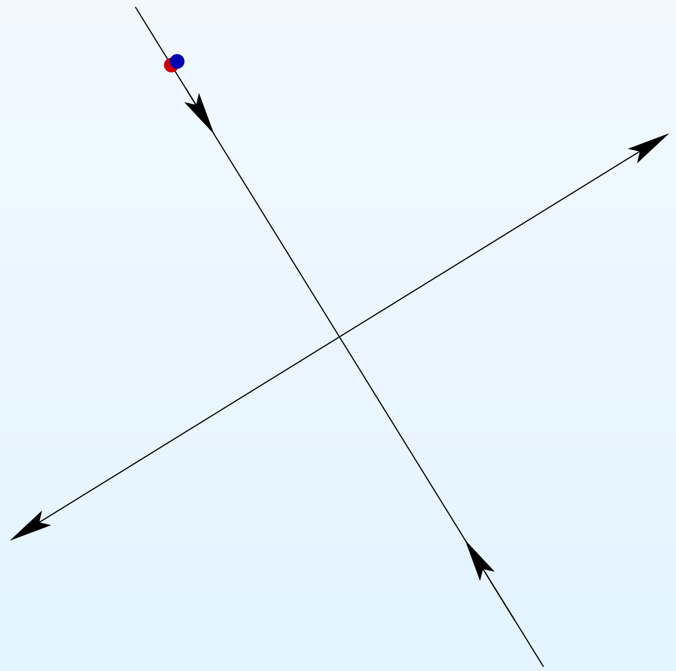
## Iterations of a linear transformation




The circle is transformed to an ellipse where the directions of the axes are exactly the directions of the special vectors $\vec{u}$ and $\vec{w}$, and the lengths of the demi-axes are exactly $\lambda_1$ and $|\lambda_2| = 1/\lambda_1$. When we pass to iterates $F \circ F \circ \cdots \circ F$ of the linear transformation $F$, the directions of the demi-axes stay unchanged while their lengths change drastically. For a sufficiently long $n$ an ellipse is basically smashed to an interval aligned along $\vec{u}$. Morally, a large iterate of the transformation $F$ projects the entire plane to the line spanned by $\vec{u}$ and then expands it by an enormous dilatation.

## Divergence of trajectories

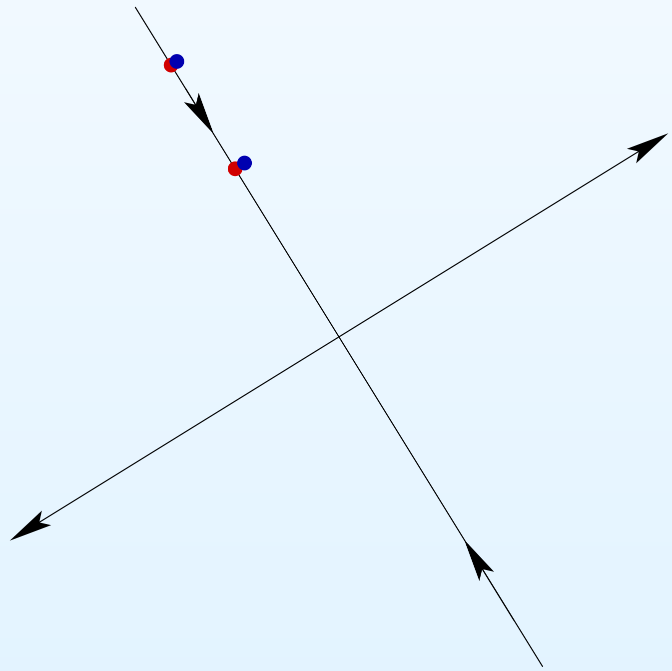
Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$.



## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

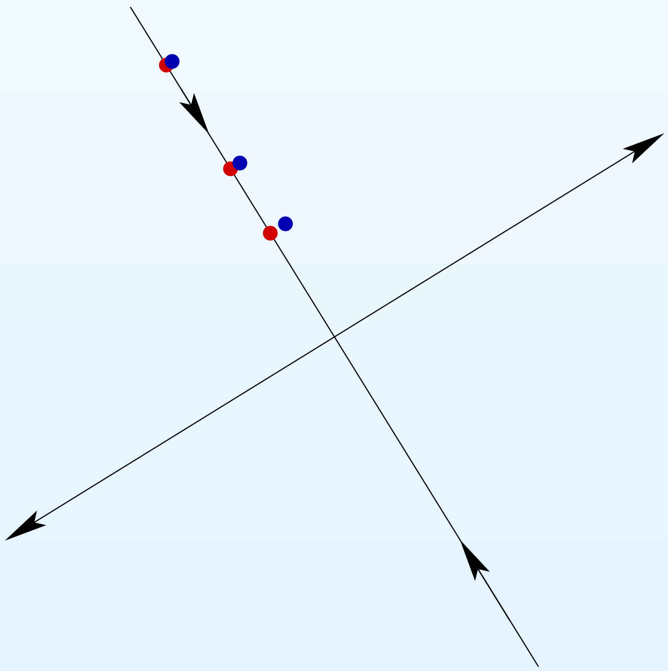
$$\underbrace{F(F(\dots F}_{2n}(\vec{v})\dots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$



## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

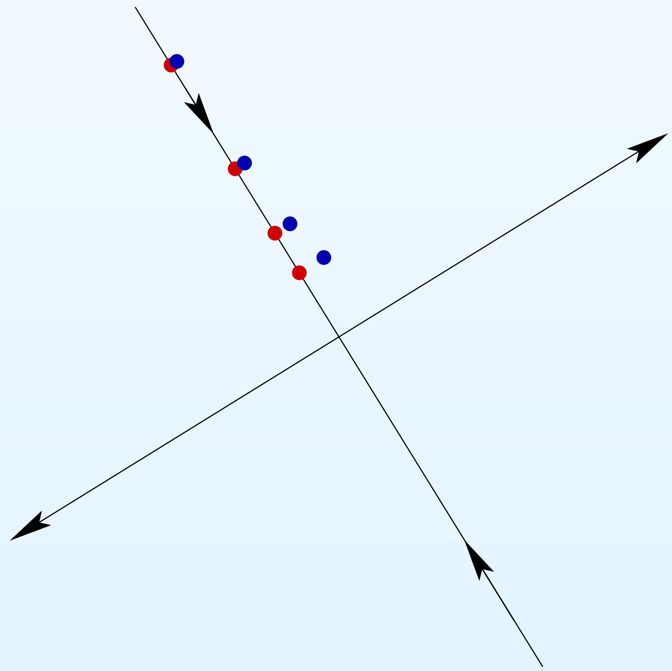
$$\underbrace{F(F(\dots F}_{2n}(\vec{v})\dots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$



## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

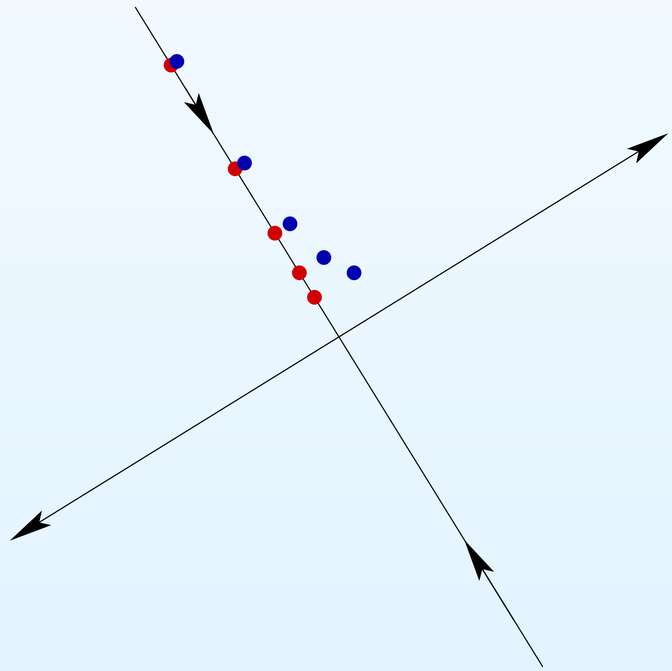
$$\underbrace{F(F(\ldots F}_{2n}(\vec{v})\ldots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$



## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

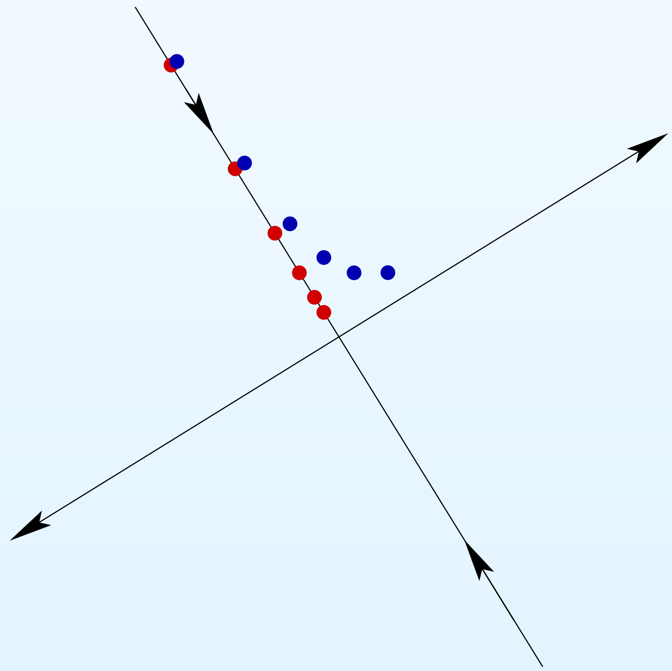
$$\underbrace{F(F(\dots F}_{2n}(\vec{v})\dots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$



## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

$$\underbrace{F(F(\ldots F}_{2n}(\vec{v})\ldots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$

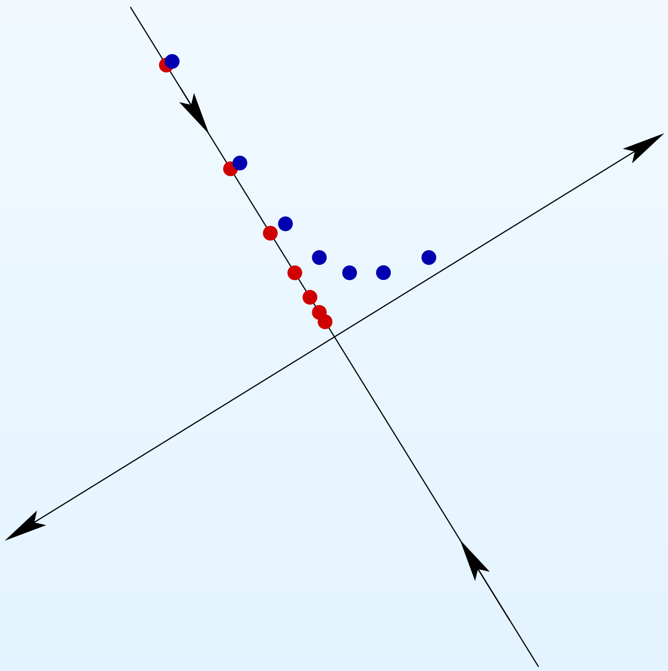


For large $n$ the extra term caused by a little error becomes dominating!

## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

$$\underbrace{F(F(\dots F}_{2n}(\vec{v})\dots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$

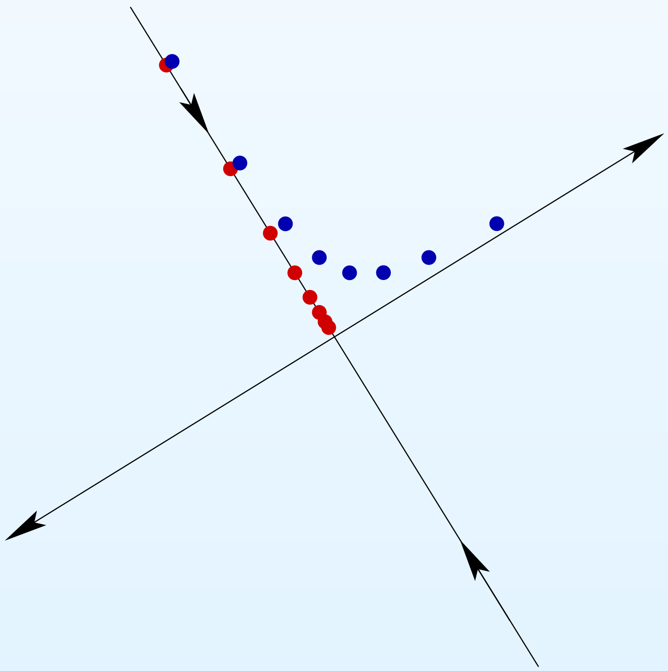


For large $n$ the extra term caused by a little error becomes dominating!

## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

$$\underbrace{F(F(\ldots F}_{2n}(\vec{v})\ldots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$

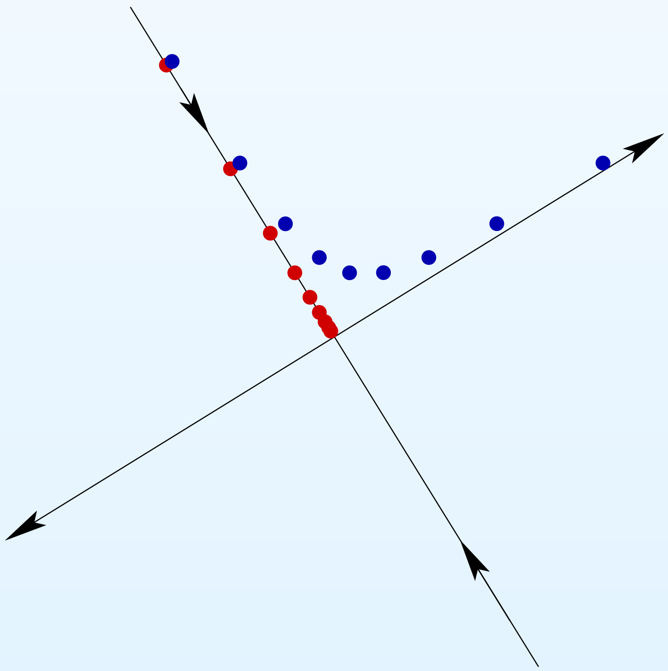


For large $n$ the extra term caused by a little error becomes dominating!

## Divergence of trajectories

Our software memorizes the initial data with a tiny error $\varepsilon$ : instead of $\vec{w}$ we, actually, have $\vec{w} + \varepsilon\vec{u}$. After $2n$ iterations, the vector $\vec{v} = \varepsilon\vec{u} + \vec{w}$ with $0 < |\varepsilon| \ll 1$ is mapped to the vector

$$\underbrace{F(F(\ldots F}_{2n}(\vec{v})\ldots)) = \varepsilon\lambda_1^{2n}\vec{u} + \lambda_2^{2n}\vec{w}$$



For large $n$ the extra term caused by a little error becomes dominating!

## Butterfly effect

We have observed a phenomenon of *instability* of a trajectory with respect to the stating data. Arbitrary tiny error makes the trajectory escape to infinity instead of landing to the origin.

This effect is known as a “butterfly effect” : “Does the Flap of a Butterfly’s Wings in Brazil Set a Tornado in Texas?”

## Butterfly effect

We have observed a phenomenon of *instability* of a trajectory with respect to the stating data. Arbitrary tiny error makes the trajectory escape to infinity instead of landing to the origin.

This effect is known as a “butterfly effect” : “Does the Flap of a Butterfly’s Wings in Brazil Set a Tornado in Texas?”

## Butterfly effect

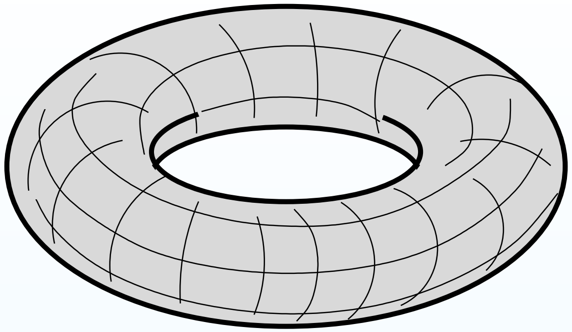
We have observed a phenomenon of *instability* of a trajectory with respect to the stating data. Arbitrary tiny error makes the trajectory escape to infinity instead of landing to the origin.

This effect is known as a “butterfly effect” : “Does the Flap of a Butterfly’s Wings in Brazil Set a Tornado in Texas?”

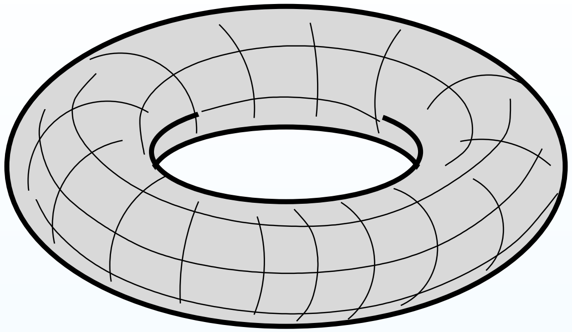
In dynamical systems one studies the behavior of transformations after many iterations (or flows after a long time, etc.)

# Chaotic systems

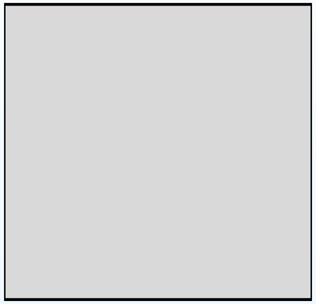
# Map from a torus to itself

The surface $\mathbb{T}^2$ at the picture is called a *torus*: 

## Map from a torus to itself

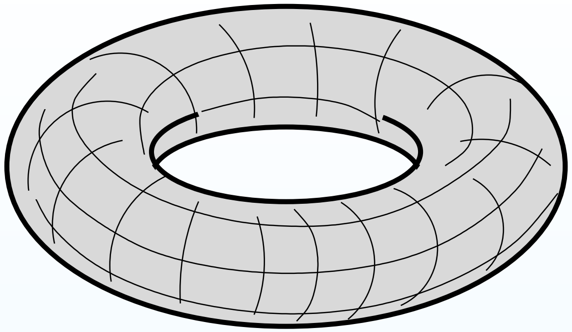


The surface $\mathbb{T}^2$ at the picture is called a *torus*:
It can be glued from a unit square:

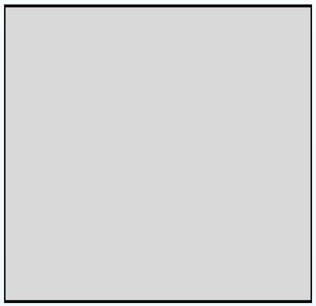


# Map from a torus to itself

The surface $\mathbb{T}^2$ at the picture is called a *torus*:



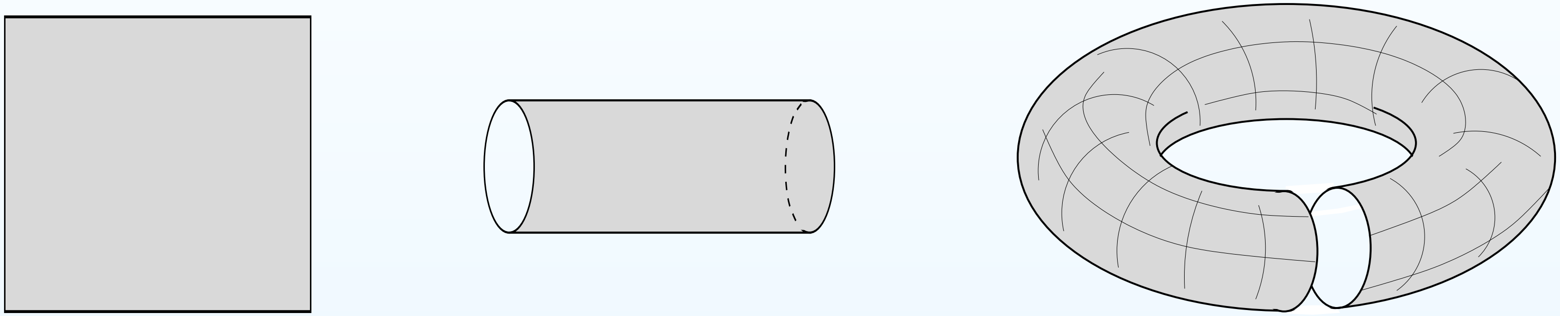
It can be glued from a unit square:



Identifying the horizontal sides we first get a cylinder.

## Map from a torus to itself

The surface $\mathbb{T}^2$ at the picture is called a *torus*:
It can be glued from a unit square:

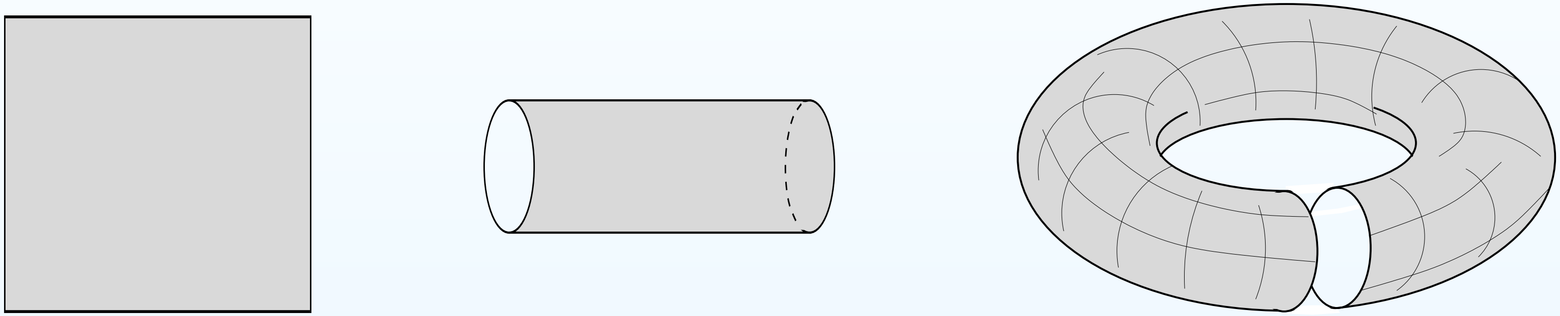


Identifying the horizontal sides we first get a cylinder. Identifying next the vertical sides (they became meanwhile the boundary components of the cylinder) we get a torus.

## Map from a torus to itself

The surface $\mathbb{T}^2$ at the picture is called a *torus*:

It can be glued from a unit square:

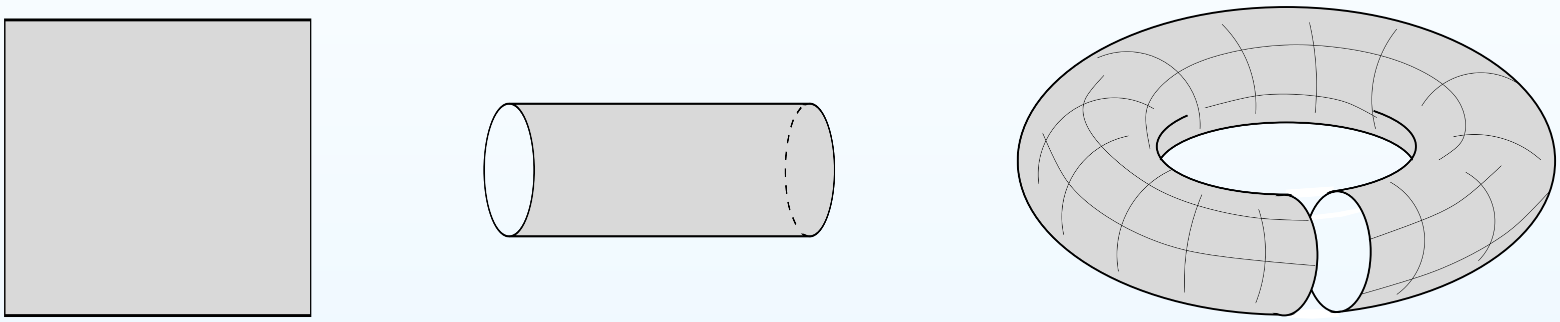


Identifying the horizontal sides we first get a cylinder. Identifying next the vertical sides (they became meanwhile the boundary components of the cylinder) we get a torus.

More algebraically, we can consider the torus $\mathbb{T}^2$ as a quotient of the plane $\mathbb{R}^2$ over the action of the group $\mathbb{Z} \oplus \mathbb{Z}$ of integer translations.

# Map from a torus to itself

The surface $\mathbb{T}^2$ at the picture is called a *torus*:
It can be glued from a unit square:



Identifying the horizontal sides we first get a cylinder. Identifying next the vertical sides (they became meanwhile the boundary components of the cylinder) we get a torus.

More algebraically, we can consider the torus $\mathbb{T}^2$ as a quotient of the plane $\mathbb{R}^2$ over the action of the group $\mathbb{Z} \oplus \mathbb{Z}$ of integer translations.
Considering the numbers $x, y, x+y$ modulo integers it is easy to check that the transformation

$$F : (x, y) \mapsto (y, x+y)\,.$$

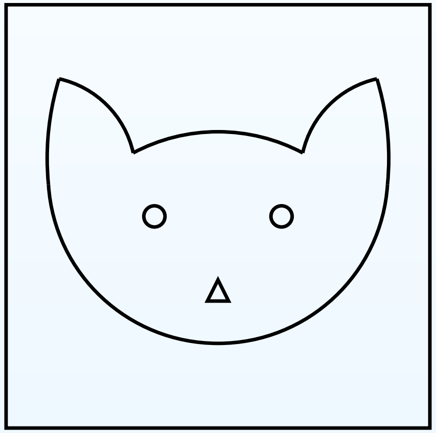
induces a well-defined continuous map of the torus $\mathbb{T}^2$ to itself.

## Arnold's CAT (Conservative Automorphism of a Torus)

To illustrate the action of the map $F$ on the torus we can draw a CAT (following Vladimir Arnold) in the square and observe how it is transformed under $F$.

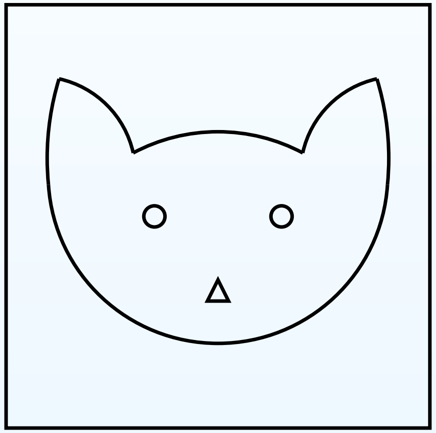
## Arnold's CAT (Conservative Automorphism of a Torus)

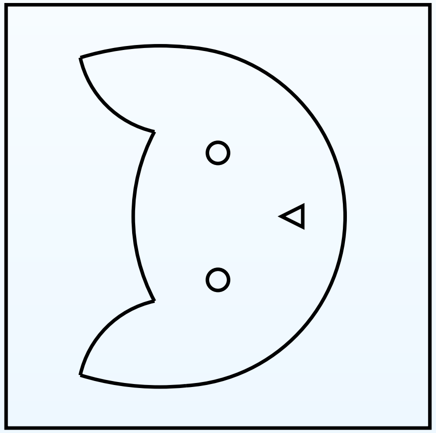
To illustrate the action of the map $F$ on the torus we can draw a CAT (following Vladimir Arnold) in the square and observe how it is transformed under $F$.



## Arnold's CAT (Conservative Automorphism of a Torus)

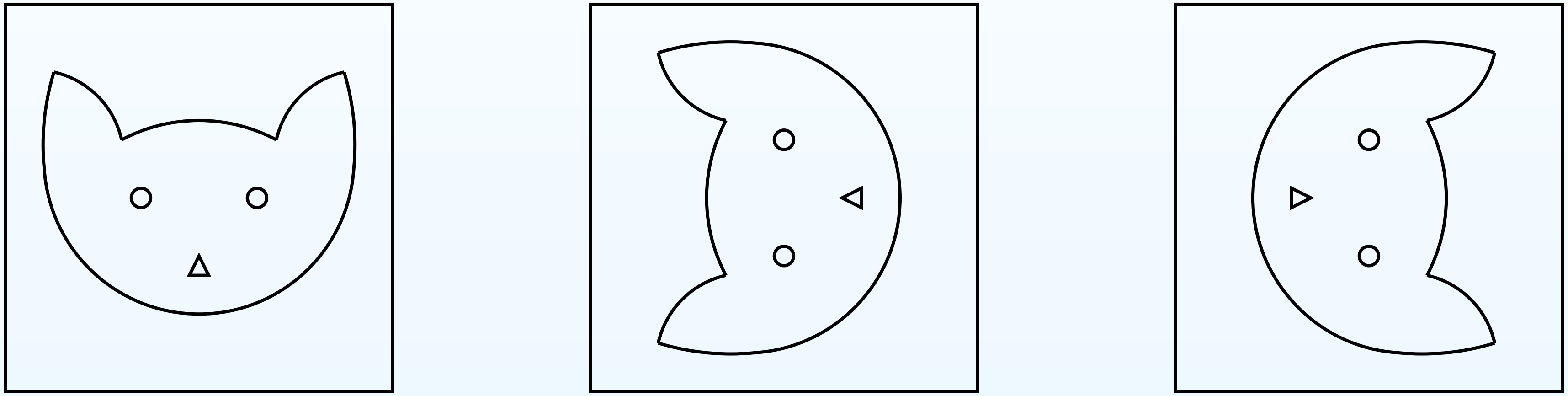
To illustrate the action of the map $F$ on the torus we can draw a CAT (following Vladimir Arnold) in the square and observe how it is transformed under $F$.





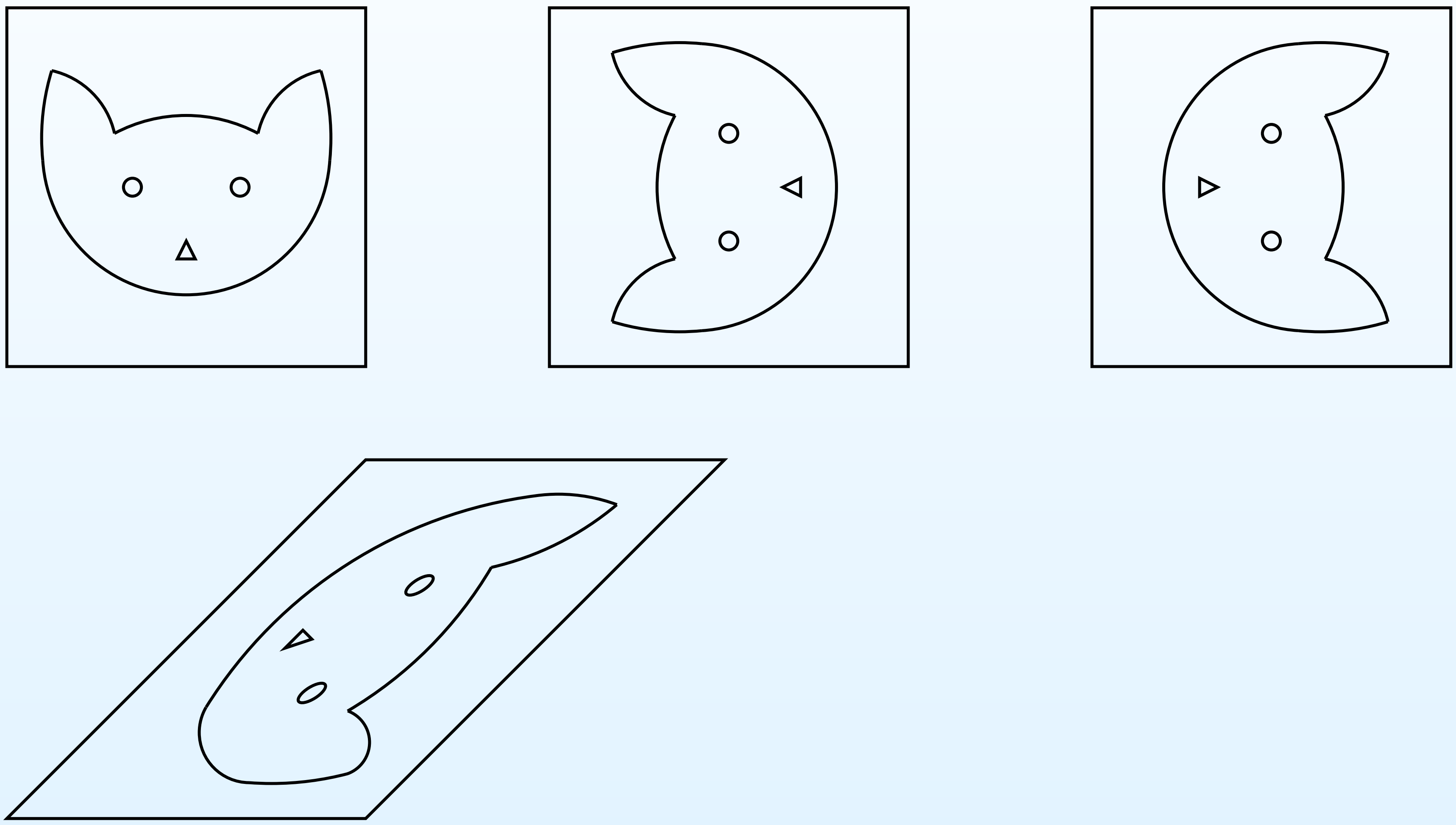
## Arnold’s CAT (Conservative Automorphism of a Torus)

To illustrate the action of the map $F$ on the torus we can draw a CAT (following Vladimir Arnold) in the square and observe how it is transformed under $F$.



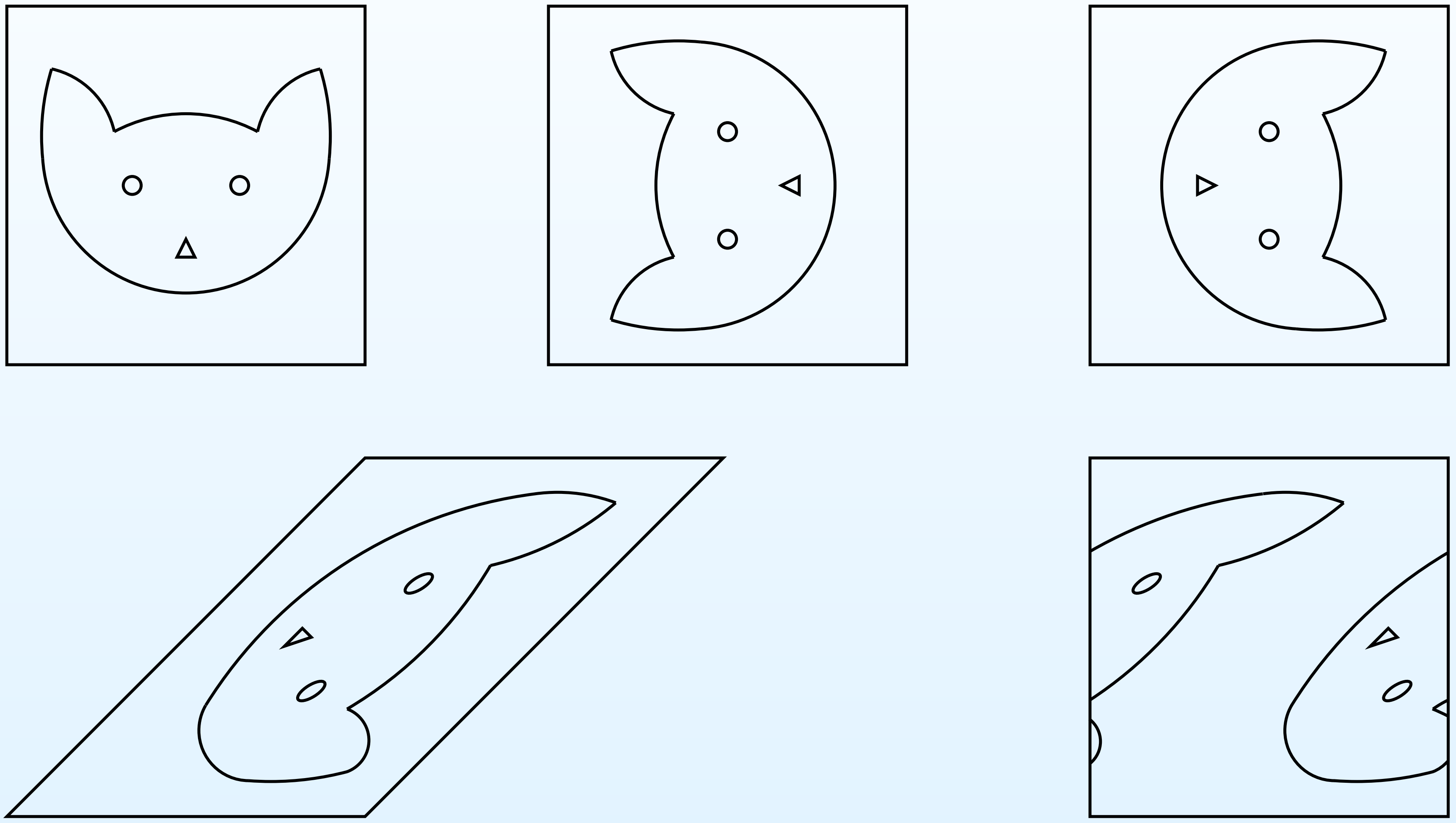
# Arnold's CAT (Conservative Automorphism of a Torus)

To illustrate the action of the map $F$ on the torus we can draw a CAT (following Vladimir Arnold) in the square and observe how it is transformed under $F$.



# Arnold's CAT (Conservative Automorphism of a Torus)

To illustrate the action of the map $F$ on the torus we can draw a CAT (following Vladimir Arnold) in the square and observe how it is transformed under $F$.



## Instability of trajectories

Let us launch an orbit $P_0, P_1 = F(P_0), P_2 = F(F(P_0)), \ldots$ of the map $F$ from some point $P_0 \in \mathbb{T}^2$. How it would be followed by an orbit $Q_0, Q_1, Q_2, \ldots$ launched from a nearby point $Q_0$?

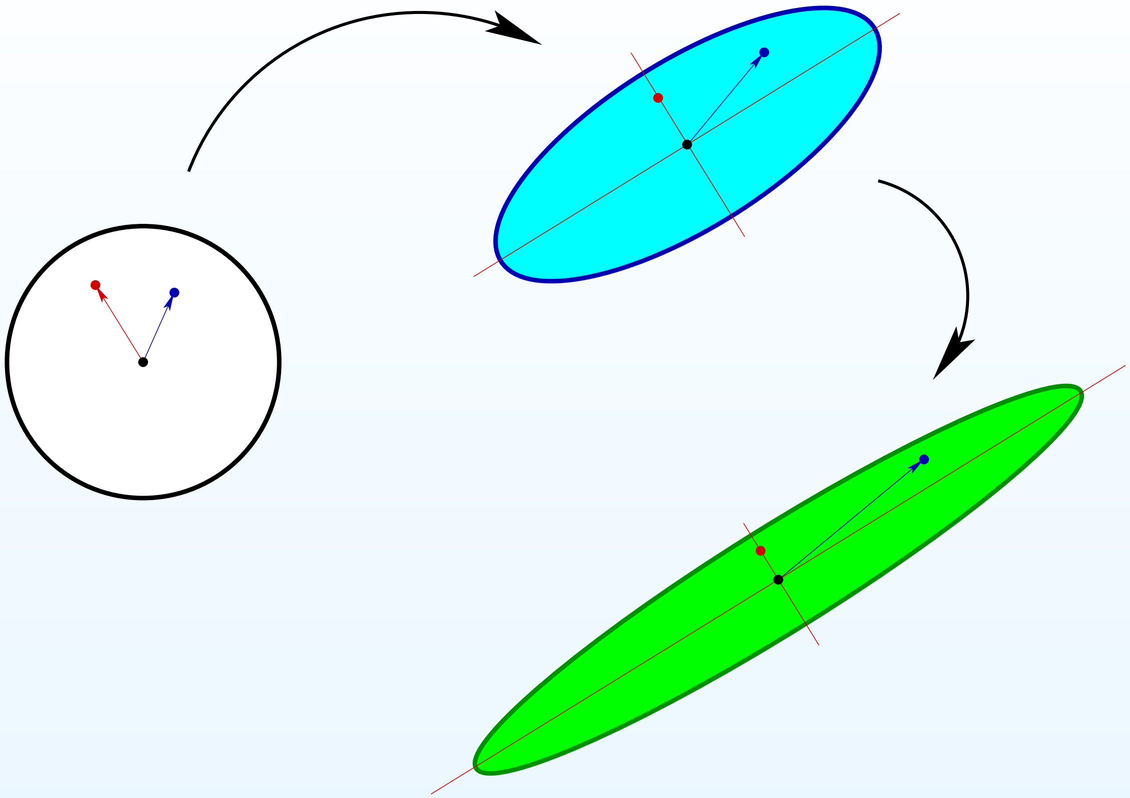
Up to a translation, a small neighborhood of $P_0$ is mapped to its image by the linear transformation $F$. We have already studied the behavior of iterates of $F$.

## Instability of trajectories

Let us launch an orbit $P_0, P_1 = F(P_0), P_2 = F(F(P_0)), \ldots$ of the map $F$ from some point $P_0 \in \mathbb{T}^2$. How it would be followed by an orbit $Q_0, Q_1, Q_2, \ldots$ launched from a nearby point $Q_0$?

Up to a translation, a small neighborhood of $P_0$ is mapped to its image by the linear transformation $F$. We have already studied the behavior of iterates of $F$.

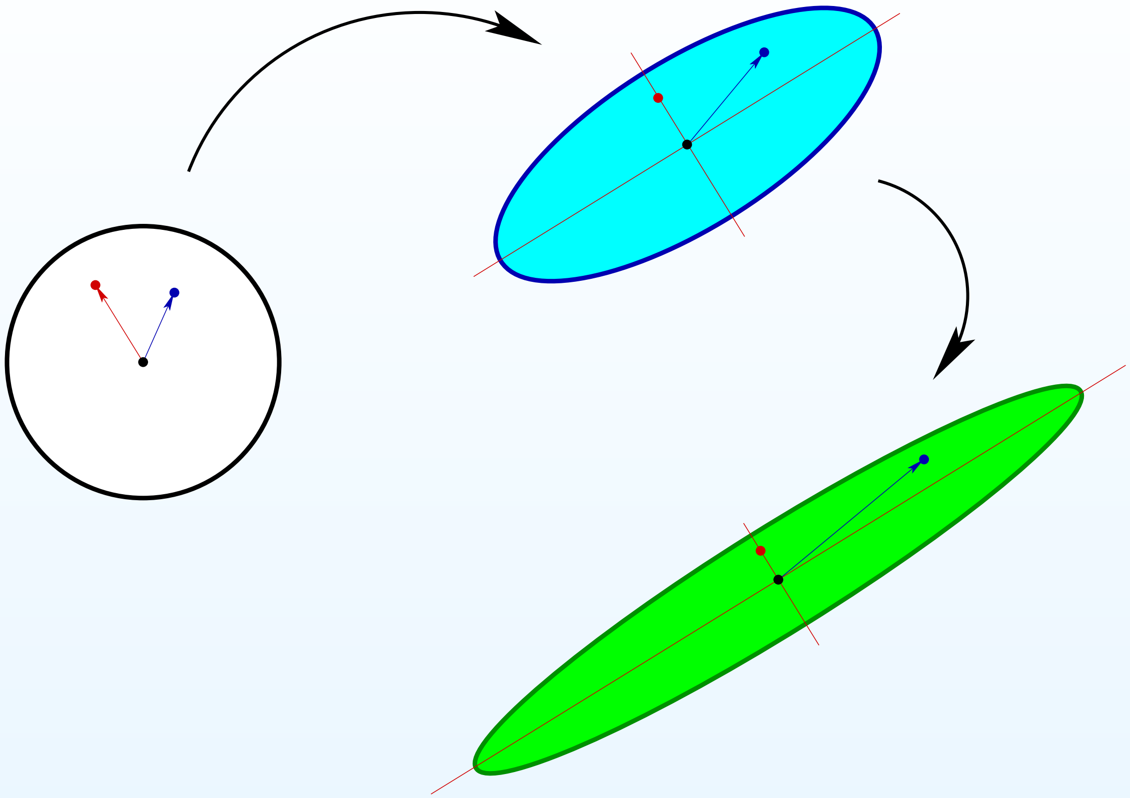
## Instability of trajectories. Chaos



For most of the nearby starting points the orbits would diverge with the exponential rate of divergence like the blue orbit. However, if we move the initial point exactly in the direction of the contracting vector $\vec{w}$ the orbits would converge with the exponential rate of convergence like the red orbit.

Such behavior is called *chaotic*: orbits launched at two very close points diverge rapidly and after a short time we do not see any relation between them.

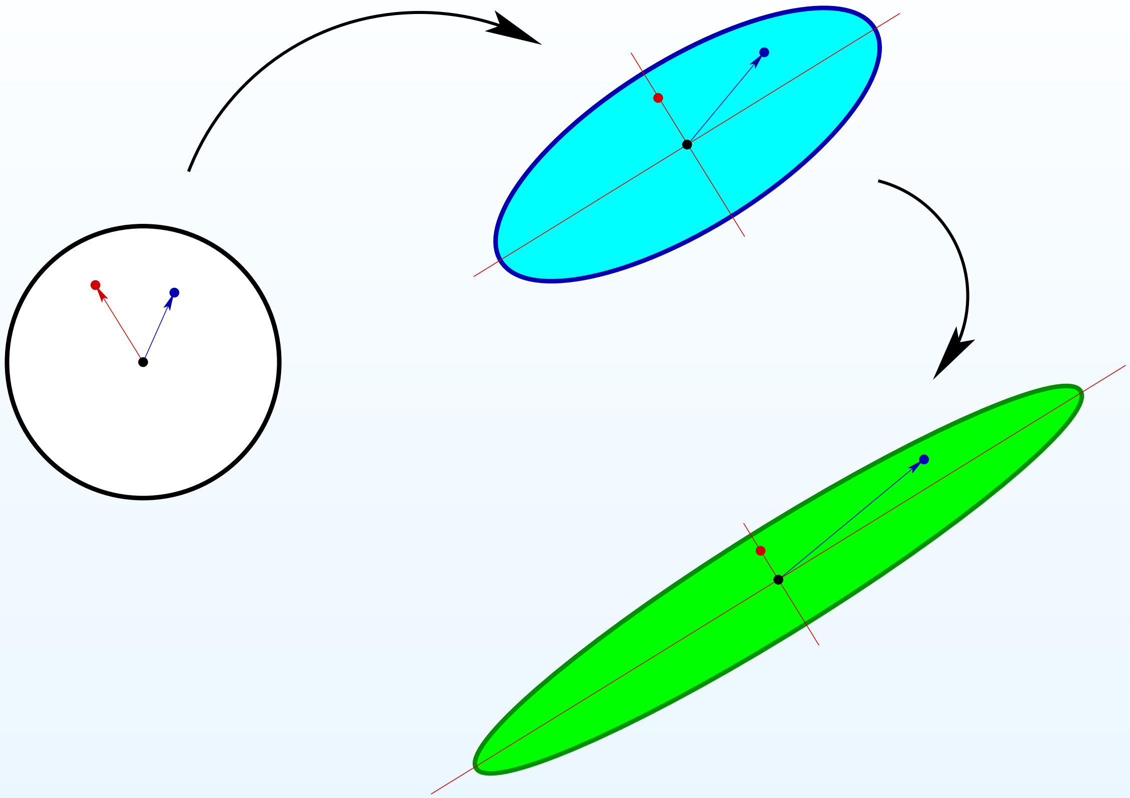
## Instability of trajectories. Chaos



For most of the nearby starting points the orbits would diverge with the exponential rate of divergence like the blue orbit. However, if we move the initial point exactly in the direction of the contracting vector $\vec{w}$ the orbits would converge with the exponential rate of convergence like the red orbit.

Such behavior is called *chaotic*: orbits launched at two very close points diverge rapidly and after a short time we do not see any relation between them.

## Instability of trajectories. Chaos



For most of the nearby starting points the orbits would diverge with the exponential rate of divergence like the blue orbit. However, if we move the initial point exactly in the direction of the contracting vector $\vec{w}$ the orbits would converge with the exponential rate of convergence like the red orbit.

Such behavior is called *chaotic*: orbits launched at two very close points diverge rapidly and after a short time we do not see any relation between them.

## History. Works of Henri Poincaré

One can consider the publication in 1890 of the paper of Henri Poincar “On the three body problem and equations of their dynamics” studying stability of the Solar System as a birthdate of *dynamical systems* as a domain of mathematics.

Despite a dramatic error (found by Poincaré after the paper has already received the prize of the King of Sweden Oscar II) this paper created a base of the theory and predicted numerous phenomena which were justified rigorously much later.

## History. Works of Henri Poincaré

One can consider the publication in 1890 of the paper of Henri Poincar “On the three body problem and equations of their dynamics” studying stability of the Solar System as a birthdate of *dynamical systems* as a domain of mathematics.

Despite a dramatic error (found by Poincaré after the paper has already received the prize of the King of Sweden Oscar II) this paper created a base of the theory and predicted numerous phenomena which were justified rigorously much later.

## Ergodic Theorem: chaos is very regular in average

The map $F : \mathbb{T}^2 \to \mathbb{T}^2$ has several important properties:

• It preserves area: the area of any small disc $U$ is the same as the area of the small ellipse $F(U)$ obtained as the the image of $U$.

• Any subset invariant under $F$ either has zero area or its complement has zero area. In such situation one says that the map is *ergodic*.

**Ergodic Theorem** *For any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ and for almost any starting point $P_0$ in $\mathbb{T}^2$ the mean value of the function $f$ along the orbit $P_0, \dots, \underbrace{F(F(\dots F}_{n-1}(P_0)\dots))$ coincides with the mean value of $f$ over $\mathbb{T}^2$:*

$$\lim_{n\to+\infty} \frac{1}{n}\left( f(P_0) + f(F(P_0)) + \cdots + f(\underbrace{F(\dots F}_{n-1}(P_0)\dots))\right) = \int_{\mathbb{T}^2} f(x,y)\,dx\,dy\,.$$

The theorem implies that for almost any starting point, the number of visits of a long orbit of $F$ of large length $n$ to a subset $U$ is approximately $\mathrm{area}(U) \cdot n$.

## Ergodic Theorem: chaos is very regular in average

The map $F : \mathbb{T}^2 \to \mathbb{T}^2$ has several important properties:

● It preserves area: the area of any small disc $U$ is the same as the area of the small ellipse $F(U)$ obtained as the the image of $U$.
● Any subset invariant under $F$ either has zero area or its complement has zero area. In such situation one says that the map is *ergodic*.

**Ergodic Theorem** *For any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ and for almost any starting point $P_0$ in $\mathbb{T}^2$ the mean value of the function $f$ along the orbit $P_0, \ldots, \underbrace{F(F(\ldots F}_{n-1}(P_0)\ldots))$ coincides with the mean value of $f$ over $\mathbb{T}^2$:*

$$\lim_{n\to+\infty} \frac{1}{n}\left(f(P_0) + f(F(P_0)) + \cdots + f(\underbrace{F(\ldots F}_{n-1}(P_0)\ldots))\right) = \int_{\mathbb{T}^2} f(x,y)dx\,dy\,.$$

The theorem implies that for almost any starting point, the number of visits of a long orbit of $F$ of large length $n$ to a subset $U$ is approximately $\mathrm{area}(U) \cdot n$.

## Ergodic Theorem: chaos is very regular in average

The map $F : \mathbb{T}^2 \to \mathbb{T}^2$ has several important properties:

● It preserves area: the area of any small disc $U$ is the same as the area of the small ellipse $F(U)$ obtained as the the image of $U$.

● Any subset invariant under $F$ either has zero area or its complement has zero area. In such situation one says that the map is *ergodic*.

**Ergodic Theorem** *For any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ and for almost any starting point $P_0$ in $\mathbb{T}^2$ the mean value of the function $f$ along the orbit $P_0, \ldots, \underbrace{F(F(\ldots F}_{n-1}(P_0)...))$ coincides with the mean value of $f$ over $\mathbb{T}^2$:*

$$\lim_{n\to+\infty} \frac{1}{n} \left( f(P_0) + f(F(P_0)) + \cdots + f(\underbrace{F(\ldots F}_{n-1}(P_0)...)) \right) = \int_{\mathbb{T}^2} f(x,y)dx\,dy\,.$$

The theorem implies that for almost any starting point, the number of visits of a long orbit of $F$ of large length $n$ to a subset $U$ is approximately $\mathrm{area}(U) \cdot n$.

## Ergodic Theorem: chaos is very regular in average

The map $F : \mathbb{T}^2 \to \mathbb{T}^2$ has several important properties:

● It preserves area: the area of any small disc $U$ is the same as the area of the small ellipse $F(U)$ obtained as the the image of $U$.
● Any subset invariant under $F$ either has zero area or its complement has zero area. In such situation one says that the map is *ergodic*.

**Ergodic Theorem** *For any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ and for almost any starting point $P_0$ in $\mathbb{T}^2$ the mean value of the function $f$ along the orbit $P_0, \dots, \underbrace{F(F(\dots F}_{n-1}(P_0)...))$ coincides with the mean value of $f$ over $\mathbb{T}^2$:*

$$\lim_{n\to+\infty} \frac{1}{n} \left( f(P_0) + f(F(P_0)) + \cdots + f(\underbrace{F(\dots F}_{n-1}(P_0)...)) \right) = \int_{\mathbb{T}^2} f(x,y) dx\, dy\,.$$

The theorem implies that for almost any starting point, the number of visits of a long orbit of $F$ of large length $n$ to a subset $U$ is approximately $\mathrm{area}(U) \cdot n$.

## Ergodic Theorem: chaos is very regular in average

The map $F : \mathbb{T}^2 \to \mathbb{T}^2$ has several important properties:

• It preserves area: the area of any small disc $U$ is the same as the area of the small ellipse $F(U)$ obtained as the the image of $U$.

• Any subset invariant under $F$ either has zero area or its complement has zero area. In such situation one says that the map is *ergodic*.

**Ergodic Theorem** *For any continuous function $f : \mathbb{T}^2 \to \mathbb{R}$ and for almost any starting point $P_0$ in $\mathbb{T}^2$ the mean value of the function $f$ along the orbit $P_0, \dots, \underbrace{F(F(\dots F}_{n-1}(P_0)\dots))$ coincides with the mean value of $f$ over $\mathbb{T}^2$:*

$$\lim_{n\to+\infty} \frac{1}{n} \left( f(P_0) + f(F(P_0)) + \dots + f(\underbrace{F(\dots F}_{n-1}(P_0)\dots)) \right) = \int_{\mathbb{T}^2} f(x,y)dx\,dy\,.$$

The theorem implies that for almost any starting point, the number of visits of a long orbit of $F$ of large length $n$ to a subset $U$ is approximately $\operatorname{area}(U) \cdot n$.

# Billiards in polygons

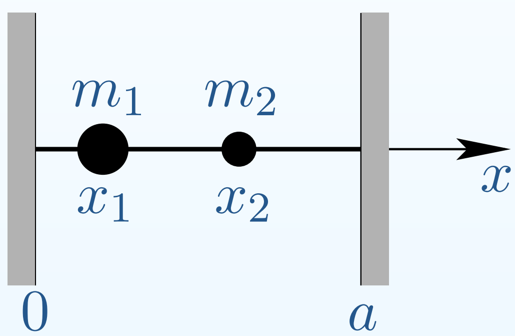
## Billiards

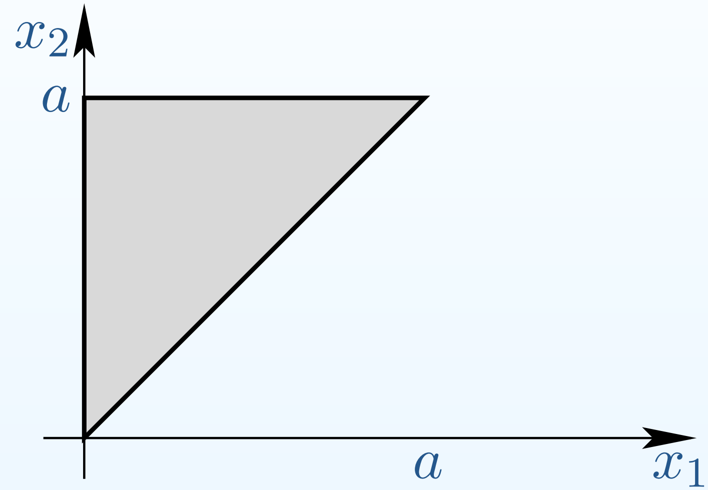
Following Moon Duchin let us play billiard in a polygon which might be more sophisticated than a usual rectangle.

Actually, we assume that a ball is very small, the wall do not have any holes, that there is no friction, and that the reflections are ideal and follow the rules of optic.

## Motivation to study billiards: gas of two molecules in a one-dimensional chamber

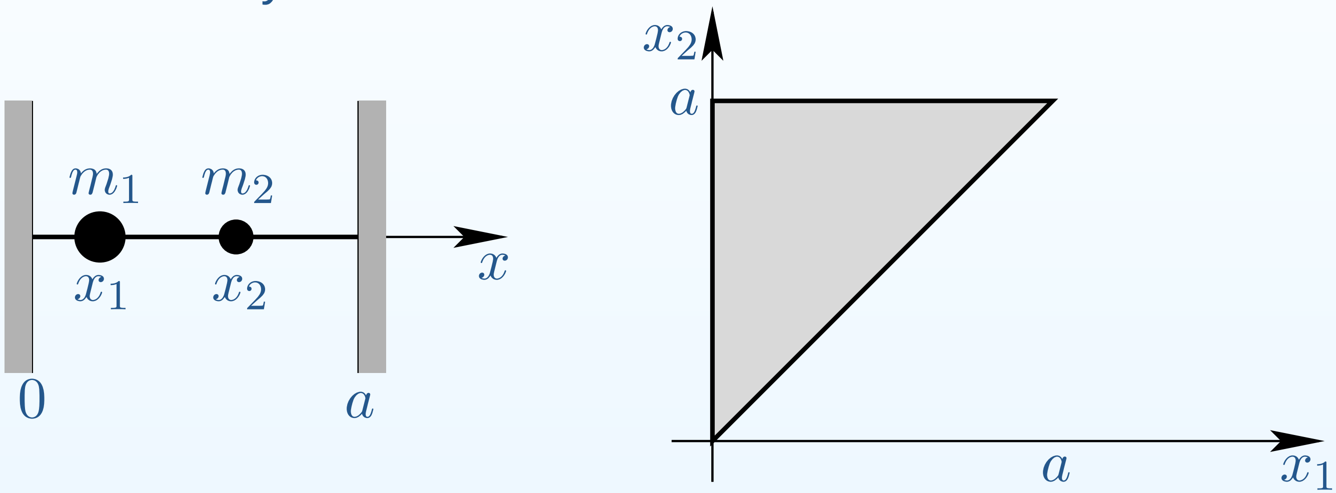
Consider two elastic balls ("molecules") sliding along a rod. They are bounded from two sides by solid walls. All collisions are ideal — without loss of energy.



## Motivation to study billiards: gas of two molecules in a one-dimensional chamber

Consider two elastic balls (“molecules”) sliding along a rod. They are bounded from two sides by solid walls. All collisions are ideal — without loss of energy.



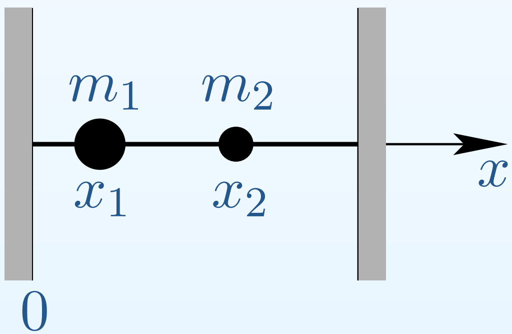

Neglecting the sizes of the balls we can describe the configuration space of our system using coordinates $0 < x_1 \leq x_2 \leq a$ of the balls, where $a$ is the distance between the walls. This gives a right isosceles triangle.

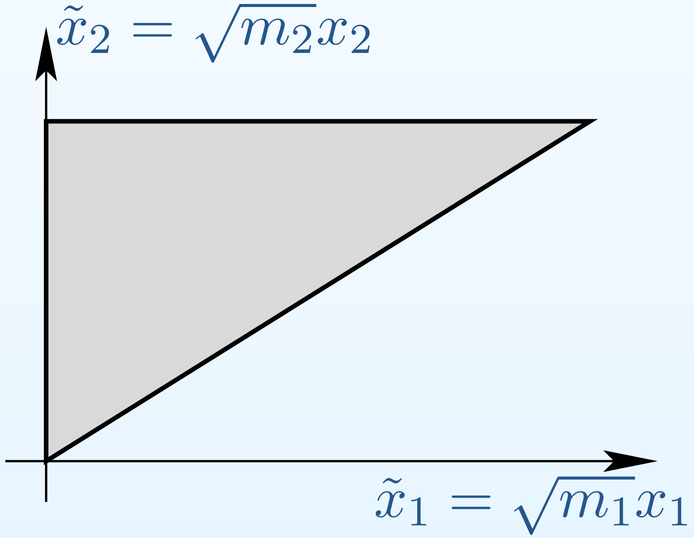
## Gas of two molecules

Rescaling the coordinates by square roots of masses

$$\begin{cases} \tilde{x}_1 & = \sqrt{m_1} x_1 \\ \tilde{x}_2 & = \sqrt{m_2} x_2 \end{cases}$$

we get a new right triangle $\Delta$ as a configuration space.



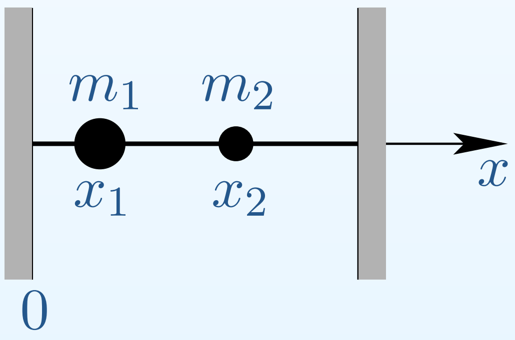



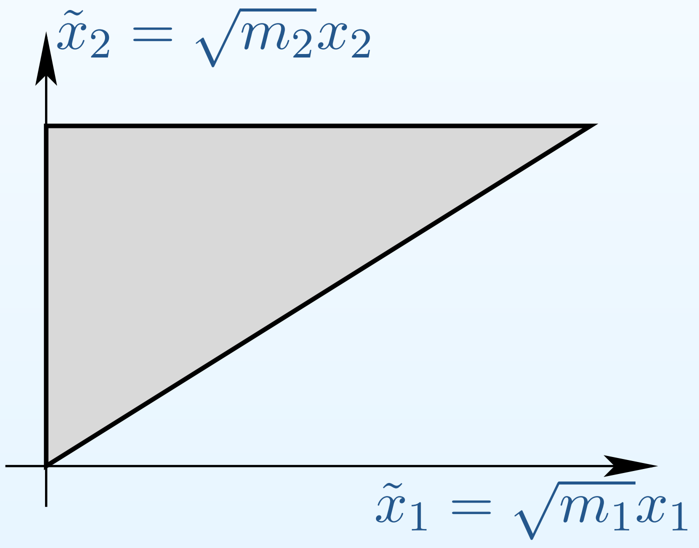
## Gas of two molecules

Rescaling the coordinates by square roots of masses

$$\begin{cases} \tilde{x}_1 & = \sqrt{m_1} x_1 \\ \tilde{x}_2 & = \sqrt{m_2} x_2 \end{cases}$$

we get a new right triangle $\Delta$ as a configuration space.

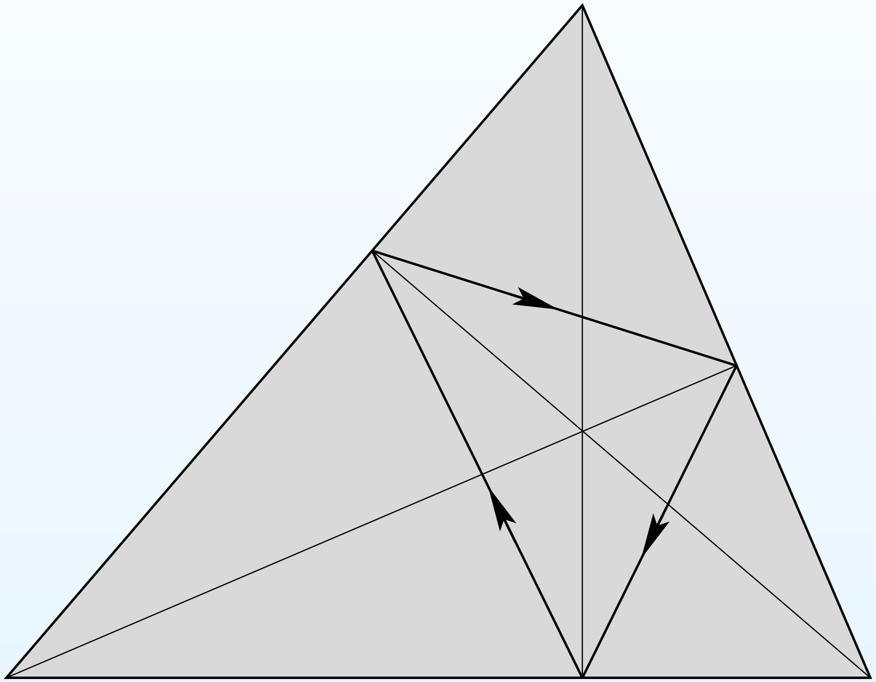



**Lemma** *In coordinates $(\tilde{x}_1, \tilde{x}_2)$ trajectories of the system of two balls on a rod correspond to billiard trajectories in the triangle $\Delta$.*

## Closed billiard trajectories

It is easy to find a periodic trajectory in an acute triangle:



**Exercise.** Show that the broken line joining the base points of the heights in an acute triangle is a closed billiard trajectory (called *Fagnano trajectory*). Show that it is an inscribed triangle of the minimal possible perimeter.

## Challenge

It is difficult to believe, but for an obtuse triangle the problem is open:

**Open Problem.** *Is there at least one periodic trajectory in any obtuse triangle?*

## Challenge

It is difficult to believe, but for an obtuse triangle the problem is open:

**Open Problem.** *Is there at least one periodic trajectory in any obtuse triangle?*

The answer might be affirmative (for triangles with obtuse angle at most $100^o$ R. Schwartz has verified it by a rigorous heavily computer-assisted proof).

## Challenge

It is difficult to believe, but for an obtuse triangle the problem is open:

**Open Problem.** *Is there at least one periodic trajectory in any obtuse triangle?*

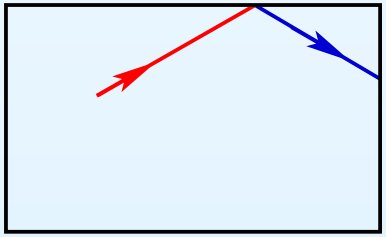
The answer might be affirmative (for triangles with obtuse angle at most $100^o$ R. Schwartz has verified it by a rigorous heavily computer-assisted proof). But even if it is affirmative, the natural question "And how many?.." is completely and desperately open already for acute triangles.

**Open Problem.** *Estimate the number $N(\Pi, L)$ of periodic trajectories of length at most $L$ in a polygon $\Pi$ as $L \to +\infty$.*

**Open Problem.** *Is the billiard flow ergodic for almost any triangle?*

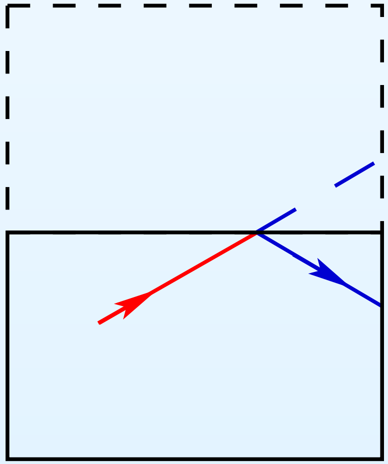
## From a billiard to a surface

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



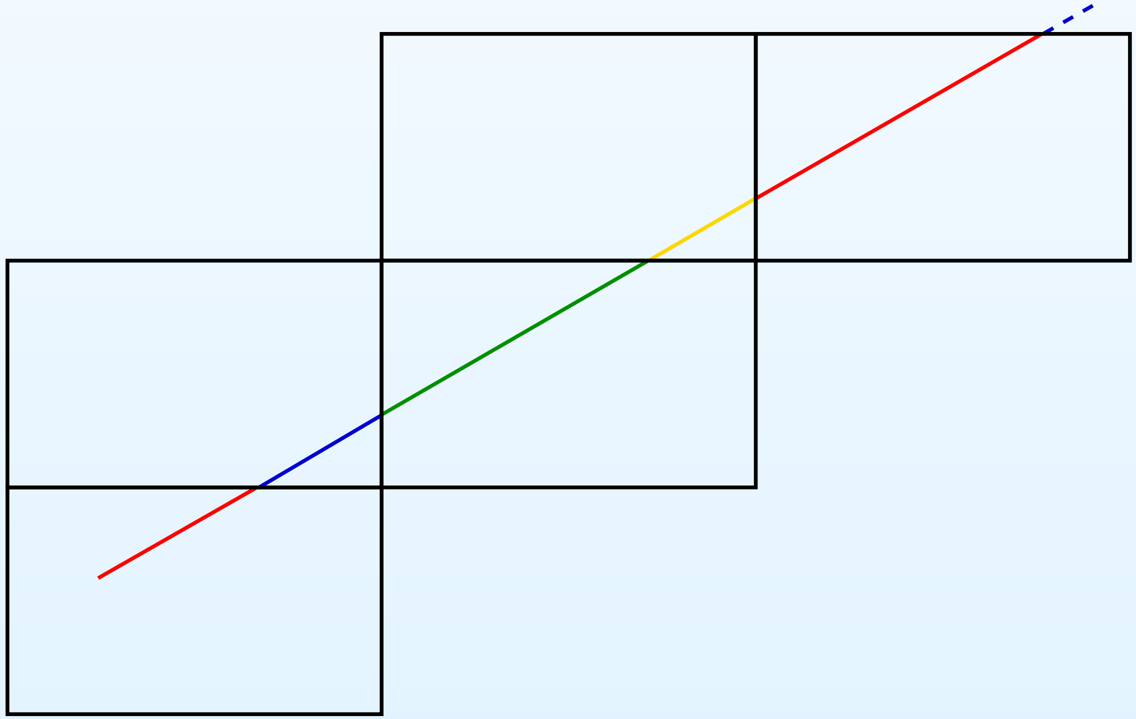
## From a billiard to a surface

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



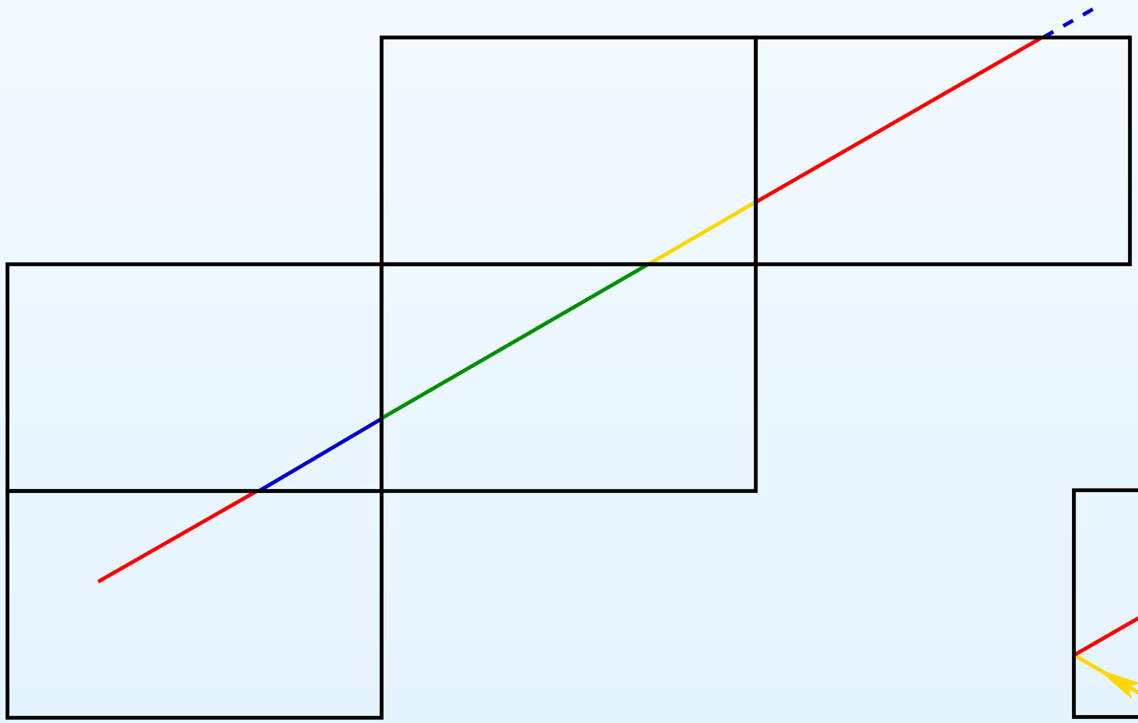
## From a billiard to a surface

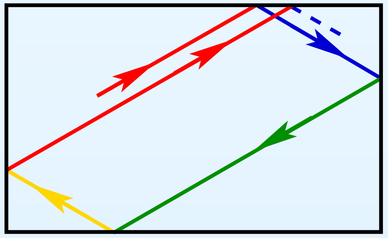
Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



## From a billiard to a surface

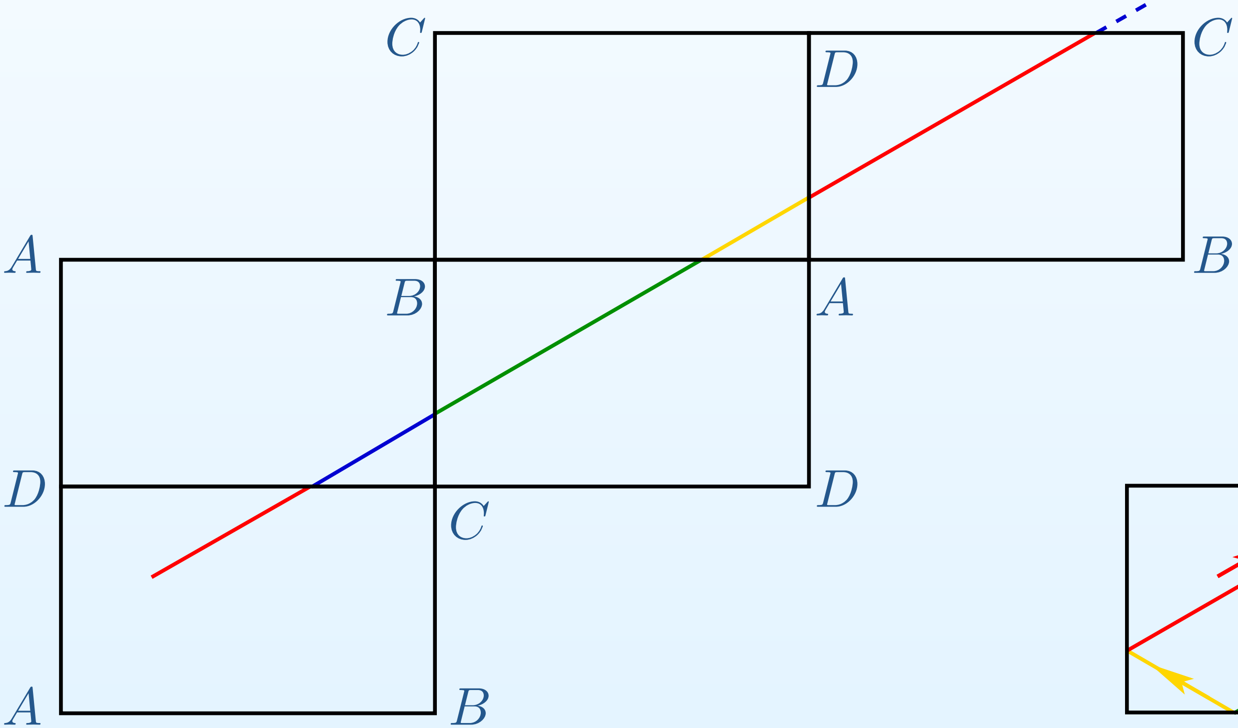
Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation.

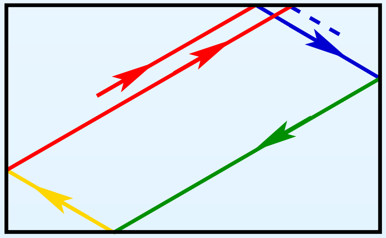




# From a billiard to a surface

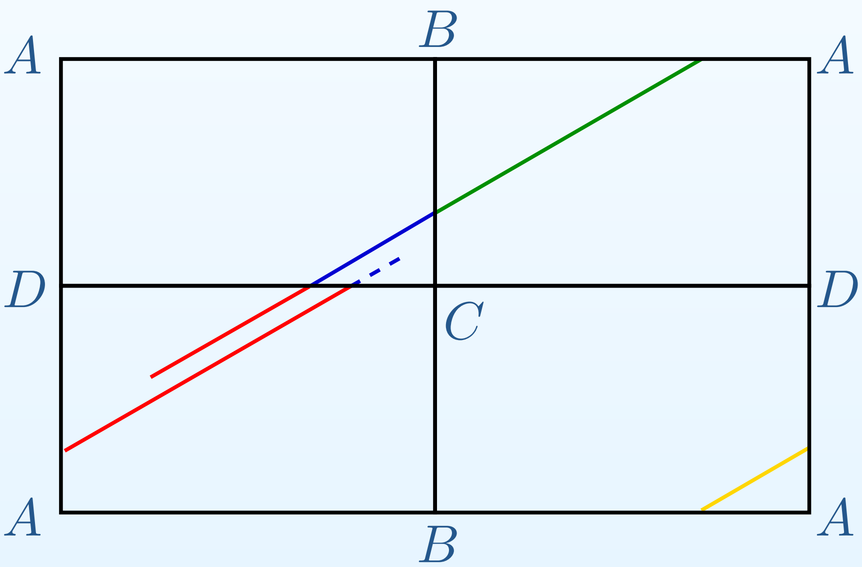
Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation.

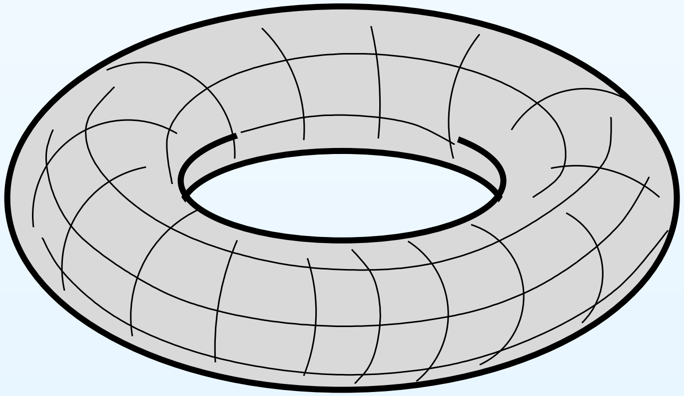


# From a billiard to a surface

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



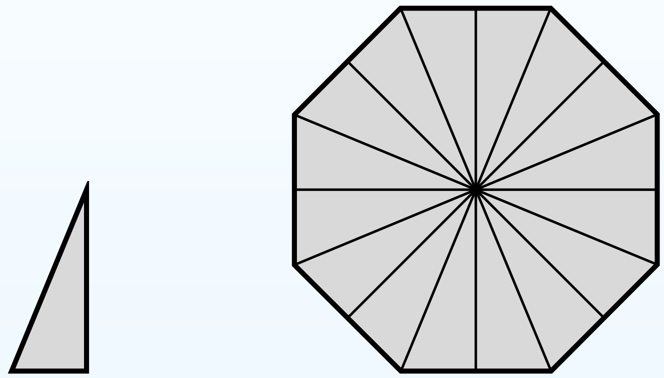
Identifying the equivalent patterns by a parallel translation we obtain a torus; the billiard trajectory unfolds to a "straight line" on the corresponding torus.

## Billiards in rational polygons

One can apply a similar unfolding construction to any polygon with angles which are rational multiples of $\pi$ to pass from a billiard to a flat surface.

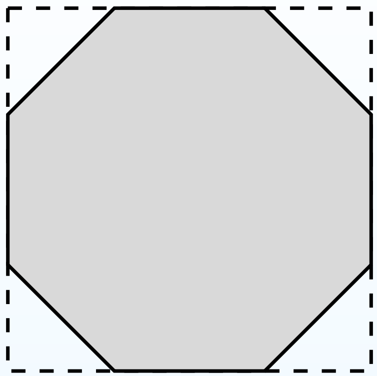
## Billiards in rational polygons

One can apply a similar unfolding construction to any polygon with angles which are rational multiples of $\pi$ to pass from a billiard to a flat surface.



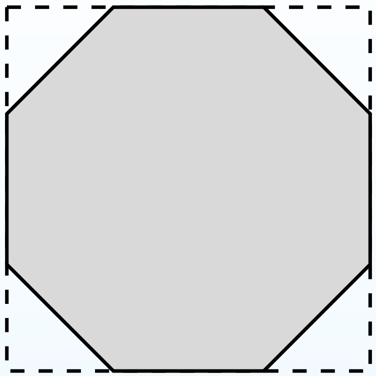
Consider, for example the triangle with angles $\pi/8, 3\pi/8, \pi/2$. It is easy to check that a generic trajectory of such billiard moves at any time in one of $16$ directions (compared to $4$ for a rectangle). We can unfold the triangle to a regular octagon glued from $16$ copies of the triangle. Identifying opposite sides of the octagon we get a flat surface. All “straight lines” on this surface project to the initial billiard trajectories.

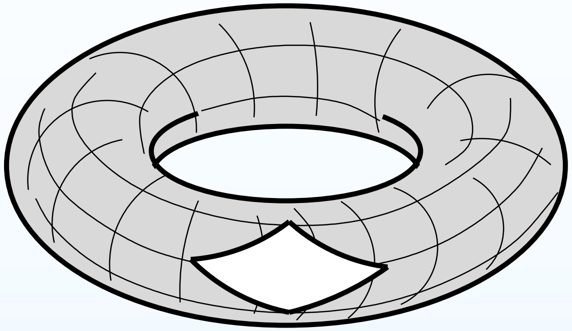
## Flat surface more complicated than a torus



Identifying the pair of horizontal sides and then the pair of vertical sides of a regular octagon we get a torus with a single square hole. Identifying two opposite sides of the hole we get a torus with two distinct holes. Identifying the resulting holes we get a flat torus... with an extra handle.

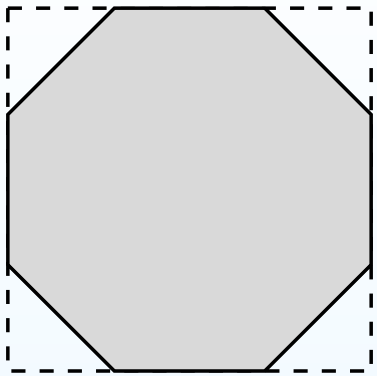
## Flat surface more complicated than a torus

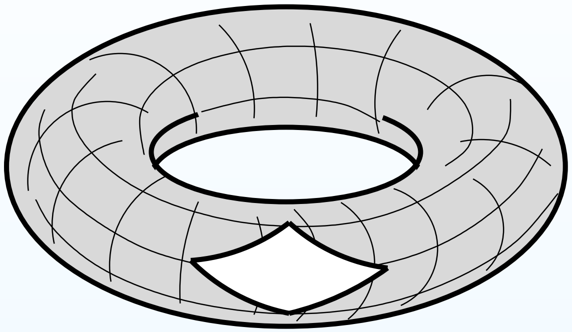


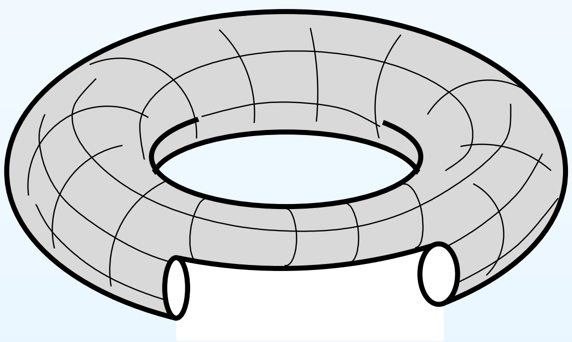


Identifying the pair of horizontal sides and then the pair of vertical sides of a regular octagon we get a torus with a single square hole. Identifying two opposite sides of the hole we get a torus with two distinct holes. Identifying the resulting holes we get a flat torus... with an extra handle.

# Flat surface more complicated than a torus

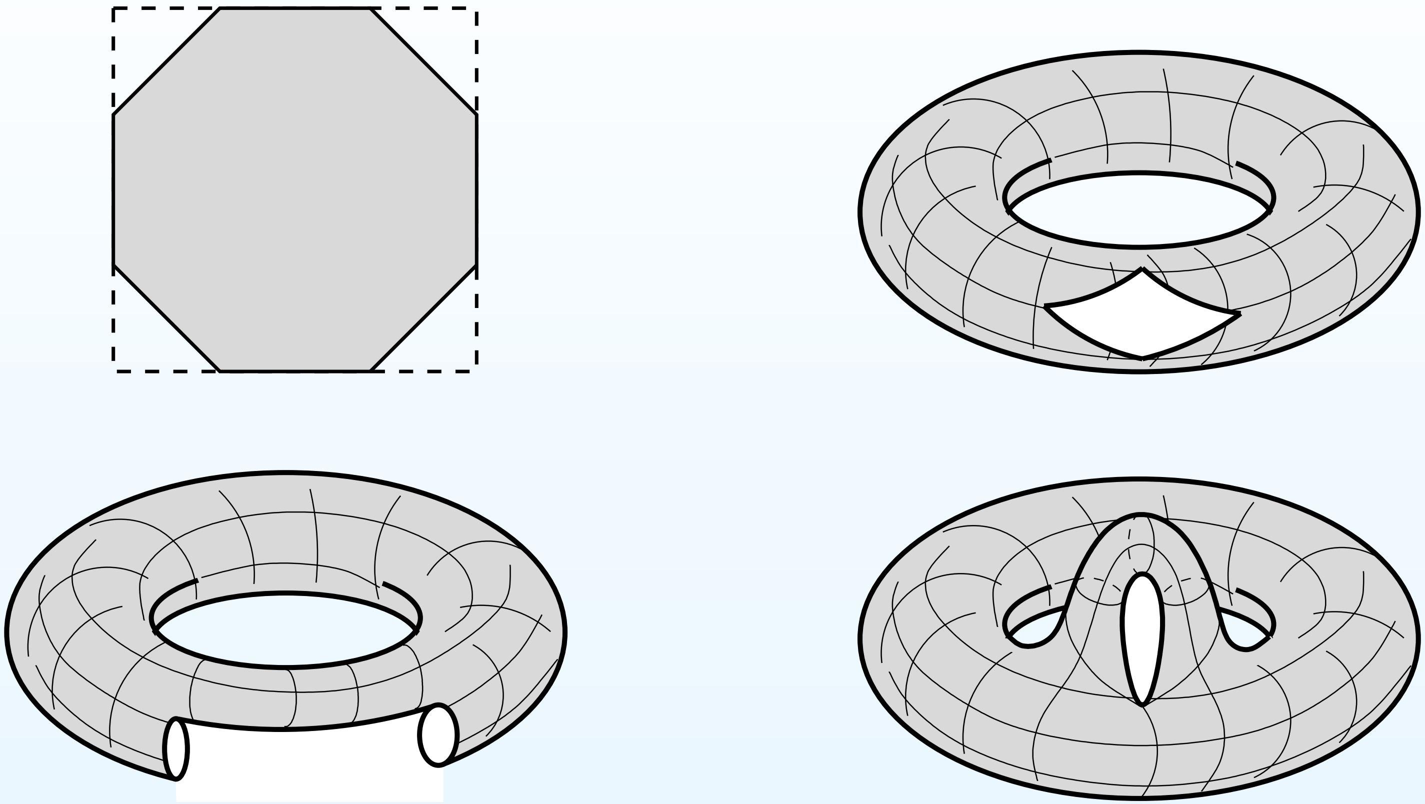






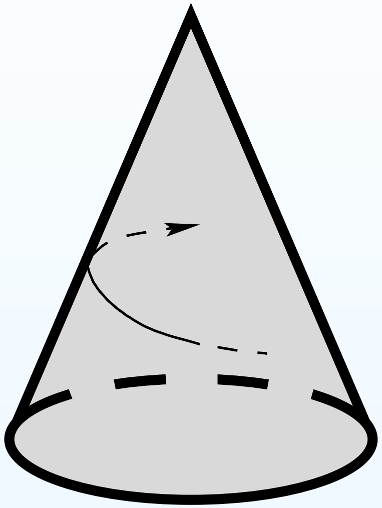
Identifying the pair of horizontal sides and then the pair of vertical sides of a regular octagon we get a torus with a single square hole. Identifying two opposite sides of the hole we get a torus with two distinct holes. Identifying the resulting holes we get a flat torus... with an extra handle.

# Flat surface more complicated than a torus



Identifying the pair of horizontal sides and then the pair of vertical sides of a regular octagon we get a torus with a single square hole. Identifying two opposite sides of the hole we get a torus with two distinct holes. Identifying the resulting holes we get a flat torus... with an extra handle.

## Geodesics on a cone



**Exercise.** One can wrap a large piece of paper into a cone. Straight lines on the plane become so-called *geodesic lines* on the cone. Launch a straight ray in a direction different from the direction of the vertex of the cone. Would it infinitely spiral around the vertex or it would come back? If you think that it would come back, count the number of turns and count how close to the vertex would it get. Can one create oscillations for some particular values of parameters?

## Magic Wand Theorem

Developing ideas of Bill Thurston, American mathematicians Howard Masur and William Veech have studied in early 80's a natural dynamical system of the *moduli space of all flat surfaces* to describe dynamics on an individual flat surface. Couple of years ago Alex Eskin, Amir Mohammadi, Maryam Mirzakhani, and Simion Filip have proved fantastic results about this dynamical system proving that *absolutely all complex geodesics in the moduli space* behave in some specific sense very nicely: they cannot have any fractal behavior which is very common in dynamical systems.

## Fields Medal

At the last International Congress of Mathematics Maryam Mirzakhani has received a Fields Medal for *“for her exceptional contributions to dynamics and geometry of Riemann surfaces and their moduli spaces”* becoming the first woman to receive the Fields Medal.

# Billiard in a polygon : an artistic image

Varvara Stepanova. Billiard players. Thyssen museum, Madrid