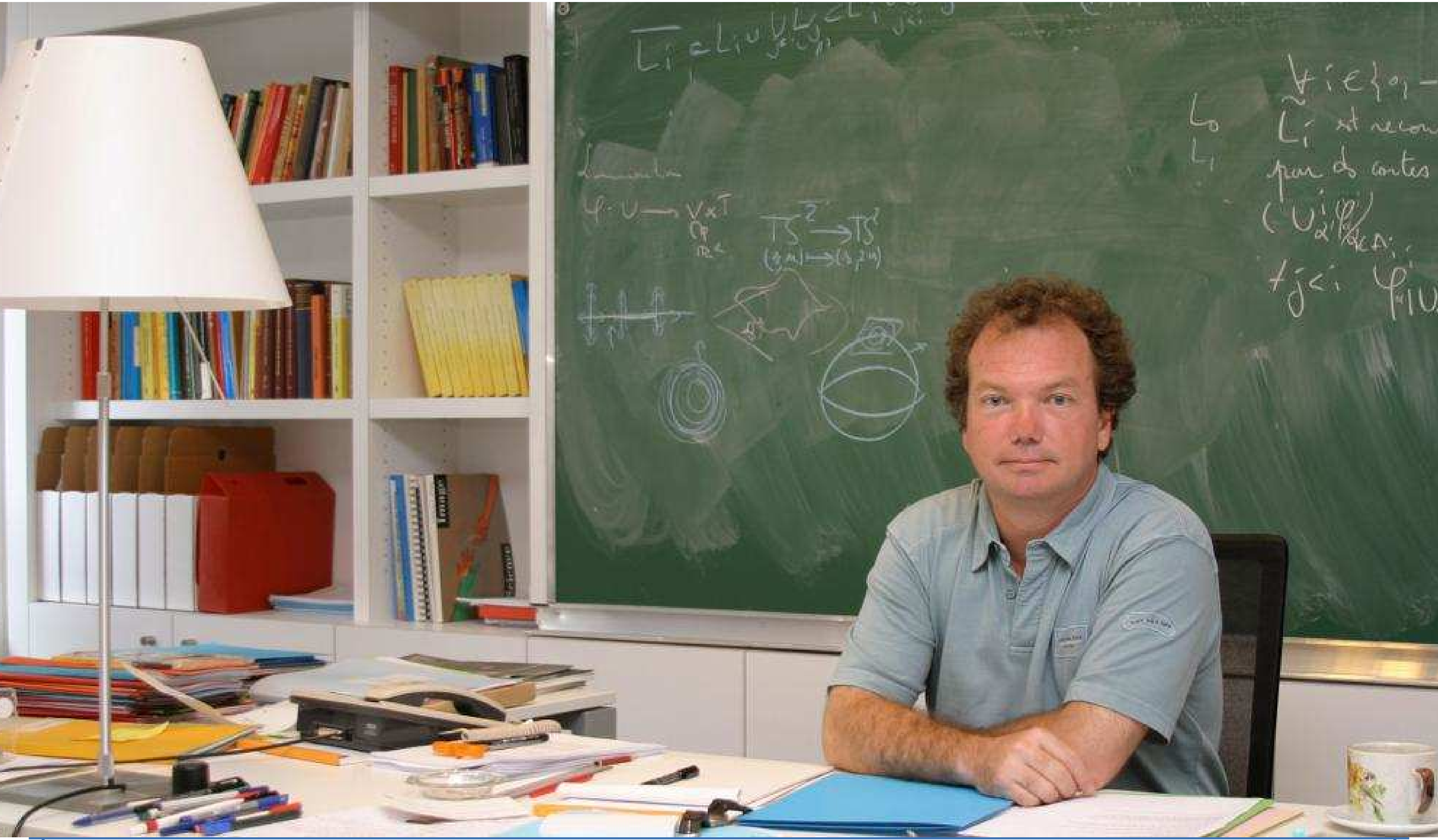
Equidistribution of square-tiled surfaces, meanders, and Masur-Veech volumes

Anton Zorich
(joint work with V. Delecroix, E. Goujard, P. Zograf)

Conference in memory of Jean-Christophe Yoccoz
Collège de France
May 31, 2017



Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials

Period coordinates and Masur–Veech volume

The moduli space $\mathcal{H}(m_1, \dots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \dots, m_n$, where $\sum m_i = 2g - 2$, is modelled on the vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

The following homogeneous function is defined on every stratum $\mathcal{H}(m_1, \dots, m_n)$:

$$\text{area}(C, \omega) = \frac{i}{2} \int_C \omega \wedge \bar{\omega} = \frac{i}{2} \sum_{i=1}^{g} (A_i \bar{B}_i - \bar{A}_i B_i) .$$

and $\mathrm{area}(C, t \cdot \omega) = |t|^2 \cdot \mathrm{area}(C, \omega)$. Denote by $\mathcal{H}_{\le 1}$ the subset of those (C, ω) in $\mathcal{H}(m_1, \dots, m_n)$ where $\mathrm{area}(C, \omega) \le 1$.

Definition. $\mathrm{Vol}\, \mathcal{H}(m_1, \dots, m_n) := 2d \cdot \int_{\mathcal{H}_{\le 1}} d\nu$, where $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n)$ is just a conventional factor.

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}(m_1, \dots, m_n)$ or $\mathcal{Q}(m_1, \dots, m_n)$ of Abelian or quadratic differentials is finite.*

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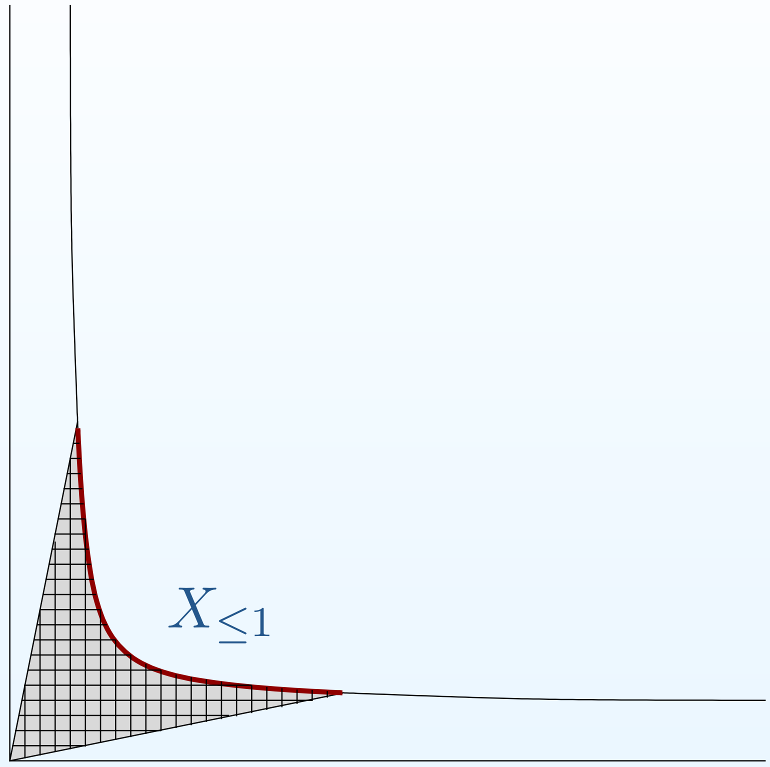
Counting volume by counting integer points in a large cone



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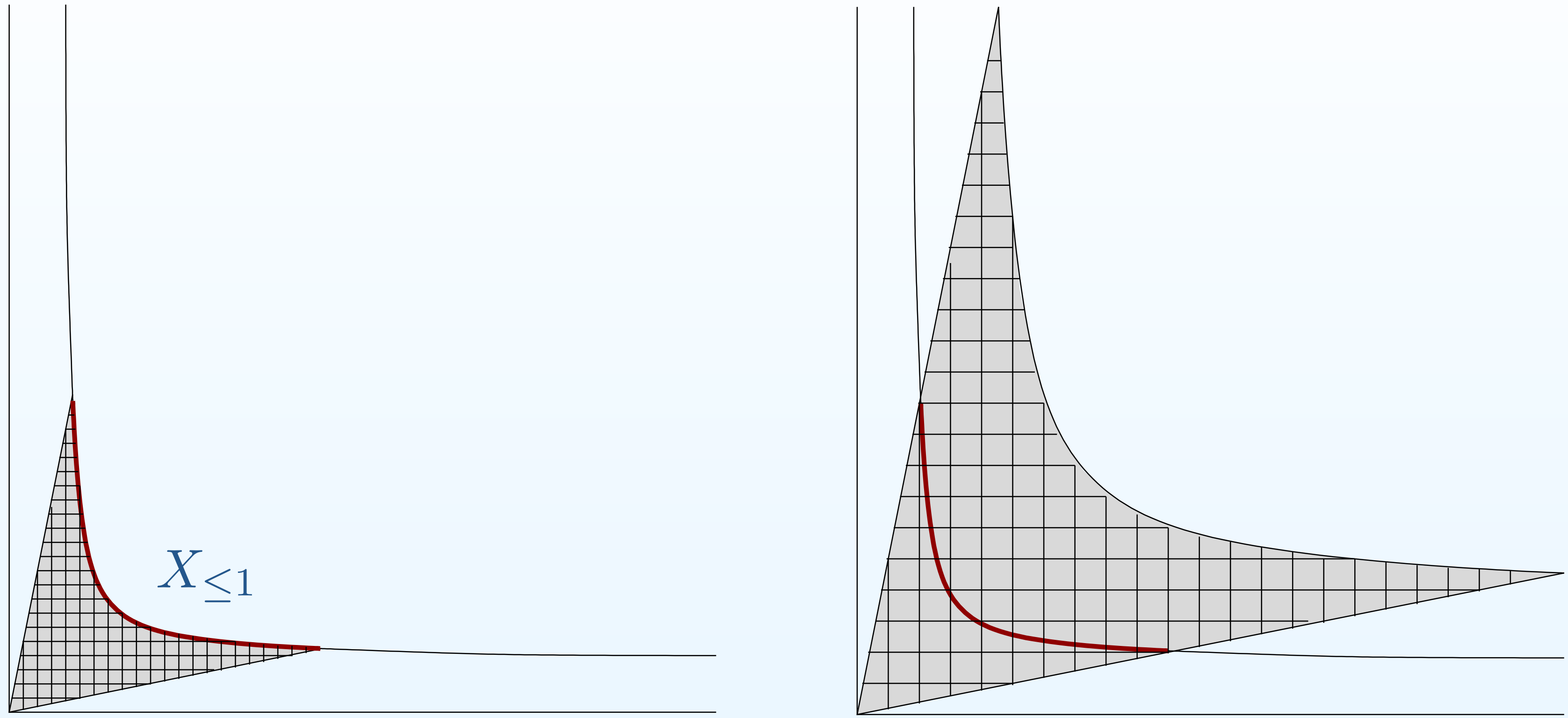

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Counting points of the ε-grid in the cone $X_{\leq 1}$ is the same as counting integer points in the larger proportionally rescaled cone $X_{\leq 1/\varepsilon}$.

Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point. Let P_1 be a zero of ω and $P \in C$ any point of the Riemann surface C. Define

$$\begin{array}{rccl} p: & P & \mapsto & \int_{P_1}^{P} \omega \pmod{\mathbb{Z} \oplus i\mathbb{Z}} \\ p: & C & \to & \mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \end{array}$$

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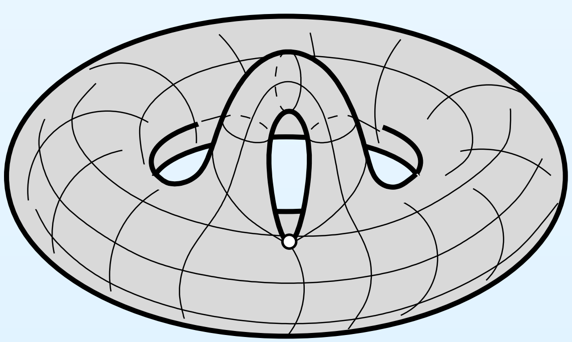
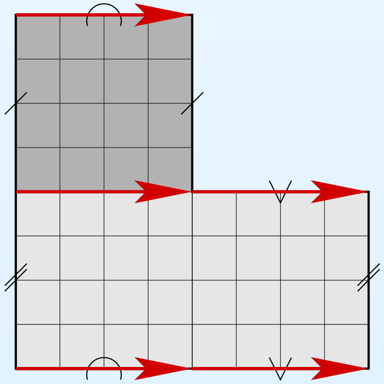
Choosing the standard unit square pattern for $\mathbb{T}$ we get induced tiling of (C, ω) by unit squares which form horizontal and vertical cylinders. The square-tiled surface of genus two in the picture has 2 maximal horizontal cylinders filled with periodic geodesics.





Contribution of k-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$0.19 \approx p_1(\mathcal{H}(3,1)) = \frac{3\,\zeta(7)}{16\,\zeta(6)}$$ ← the only quantity which is easy to compute

$$0.47 \approx p_2(\mathcal{H}(3,1)) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx p_3(\mathcal{H}(3,1)) = \frac{1}{32\,\zeta(6)}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\ &+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\ &- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\ &+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\ &- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\ &- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx p_4(\mathcal{H}(3,1)) = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big).$$

Computation of volumes

Theorem (A. Eskin, A. Okounkov, R. Pandharipande). *For every connected component $\mathcal{H}^c(d_1, \dots, d_n)$ of every stratum, the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled} \\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form. Volume $\operatorname{Vol} \mathcal{H}^c(d_1, \dots, d_n)$ of every connected component of every stratum is a rational multiple $\frac{p}{q} \cdot \pi^{2g}$ of π^{2g}, where g is the genus.

A. Eskin implemented this theorem to an algorithm allowing to compute $\frac{p}{q}$ for all strata up to genus 10 and for some strata (like the principal one) up to genus 200. Based on these calculations we developed a conjecture on a very simple asymptotic formula for volumes in large genera.

D. Chen, M. Möller, D. Zagier have recently constructed more general generation function, which englobes all genera at once. In particular, they can compute the volume of the principal stratum up to genus 2000 and prove the conjecture on large genus volume asymptotics.

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Volumes of the moduli spaces of quadratic differentials

A. Eskin, A. Okounkov and R. Pandharipande proved analogous result for the Masur–Veech volumes $\operatorname{Vol}\mathcal{Q}(d_1,\dots,d_k)$ of the moduli spaces of meromorphic quadratic differentials with at most simple poles. However, with exception for genus 0 the implementation is more painful: values for the first 300 low-dimensional strata in $g>0$ were obtained only in 2015 by E. Goujard.

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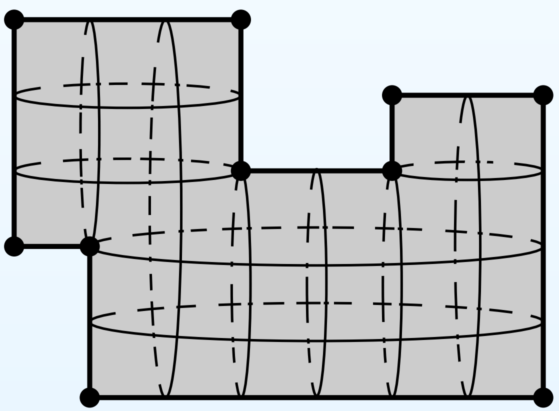

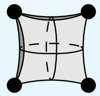

Pillowcase covers play the role of square-tiled surfaces: they represent integer points in period coordinates. Pillowcase covers are branched covers over four points of the sphere (pillow). The pillowcase cover as in the picture lives in the stratum $\mathcal{Q}(1,1,1,\underbrace{-1,\dots,-1}_{7}) =: \mathcal{Q}(1^3,-1^7)$.

Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics.
An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes $\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})$.

Let $$v(n) := \frac{n!!}{(n+1)!!}\cdot\pi^n\cdot\begin{cases}\pi & \text{when } n\geq -1 \text{ is odd}\\ 2 & \text{when } n\geq 0 \text{ is even}\end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1)=1$ and $v(0)=2$.

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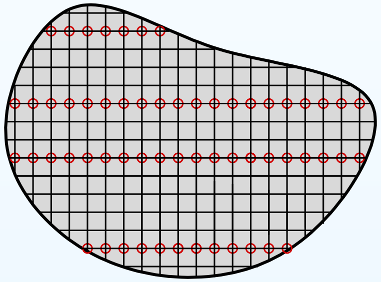
Theorem (J. Athreya, A. Eskin, A. Z., 2014 ; conjectured by M. Kontsevich about 2003) *The volume of any stratum $\mathcal{Q}(d_1,\dots,d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. when $d_i\in\{-1\,;\,0\}\cup\mathbb{N}$ for $i=1,\dots,k$, and $\sum_{i=1}^k d_i=-4$) is equal to*

$$\operatorname{Vol}\mathcal{Q}(d_1,\dots,d_k) = 2\pi\cdot\prod_{i=1}^{k} v(d_i)\,.$$

Asymptotic equidistribution

Equidistribution Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \dots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L})/\operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula! Proof: Moore ergodicity theorem.

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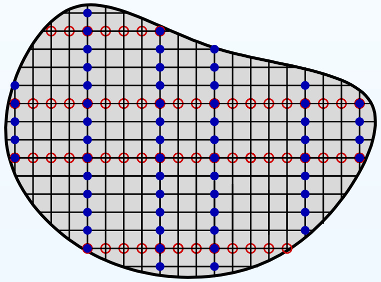

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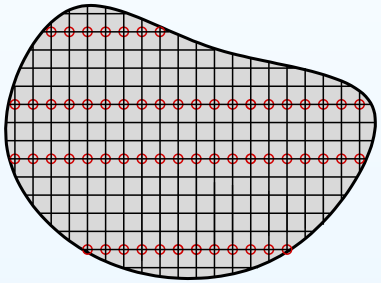
Experimental evaluation of volumes

The Equidistribution Theorem allows to compute approximate values of volumes experimentally. Choose some ball B (or some box) in the stratum. Consider a sufficiently small grid in it and collect statistics of frequency $p_1(B)$ of 1-cylinder square-tiled surfaces (pillow-case covers) in our grid in B.



Now compute the **absolute** contribution $c_1(\mathcal{L})$ of all 1-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{L}$; it is easier than for k-cylinder ones with $k > 2$. By the Equidistribution Theorem, the volume of the ambient stratum is $\operatorname{Vol}\mathcal{L} = \frac{c_1(\mathcal{L})}{p_1(\mathcal{L})}$.

The statistics $p_1(\mathcal{H})$ can be, actually, collected using interval exchanges, which simplifies the experiment. Approximate values of volumes were extremely useful in debugging numerous normalization factors in rigorous answers in the implementation by E. Goujard of the method of Eskin–Okounkov.

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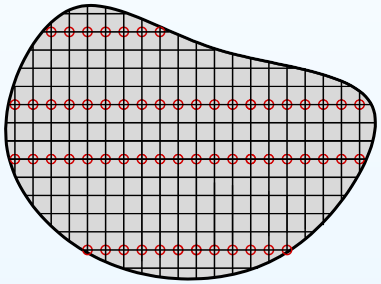

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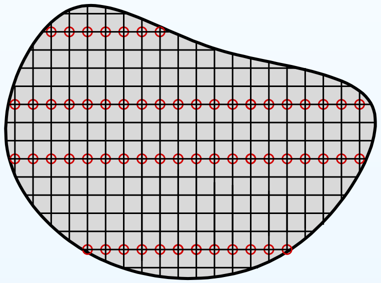

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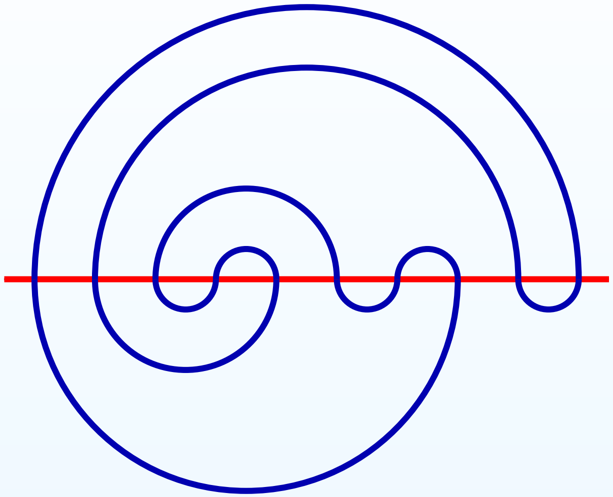
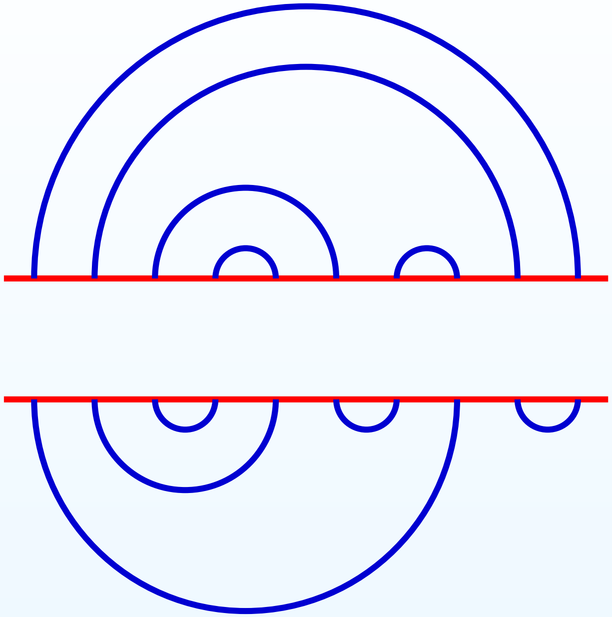
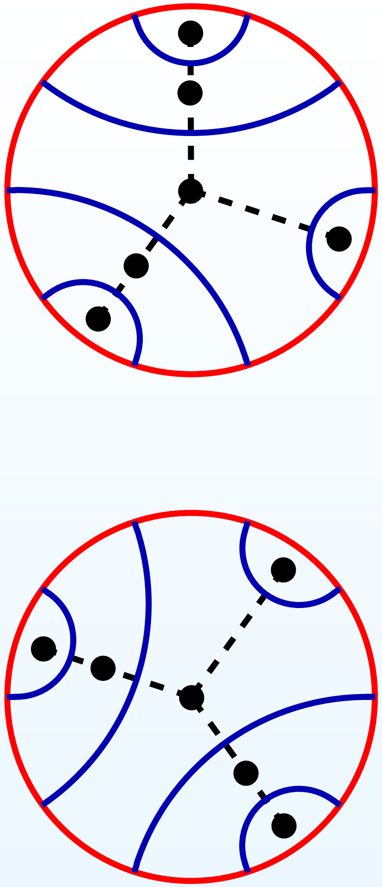
Meanders

Meanders and arc systems







A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line.

According to S. Lando and A. Zvonkin the notion “meander” was suggested by V. Arnold though meanders were studied already by H. Poincaré.

Meanders appear in various contexts, in particular in physics (P. Di Francesco, O. Golinelli, E. Guitter).

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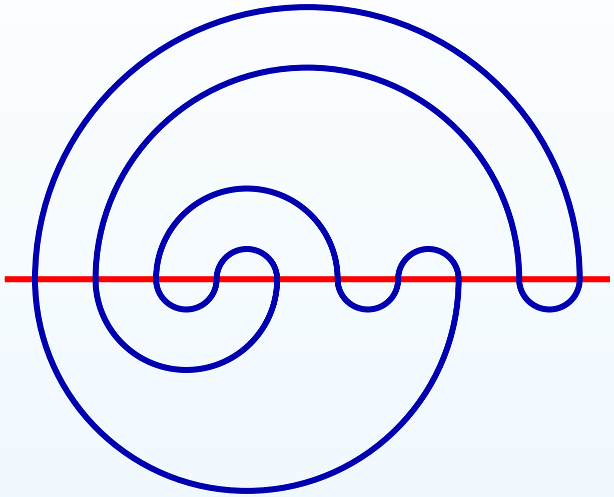
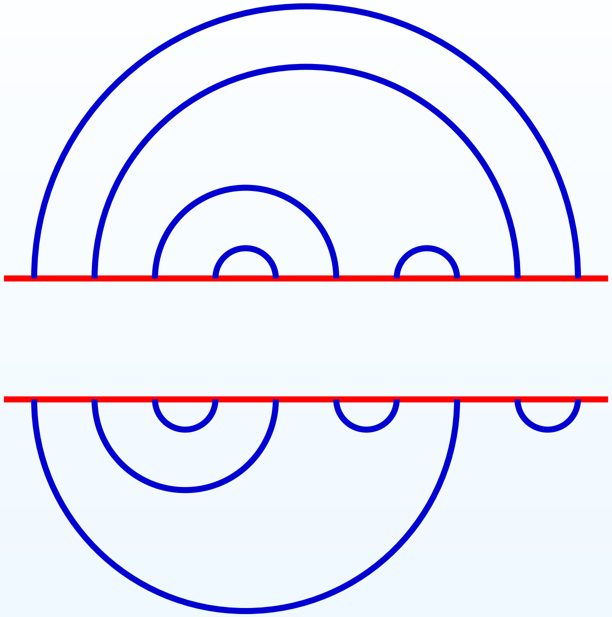
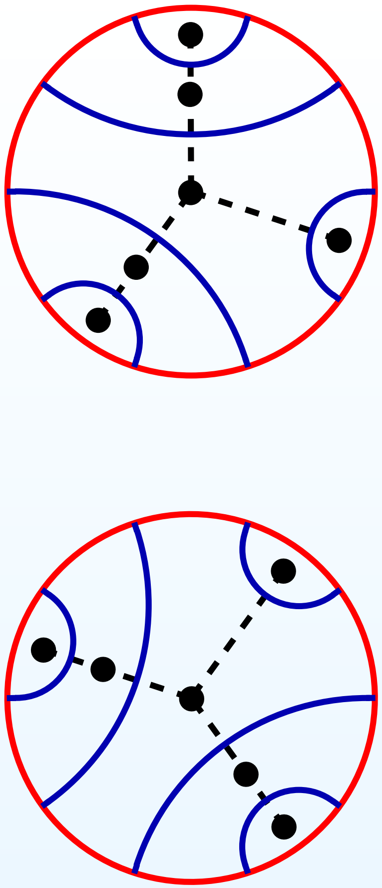




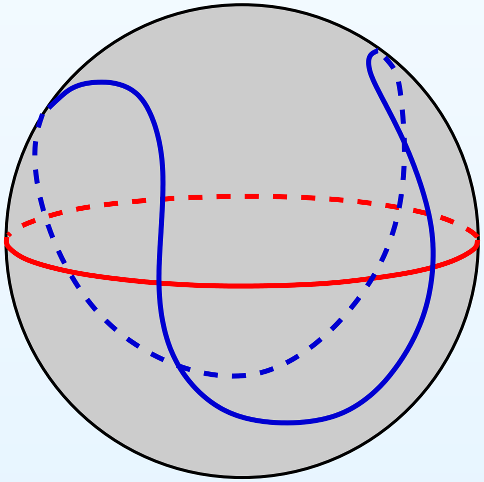


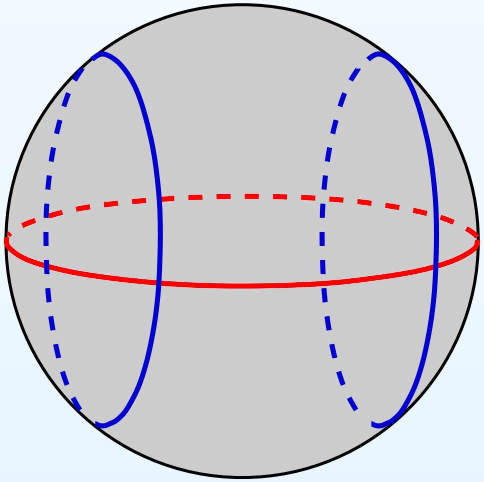
A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Compactifying the plane on the left with one point at infinity, or gluing two arc systems together we get an ordered pair of smooth simple transversally intersecting closed curves on the sphere.

Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a curve with several connected components





Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

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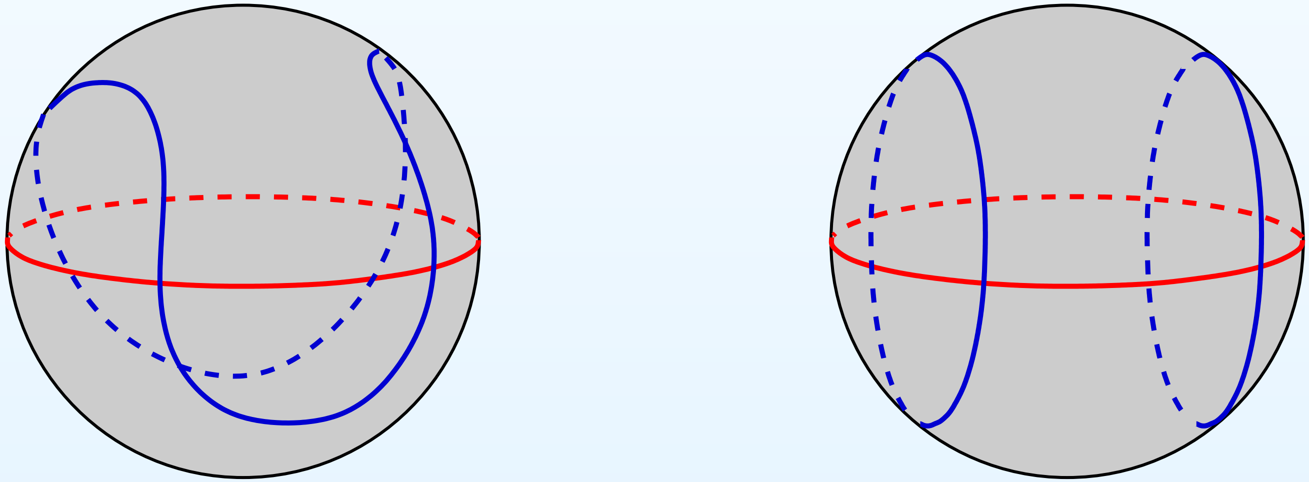

Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

Asymptotic frequency of meanders

Consider arc system with the same number $n \leq N$ of arcs on a labeled pair of oriented discs having $\mathcal{T}_{top}$ and $\mathcal{T}_{bottom}$ as reduced dual trees. We draw $\mathcal{T}_{top}$ on the northern hemisphere and $\mathcal{T}_{bottom}$ on the southern hemisphere. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. We consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{top};\, n\text{-arc system of type } \mathcal{T}_{bottom};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$p_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$

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$$p_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$

Theorem. *This ratio has a limit as $N \to +\infty$ which depends only on the vertex type $\nu = [1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \dots]$ of the graph $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$, where ν_j encodes the total number of vertices of valence $j+2$ in $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ for $j \in \mathbb{N}$. The limit is given by closed formula.*

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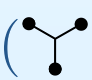
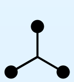
Example. The fact that this asymptotic frequency is nonzero is already somehow unexpected. For example, the following asymptotic frequency is not even so small:

$p_1($  $,$  $) = \frac{280}{\pi^6} \approx 0.291245\,,$

Fixing the number of vertices of valence one

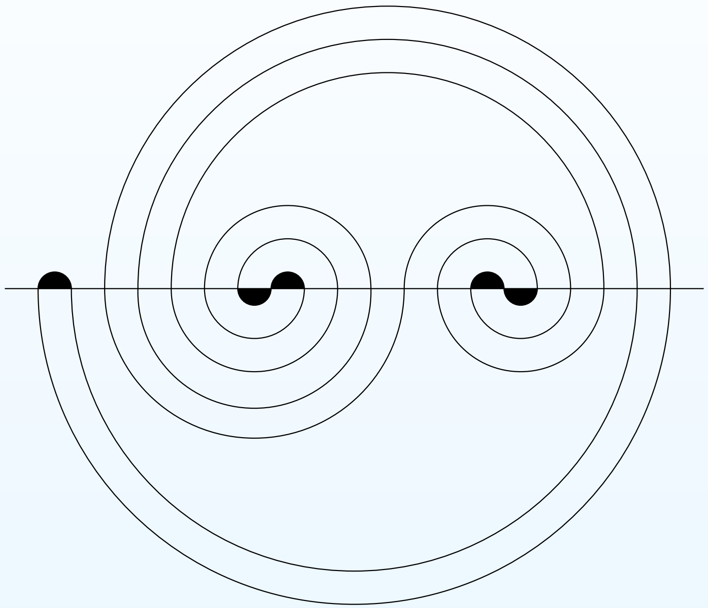
Theorem. *For any pair of planar trees having the total number p of leaves (vertices of valence one) the following limit exists:*

$$\lim_{N\to+\infty} p_{connected}(p;N) \;=\; p_1(\mathcal{Q}(1^{p-4},-1^p)) \;=$$

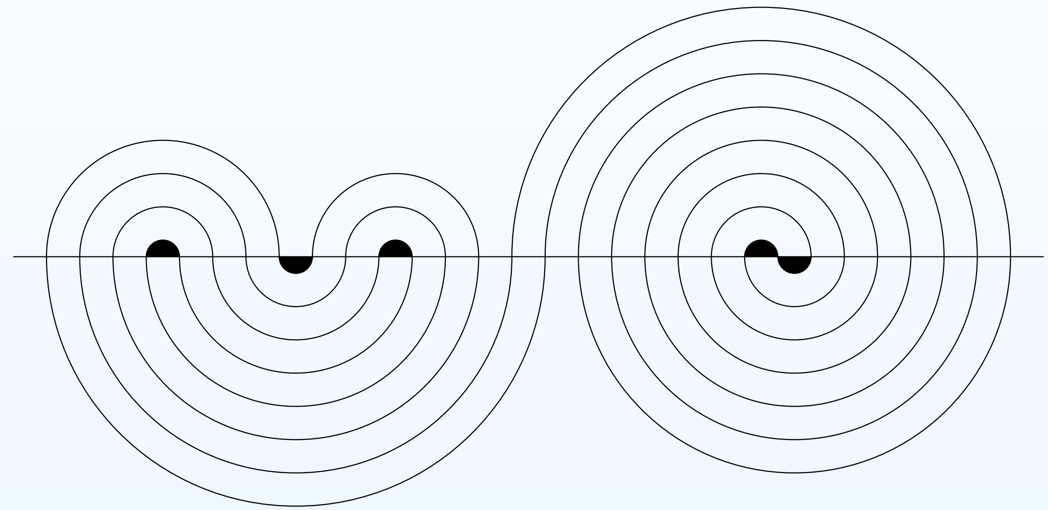
$$=\; \frac{cyl_1(\mathcal{Q}(1^{p-4},-1^p))}{\operatorname{Vol}\mathcal{Q}(1^{p-4},-1^p)} \;=\; \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3}\cdot\binom{2p-4}{p-2}.$$

Meanders with and without maximal arc

These two meanders have 5 minimal arcs (“pimples”) each.



Meander with a maximal arc (“rainbow”) contributes to $\mathcal{M}_5^+(N)$



Meander without maximal arc contributes to $\mathcal{M}_5^-(N)$

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc (“rainbow”) and having at most $2N$ crossings with the horizontal line and exactly p minimal arcs (“pimples”). We consider p as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

Counting formulae for meanders

Theorem. *For any fixed p the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with p minimal arcs (pimples) and with at most $2N$ crossings have the following asymtotics as $N \to +\infty$:*

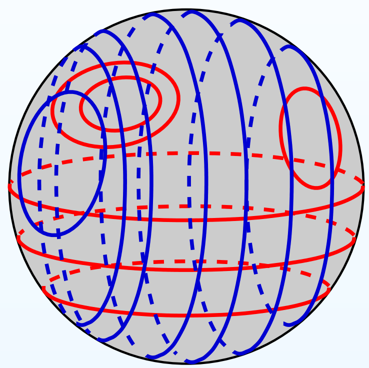
$$\mathcal{M}_p^+(N) = 2(p+1) \cdot \frac{cyl_{1,1}\big(\mathcal{Q}(1^{p-3}, -1^{p+1})\big)}{(p+1)!\,(p-3)!} \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4}) =$$
$$= \frac{2}{p!\,(p-3)!} \left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4})\,.$$

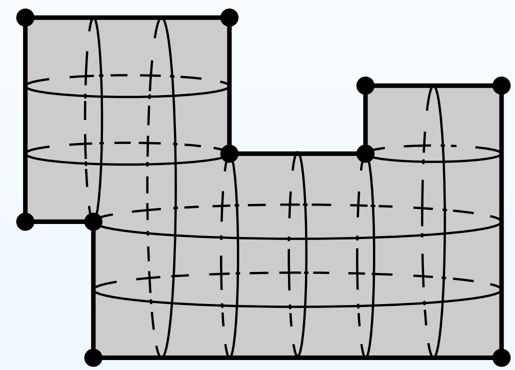
$$\mathcal{M}_p^-(N) = \frac{2cyl_{1,1}\big(\mathcal{Q}(0, 1^{p-4}, -1^{p})\big)}{p!\,(p-4)!} \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5}) =$$
$$= \frac{4}{p!\,(p-4)!} \left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,.$$

Note that $\mathcal{M}_p^+(N)$ grows as N^{2p-4} while $\mathcal{M}_p^-(N)$ grows as N^{2p-5}.

Proof

Step 1. There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





Step 2. Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height 1. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $cyl_{1,1}(\mathcal{Q}) = cyl_1^2(\mathcal{Q}) / \operatorname{Vol}(\mathcal{Q})$, where $cyl_1(\mathcal{Q})$ is easy to compute and $\operatorname{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula.

Step 3. Fixing the number of minimal arcs (“pimples”) we fix the number of simple poles p of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

Large genus asymptotics

Conjecture on asymptotics of volume for large genera

Let $\mathbf{m} = (m_1, \dots, m_n)$ be an unordered partition of an even number $2g-2$, $|\mathbf{m}| = m_1 + \dots + m_n = 2g-2$. Denote by Π_{2g-2} the set of all partitions.

Conjecture on Asymptotics of Volumes (A. Eskin, A. Z., 2003). *For any $\mathbf{m} \in \Pi_{2g-2}$ one has*

$$\operatorname{Vol}\mathcal{H}(m_1, \dots, m_n) = \frac{4}{(m_1+1)\cdot \dots \cdot (m_n+1)} \cdot (1 + \varepsilon(\mathbf{m}))\,,$$

$$\text{where } |\varepsilon(\mathbf{m})| \le \frac{\text{const}}{\sqrt{g}}\,.$$

Theorem on asymptotics of volume for large genera

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Theorem on Asymptotics of Volumes (M. Möller, D. Zagier; parly with D. Chen; 2015–2017). *For any* $\mathbf{m} \in \Pi_{2g-2}$ *one has*

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Contribution of 1-cylinder diagrams. Equivalent conjecture

Theorem. *The contribution c_1 of 1-cylinder square-tiled surfaces to the volume $\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)$ of any nonhyperelliptic stratum of Abelian differentials satisfies*

$$\frac{\zeta(d)}{d+1}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\le c_1\le\frac{\zeta(d)}{d-\frac{10}{29}}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\,,$$

where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

Corollary. *Conjecture on volume asymptotics is (basically) equivalent to the following statement: the relative contribution of 1-cylinder square-tiled surfaces to the volume of the stratum is of the order $1/($dimension of the stratum$)$ when $g\gg 1$,*

$$d\cdot\frac{c_1(\mathcal{H}(m_1,\dots,m_n))}{\operatorname{Vol}(\mathcal{H}(m_1,\dots,m_n))}=d\cdot p_1(\mathcal{H}(m_1,\dots,m_n))\to 1\ \text{as}\ g\to+\infty\,,$$

where convergence is uniform for all strata in genus g.

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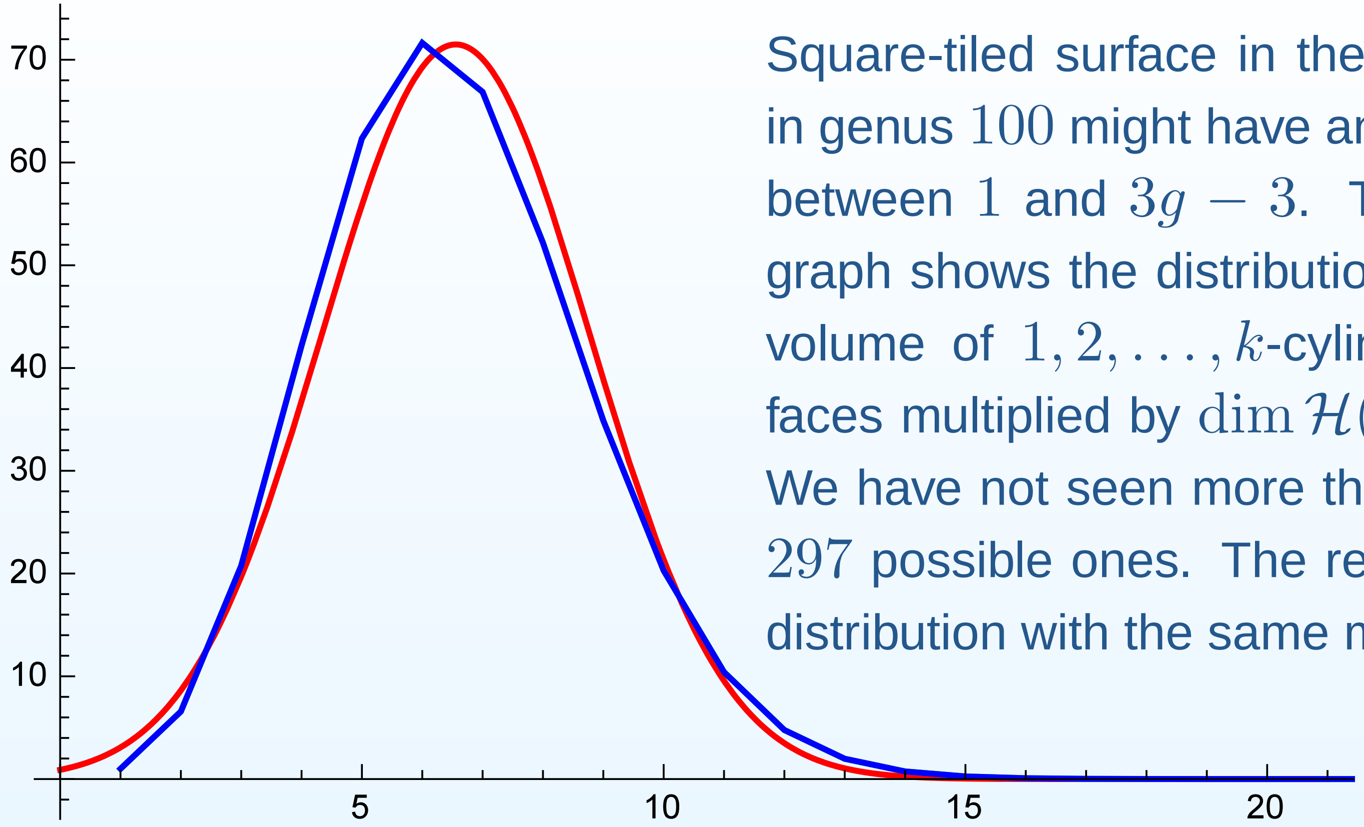
Frequencies of $1, \ldots, k$-cylinder square-tiled surfaces in large genera

Square-tiled surface in the stratum $\mathcal{H}(1, \ldots, 1)$ in genus 100 might have any number of cylinders between 1 and $3g-3$. How often a "random" square-tiled surface has $1, 2, ..., 297$-cylinders? What distribution do you expect?

Frequencies of $1, \ldots, k$-cylinder square-tiled surfaces in large genera



Square-tiled surface in the stratum $\mathcal{H}(1, \ldots, 1)$ in genus 100 might have any number of cylinders between 1 and $3g-3$. The experimental blue graph shows the distribution of the impact to the volume of $1, 2, \ldots, k$-cylinder square-tiled surfaces multiplied by $\dim \mathcal{H}(1, \ldots, 1) = 4g-3$. We have not seen more than 23 cylinders out of 297 possible ones. The red graph is the normal distribution with the same mean and variance.

Conjecture. *For any nonhyperellyptic component of a stratum of Abelian differentials, the mean of the distribution is asymptotically located at* $\mathrm{const} + \log($*dimension of the stratum*$)$, *where* const *is a universal constant.*

Suspicion. *The distribution tends to a universal limiting distribution.*

Pure speculation. *If it is true, is it some known distribution (like Tracy–Widom distribution)? Is it also related to Airy function and to Painlevé equation?*

Strata of different genera but of the same dimension.

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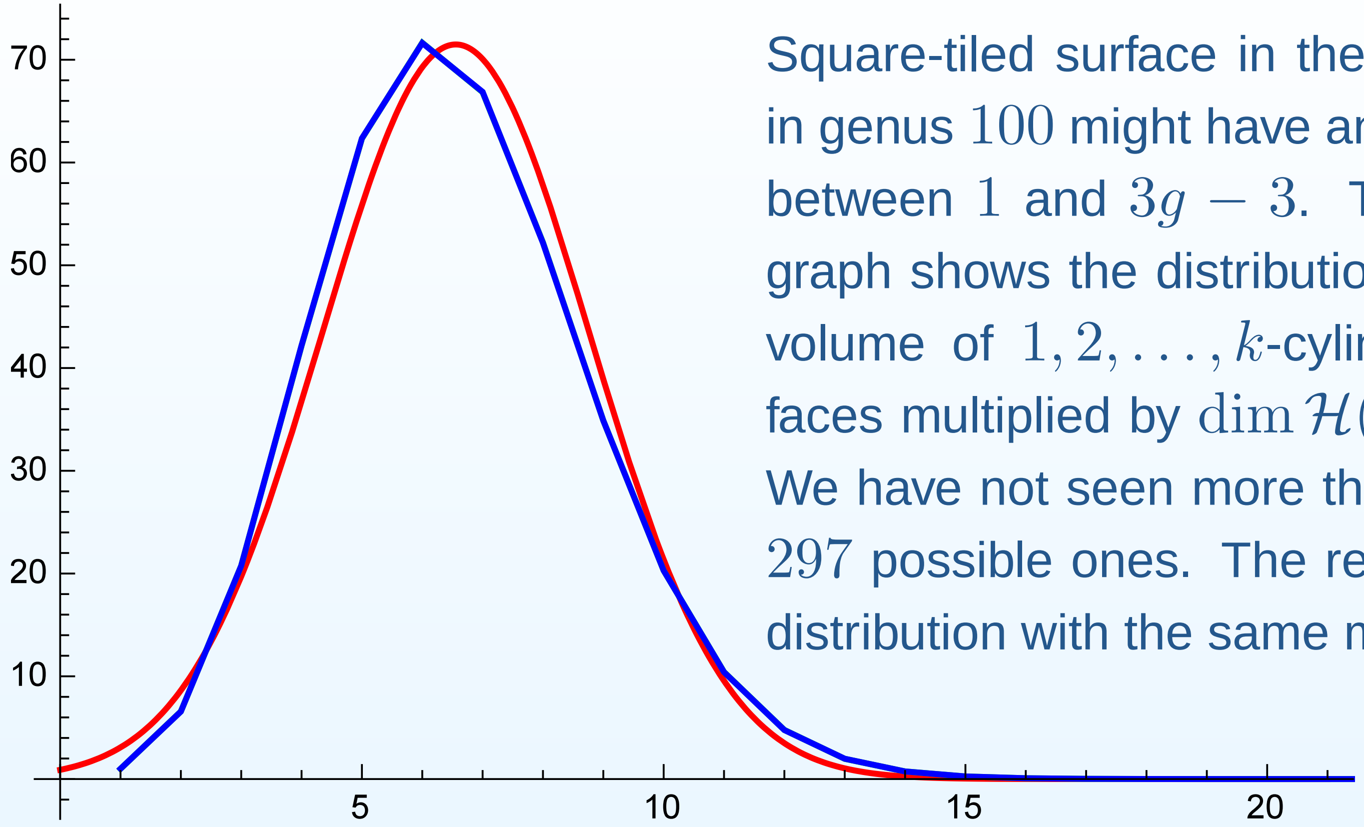

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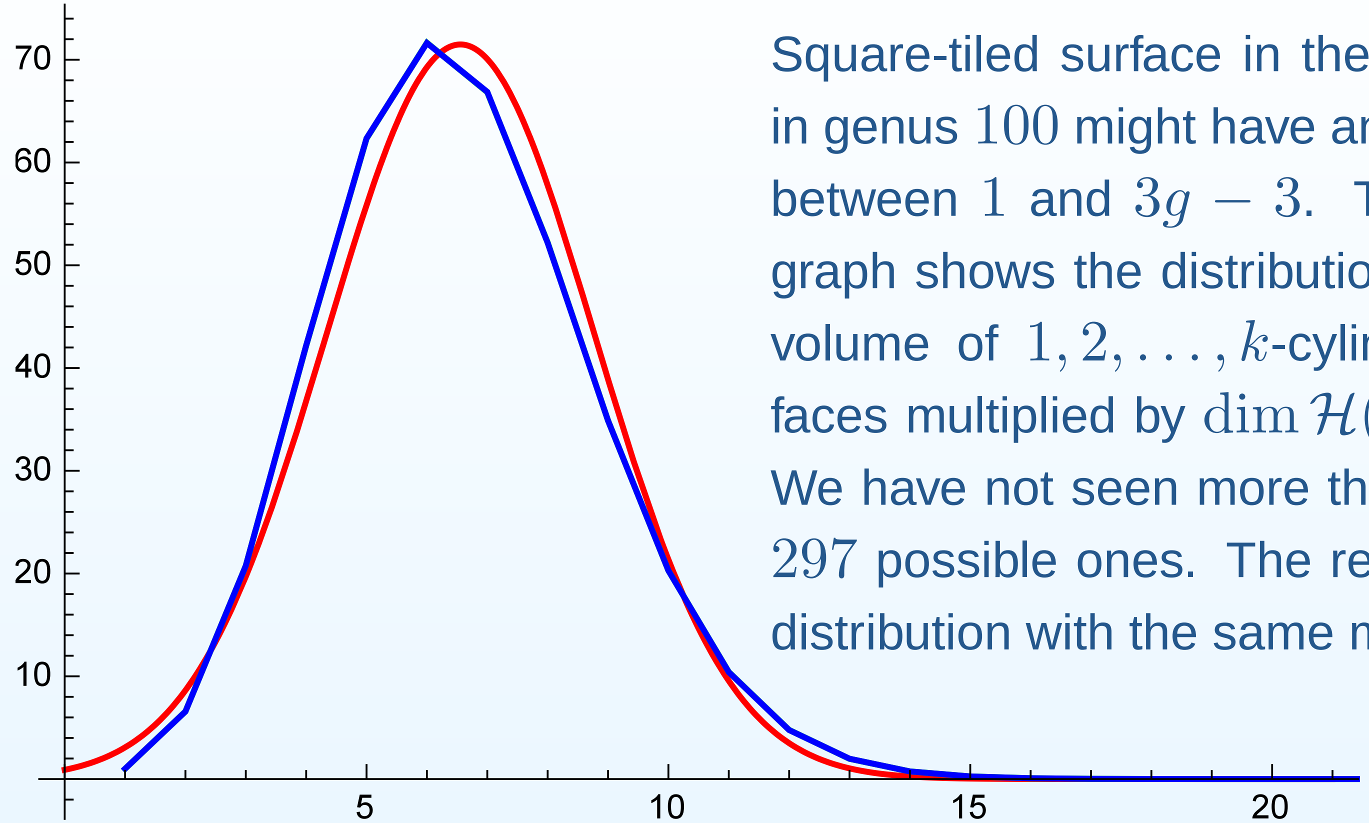

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Masur–Veech volumes and Mirzakhani–Weil–Petersson volumes

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.

Define the homogeneous polynomial $N_{g,n}(b_1^2, \ldots, b_n^2)$ of degree $3g-3+n$ in variables $b_1^2, \ldots, b_n^2$:

$$N_{g,n}(b_1^2, \ldots, b_n^2) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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The polynomial $N_{g,n}(b_1^2, \dots, b_n^2)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $\frac{1}{2} V_{g,n}(b_1^2, \dots, b_n^2)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces.

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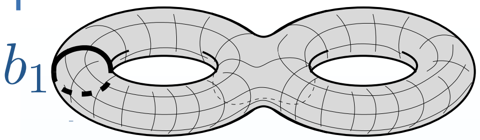
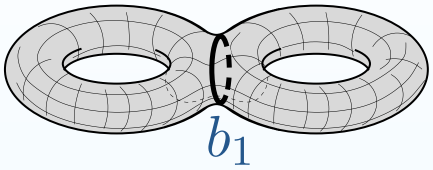
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

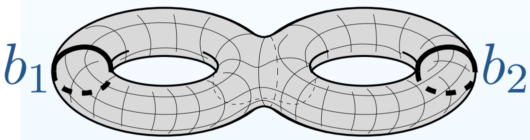
and extend it to symmetric polynomials in b_i by linearity.



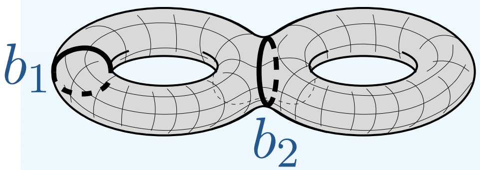
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



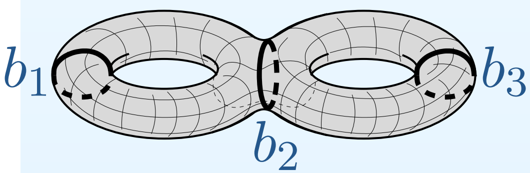
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



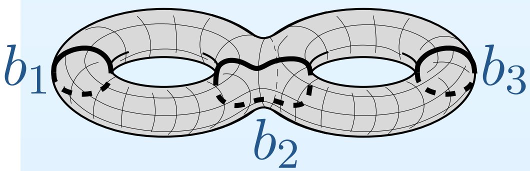
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



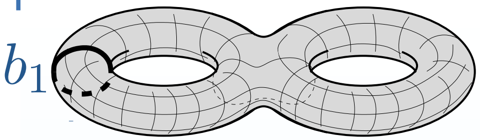
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



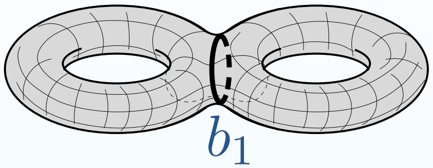
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



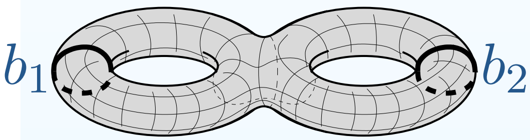
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$



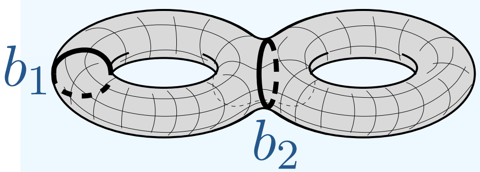
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) \qquad = \frac{1}{2} \cdot b_1 \left(\frac{1}{384}(2b_1^2)(2b_1^2) \right)$$



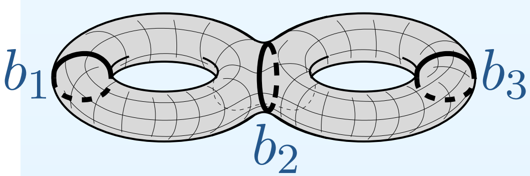
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) \qquad = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$



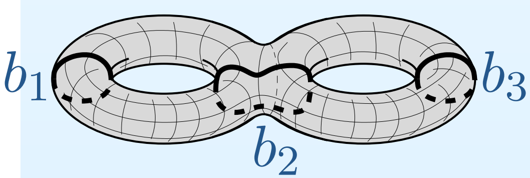
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) \qquad = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4}(2b_1^2 + 2b_2^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) \qquad = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$



$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) \qquad = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$



$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) \qquad = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

Formula for the volume

Theorem. *The Masur–Veech volume $\operatorname{Vol}\mathcal{Q}_g$ of the moduli space of holomorphic quadratic differentials has the following value:*

$$\operatorname{Vol}\mathcal{Q}_g = $$

$$= \frac{(4g-4)!}{(6g-6)!}\cdot 2^{6g-6}\cdot(12g-12)\sum_{k=1}^{3g-3}\sum_{m=1}^{2g-2}\sum_{\substack{\text{Decompositions}\\ \text{with } k \text{ cuts}\\ \text{and } m \text{ subsurfaces}}} \frac{1}{|\operatorname{Aut}|}\cdot\frac{1}{2^{m-1}}\cdot$$

$$\cdot\,\mathcal{Z}\left(b_1\cdots b_k\prod_{i=1}^{k} N_{g_i,n_i}(b_{j_1},\dots,b_{j_i}\text{adjacent to } i\text{-th surface})\right).$$

The partial sum for fixed number k of cuts gives the contribution of k-cylinder pillowcase covers.

Remark. The Weil–Peterson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials, that is we use them opposite regime when the lengths of all boundary components tend to infinity.

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$$\cdot \mathcal{Z}\left(b_1 \cdots b_k \prod_{i=1}^{k} N_{g_i,n_i}(b_{j_1}, \dots, b_{j_i} \text{adjacent to } i\text{-th surface}) \right).$$

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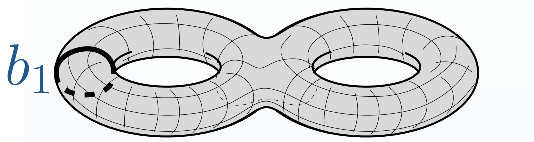
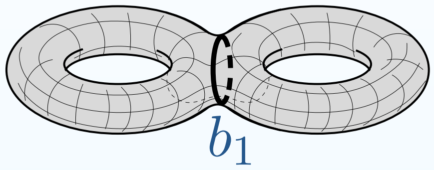
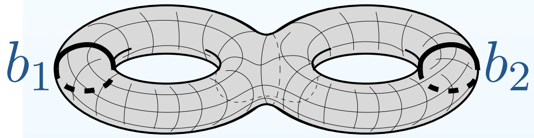
Volume of $\mathcal{Q}_2$



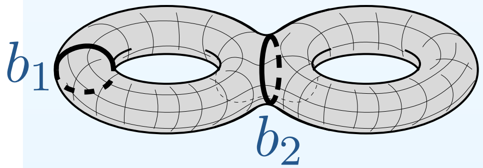
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



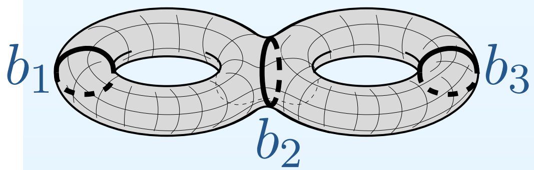
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



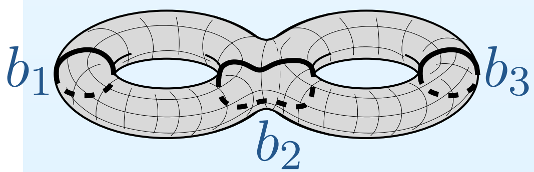
$$\frac{1}{16}(b_1^3 b_2 + + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



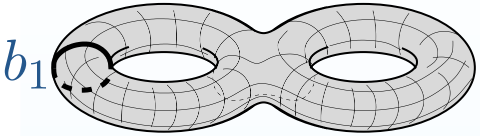
$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



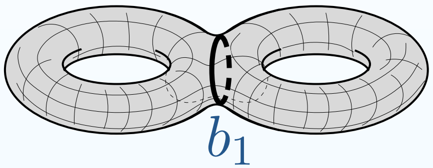
$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\Big(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\Big)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

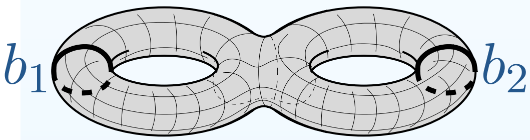
Volume of $\mathcal{Q}_2$



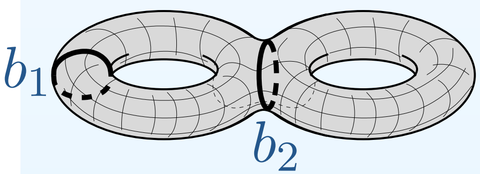
$$\frac{1}{192} \cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192} \cdot \big(5! \cdot \zeta(6)\big) = \frac{1}{1512} \cdot \pi^6$$



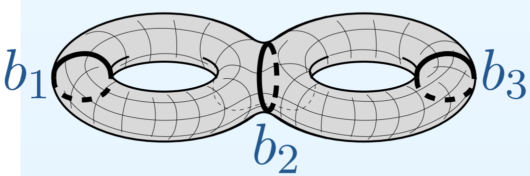
$$\frac{1}{9216} \cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216} \cdot \big(5! \cdot \zeta(6)\big) = \frac{1}{72576} \cdot \pi^6$$



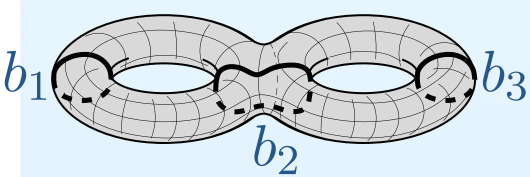
$$\frac{1}{16}(b_1^3 b_2 + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16} \cdot 2\big(1! \cdot \zeta(2)\big) \cdot \big(3! \cdot \zeta(4)\big) = \frac{1}{720} \cdot \pi^6$$



$$\frac{1}{192} \cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192} \cdot \big(1! \cdot \zeta(2)\big) \cdot \big(3! \cdot \zeta(4)\big) = \frac{1}{17280} \cdot \pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16} \cdot \big(1! \cdot \zeta(2)\big)^3 = \frac{1}{3456} \cdot \pi^6$$



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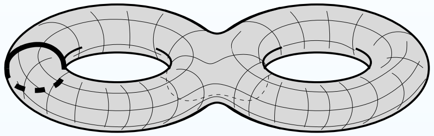
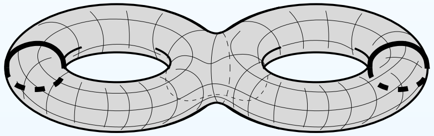
$$\operatorname{Vol} \mathcal{Q}_2 = \frac{128}{5} \cdot \Big(\frac{1}{1512} + \frac{1}{72576} + \frac{1}{720} + \frac{1}{17280} + \frac{1}{3456} + \frac{1}{5184}\Big) \cdot \pi^6 = \frac{1}{15}\pi^6 .$$

Simplest decompositions: just cut k out of g handles



$\mathbb{G}_1$

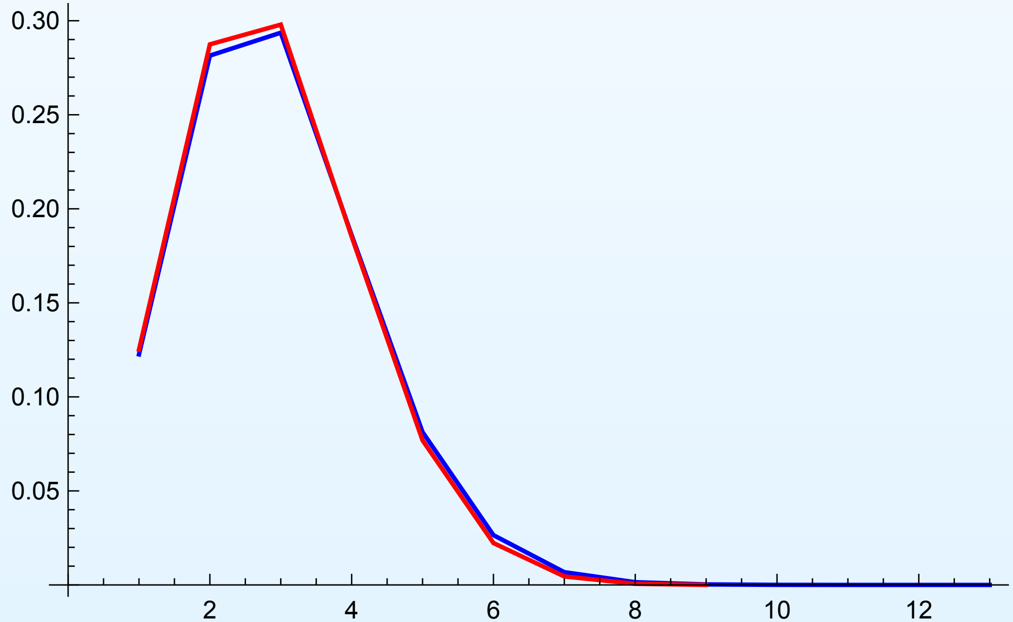


$\mathbb{G}_2$

Conjectural asymptotics of $\operatorname{Vol}\mathcal{Q}_g$ for large genus

Conjecture. *When genus g tends to infinity, the following properties hold:*

- *For every number $1 \le k \le g$ of cuts the contribution $\operatorname{Vol}(\mathbb{G}_k)$ of the partition which chops k distinct handles dominates the contribution of k-cylinder pillowcase covers to the volume $\operatorname{Vol}\mathcal{Q}_g$.*
- $\lim_{g\to+\infty} \frac{\sum_{k=1}^{g}\operatorname{Vol}(\mathbb{G}_k)}{\operatorname{Vol}\mathcal{Q}_g} = 1\,.$
- $\operatorname{Vol}\mathcal{Q}_g = const \cdot \left(\frac{8}{3}\right)^{4g-4}\left(1+O\left(\frac{1}{g}\right)\right)$, where $const \approx 0.8\,.$



Genus $g=9$. Relative contributions of *all* k-cylinder pillowcase covers in blue and $\operatorname{Vol}(\mathbb{G}_k)/\left(\sum_{k=1}^{9}\operatorname{Vol}(\mathbb{G}_k)\right)$ for $k=1,\dots,9$ in red.