Picture by François Labourie taken at CIRM

Flat and hyperbolic enumerative geometry

Anton Zorich (with V. Delecroix, E. Goujard and P. Zograf)

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in honour of Maryam Mirzakhani

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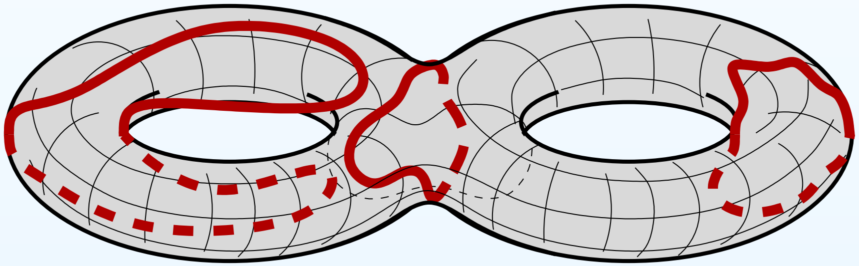
Mirzakhani's count of simple closed geodesics

Multicurves

Consider a finite collection of pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g.n}$ of genus g with n punctures.

For any hyperbolic metric X on $S_{g,n}$ and for any simple closed curve γ_i there exists a unique geodesic representative in the free homotopy class of γ_i.



Fact. *For any hyperbolic metric X and any collection $\gamma_1, \ldots, \gamma_k$ of pairwise non-intersecting simple closed curves, their geodesic representatives do not self-intersect and do not pairwise intersect either.*

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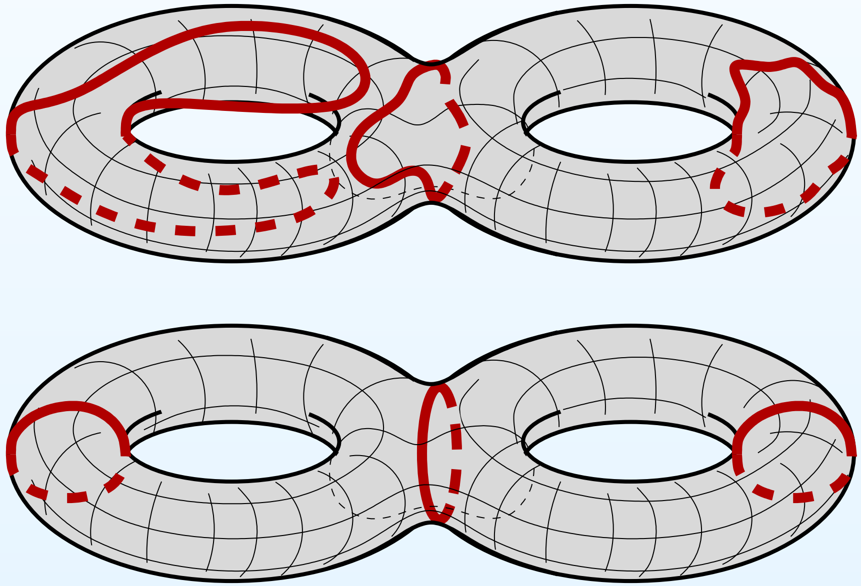

Fact. *For any hyperbolic metric X and any collection $\gamma_1, \ldots, \gamma_k$ of pairwise non-intersecting simple closed curves, their geodesic representatives do not self-intersect and do not pairwise intersect either.*

Multicurves

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

We say that two multicurves γ, ρ have the same *topological type* $[\gamma] = [\rho]$ *if and only if they belong to the same orbit of the mapping class group:* $\rho \in \mathrm{Mod}_{g,n} \cdot \gamma$.

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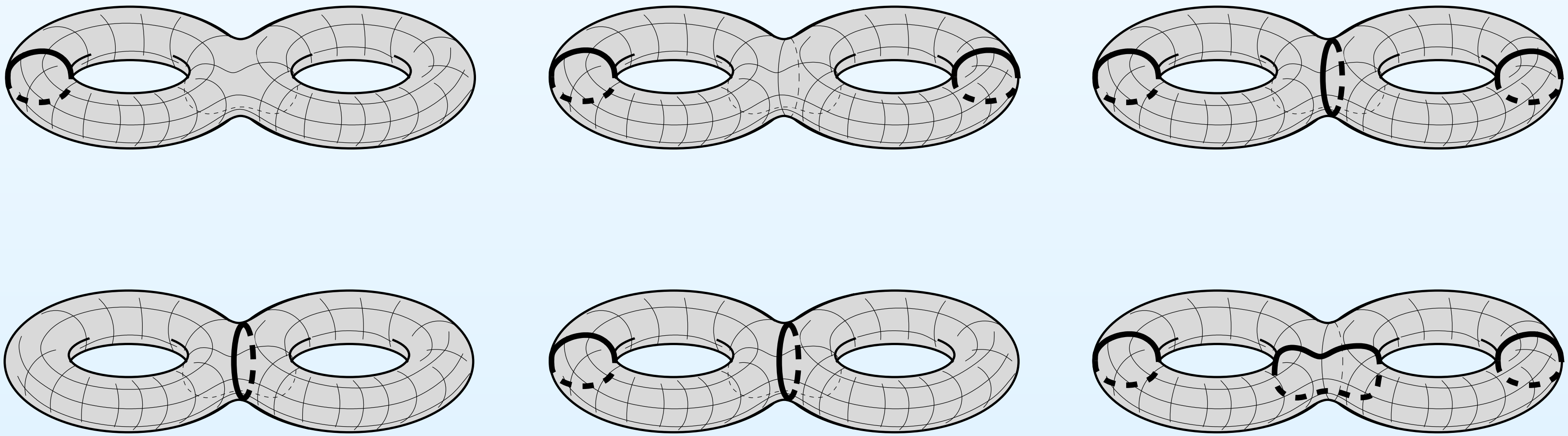
Example: primitive multicurves on a surface of genus two

The picture below illustrates all possible types of primitive multicurves on a surface of genus two without punctures.

Note that contracting all components of a multicurve we get a “stable curve” — a Riemann surface degenerated in one of the several regular ways. In this way the “topological types of primitive multicurves” on a smooth surface $S_{g,n}$ of genus g with n punctures are in the natural bijective correspondence with boundary divisors of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of pointed complex curves.



Frequencies of multicurves

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *the number* $s_X(L,\gamma)$ *of simple closed geodesic multicurves on* X *of topological type* $[\gamma]$ *and of hyperbolic length at most* L *has the following asymtotocs:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

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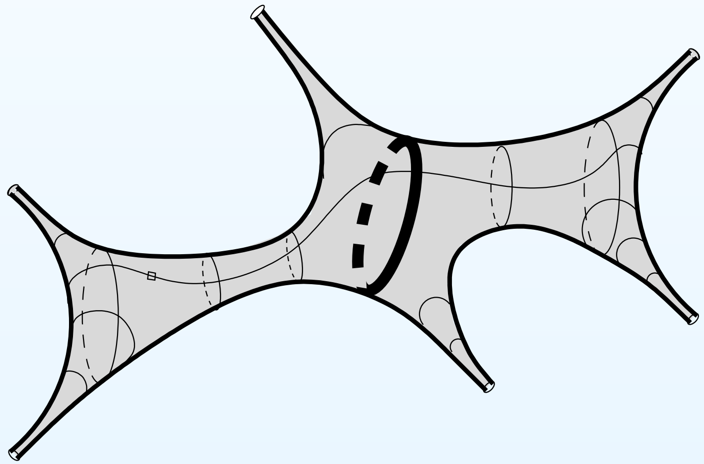
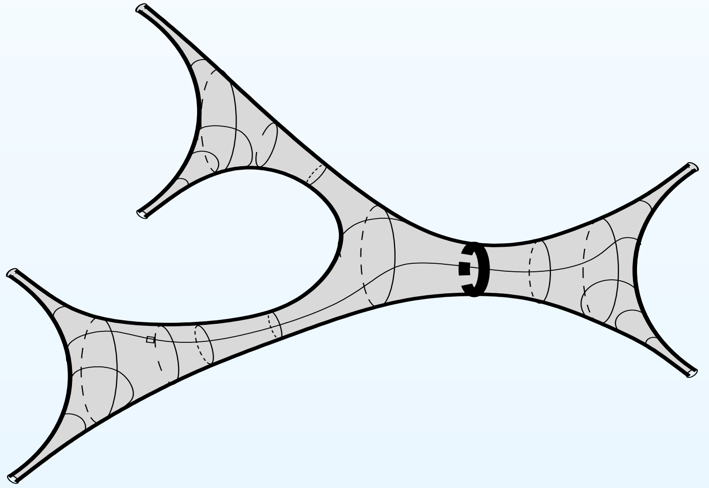
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$*, and any two rational multicurves* γ_1, γ_2 *on a smooth surface* $S_{g,n}$ *considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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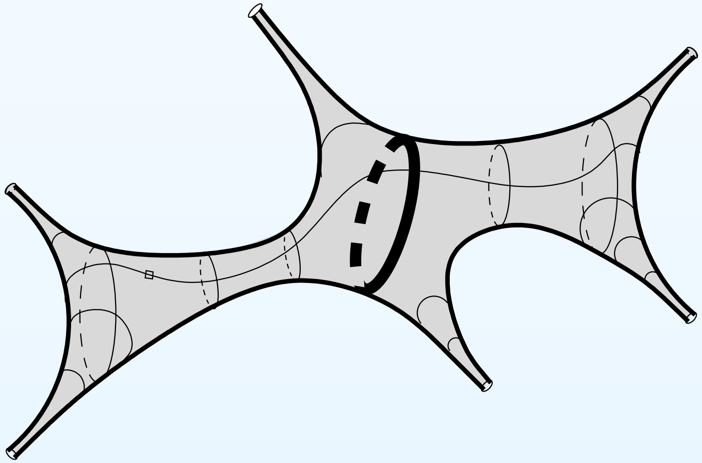

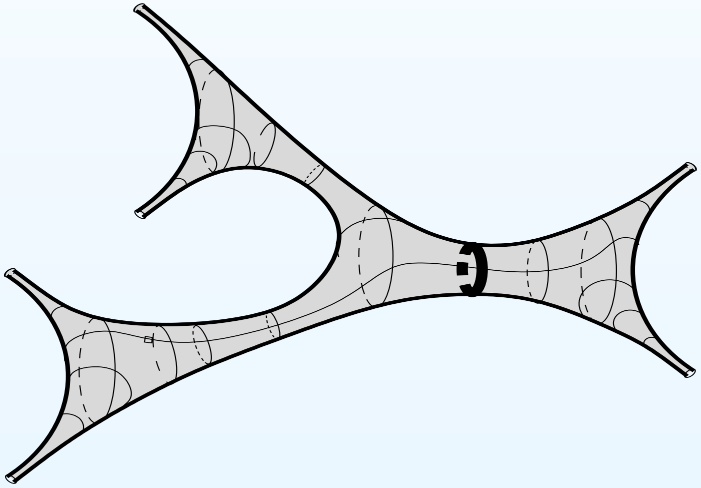

Example (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell and S. Schleimer; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

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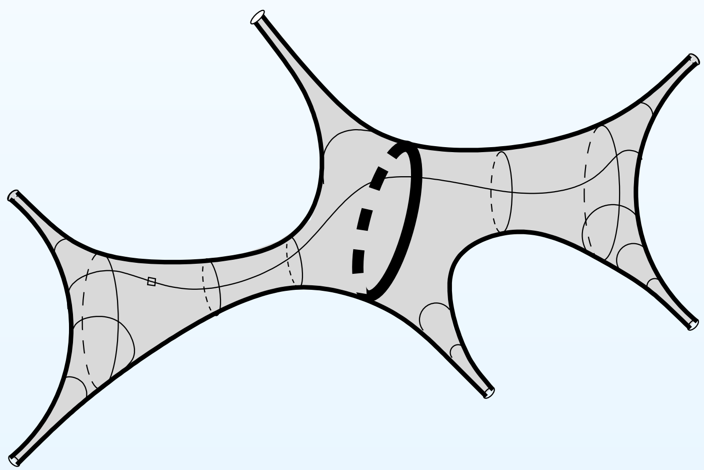

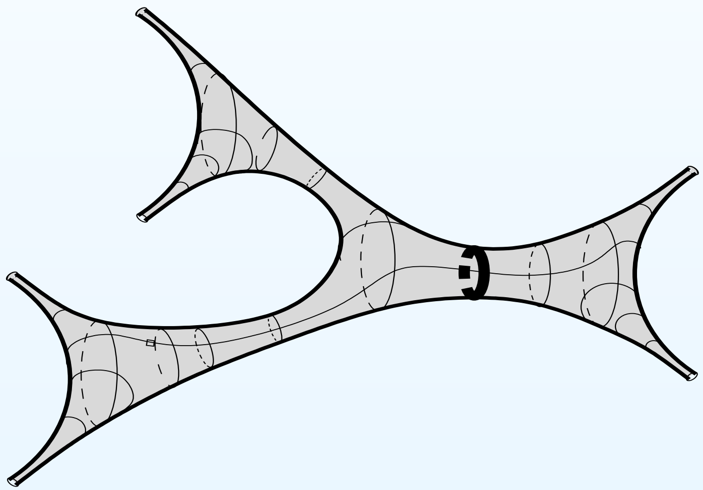

In this sense one can say that for any hyperbolic metric X on a sphere with 6 cusps, a long simple closed geodesic separates the cusps as $(3+3)$ with probability $\frac{4}{7}$ and as $(2+4)$ with probability $\frac{3}{7}$.

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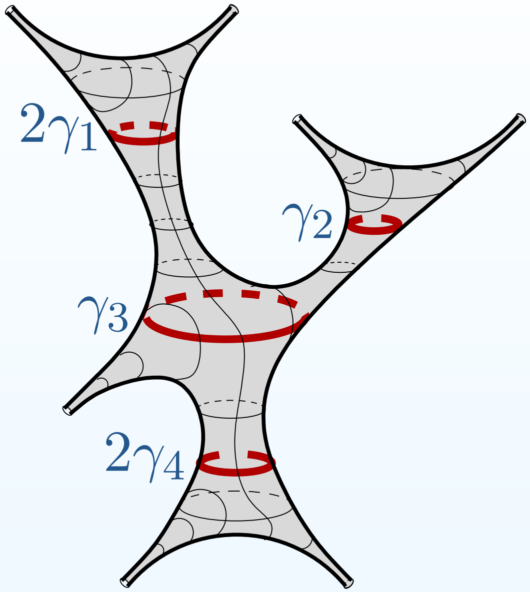
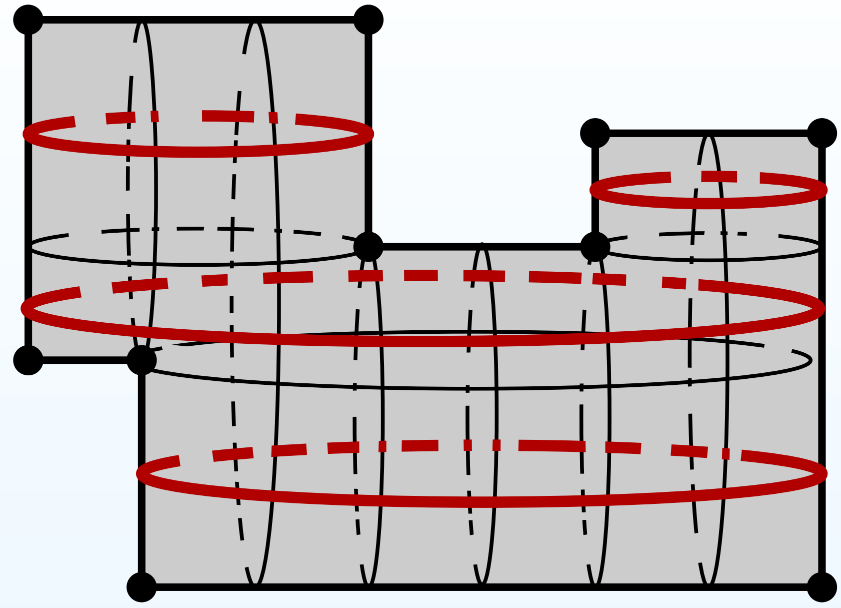
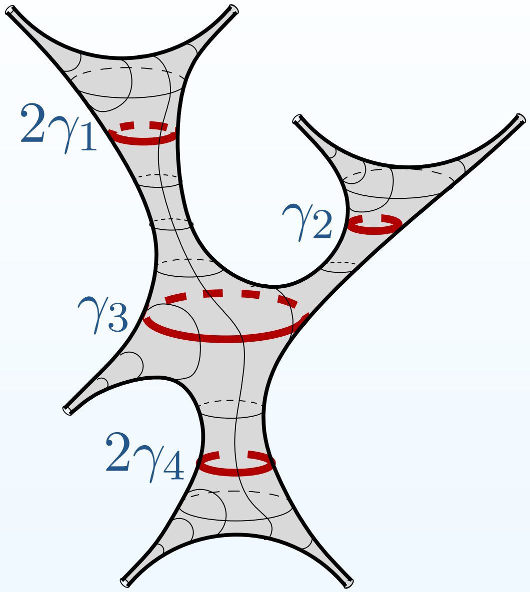
Hyperbolic and flat geodesic multicurves

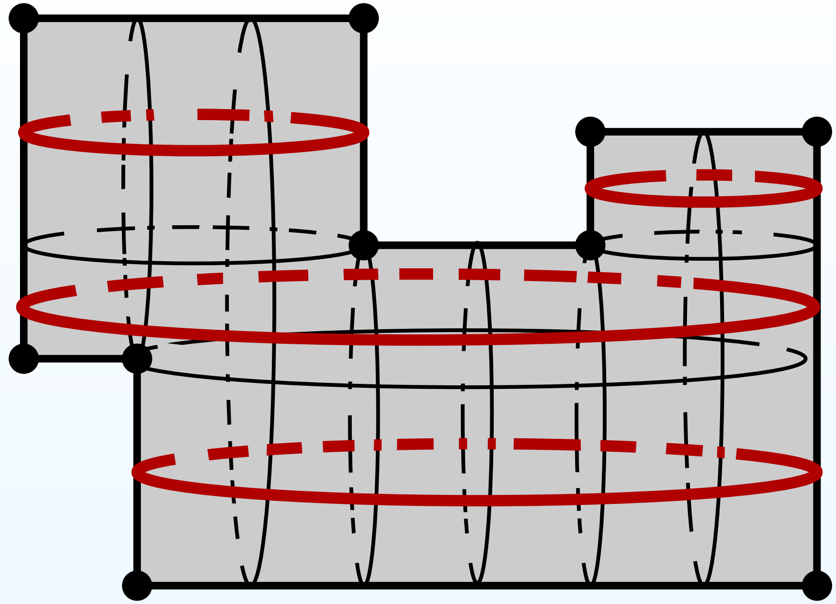




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves





Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of **hyperbolic** multicurves coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Masur–Veech versus Weil–Petersson volume

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \ldots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\, \mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b)\,.$$

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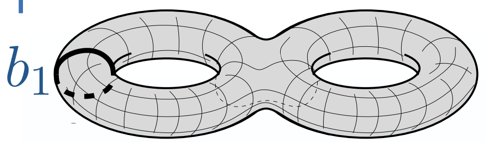
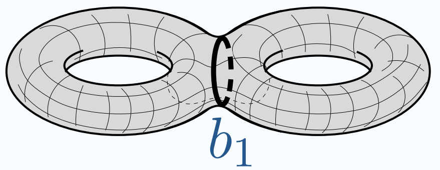
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

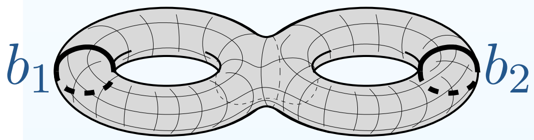
and extend it to symmetric polynomials in b_i by linearity.



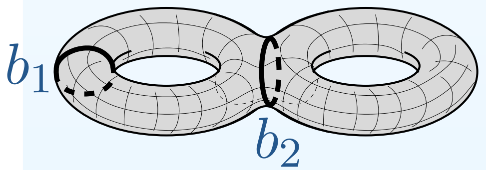
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



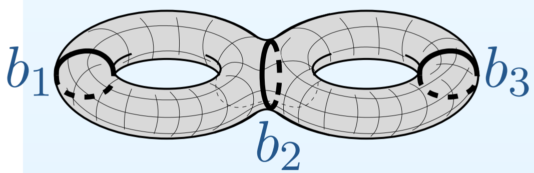
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



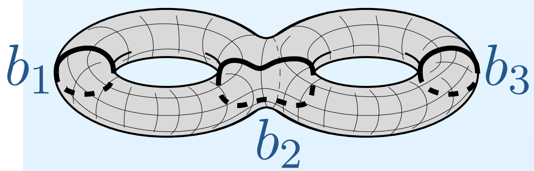
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



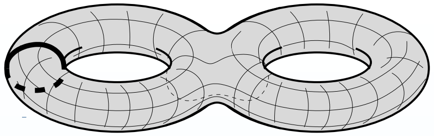
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



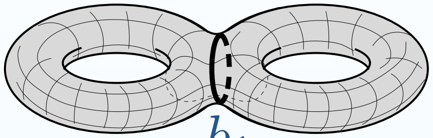
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



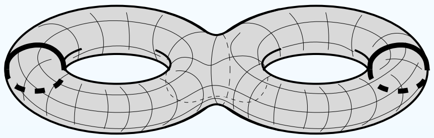
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$



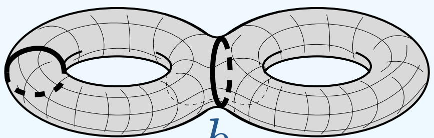
$$\textcolor{red}{\tfrac{1}{2}} \cdot \textcolor{blue}{1} \cdot b_1 \cdot N_{1,2}(b_1, b_1) = \tfrac{1}{2} \cdot b_1 \left(\tfrac{1}{384}(2b_1^2)(2b_1^2) \right)$$



$$\textcolor{red}{\tfrac{1}{2}} \cdot \textcolor{blue}{\tfrac{1}{2}} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) = \tfrac{1}{4} \cdot b_1 \left(\tfrac{1}{48} b_1^2 \right) \left(\tfrac{1}{48} b_1^2 \right)$$

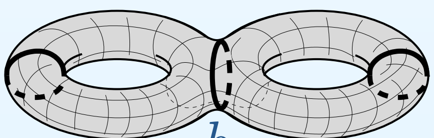


$$\textcolor{red}{\tfrac{1}{8}} \cdot \textcolor{blue}{1} \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) = \tfrac{1}{8} \cdot b_1 b_2 \cdot \left(\tfrac{1}{4}(2b_1^2 + 2b_2^2) \right)$$



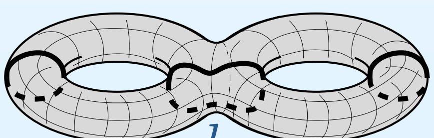
$$\textcolor{red}{\tfrac{1}{2}} \cdot \textcolor{blue}{\tfrac{1}{2}} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

$$\cdot N_{1,1}(b_2) = \tfrac{1}{4} \cdot b_1 b_2 \cdot \left(1 \right) \cdot \left(\tfrac{1}{48} b_2^2 \right)$$



$$\textcolor{red}{\tfrac{1}{8}} \cdot \textcolor{blue}{\tfrac{1}{2}} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

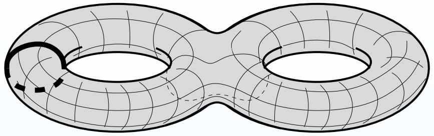
$$\cdot N_{0,3}(b_2, b_3, b_3) = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$



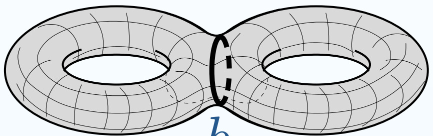
$$\textcolor{red}{\frac{1}{12}} \cdot \textcolor{blue}{\frac{1}{2}} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$

$$\cdot N_{0,3}(b_1, b_2, b_3) = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

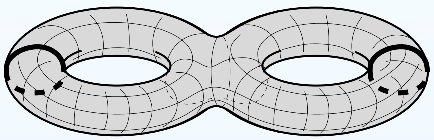
Volume of $\mathcal{Q}_2$

b_1 

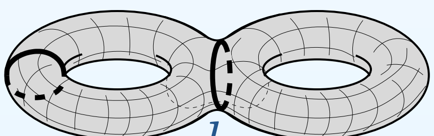
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$


b_1

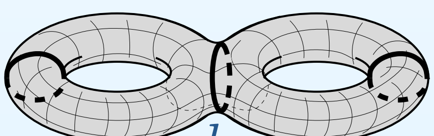
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$

b_1  b_2

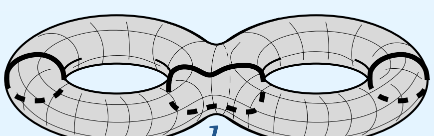
$$\frac{1}{16}(b_1^3 b_2 + +b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$

b_1 
b_2

$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$

b_1  b_3
b_2

$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$

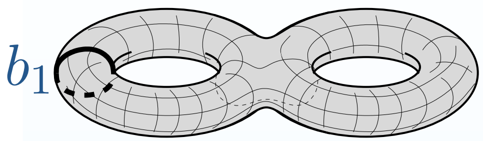
b_1  b_3
b_2

$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

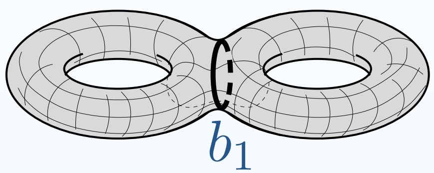
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6 .$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

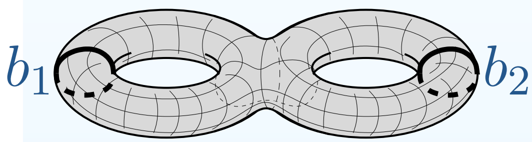
Volume of $\mathcal{Q}_2$



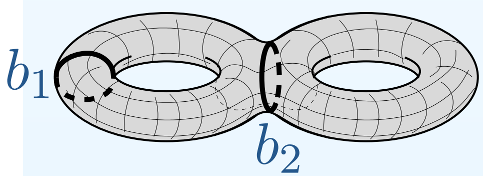
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



$$\frac{1}{16}(b_1^3 b_2 + {} + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



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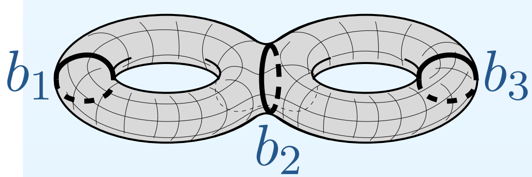

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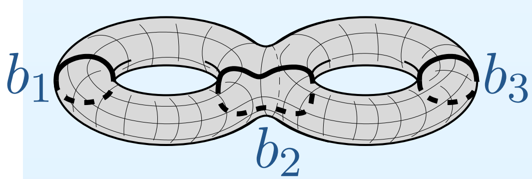

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Volume of $\mathcal{Q}_{g,n}$

Theorem. (Delecroix, Goujard, Zograf, Zorich) *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs } \Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of } \Gamma - 1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k *of edges gives the contribution of* k*-cylinder square-tiled surfaces.*

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Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

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When any of g, n grow, the number of graphs grows very fast. Also the correlators are computed only inductively. Thus, the formula gives effective answer only in a limited number of cases.

Miracles: two exceptions which admit simple closed answer: when $g=0$ and n is arbitrary (rigorous); when $g\gg 1$ and $n=0$ (conjectural asymptotic value).

Remark. A. Eskin conceptually knew this 20 years ago but has chosen other way...

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M. Mirzakhani defined the Masur–Veech measure on $\mathcal{Q}_g$ as pushforward measure from $\mathcal{M}_g\times\mathcal{ML}(X)$. Then the volume of $\mathcal{Q}_g$ is the average volume of the “unit ball” $B_X=\{\lambda\in\mathcal{ML}(X) : \ell_X(\lambda)\le 1\}$ measured in Thurston’s measure. The relation to standard normalization remained, however, unknown.

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Maryam’s counting of volumes. We measure areas of figures in $\mathbb{R}^2$ using $\mathbb{Z}^2$-lattice. We could use points with coprime integer coordinates as a *lattice*. The area of a unit disk would be $\frac{6}{\pi}$ and the area of a unit square would be $\frac{6}{\pi^2}$.

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Theorem (Delecroix, Goujard, Zograf, Zorich; independent proof by Herrera; another independent proof by Monin and Telpukhovskiy)

$$\operatorname{Vol}\mathcal{Q}_{g,n} = 2\cdot(6g-6+2n)\cdot(4g-4+n)!\cdot 2^{4g-3+n}\cdot\int_{\mathcal{M}_{g,n}}\mu_{\mathrm{Th}}(B_X)\,dX\,.$$

This factor (analogous to $\frac{6}{\pi^2}$ for the density lattice coprime integers) is the last missing element in item 5 of Maryam’s 2003 research program.

Idea of the proof: Kontsevich's count of metric ribbon graphs

Each horizontal layer containing zeroes or poles of a square-tiled surface can be seen as a metric ribbon graph. When the associate quadratic differential has only simple zeroes, the metric ribbon graph is trivalent.

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Theorem (Kontsevich). *Consider a collection of positive integers $b_1, \dots, b_n$ such that $\sum_{i=1}^n b_i$ is even. The weighted count of genus g connected trivalent metric ribbon graphs Γ with integer edges and with n labeled boundary components of lengths $b_1, \dots, b_n$ is equal to $N_{g,n}(b_1, \dots, b_n)$ up to the lower order terms:*

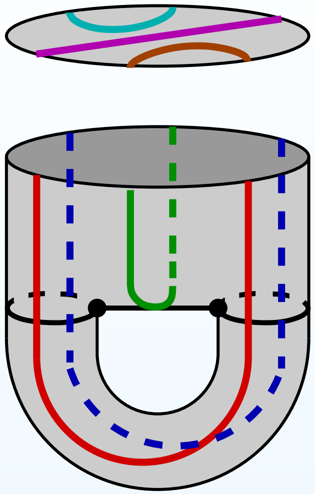
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \dots, b_n) = N_{g,n}(b_1, \dots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ denote the set of (nonisomorphic) trivalent Ribbon graphs Γ of genus g and with n boundary components.

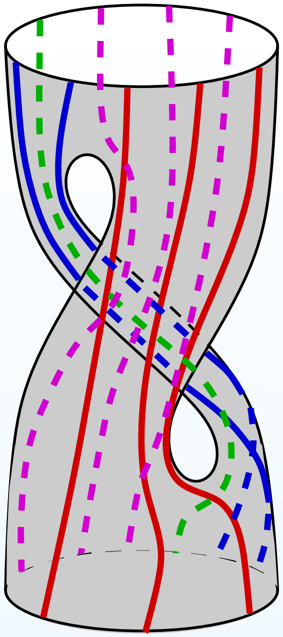
This Theorem is an important part of Kontsevich's proof of Witten's conjecture.

From braids on surfaces to multicurves

Arc systems and braids on surfaces



Arc system with two bigons

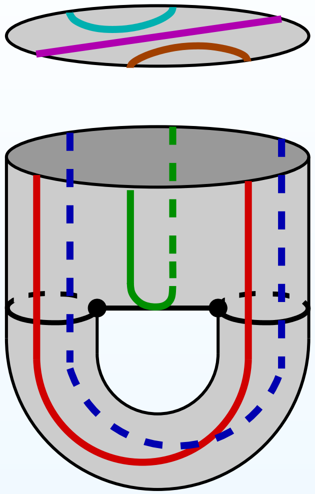


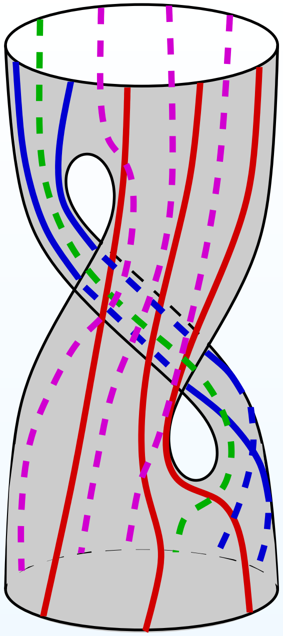
Braid

Arc system. Consider a smooth surface with two boundary components. Draw N non selfintersecting and non pairwise intersecting strands on the surface placing all endpoints at the boundary in such way that there are N endpoints at one component and N endpoints at the other.

Braid. If the surface is connected, we can let all the strands go from one component to another. These kind of braids mimic gradient flow from one regular level of a Morse function on a surface to another regular level.

From arc systems and braids to multicurves





Identifying pairs of boundary components in a way which matches the endpoints of the strands on each side we get a multicurve on a closed surface.

Question 1. *With what probability the resulting multicurve chops the surface into* $j = 1, 2, \ldots, 2g - 2$ *connected components?*

Question 2. *With what probability the resulting multicurve has* $k = 1, 2, \ldots, 3g - 3$ *(primitive) connected components? How often simple closed curves show up?*

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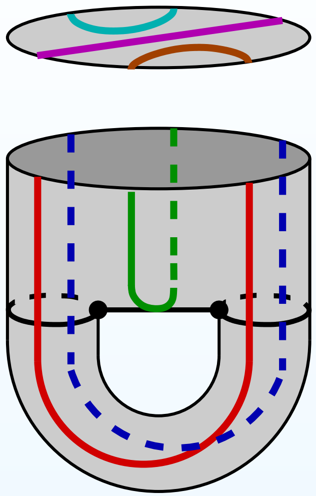

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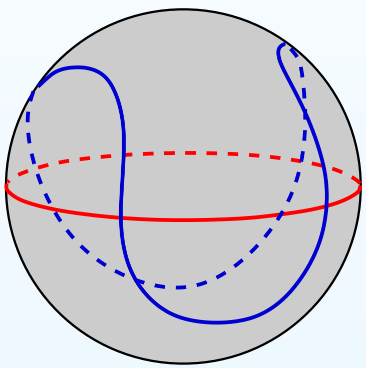
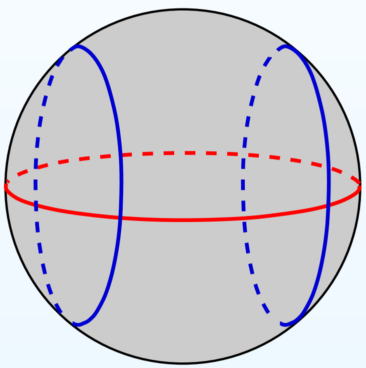
Theorem. *The corresponding probabilities are exactly the same as the frequencies of "unconditional" multicurves considered before. They depend only on the parameters* g *and* p *of the resulting surface* $S_{g,p}$.

Matching arc systems on a pair of discs

The problem is quite meaningful even for a pair of arc systems on a pair of discs. Depending on the twist chosen to identify a pair of arc systems we get a multicurve with one or several connected components.





Fix the numbers of bigons $p_1 \geq 2$ and $p_2 \geq 2$ on each of the discs.

Theorem. *The frequency $\mathrm{P}_1(p_1, p_2; N)$ of simple closed curves obtained by all possible identifications of all arc systems with at most N arcs and having p_1 bigons on one disc and with p_2 bigons on the other disc has the following limit:*

$$\lim_{N\to+\infty} \mathrm{P}_1(p_1, p_2; N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2} \qquad \text{where } p = p_1 + p_2\,.$$

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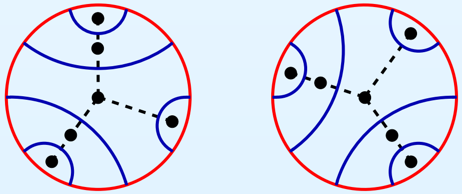
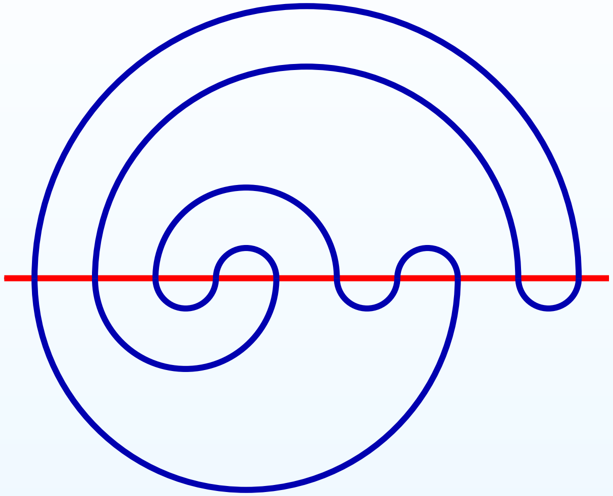
$$\lim_{N\to+\infty} \mathrm{P}_1(2,4;N) = \mathrm{P}_1(3,3;N) = \frac{280}{\pi^6} \approx 0.291245$$



Application: count of meanders



A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line. The notion “meander” was suggested by Arnold though meanders were studied already by Poincaré. They appear in various contexts, in particular in physics (P. Di Francesco, O. Golinelli, E. Guitter).

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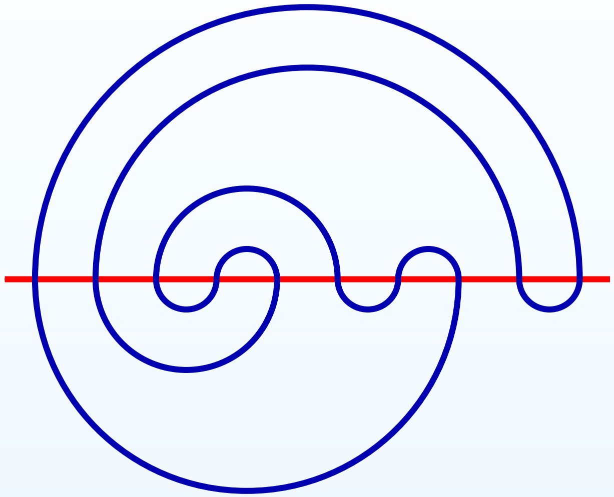

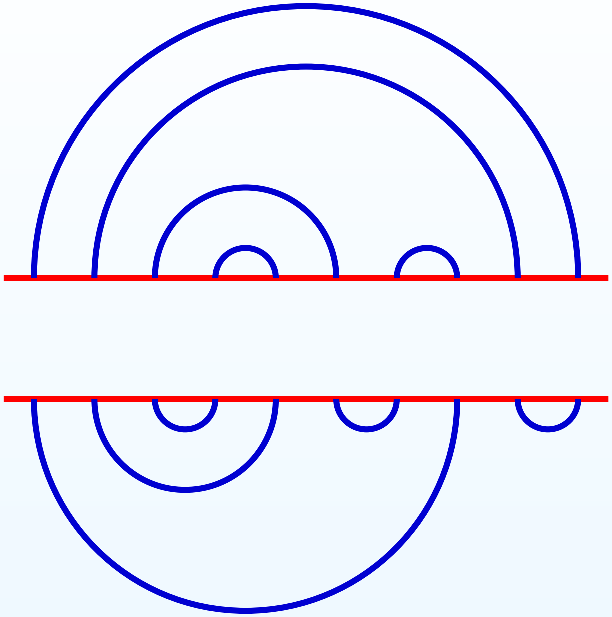

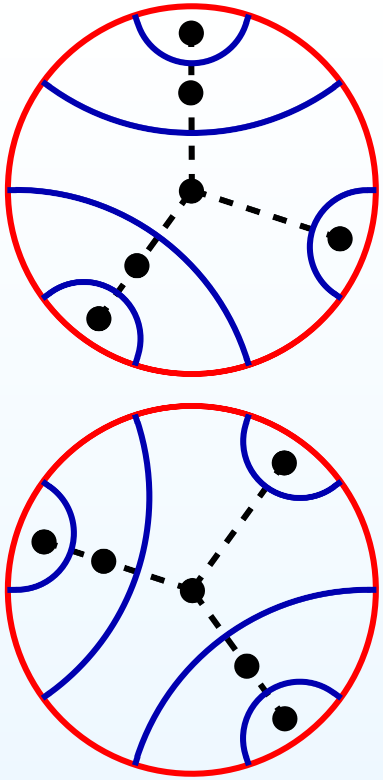

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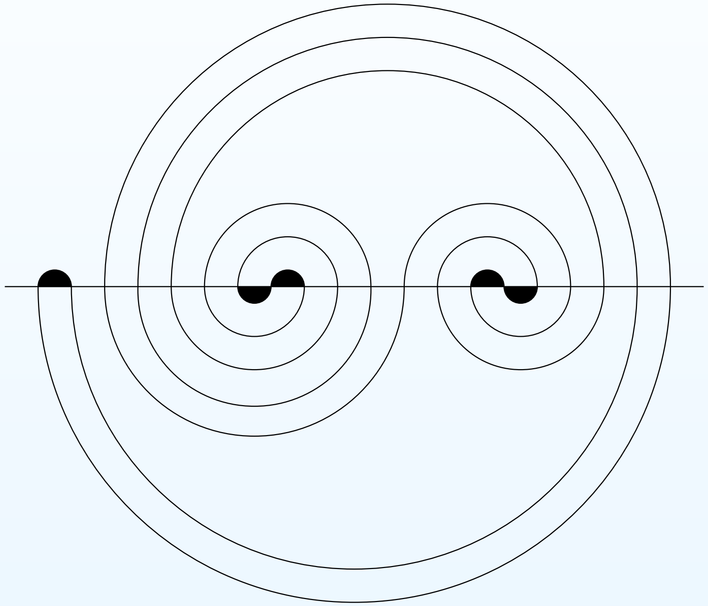
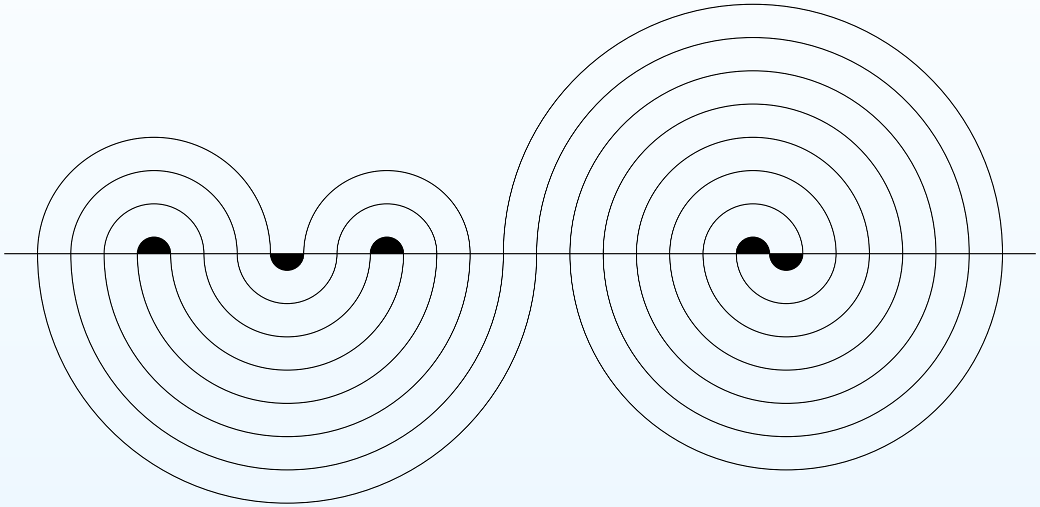
Each meander defines a pair of arc systems on a disc. It is easy to count arc systems on a disc. Thus, the frequency of connected curves (versus multicurves) obtained after gluing a pair of arc systems provides us with the asymptotic count of meanders with fixed number p of bigons.

Meanders with and without maximal arc

These two meanders have 5 minimal arcs (“pimples”) each.



Meander with a maximal arc (“rainbow”) contributes to $\mathcal{M}_5^+(N)$



Meander without maximal arc contributes to $\mathcal{M}_5^-(N)$

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc (“rainbow”) and having at most $2N$ crossings with the horizontal line and exactly p minimal arcs (“pimples”). We consider p as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

Counting formulae for meanders

Theorem. *For any fixed p the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with p minimal arcs (pimples) and with at most $2N$ crossings have the following asymtotics as $N \to +\infty$:*

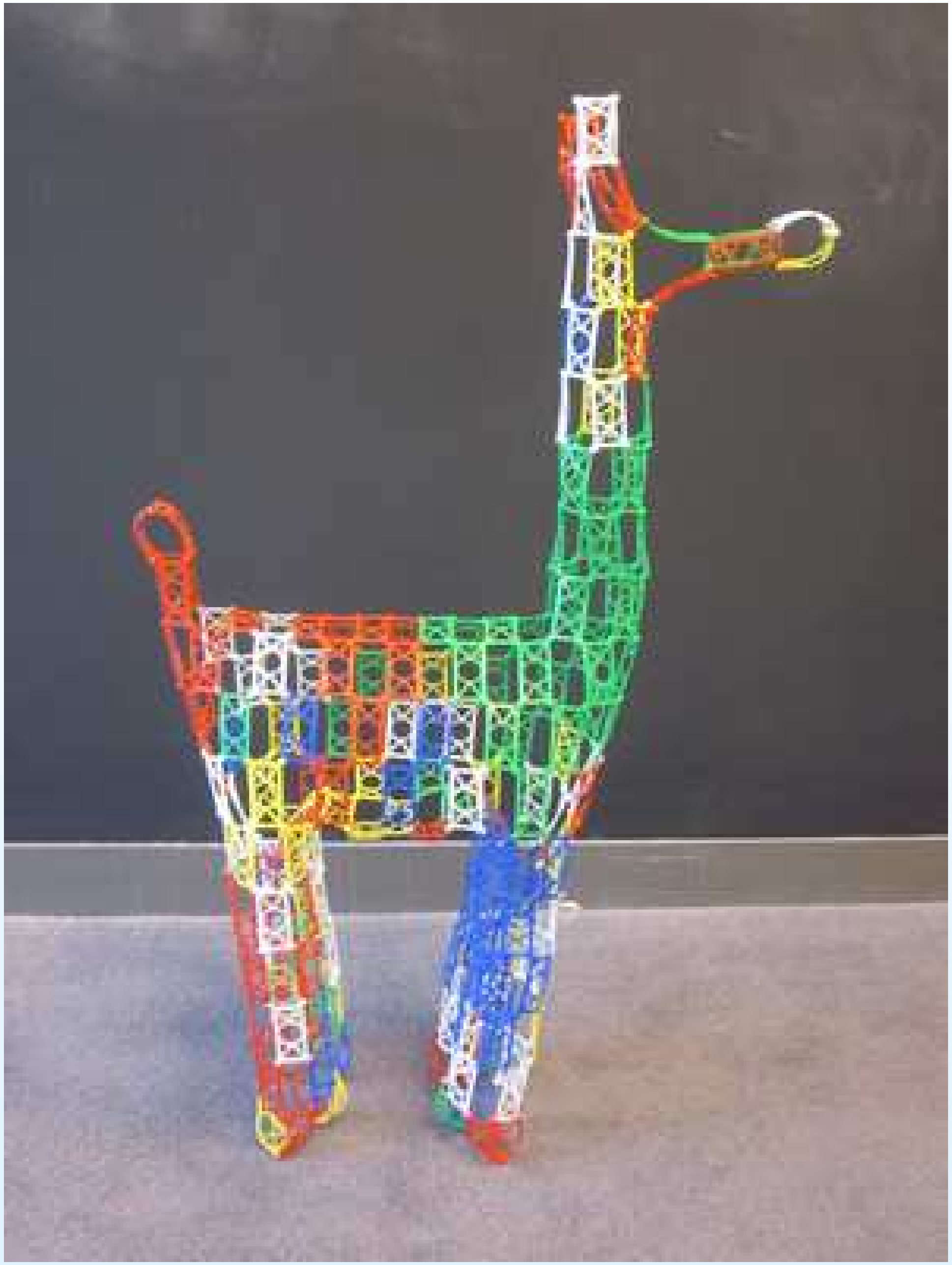
$$\mathcal{M}_p^+(N) = \frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4})\,.$$

$$\mathcal{M}_p^-(N) = \frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,.$$

Note that $\mathcal{M}_p^+(N)$ grows as N^{2p-4} while $\mathcal{M}_p^-(N)$ grows as N^{2p-5}.

Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials

Counting square-tiled surfaces



Brief history of evaluation of volumes

Theorem (A. Eskin, A. Okounkov, R. Pandharipande). *For every connected component $\mathcal{H}^c(d_1, \dots, d_n)$ of every stratum, the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled}\\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form. The Masur–Veech volume of every connected component of every stratum is a rational multiple of π^{2g}, where g is the genus.

A. Eskin implemented this theorem (around 2002) to an algorithm allowing to compute volumes for all strata up to genus 10 and for some strata (like the principal one) up to genus 200. Based on these calculations we developed a conjecture on a very simple asymptotic formula for volumes in large genera.

D. Chen, M. Möller, D. Zagier can compute the volume of the principal stratum up to genus 2000 and more; in 2017 they proved our conjecture with Eskin on large genus volume asymptotics for the principal stratum.

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Brief history of evaluation of volumes

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Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus 0.

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

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Theorem (J. Athreya, A. Eskin, A. Z., 2014 ; conjectured by M. Kontsevich about 2003) *The volume of any stratum $\mathcal{Q}(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{CP}^1$ (i.e. when $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \dots, k$, and $\sum_{i=1}^{k} d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

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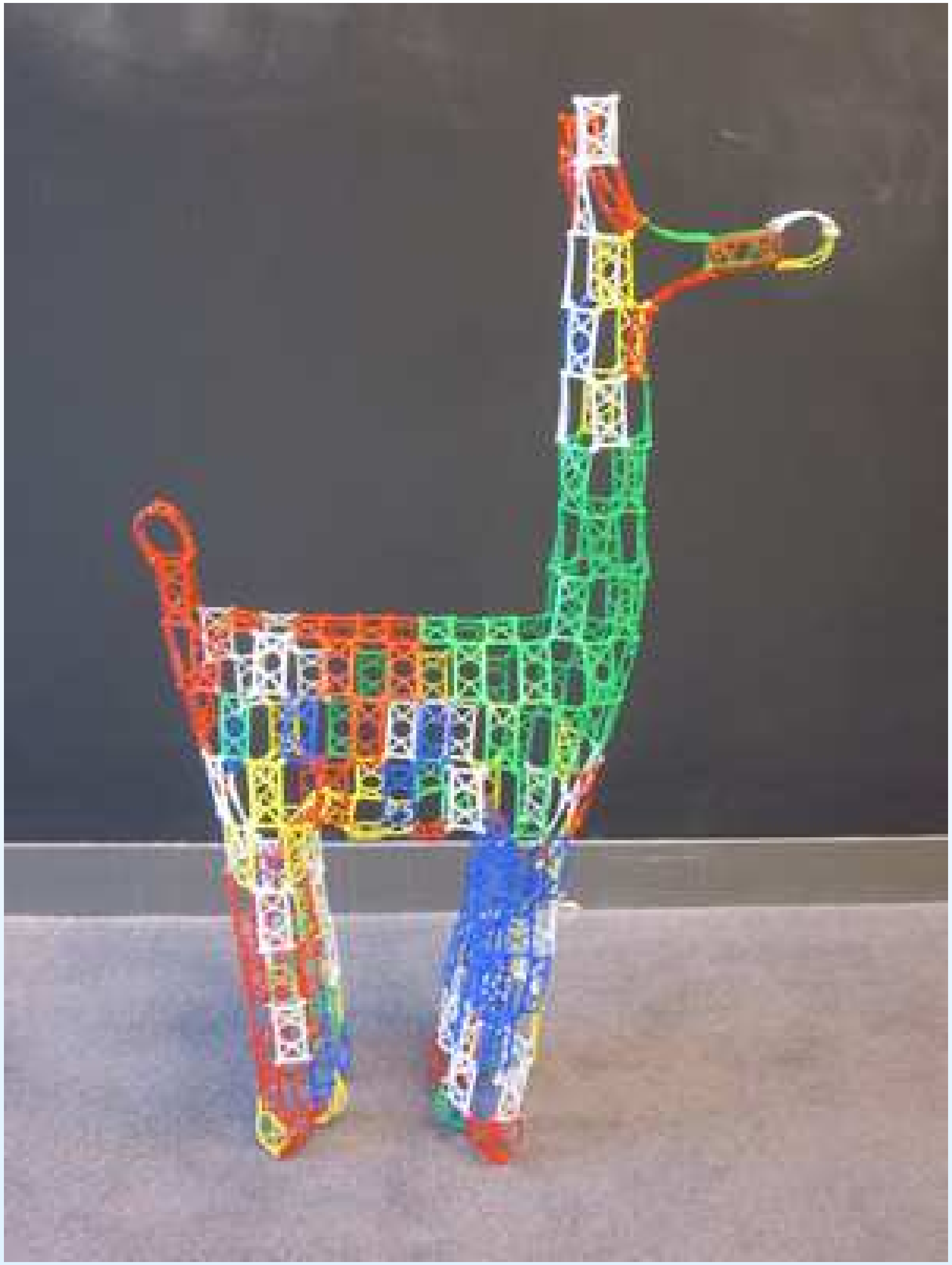
$$\operatorname{Vol} \mathcal{Q}_{0,n} = 2\pi \cdot \left(\frac{\pi^2}{2}\right)^{n-4} .$$

This is one of the miraculous identities: applying formula based on Mirzakhani or Kontsevich polynomials one gets ENORMOUS sum over labeled trees.

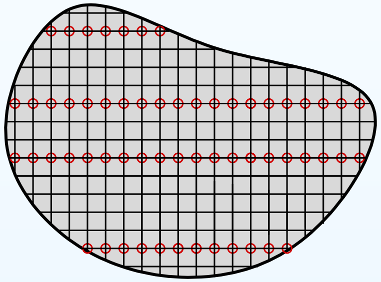
Non-correlation of vertical and horizontal foliations on square-tiled surfaces

Cylinder decomposition of a square-tiled surface



Equidistribution and Non-correlation Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L}) / \operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula!

Proof: Maryam's counting of volumes + Moore ergodicity theorem.

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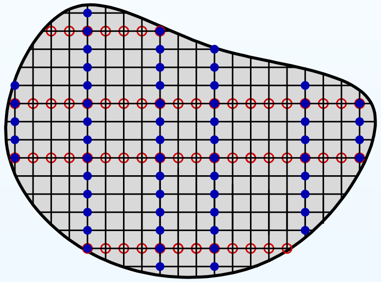

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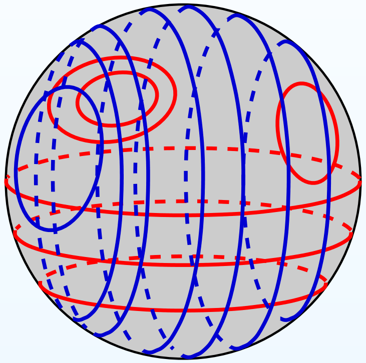
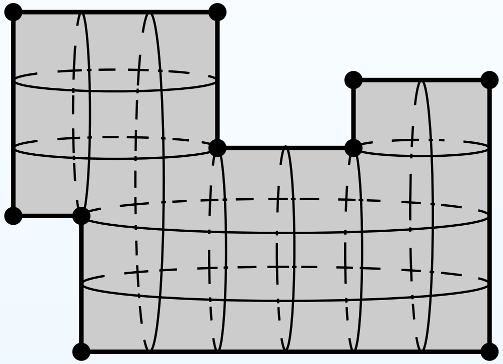
How to count meanders

Step 1. There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





Step 2. Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height 1. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

Step 3. Fixing the number of minimal arcs (“pimples”) we fix the number of simple poles p of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

Hyperbolic metric endowes a multicurve with canonical shape. A pair of multicurves canonically defines a hyperbolic metric. Discrete analog of Hubbard–Masur Theorem.

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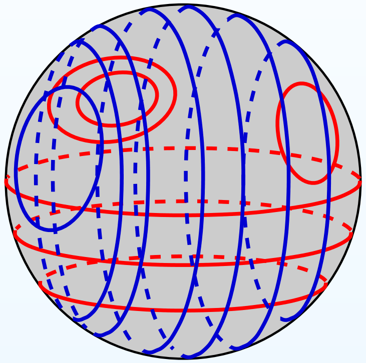

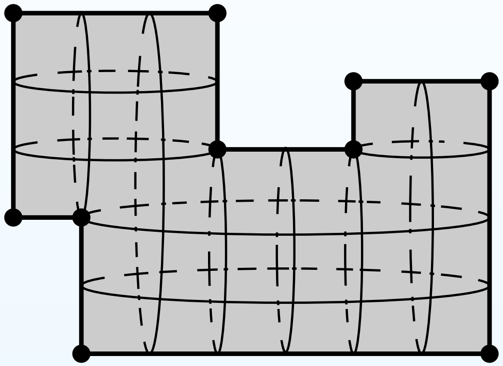

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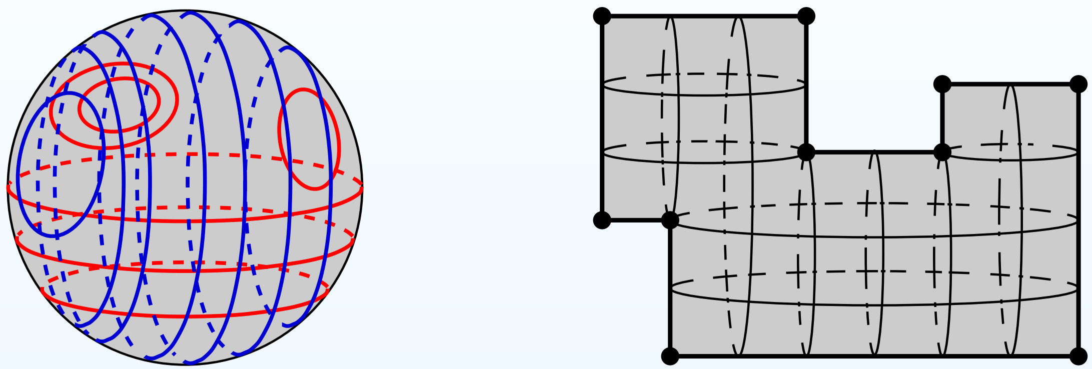

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General philosophy

● Pairs of transverse multicurves correspond to square-tiled surfaces. Thus, count of all pairs of transverse multicurves is equivalent to count of Masur–Veech volumes.

● Count of arc systems, braids, ribbon graphs, pairs: simple closed curve plus transverse multicurve, one-cylinder square-tiled surfaces is another group of (somehow equivalent) problems, which usually admits a more efficient solution.

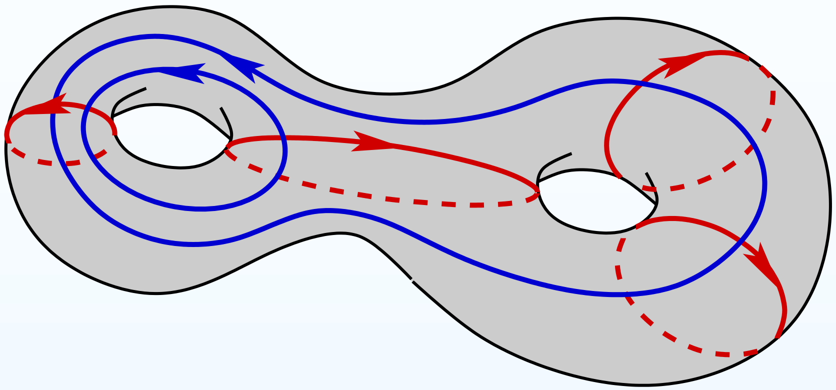
● Consider the following three counting problems:

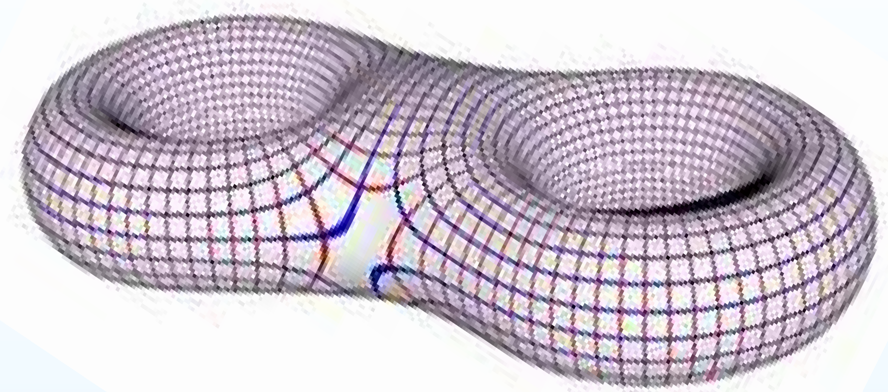
1. count of all square-tiled surfaces (i.e. Masur–Veech volume Vol);
2. count of horizontally one-cylinder square-tiled surfaces (i.e. c_1);
3. count of horizontally and vertically square-tiled surfaces (i.e. $c_{1,1}$).

By non-correlation, $c_{1,1} = \dfrac{c_1^2}{\mathrm{Vol}}$. Count of c_1 usually admits a relatively efficient solution. Hence, as soon as we know the appropriate Masur–Veech volume, we know $c_{1,1}$, and hence we can count meanders, pairs of transverse simple closed curves etc.

Braids on surfaces; positively intersecting multicurves; Masur–Veech volumes of strata of Abelian differentials

Pairs of positively intersecting multicurves





Picture created by Jian Jiang

Questions.

● *With what probability a random square-tiled translation surface has* $k = 1, 2, \ldots, 3g-3$ *maximal horizontal cylinders? How often it has a single horizontal cylinder?*

● *How strongly these quantities depend on the ambient stratum? On the genus?*

● *Do we get the same distribution as for random square-tiled quadratic differentials (random multicurves)?*

● *What is the shape of a random square-tiled surface of large genus?*

Contribution of k-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$0.19 \approx \mathrm{P}_1(\mathcal{H}(3,1)) = \frac{3\,\zeta(7)}{16\,\zeta(6)} \leftarrow \text{ the only quantity which is easy to compute}$$

$$0.47 \approx \mathrm{P}_2(\mathcal{H}(3,1)) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx \mathrm{P}_3(\mathcal{H}(3,1)) = \frac{1}{32\,\zeta(6)}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\
&+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\
&- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\
&+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\
&- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\
&- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx \mathrm{P}_4(\mathcal{H}(3,1)) = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big).$$

State of the art

We know the similar distribution of frequencies of square-tiled surfaces with respect to the number of horizontal cylinders only for several low-dimensional strata. However, we have a conjecture on this distribution for large genera based on extensive numerical experiments.

Conjecture. *The distribution of square-tiled surfaces by the number k of maximal horizontal cylinders tends to the Poisson distribution $\frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}$ with parameter $\lambda = \log(d) + \gamma$, where $\gamma \approx 0.577$ is the Euler constant. Here $d = \dim_{\mathbb{C}} \mathcal{H}$ is the dimension of the ambient stratum.*

According to this conjecture *a random translation square-tiled surface has about $\log d + 1 + \gamma$ cylinders and very rarely more that $3 \log d$ cylinders.*

Remark. Conjecturally, the distribution is the same for all strata (except hyperelliptic components) in different genera but of the same dimension.

Remark. The parameter λ in the analogous conjectural distribution for $\mathcal{Q}(1,\ldots,1)$ is different. It is $\dfrac{\log d + \gamma}{2}$, where $d = \dim \mathcal{Q}(1,\ldots,1)$.

Count of 1-cylinder square-tiled surfaces

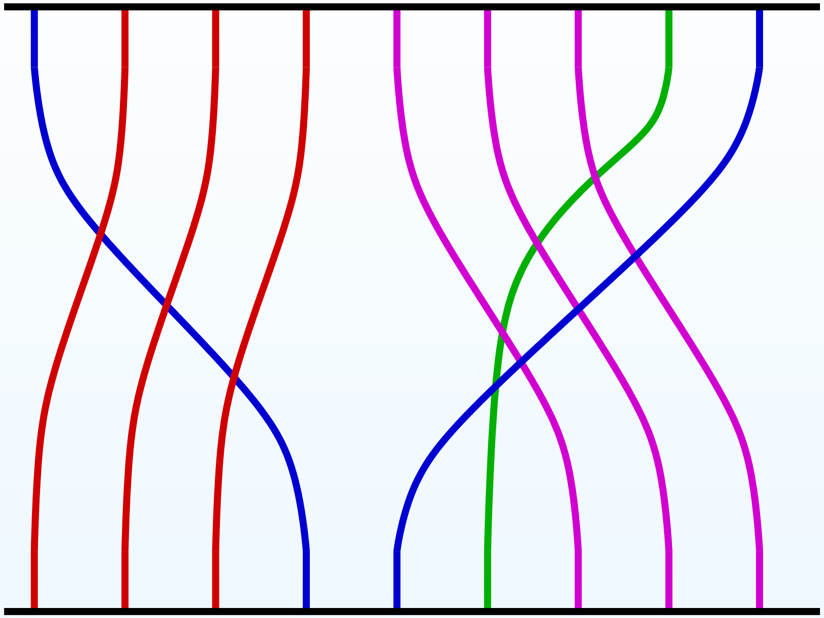
We have rather comprehensive information about Masur–Veech volumes of strata of Abelian differentials. Namely, in all low genera we know them explicitly. In higher genera the volume of the principal stratum $\mathcal{H}(1,\dots,1)$ can be computed exactly up to very high genus. When $g \to +\infty$ it can be computed approximately with arbitrary precision by the work of Chen–Möller–Zagier. Less precise but universal formula for all strata proving our conjecture with Eskin

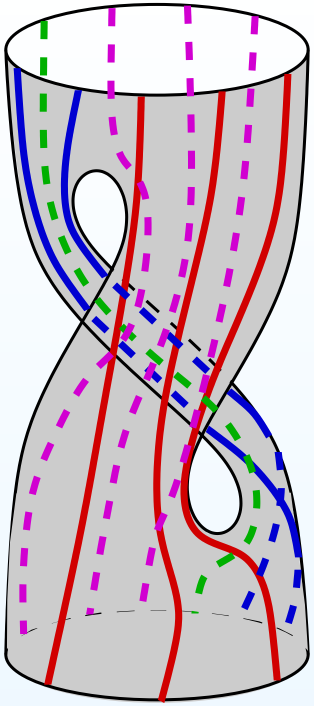
$$\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n) \sim \frac{4}{(m_1+1)\dots(m_n+1)}\,.$$

was recently proved by Aggarwal.

By the general philosophy, to compute braids, frequencies of simple closed curves, numbers of pairs of positively intersecting simple closed curves, etc in this context, one has to compute contribution of 1-cylinder square-tiled surfaces. As in the previous cases, this problem admits a solution.

Counting braids and positive meanders





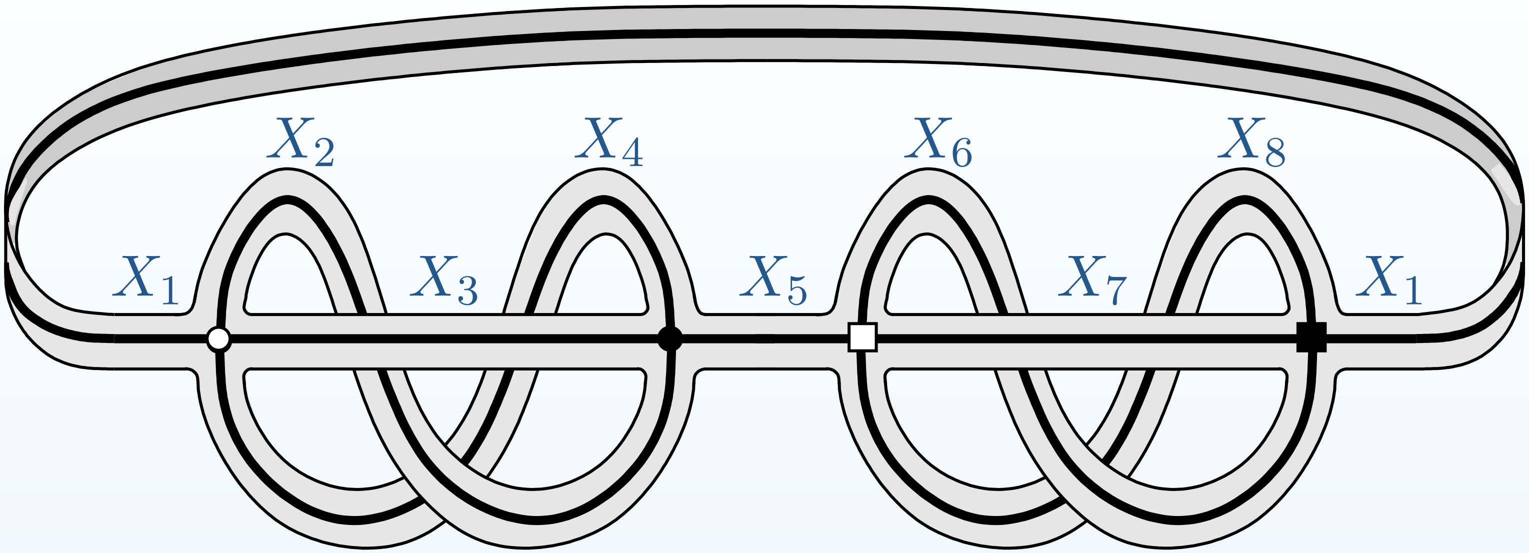
Theorem. *The relative contribution* $p_1(\mathcal{H}(m_1,\dots,m_n))$ *of* 1*-cylinder square-tiled surfaces to the Masur–Veech volume* $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ *is of the order* $\frac{1}{d}$*, where* $d=\dim\mathcal{H}(m_1,\dots,m_n)$*.*

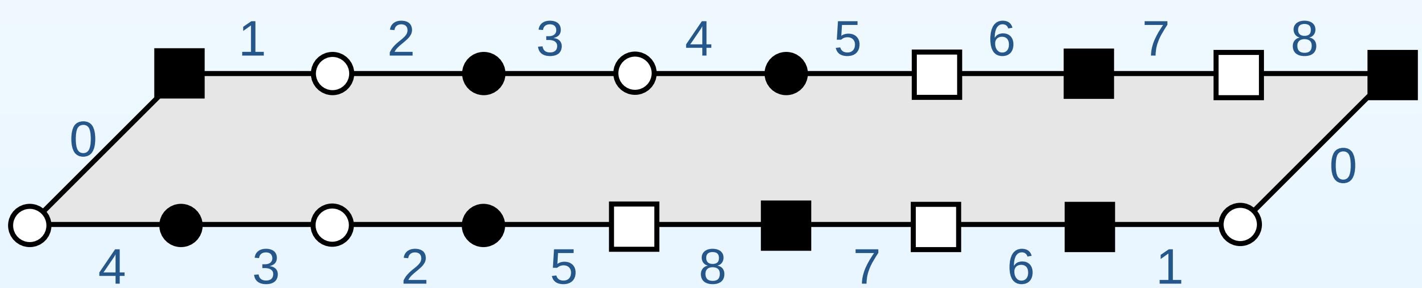
In particular, the probability to get a connected curve from a braid on a surface of genus g *is of the order* $\frac{1}{4g}$*.*

Pairs of positively intersecting transverse simple closed curves have frequency $\frac{1}{16g^2}$ *among all positively intersecting pairs of multicurves.*

1-cylinder square-tiled surfaces and permutations

1-cylinder surface as a pair of permutations





$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8$$

$$4 \to 3 \to 2 \to 5 \to 8 \to 7 \to 6 \to 1$$

$$c_{bot} \cdot c_{top}^{-1} = (1,3)(2,4)(5,7)(6,8)$$

Frobenius formula

The count of 1-cylinder N-square-tiled surfaces in the stratum $\mathcal{H}(m_1,\dots,m_n)$ is reduced to the count of solutions of the following equation for permutations:

$$(N\text{- cycle})\cdot(N\text{- cycle}) = \text{product of cycles of lengths } m_1+1,\dots,m_n+1\,.$$

Frobenius formula expresses this number in terms of characters of the exterior powers of the standard representation $\mathbf{St}_n$ of the symmetric group S_n:

$$\chi_j(g) := \operatorname{tr}(g,\pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0\le j\le n-1)\,.$$

Theorem. *The absolute contribution $c_1(\mathcal{H}(m_1,\dots,m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ equals*

$$c_1 = \frac{2}{(d-1)!}\cdot\prod_k \frac{1}{(k+1)^{\mu_k}}\cdot\sum_{j=0}^{d-2} j!\,(n-1-j)!\,\chi_j(\nu)\,.$$

Here $d=\dim\mathcal{H}(m_1,\dots,m_n)$; $\nu\in S_n$ is any permutation with decomposition into cycles of lengths $(m_1+1),\dots,(m_n+1)$; μ_i is the number of zeroes of order i, i.e. the multiplicity of the entry i in the multiset $\{m_1,\dots,m_n\}$.

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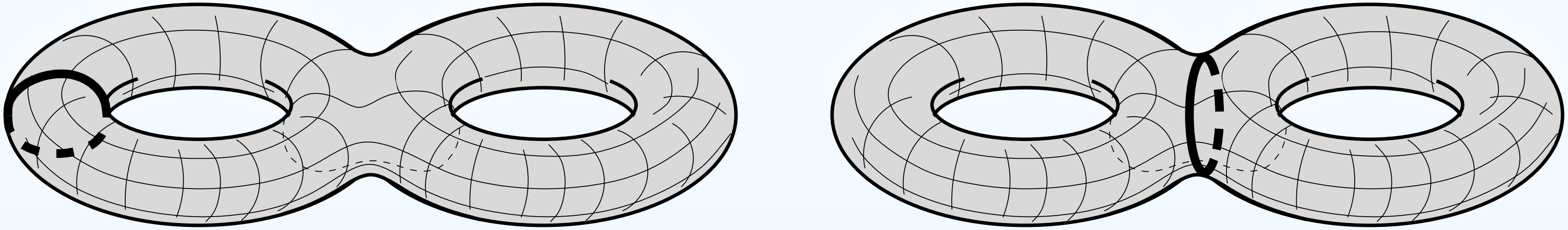
$$\frac{1}{d+1} \cdot \frac{4}{(m_1+1)\dots(m_n+1)} \le c_1(\mathcal{H}) \le \frac{1}{d - \frac{10}{29}} \cdot \frac{4}{(m_1+1)\dots(m_n+1)}$$

We were able to replace the formula in characters by this much more efficient estimate using the results of Zagier.

Shape of a random multicurve on a surface of large genus.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus $g = 2$; no cusps.*



$$\lim_{L \to +\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{6}$$

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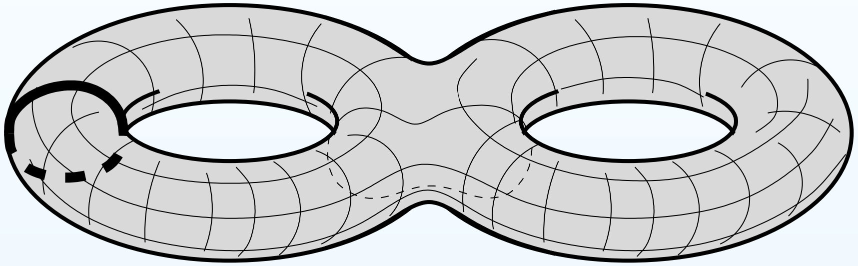
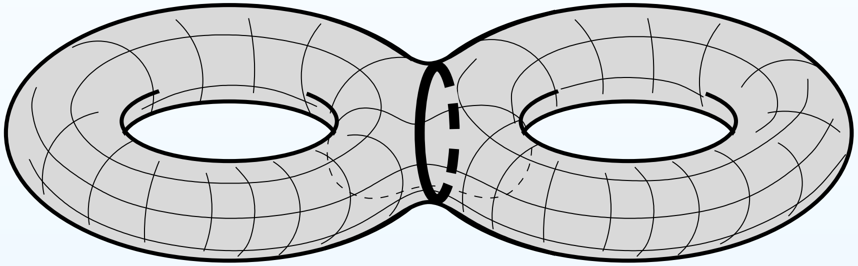


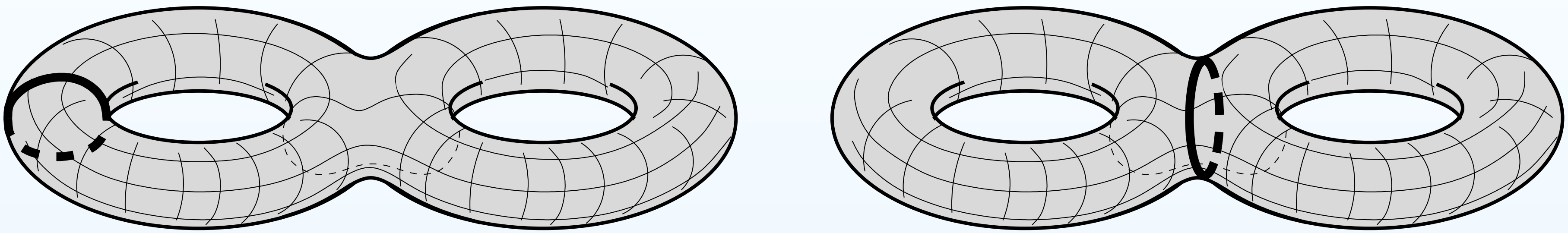


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Maryam's calculation.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus $g = 2$; no cusps.*



$$\lim_{L\to+\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in Maryam's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell; also nailed by C. Ball.

Random simple closed curve (almost) never separates

Theorem. *A random simple closed curve on a surface of large genus does not separate the surface. Namely, the probability that it separates decays exponentially with the rate* 4^{-g}*: the ratio of frequencies of non-separating over separating simple closed geodesics on a closed surface of genus* g *satisfies:*

$$\lim_{g\to+\infty} \frac{1}{g} \log \frac{c(\gamma_{non-sep}, g)}{c(\gamma_{sep}, g)} = \log 4$$

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Idea of the proof. Frequencies of separating simple closed curves are expressed in terms of the intersection numbers which admit closed expression:

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g\, g!}\,.$$

Frequencies of non-separating simple closed curves are expressed in terms of

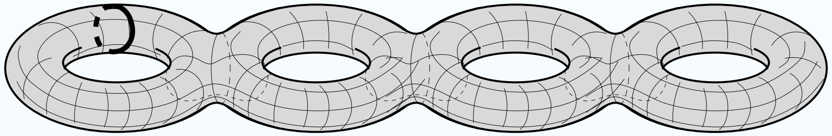
$$\int_{\overline{\mathcal{M}}_{g,2}} \psi_1^{k} \psi_2^{3g-1-k}$$

for which we obtain large genus asymptotics uniform for all k in fixed genus g.

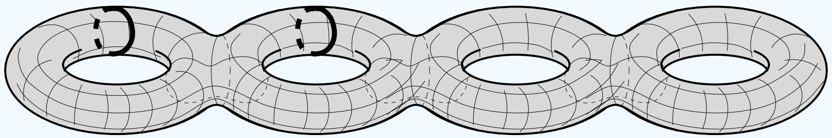
The study of large genus asymptotics of correlators was initiated by Maryam and P. Zograf.

Conjectural shape of a random multicurve

Conjecture. *A randomly chosen multicurve on a surface of large genus g very rarely separates the surface. It has one of the following shapes:*

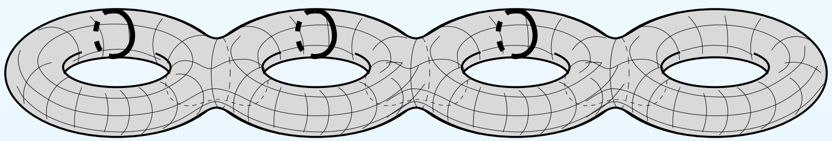


Γ_1



Γ_2

$\dots$ $\dots$



$\Gamma_{2\log g}$

• *Usually it has about $1 + \frac{\log(6g-6)+\gamma}{2}$ connected components. It has more than $2\log(g)$ components exceptionally rarely.*

• *The distribution of frequencies $c(\Gamma_k, g)$ of multicurves with exactly k components tends to the Poisson distribution $\frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}$ with parameter $\lambda = \frac{\log(6g-6)+\gamma}{2}$, where $\gamma \approx 0.577$ is the Euler constant.*