Equidistribution of square-tiled surfaces, meanders, and Masur-Veech volumes

Anton Zorich
(joint work with V. Delecroix, E. Goujard, P. Zograf)

The Mathematical Legacy of Maryam Mirzakhani

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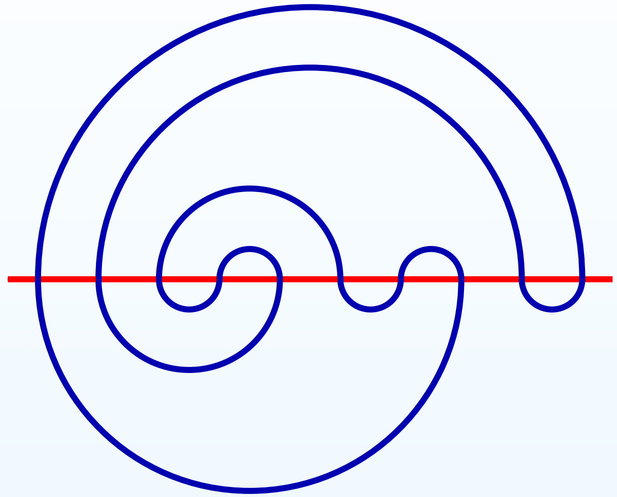
Meanders

Meanders and arc systems



A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line.

According to S. Lando and A. Zvonkin the notion “meander” was suggested by V. Arnold though meanders were studied already by H. Poincaré.

Meanders appear in various contexts, in particular in physics (P. Di Francesco, O. Golinelli, E. Guitter).

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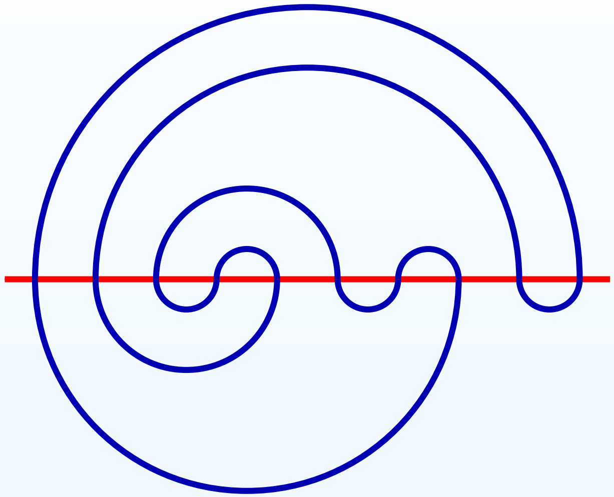
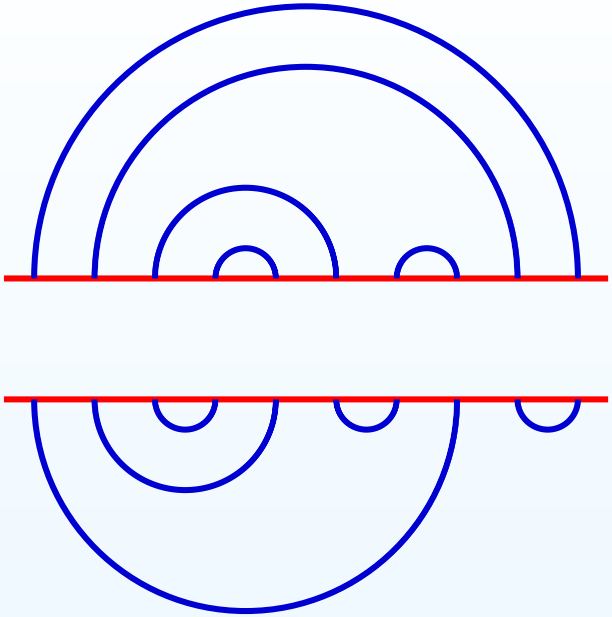
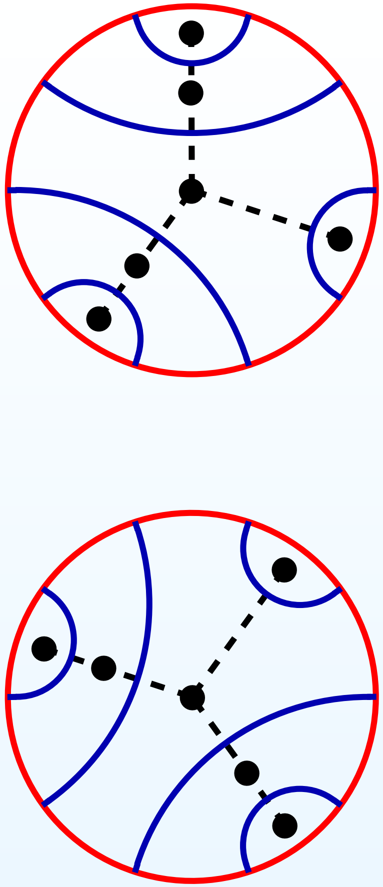




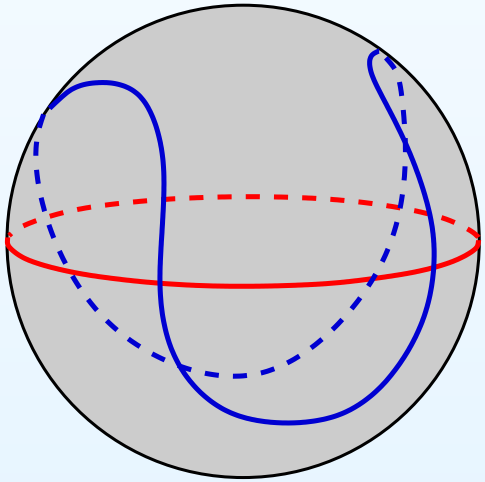


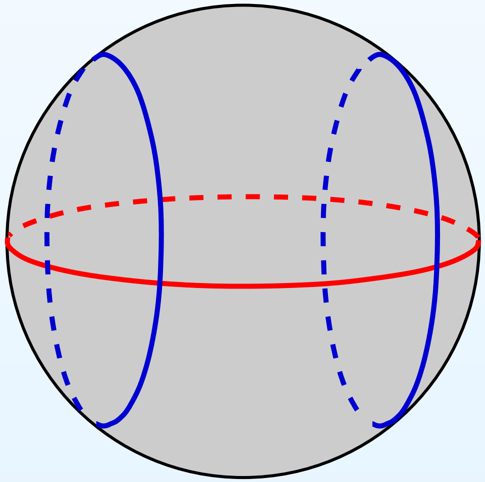
A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Compactifying the plane (left picture) with one point at infinity, or gluing together arc systems on the two discs (right picture) we get an ordered pair of smooth simple transversally intersecting closed curves on the sphere.

Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a curve with several connected components





Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

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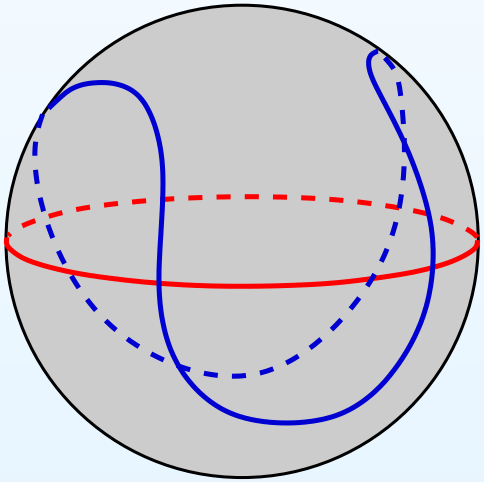

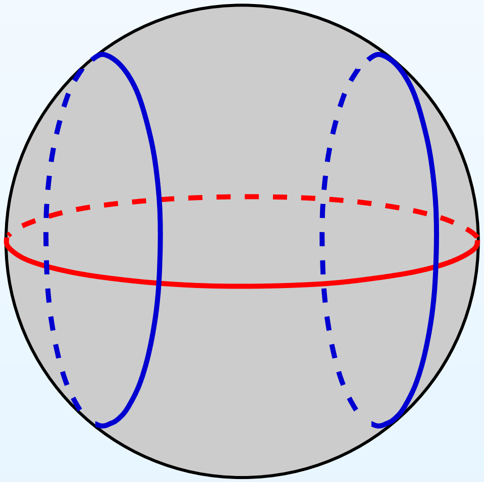

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Asymptotic frequency of meanders

Consider arc system with the same number $n \leq N$ of arcs on a labeled pair of oriented discs having $\mathcal{T}_{top}$ and $\mathcal{T}_{bottom}$ as reduced dual trees. We draw $\mathcal{T}_{top}$ on the northern hemisphere and $\mathcal{T}_{bottom}$ on the southern hemisphere. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. We consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{top};\, n\text{-arc system of type } \mathcal{T}_{bottom};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$p_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$

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Question. *What is the asymptotics of the quantity*

$$p_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N) \quad \textit{as } N \to +\infty?$$

Does it behave like $N^{-\alpha}$? Like $\exp(-\beta N)$? Describe how α (respectively β) depend on $\mathcal{T}_{top}, \mathcal{T}_{bottom}$.

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Answer (V. Delecroix, E. Goujard, P. Zograf, A. Zorich). *For any pair of trees $\mathcal{T}_{top}, \mathcal{T}_{bottom}$ the quantity $p_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N)$ admits a strictly positive limit as $N \to +\infty$.*

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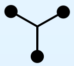
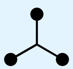
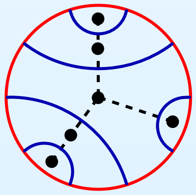
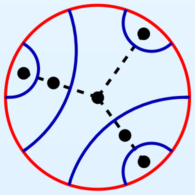
Example. The fact that this asymptotic frequency is nonzero is already unexpected. For example, the following asymptotic frequency equals:

$\lim_{N\to+\infty} p_{connected}($  $,$  $, N) = \frac{280}{\pi^6} \approx 0.291245$





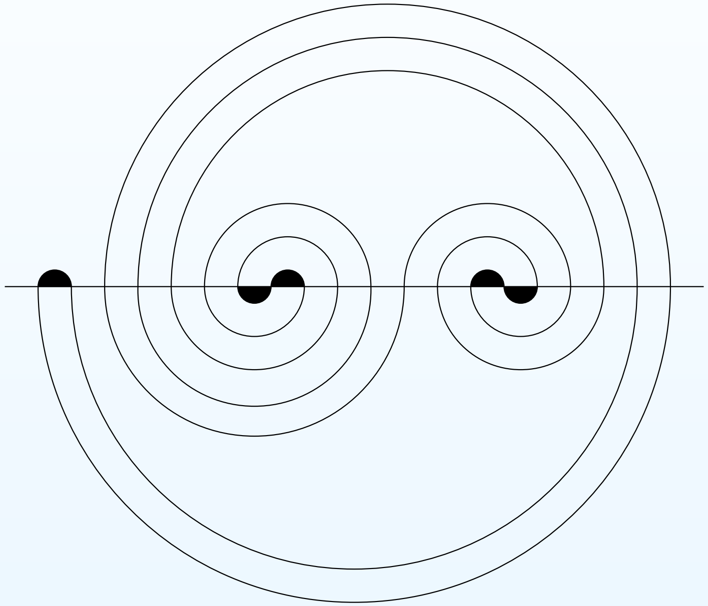
Fixing the number of vertices of valence one

Theorem. *Let* $p \geq 4$*. The frequency* $p_{connected}(p;N)$ *of meanders obtained by all possible identifications of all arc systems with at most* N *arcs represented by all possible pairs of plane trees having the total number* p *of leaves (vertices of valence one) has the following limit:*

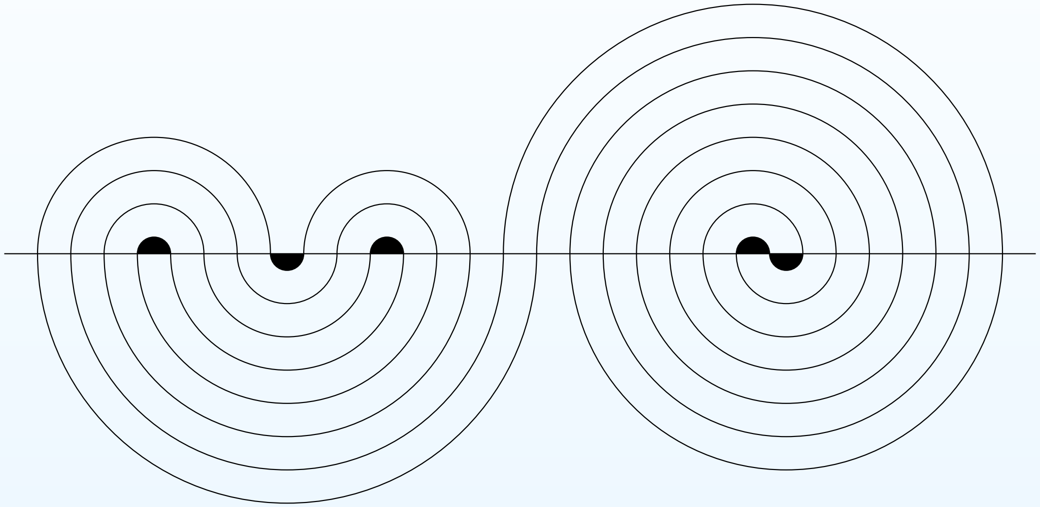
$$\lim_{N\to+\infty} p_{connected}(p;N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}\,.$$

Meanders with and without maximal arc

These two meanders have 5 minimal arcs (“pimples”) each.



Meander with a maximal arc (“rainbow”) contributes to $\mathcal{M}_5^+(N)$



Meander without maximal arc contributes to $\mathcal{M}_5^-(N)$

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc (“rainbow”) and having at most $2N$ crossings with the horizontal line and exactly p minimal arcs (“pimples”). We consider p as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

Counting formulae for meanders

Theorem. *For any fixed p the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with p minimal arcs (pimples) and with at most $2N$ crossings have the following asymtotics as $N \to +\infty$:*

$$\mathcal{M}_p^+(N) = \frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4})\,.$$

$$\mathcal{M}_p^-(N) = \frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,.$$

Note that $\mathcal{M}_p^+(N)$ grows as N^{2p-4} while $\mathcal{M}_p^-(N)$ grows as N^{2p-5}.

Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials

Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1, \dots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \dots, m_n$, where $\sum m_i = 2g - 2$, is modelled on the vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1, \dots, m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2} \int_C \omega \wedge \bar{\omega} = \sum_{i=1}^{g} (A_i \bar{B}_i - \bar{A}_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S = (C, r \cdot \omega)$, where $r > 0$ and $(C, \omega) \in \mathcal{H}_1(m_1, \dots, m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu = r^{2d-1} dr\, d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n)$.

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1, \dots, m_n)$ or $\mathcal{Q}_1(m_1, \dots, m_n)$ of Abelian or quadratic differentials is finite.*

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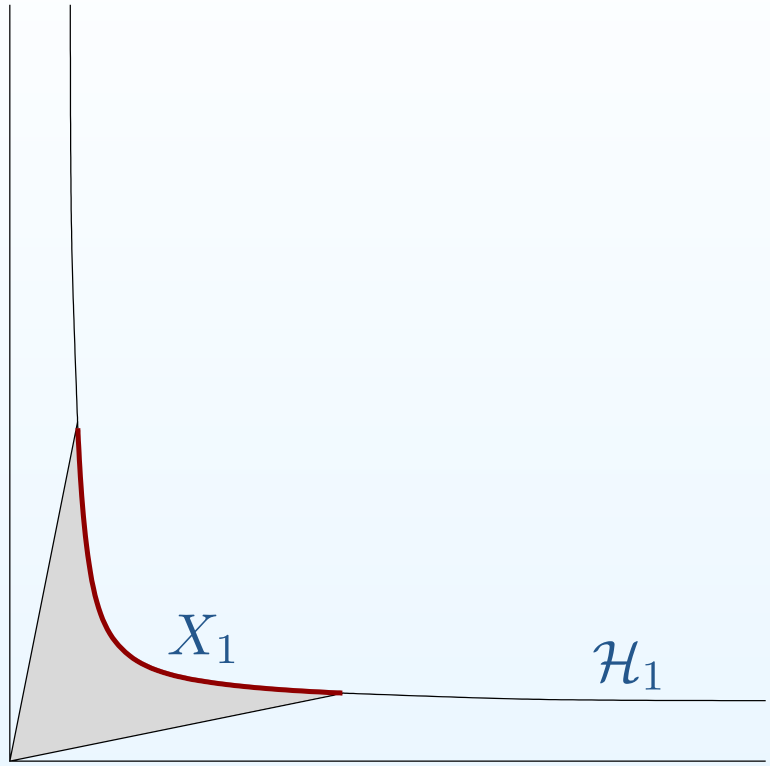
Counting volume by counting integer points in a large cone



ν_1-volume of a domain X_1 in a unit hyperboloid $\mathcal{H}_1$ is related to ν-volume of a cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ over X_1 as $\nu_1(X_1) = 2d \cdot \nu(C(X_1))$.

To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it.

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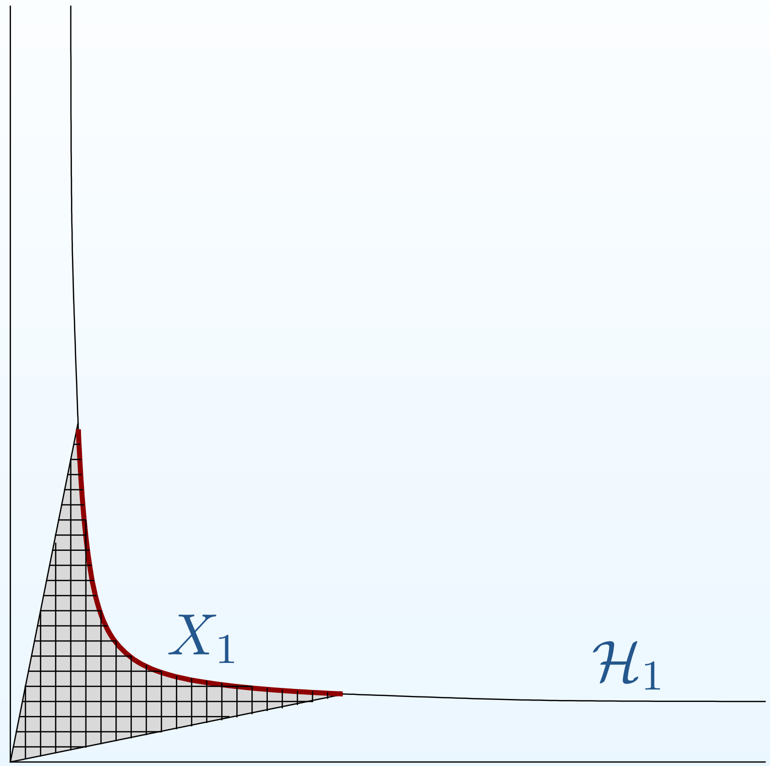

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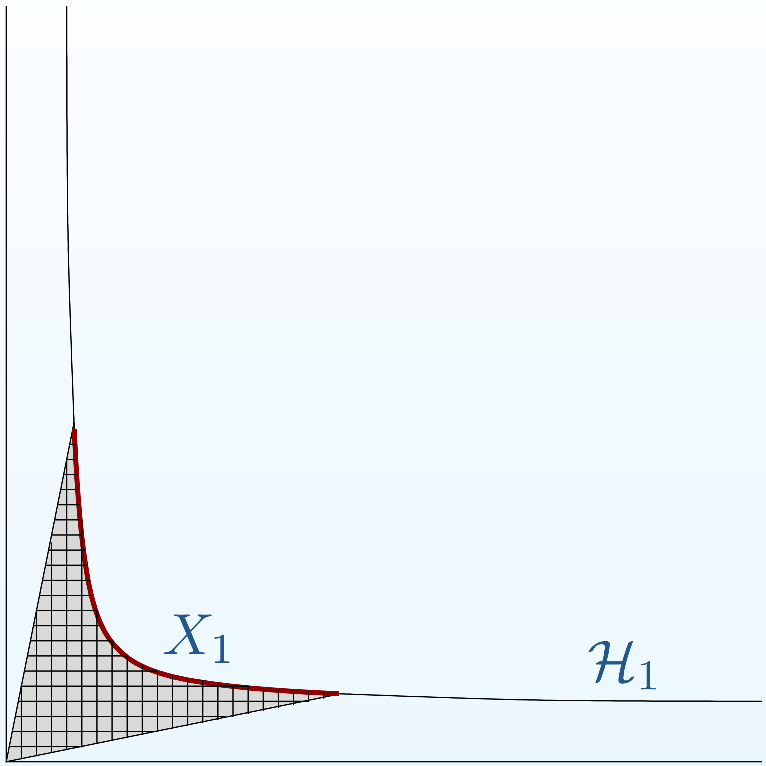

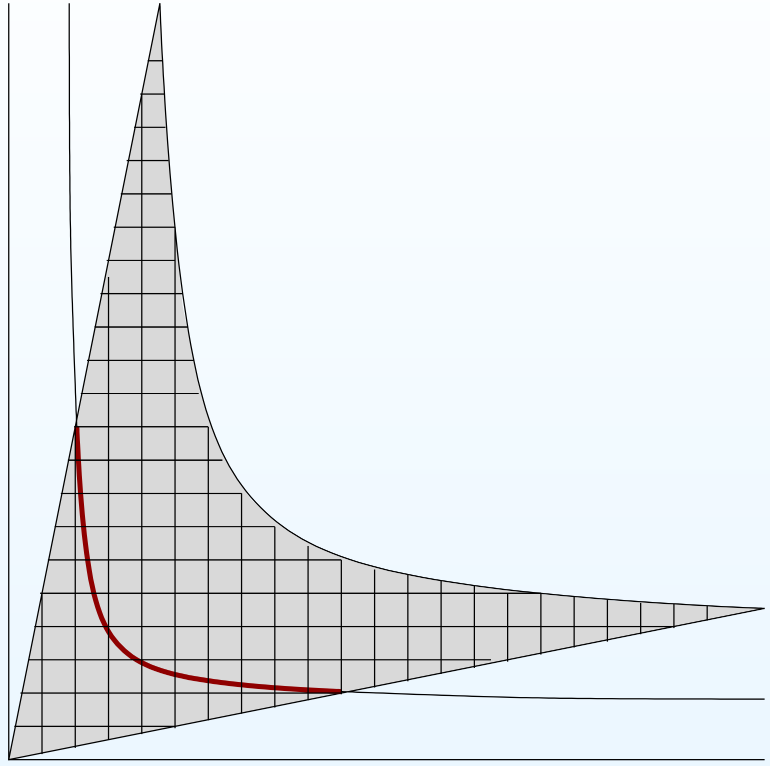

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To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it. Counting points of the $\frac{1}{N}$-grid in the cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ is the same as counting integer points in the larger proportionally rescaled cone $C_N(X_1) = \{r \cdot S | S \in X_1, r \leq N\}$.

Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}, \text{ where } P_1 \text{ is a zero of } \omega\,.$$

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Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous “pillowcase covers” over $\mathbb{C}\mathrm{P}^1$ branched at four points.

Thus, counting volumes of the strata is similar to counting analogs of Hurwitz numbers.

Brief history of evaluation of volumes

Theorem (A. Eskin, A. Okounkov, R. Pandharipande). *For every connected component $\mathcal{H}^c(d_1, \ldots, d_n)$ of every stratum, the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled} \\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form. The Masur–Veech volume of every connected component of every stratum is a rational multiple of π^{2g}, where g is the genus.

A. Eskin implemented this theorem (around 2002) to an algorithm allowing to compute volumes for all strata up to genus 10 and for some strata (like the principal one) up to genus 200. Based on these calculations we developed a conjecture on a very simple asymptotic formula for volumes in large genera.

D. Chen, M. Möller, D. Zagier can compute the volume of the principal stratum up to genus 2000 and more; in 2017 they proved our conjecture with Eskin on large genus volume asymptotics for the principal stratum.

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A. Eskin implemented this theorem (around 2002) to an algorithm allowing to compute volumes for all strata up to genus 10 and for some strata (like the principal one) up to genus 200. Based on these calculations we developed a conjecture on a very simple asymptotic formula for volumes in large genera. D. Chen, M. Möller, D. Zagier can compute the volume of the principal stratum up to genus 2000 and more; in 2017 they proved our conjecture with Eskin on large genus volume asymptotics for the principal stratum.

A. Aggarwal proved the conjecture for *all* strata several weeks ago.

A. Sauvaget recently suggested a conjectural formula for the Masur–Veech volumes of all strata of Abelian differentials through intersection numbers.

Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus 0.

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

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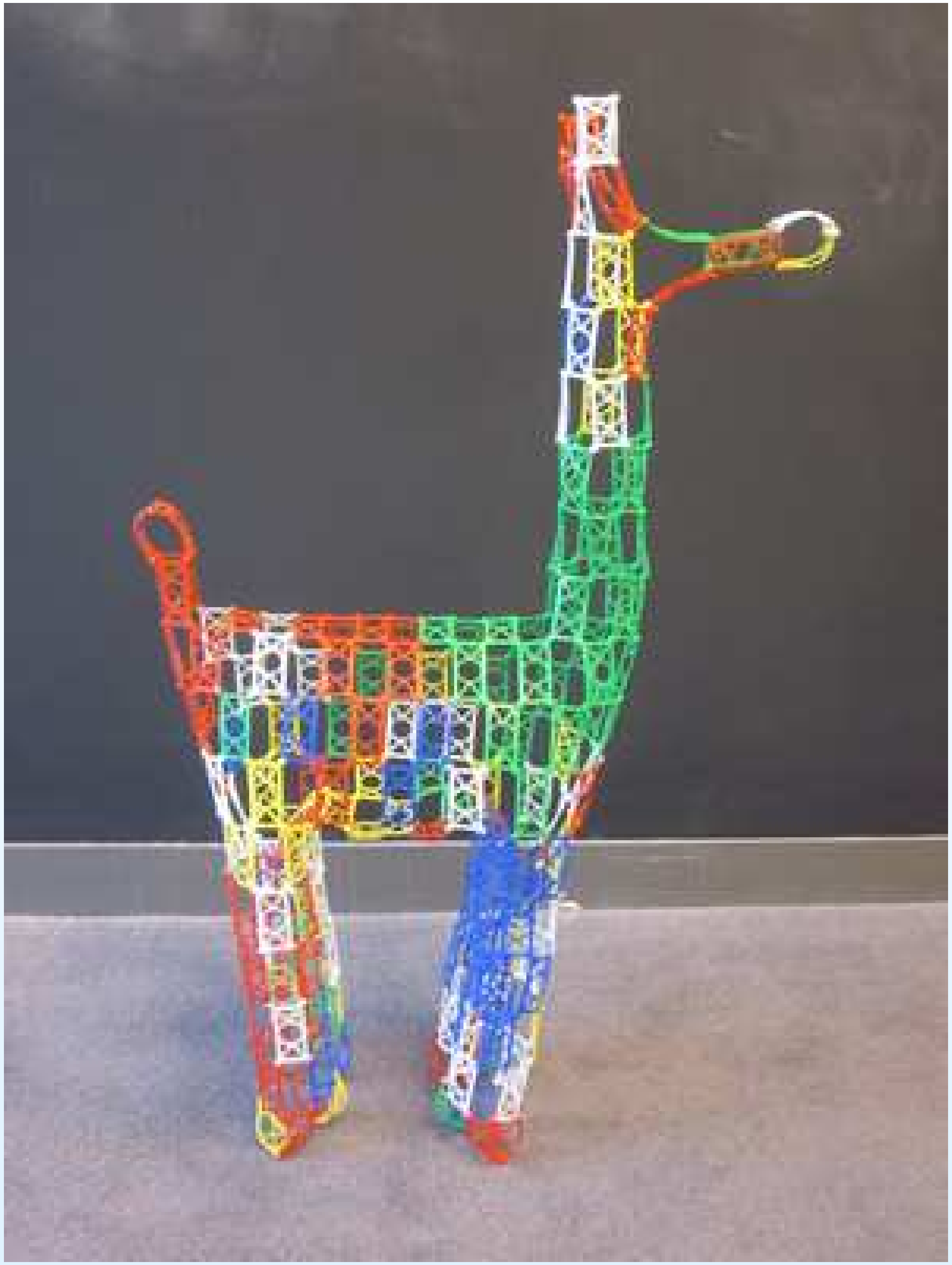
Theorem (J. Athreya, A. Eskin, A. Z., 2014 ; conjectured by M. Kontsevich about 2003) *The volume of any stratum $\mathcal{Q}(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. when $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \dots, k$, and $\sum_{i=1}^k d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

Asymptotic equidistribution

Cylinder decomposition of a square-tiled surface



Contribution of k-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$0.19 \approx p_1(\mathcal{H}(3,1)) = \frac{3\,\zeta(7)}{16\,\zeta(6)}$$ ← the only quantity which is easy to compute

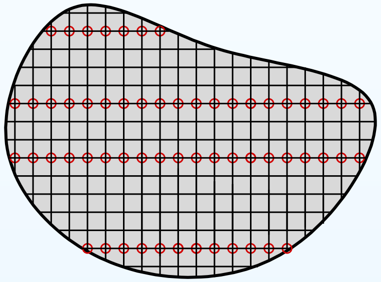
$$0.47 \approx p_2(\mathcal{H}(3,1)) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx p_3(\mathcal{H}(3,1)) = \frac{1}{32\,\zeta(6)}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\ &+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\ &- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\ &+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\ &- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\ &- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx p_4(\mathcal{H}(3,1)) = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big).$$

Equidistribution Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L})/\operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula! Proof: Moore ergodicity theorem.

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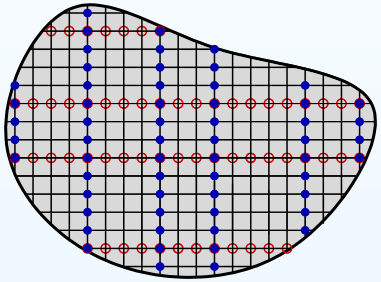

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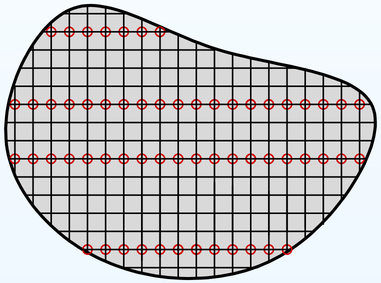
Experimental evaluation of volumes

The Equidistribution Theorem allows to compute approximate values of volumes experimentally. Choose some ball B (or some box) in the stratum. Consider a sufficiently small grid in it and collect statistics of frequency $p_1(B)$ of 1-cylinder square-tiled surfaces (pillow-case covers) in our grid in B.



Now compute the **absolute** contribution $c_1(\mathcal{L})$ of all 1-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{L}$; it is easier than for k-cylinder ones with $k > 2$. By the Equidistribution Theorem, the volume of the ambient stratum is $\operatorname{Vol}\mathcal{L} = \frac{c_1(\mathcal{L})}{p_1(\mathcal{L})}$.

The statistics $p_1(\mathcal{H})$ can be, actually, collected using interval exchanges, which simplifies the experiment. Approximate values of volumes were extremely useful in debugging numerous normalization factors in rigorous answers in the implementation by E. Goujard of the method of Eskin–Okounkov.

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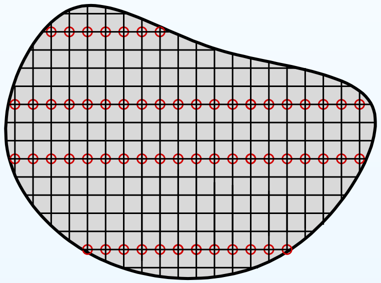

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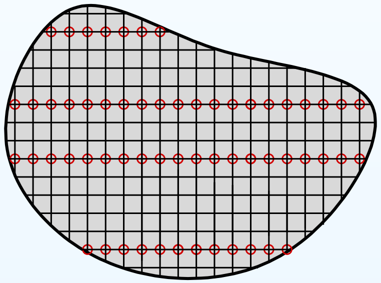

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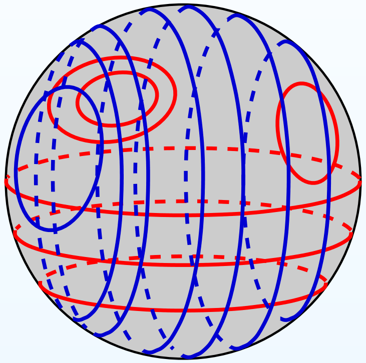
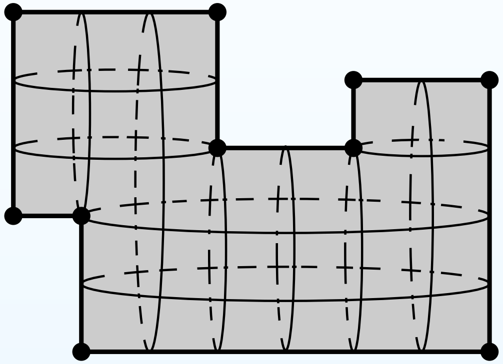
How to count meanders

Step 1. There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





Step 2. Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height 1. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $cyl_{1,1}(\mathcal{Q}) = cyl_1^2(\mathcal{Q}) / \operatorname{Vol}(\mathcal{Q})$, where $cyl_1(\mathcal{Q})$ is easy to compute and $\operatorname{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula.

Step 3. Fixing the number of minimal arcs ("pimples") we fix the number of simple poles p of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

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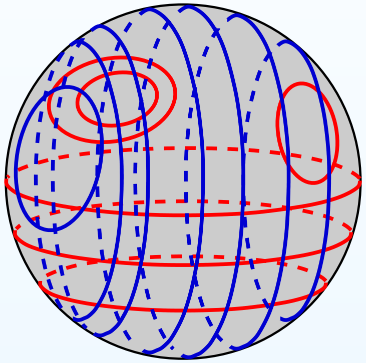

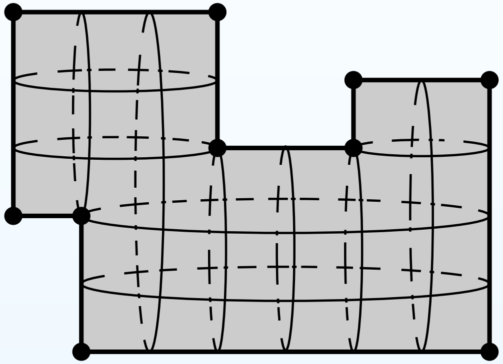

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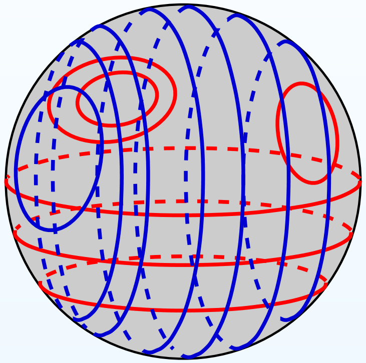

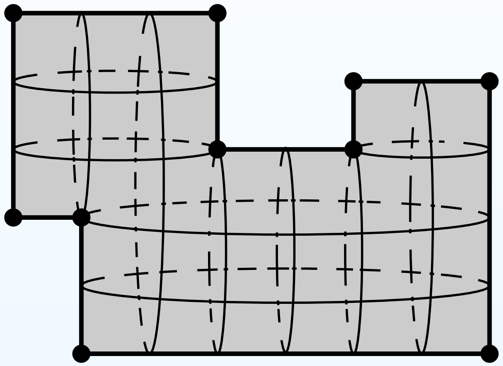

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Masur–Veech versus Mirzakhani–Weil–Petersson volumes

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1^2, \ldots, b_n^2)$ of degree $3g-3+n$ in variables $b_1^2, \ldots, b_n^2$:

$$N_{g,n}(b_1^2, \ldots, b_n^2) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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The polynomial $N_{g,n}(b_1^2, \ldots, b_n^2)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $\frac{1}{2} V_{g,n}(b_1^2, \ldots, b_n^2)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces.

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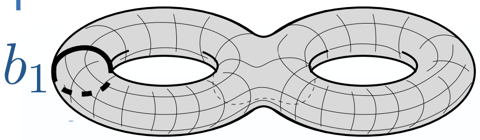
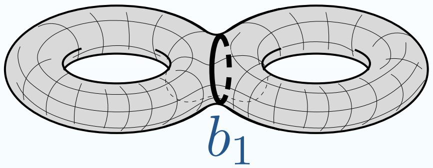
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \; : \quad \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

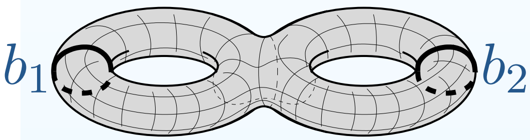
and extend it to symmetric polynomials in b_i by linearity.



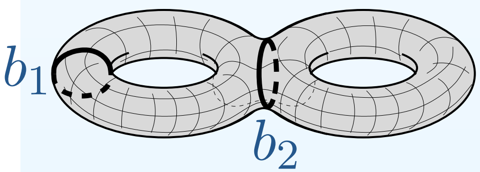
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



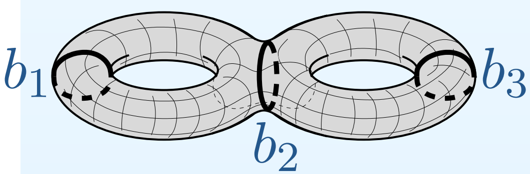
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



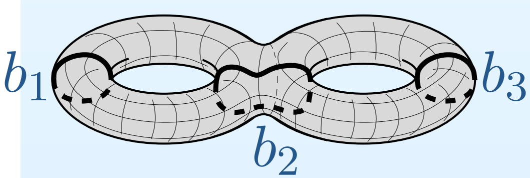
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



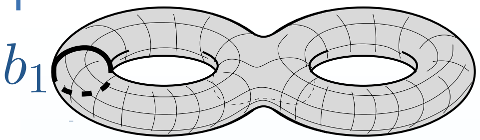
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot N_{1,1}(b_2)$$



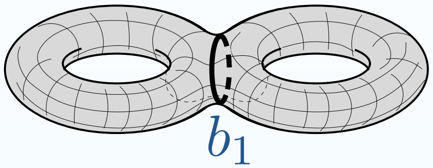
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot N_{0,3}(b_2, b_3, b_3)$$



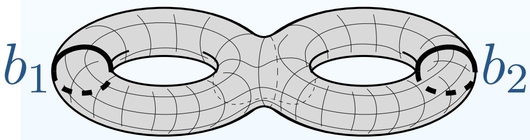
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot N_{0,3}(b_1, b_2, b_3)$$



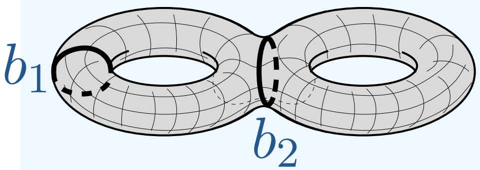
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) = \frac{1}{2} \cdot b_1 \left(\frac{1}{384}(2b_1^2)(2b_1^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$

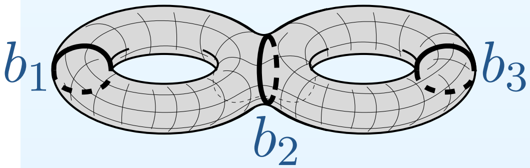


$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4}(2b_1^2 + 2b_2^2) \right)$$



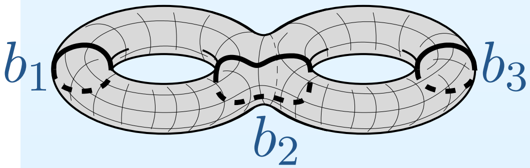
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

$$\cdot N_{1,1}(b_2) = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$



$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

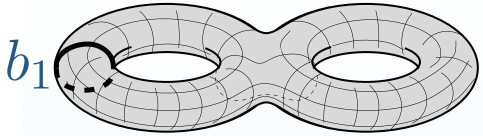
$$\cdot N_{0,3}(b_2, b_3, b_3) = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$



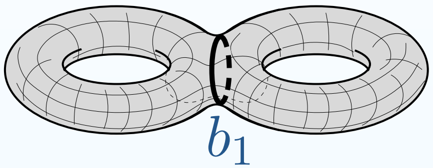
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$

$$\cdot N_{0,3}(b_1, b_2, b_3) = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

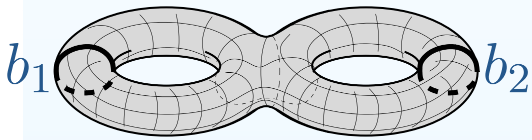
Volume of $\mathcal{Q}_2$



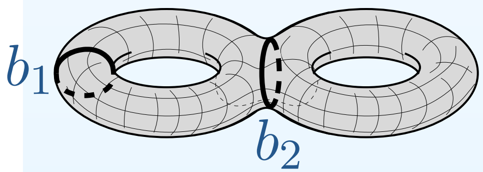
$$\frac{1}{192}\cdot b_1^5 \xmapsto{\mathcal{Z}} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



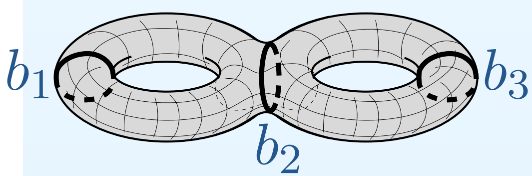
$$\frac{1}{9216}\cdot b_1^5 \xmapsto{\mathcal{Z}} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



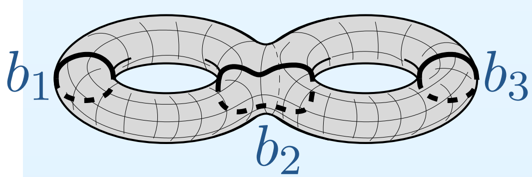
$$\frac{1}{16}(b_1^3 b_2 + {} + b_1 b_2^3) \xmapsto{\mathcal{Z}} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \xmapsto{\mathcal{Z}} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



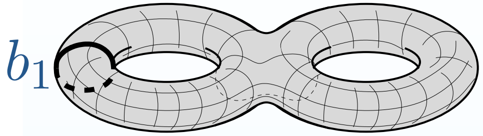
$$\frac{1}{16}b_1 b_2 b_3 \xmapsto{\mathcal{Z}} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



$$\frac{1}{24}b_1 b_2 b_3 \xmapsto{\mathcal{Z}} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

Volume of $\mathcal{Q}_2$



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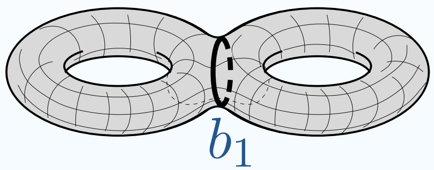

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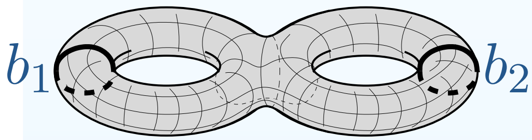

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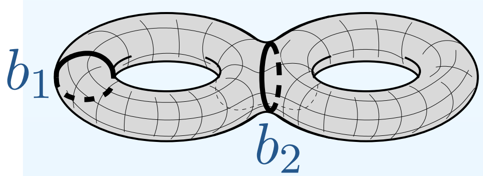

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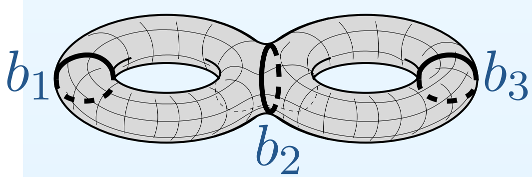

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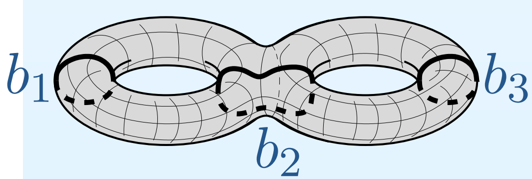

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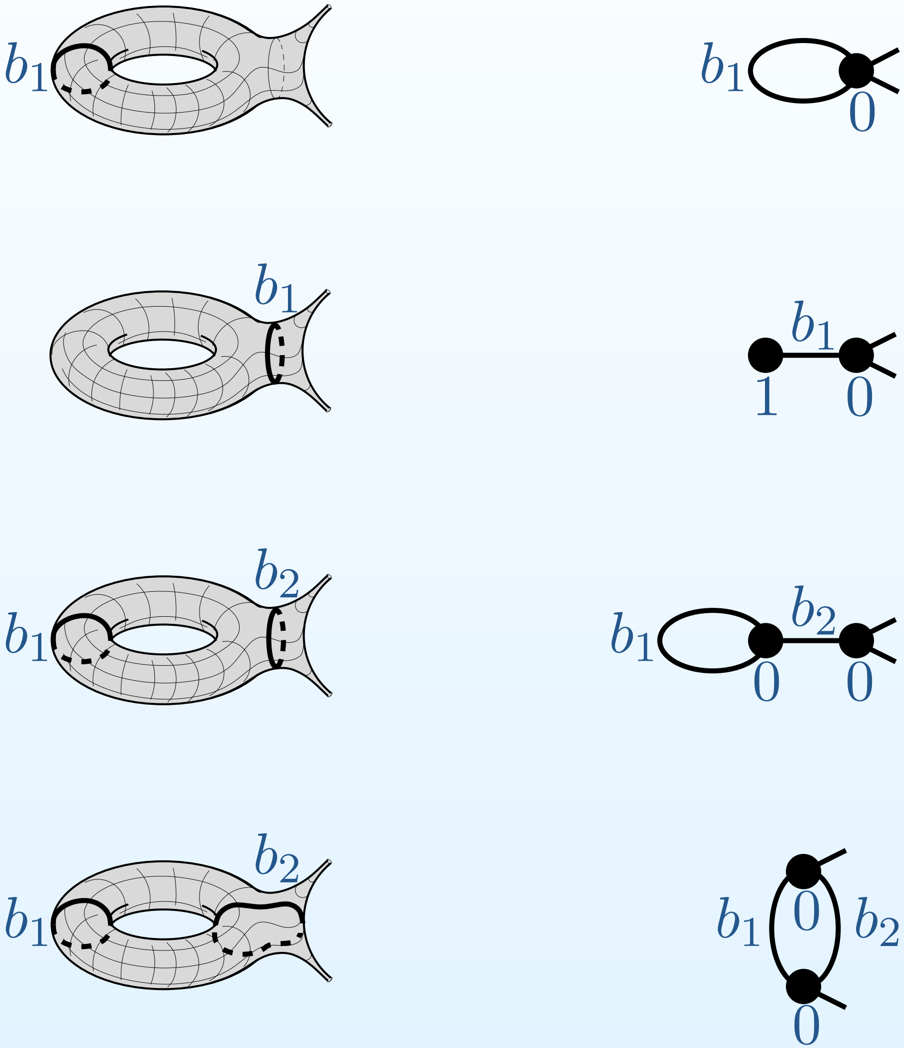
Encoding surfaces with simple curves by weighted graphs



Volume of the “unit ball” in the cotangent space to $\mathcal{M}_{g,n}$

Theorem. *The Masur–Veech volume $\operatorname{Vol}\mathcal{Q}_{g,n}$ of the moduli space of meromorphic quadratic differentials with n simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma - 1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2, \underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k of edges gives the contribution of k-cylinder square-tiled surfaces.

Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials, that is we use them opposite regime when the lengths of all boundary components tend to infinity.

$\operatorname{Vol}\mathcal{Q}_g$ is computed by Mirzakhani (up to the factor in red); for $\operatorname{Vol}\mathcal{Q}_{g,n}$ — by Delecroix, Goujard, Zograf, Zorich by different method.

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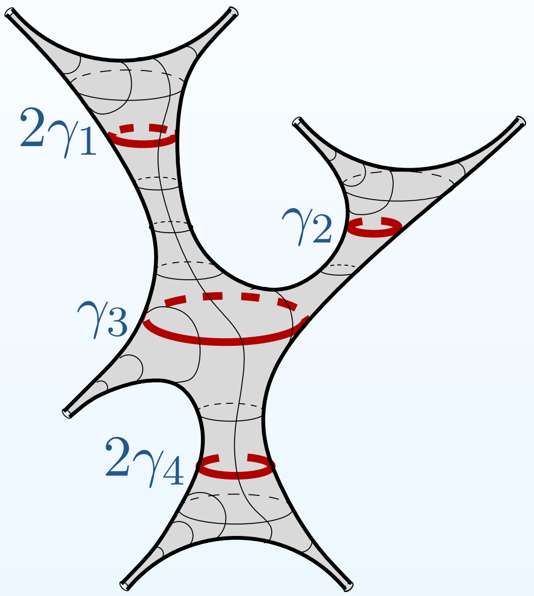
Horizontal geodesics on square-tiled surfaces

Hyperbolic and flat geodesic multicurves

Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$.



Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders.

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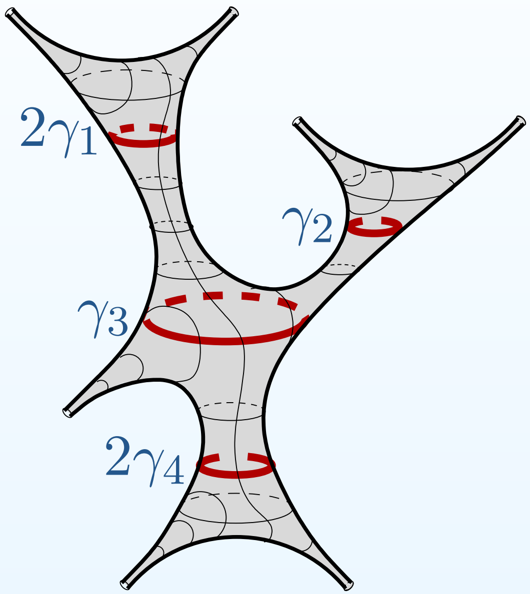

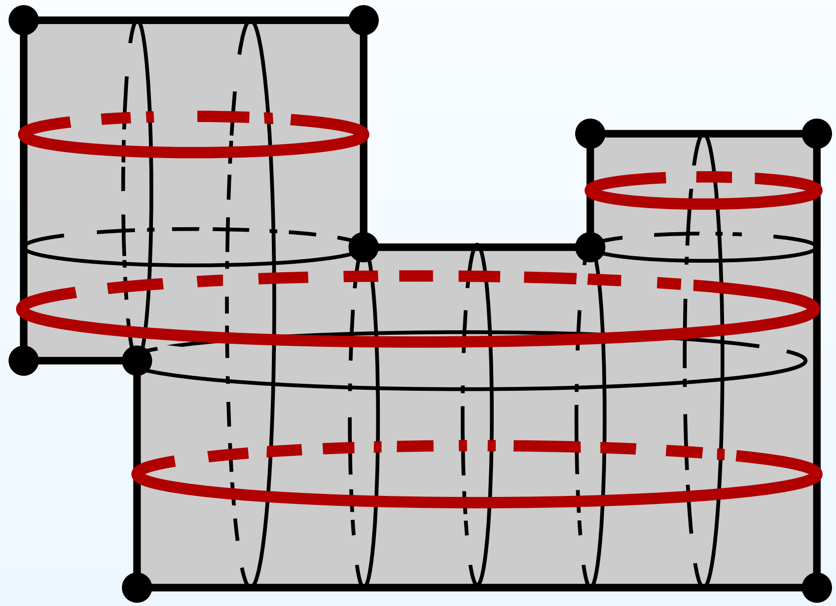

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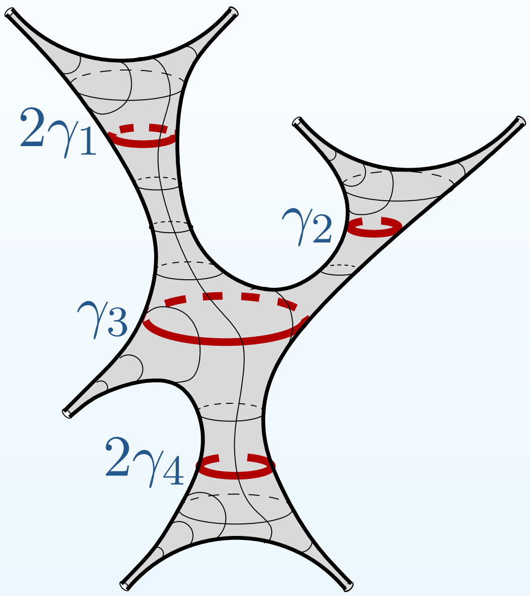

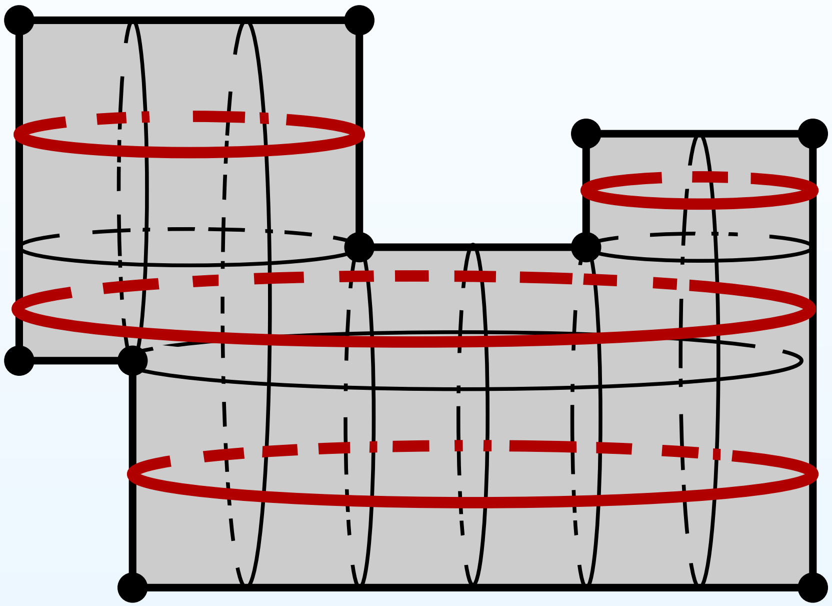

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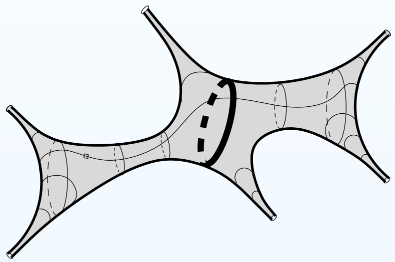
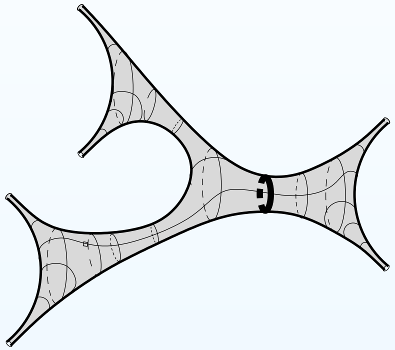
Clearly, there are plenty of square-tiled surface realizing this multicurve.

Frequencies of hyperbolic and flat simple closed geodesics

Theorem (M. Mirzakhani, 2008). *The asymptotic frequency of simple closed hyperbolic geodesics of fixed topological type does not depend on the choice of particular hyperbolic metric.*





Example 1 (M. Mirzakhani; confirmed experimentally by M. Bell and S. Schleimer)

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}$$

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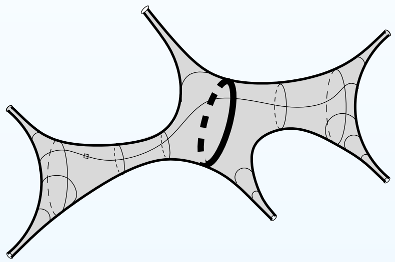

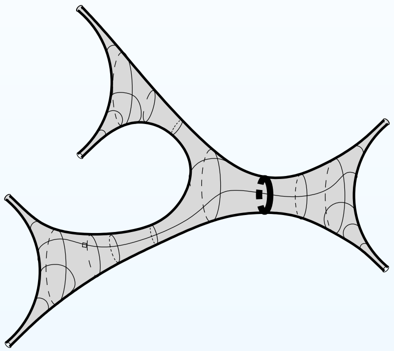

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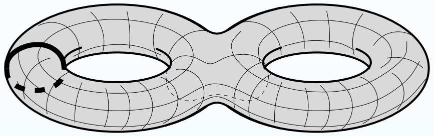
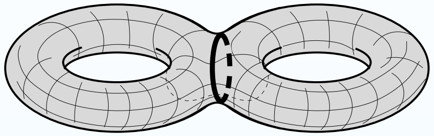
Theorem (Delecroix, Goujard, Zograf, Zorich, 2018). *For any topological class Γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\Gamma)$ of simple closed* ***hyperbolic*** *multicurves of type Γ on any hyperbolic surface $X \in \mathcal{M}_{g,n}$ coincides with the asymptotic frequency of simple closed* ***flat*** *geodesic multicurves of type Γ represented by associated square-tiled surfaces.*

Separating versus non-separating. Large genus asymptotic.

Example 2 (M. Mirzakhani, 2008). *Genus two; no cusps.*





$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

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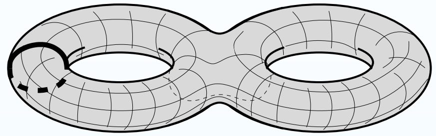

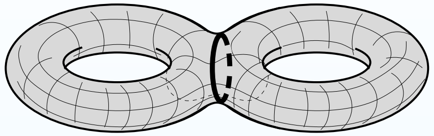

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after correction of a tiny bug in the calculation of Mirzakhani.

Separating versus non-separating. Large genus asymptotic.

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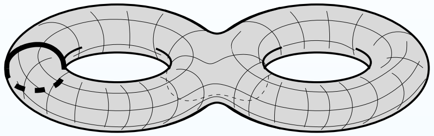

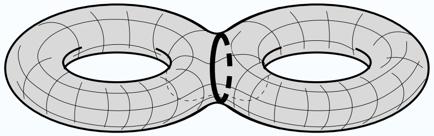

$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in the calculation of Mirzakhani. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell; also nailed by C. Ball.

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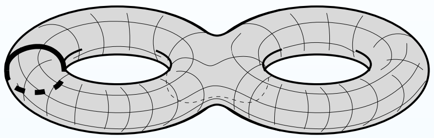

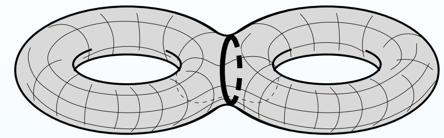

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Question. *Which simple closed geodesics are more frequent on a closed hyperbolic surface of large genus: separating or not? What is the asymptotics of the ratio of their frequencies? Does this ratio stabilize when genus grows?*

Separating versus non-separating. Large genus asymptotic.

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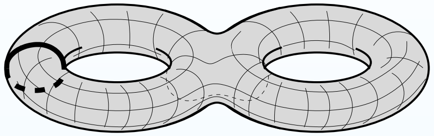

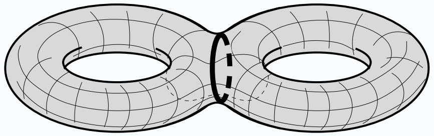

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Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018–).

$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics}(L)}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics}(L)} \sim \frac{1}{4^g}$$

Random simple closed geodesic on a closed hyperbolic surface of large genus separates the surface *extremely rarely*!