Lecture 1. Count of simple closed geodesics on Riemann surfaces (after Maryam Mirzakhani)

Anton Zorich

(Reference: M. Mirzakhani, *“Growth of the number of simple closed geodesics on hyperbolic surfaces”*, Annals of Math. (2) **168** (2008), no. 1, 97–125.)

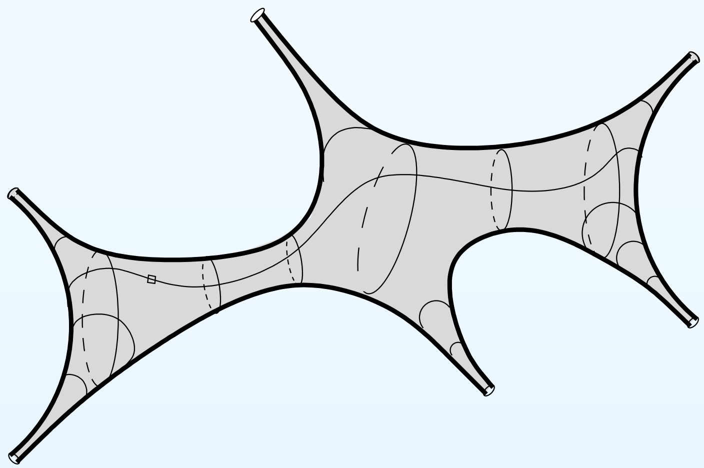
Counting Problems

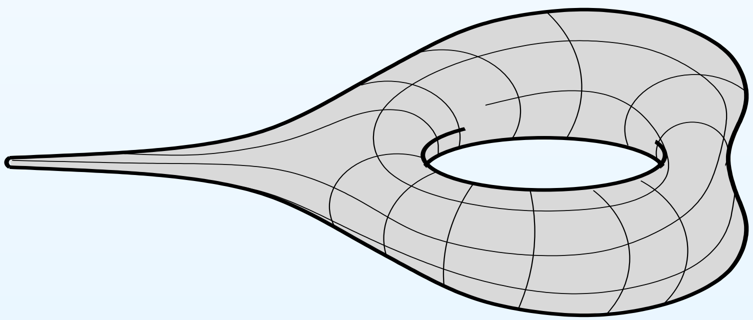
Ventotene, September 7, 2021

Hyperbolic geometry of surfaces

Hyperbolic surfaces

Any smooth orientable surface of genus $g \geq 2$ admits a metric of constant negative curvature (usually chosen to be -1), called *hyperbolic* metric.
Allowing to metric to have several singularities (cusps), one can construct a hyperbolic metric also on a sphere and on a torus.





Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

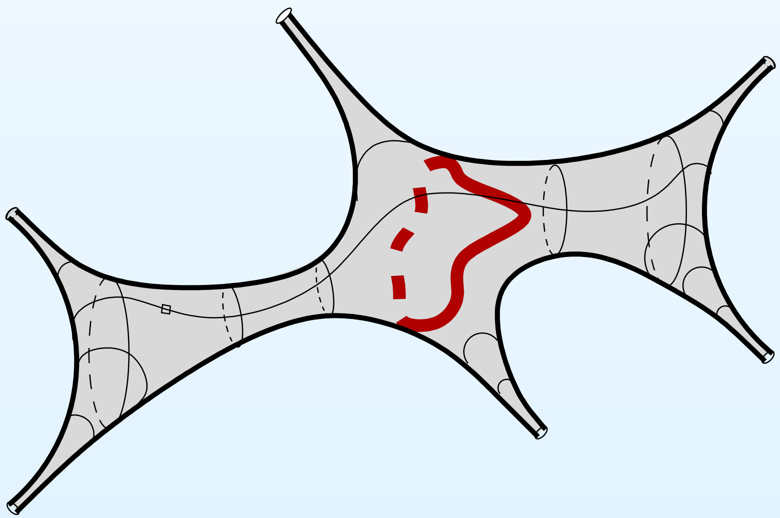
Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

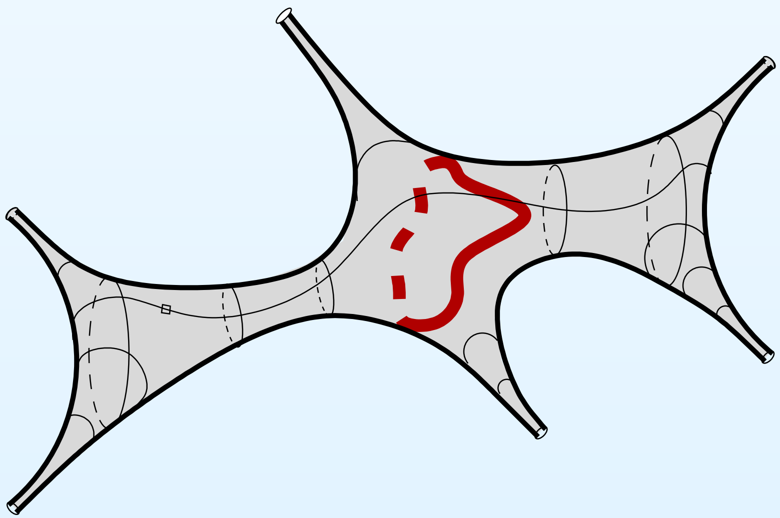


Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

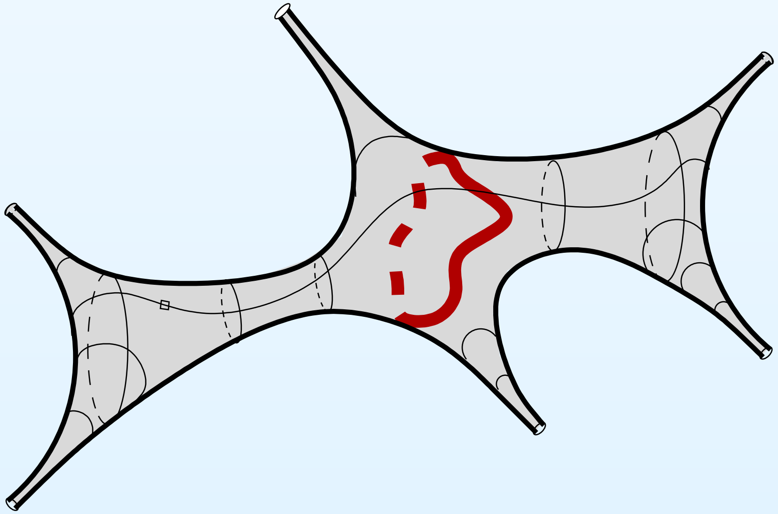


Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

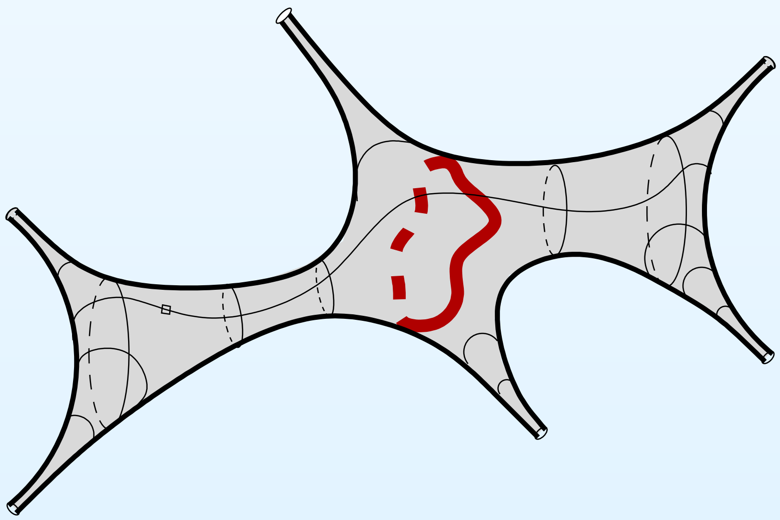


Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

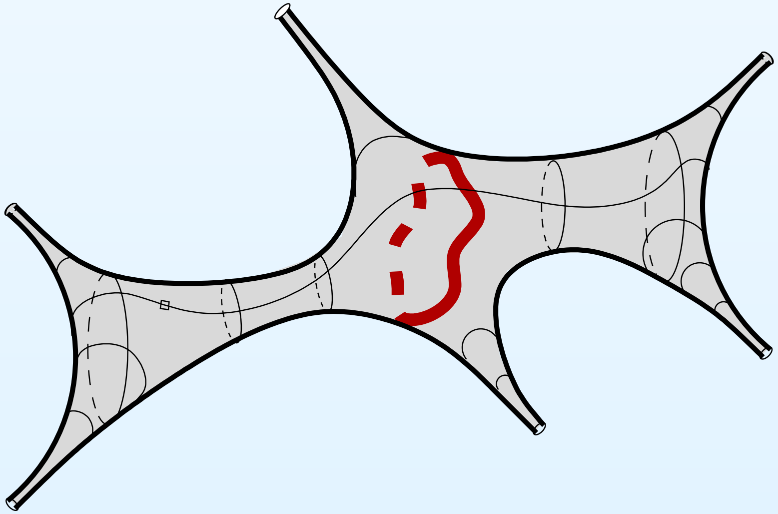


Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

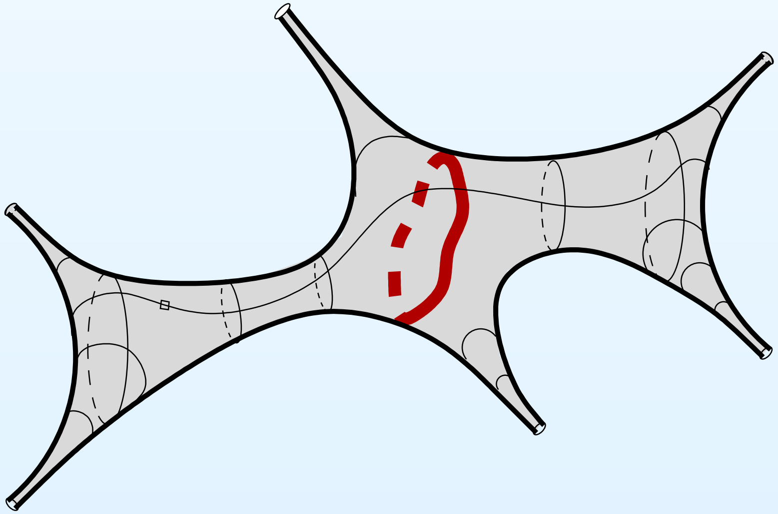


Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

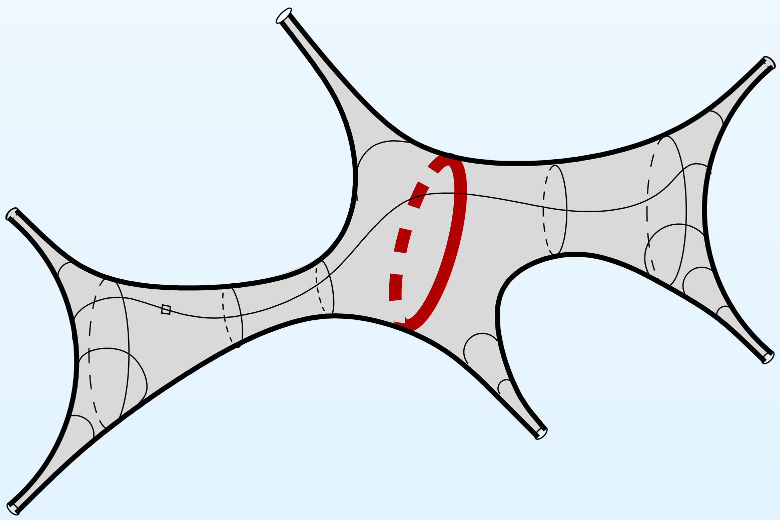


Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

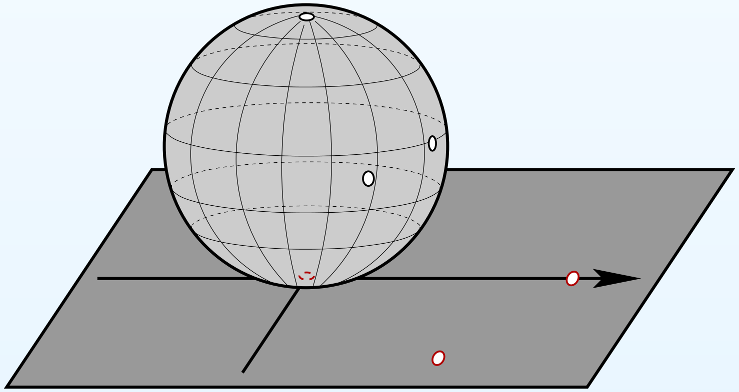
Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.



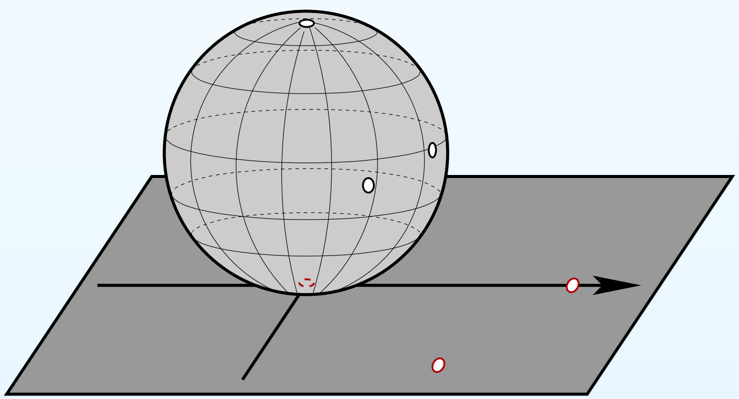
Families of hyperbolic surfaces

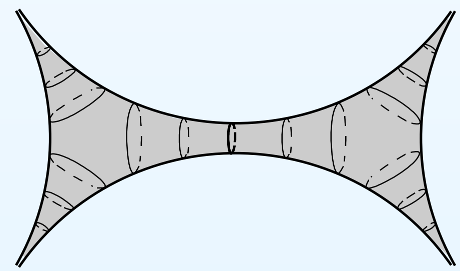
Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).



Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

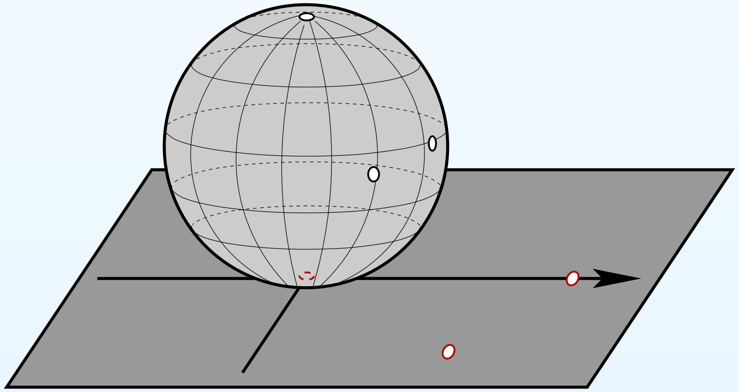


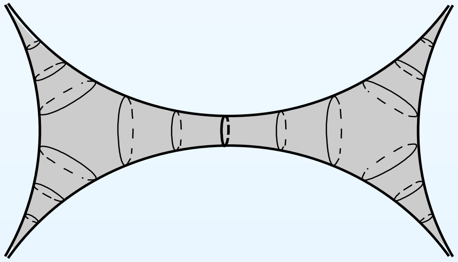


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

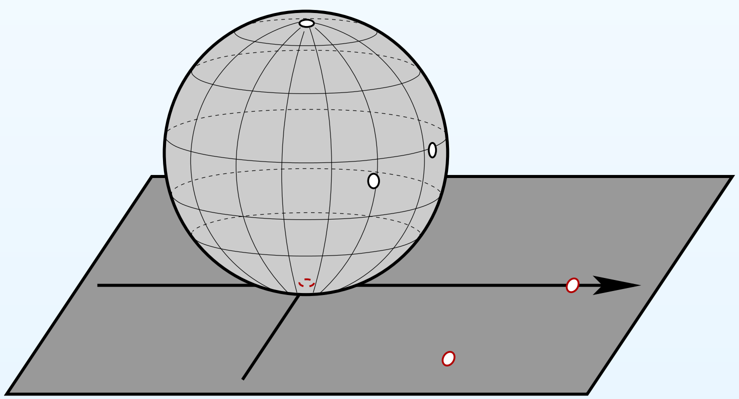


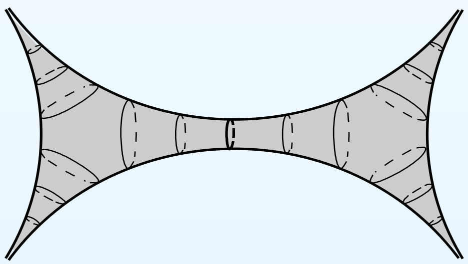


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

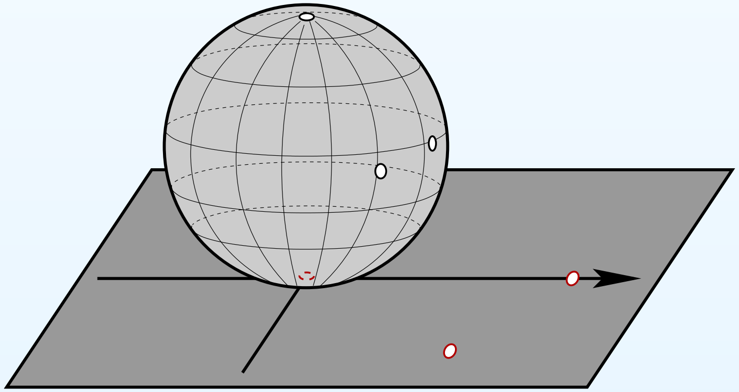


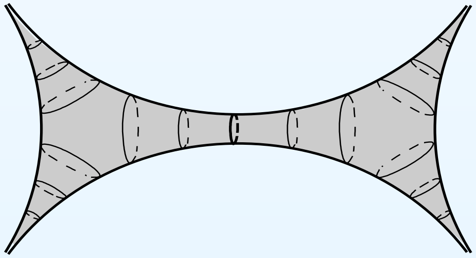


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

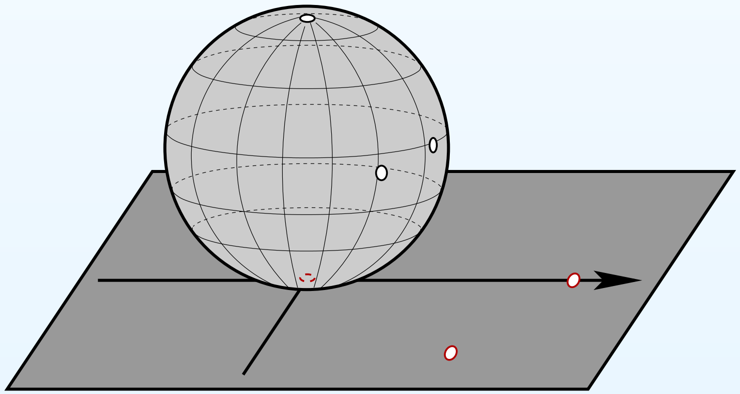


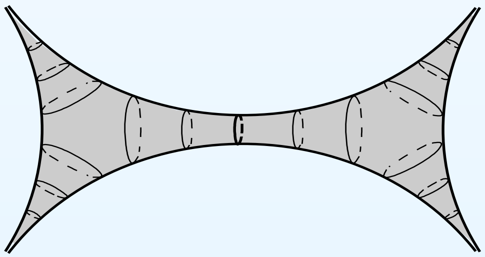


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

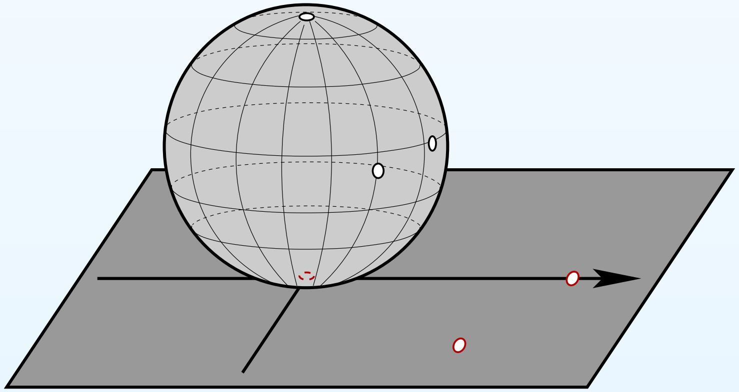


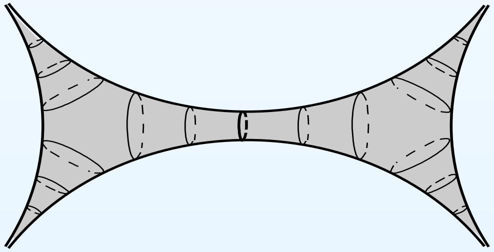


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

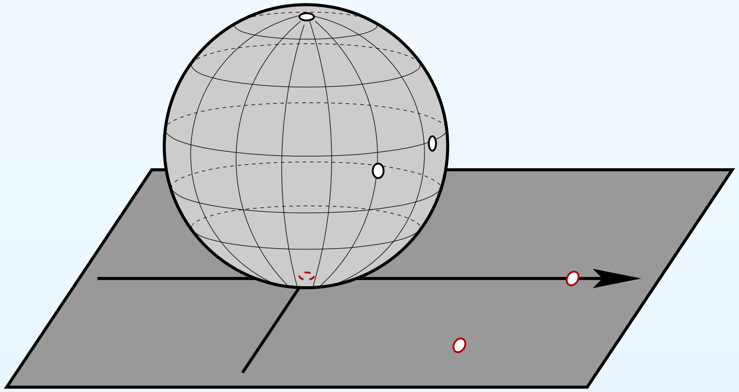


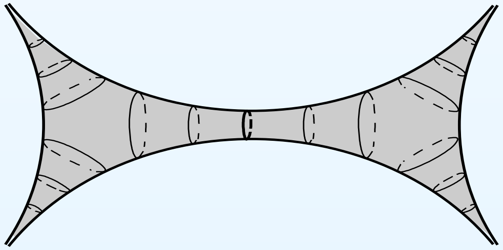


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

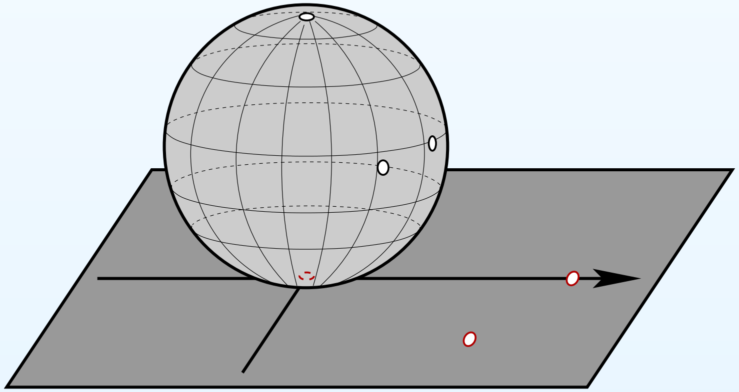


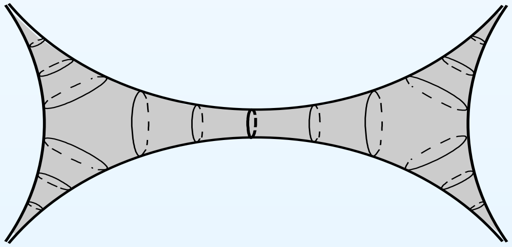


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

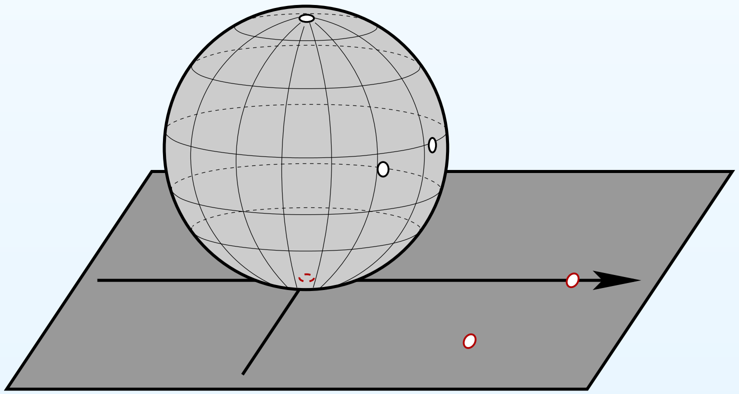


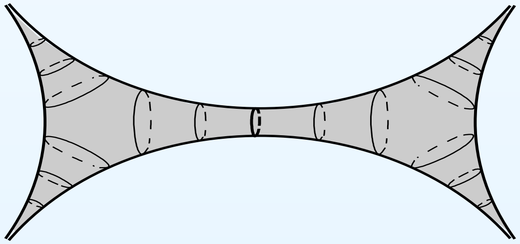


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

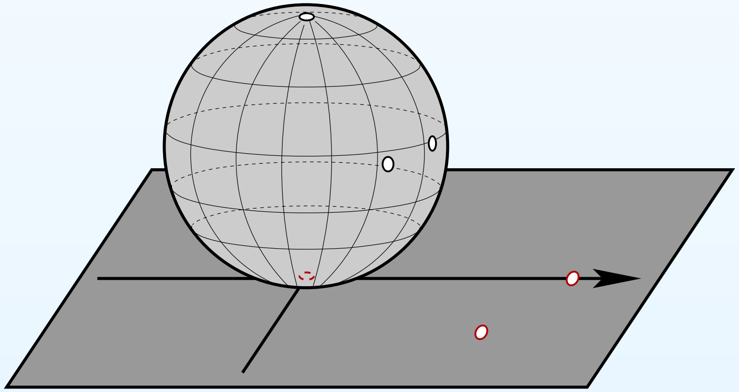


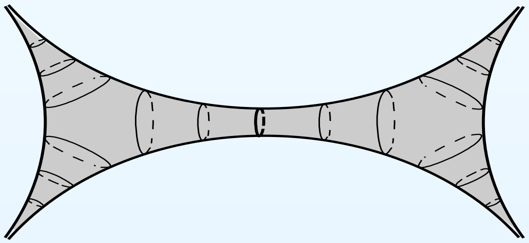


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

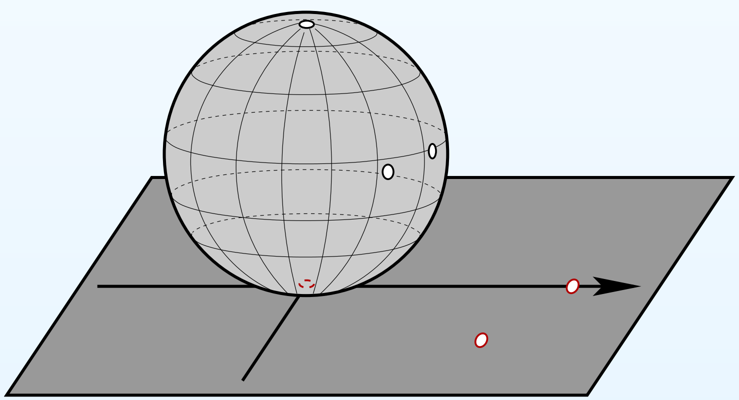


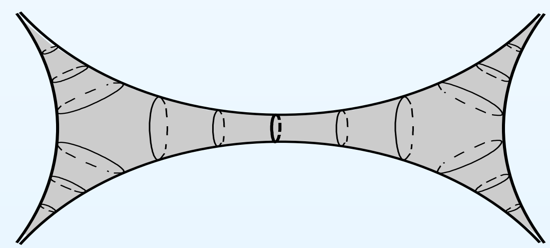


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

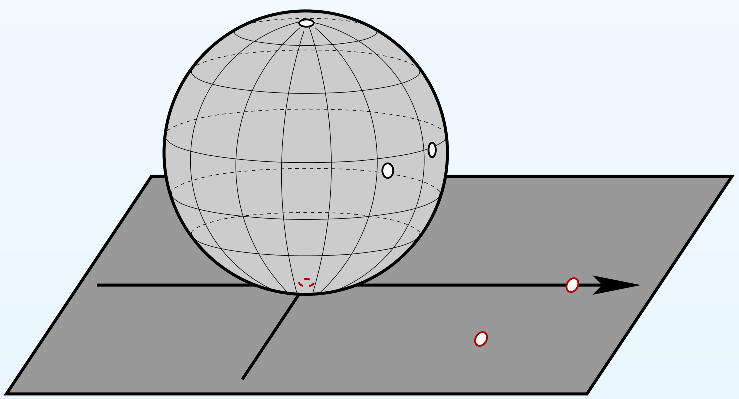


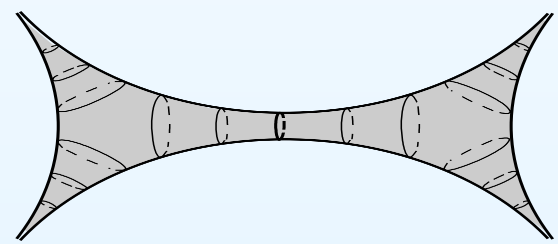


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

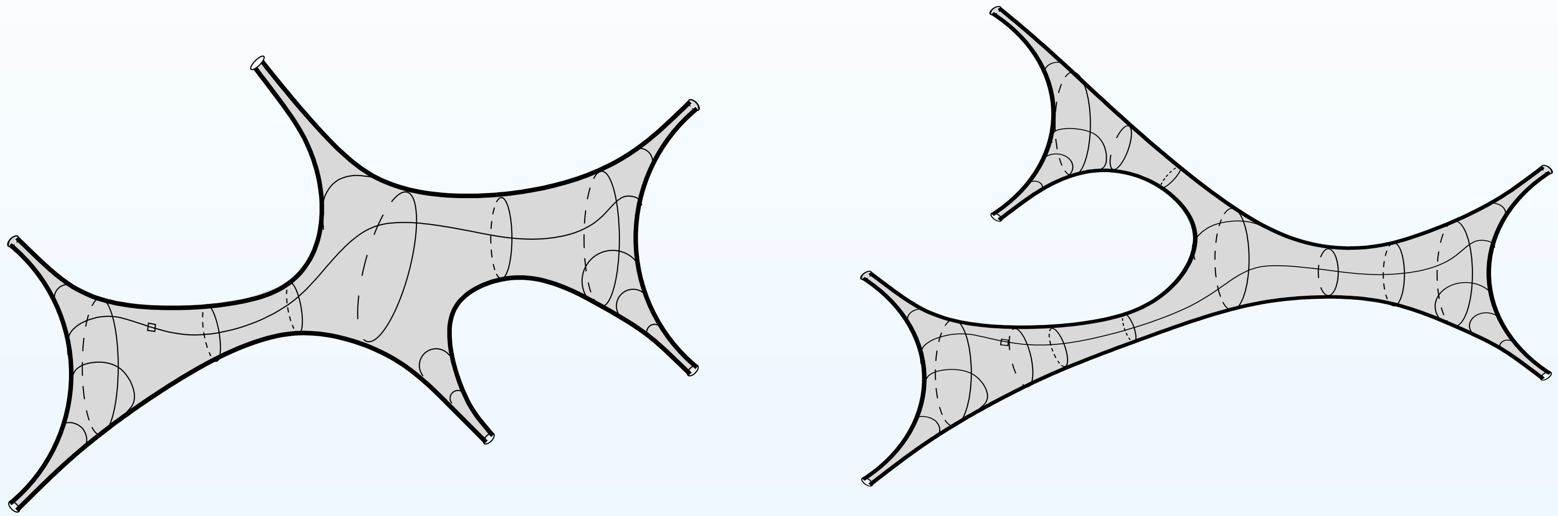




By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Moduli space $\mathcal{M}_{g,n}$

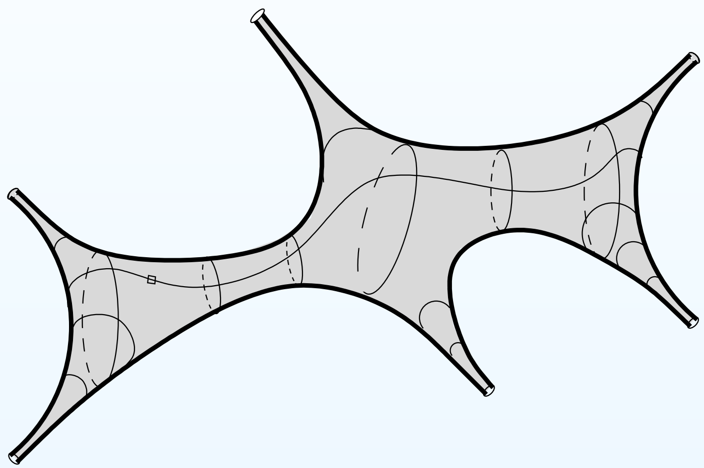
Similarly, we can consider the moduli space $\mathcal{M}_{0,n}$ of spheres with n cusps.

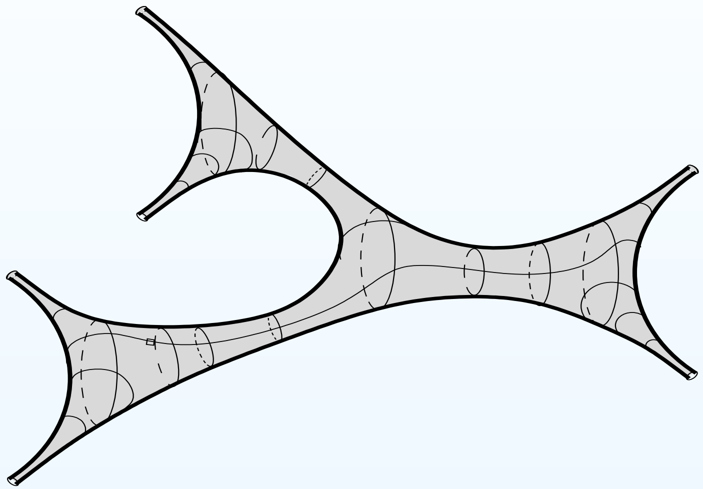


The space $\mathcal{M}_{g,n}$ of configurations of n distinct points on a smooth closed orientable Riemann surface of genus $g > 0$ is even richer. While the sphere admits only one complex structure, a surface of genus $g \geq 2$ admits complex $(3g-3)$-dimensional family of complex structures. As in the case of the Riemann sphere, complex structures on a smooth surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points. For genus $g \geq 2$ one can let $n = 0$ and consider the space $\mathcal{M}_g = \mathcal{M}_{g,0}$ of hyperbolic surfaces without cusps.

Moduli space $\mathcal{M}_{g,n}$

Similarly, we can consider the moduli space $\mathcal{M}_{0,n}$ of spheres with n cusps.





The space $\mathcal{M}_{g,n}$ of configurations of n distinct points on a smooth closed orientable Riemann surface of genus $g > 0$ is even richer. While the sphere admits only one complex structure, a surface of genus $g \geq 2$ admits complex $(3g-3)$-dimensional family of complex structures. As in the case of the Riemann sphere, complex structures on a smooth surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points. For genus $g \geq 2$ one can let $n = 0$ and consider the space $\mathcal{M}_g = \mathcal{M}_{g,0}$ of hyperbolic surfaces without cusps.

Space of multicurves

Topological types of simple closed curves

Let us say that two simple closed curves on a smooth surface have the same *topological type* if there is a diffeomorphism of the surface sending one curve to another.

It immediately follows from the classification theorem of surfaces that there is a finite number of topological types of simple closed curves. For example, if the surface does not have punctures, all simple closed curves which do not separate the surface into two pieces, belong to the same class.

One can consider more general *primitive multicurves*: collections of pairwise disjoint non-homotopic simple closed curves. For any fixed pair (g, n) the number of topological types of primitive multicurves on a surface of genus g with n punctures is also finite.

Topological types of simple closed curves

Let us say that two simple closed curves on a smooth surface have the same *topological type* if there is a diffeomorphism of the surface sending one curve to another.

It immediately follows from the classification theorem of surfaces that there is a finite number of topological types of simple closed curves. For example, if the surface does not have punctures, all simple closed curves which do not separate the surface into two pieces, belong to the same class.

One can consider more general *primitive multicurves*: collections of pairwise disjoint non-homotopic simple closed curves. For any fixed pair (g, n) the number of topological types of primitive multicurves on a surface of genus g with n punctures is also finite.

Topological types of simple closed curves

Let us say that two simple closed curves on a smooth surface have the same *topological type* if there is a diffeomorphism of the surface sending one curve to another.

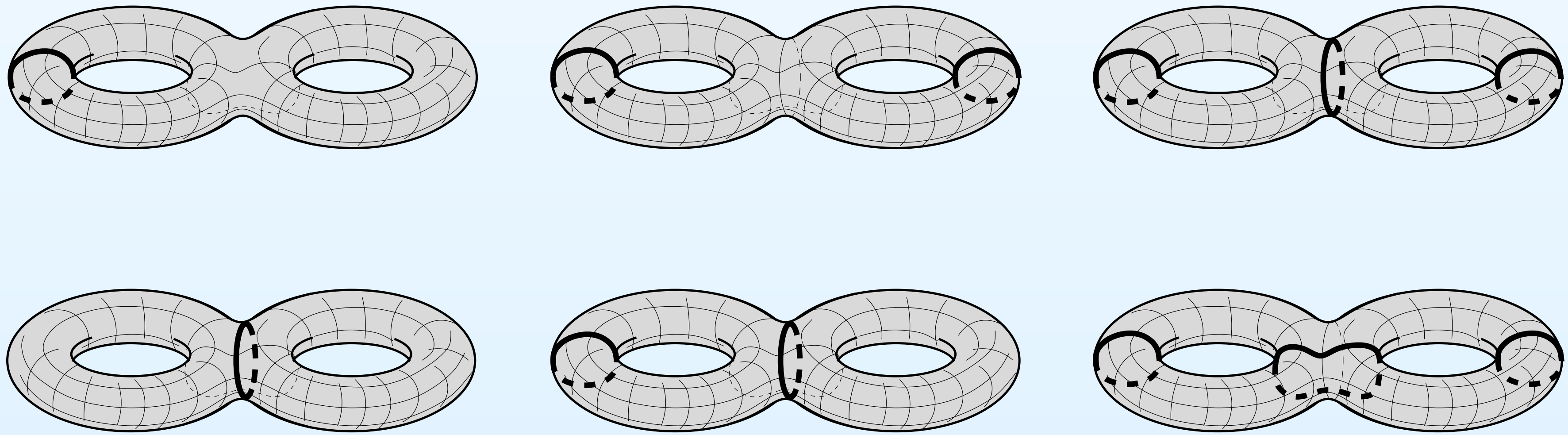
It immediately follows from the classification theorem of surfaces that there is a finite number of topological types of simple closed curves. For example, if the surface does not have punctures, all simple closed curves which do not separate the surface into two pieces, belong to the same class.

One can consider more general *primitive multicurves*: collections of pairwise disjoint non-homotopic simple closed curves. For any fixed pair (g, n) the number of topological types of primitive multicurves on a surface of genus g with n punctures is also finite.

Example: primitive multicurves on a surface of genus two

The picture below illustrates all possible types of primitive multicurves on a surface of genus two without punctures.

Note that contracting all components of a multicurve we get a “stable curve” — a Riemann surface degenerated in one of the several regular ways. In this way the “topological types of primitive multicurves” on a smooth surface $S_{g,n}$ of genus g with n punctures are in the natural bijective correspondence with boundary classes of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of pointed complex curves.



Mapping class group

The group of all diffeomorphisms of a closed smooth orientable surface of genus g quotient over diffeomorphisms homotopic to identity is called the *mapping class group* and is denoted by Mod_g.

When the surface has n marked points (punctures) we require that diffeomorphism sends marked points to marked points; the corresponding mapping class group is denoted $\mathrm{Mod}_{g,n}$.

Mapping class group

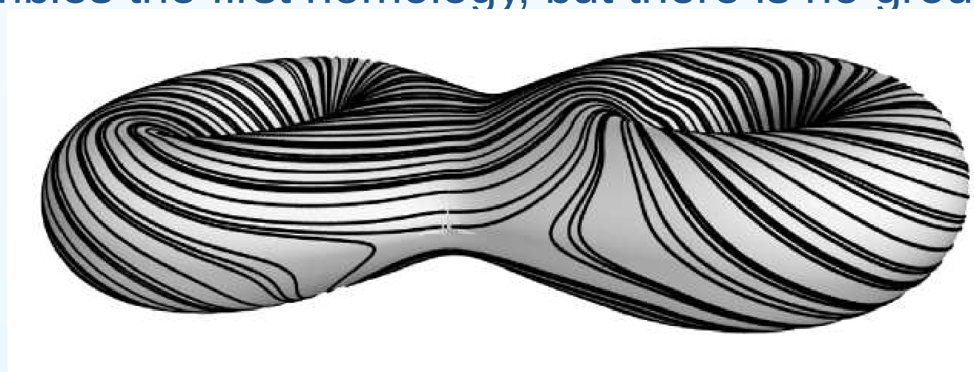
The group of all diffeomorphisms of a closed smooth orientable surface of genus g quotient over diffeomorphisms homotopic to identity is called the *mapping class group* and is denoted by Mod_g.

When the surface has n marked points (punctures) we require that diffeomorphism sends marked points to marked points; the corresponding mapping class group is denoted $\mathrm{Mod}_{g,n}$.

Simple closed multicurve, its topological type and underlying primitive multicurve

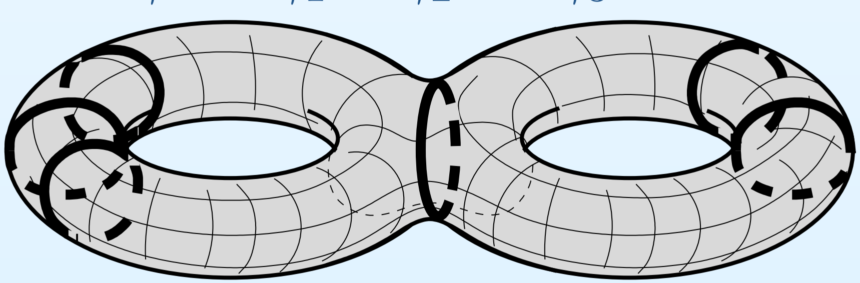
The first homology $H_1(M^2;\mathbb{Z})$ of the surface is great to study closed curves, but it ignores some interesting curves. The fundamental group $\pi_1(M^2)$ is also wonderful, but it is mainly designed to work with self-intersecting cycles. Thurston invented yet another structure to work with simple closed multicurves; in many aspects it resembles the first homology, but there is no group structure.

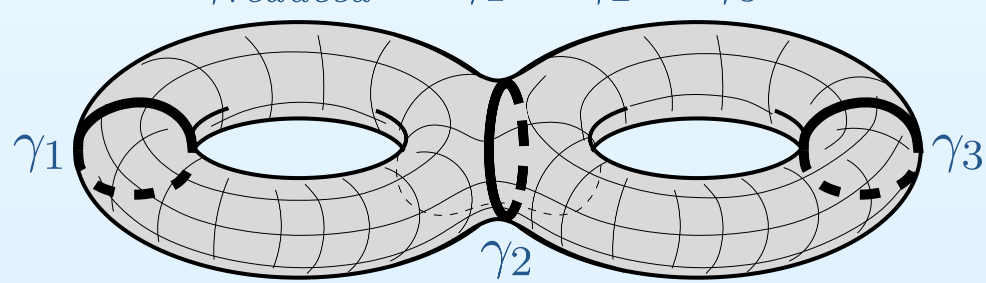
A general multicurve ρ:



the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

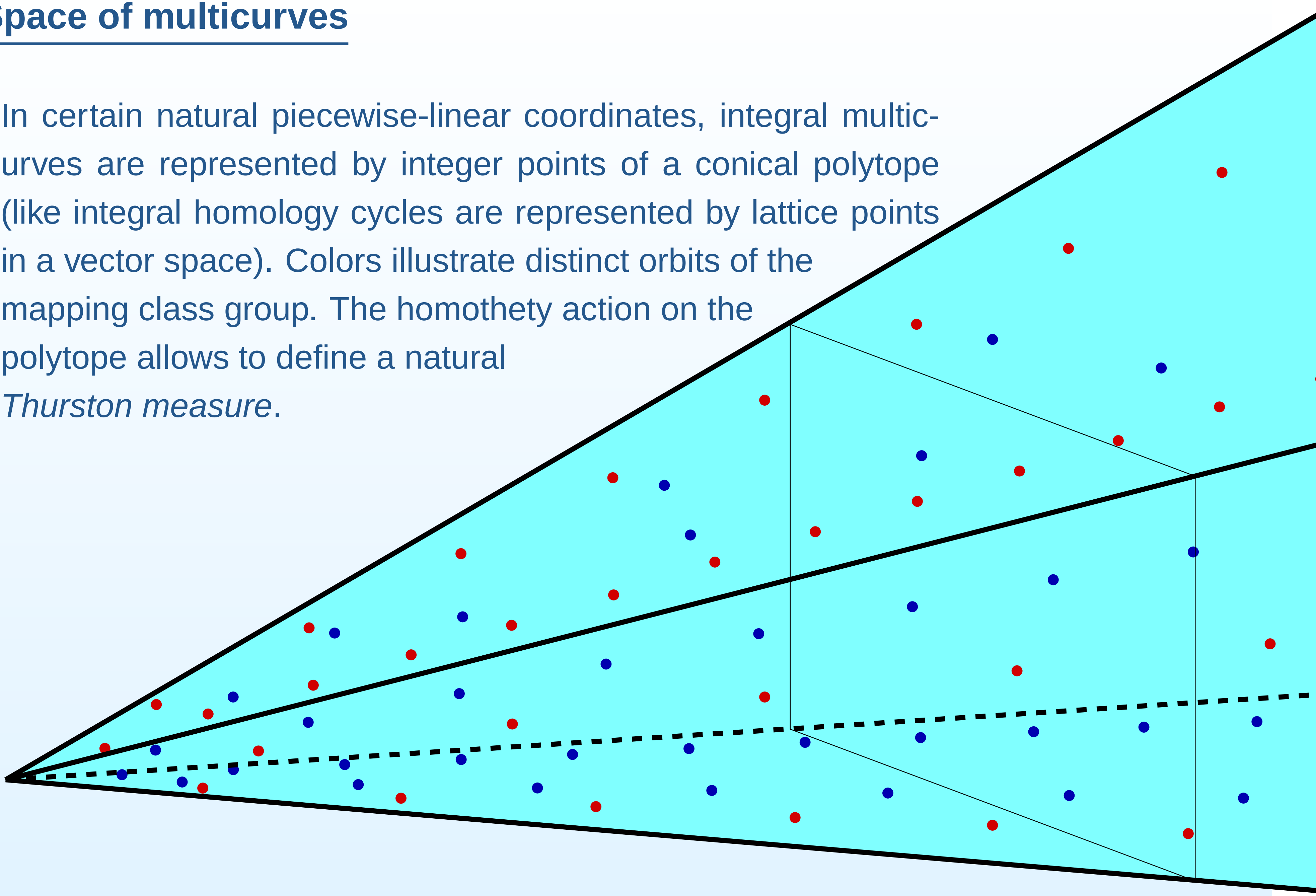
$\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ $\qquad$ $\gamma_{reduced} = \gamma_1 + \gamma_2 + \gamma_3$





Space of multicurves

In certain natural piecewise-linear coordinates, integral multicurves are represented by integer points of a conical polytope (like integral homology cycles are represented by lattice points in a vector space). Colors illustrate distinct orbits of the mapping class group. The homothety action on the polytope allows to define a natural *Thurston measure*.



Space of measured laminations $\mathcal{ML}_{g,n}$. Ergodicity of the Thurston measure

In the presence of a hyperbolic metric the integral multicurves take the shape of simple closed geodesic multicurves. Moreover, every (not necessary integral) point of the conical polytope defines a *measured geodesic lamination*. The “natural coordiantes” are, for example, the *train tracks* coordinates.

Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves, so the piecewise-linear action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the “integral lattice” $\mathcal{ML}_{g,n}(\mathbb{Z})$, and, hence, preserves the Thurston measure μ_{Th}.

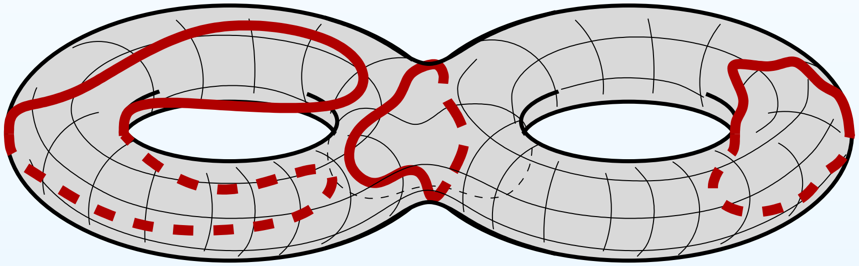
Theorem (H. Masur, 1985). *The action of* $\mathrm{Mod}_{g,n}$ *on* $\mathcal{ML}_{g,n}$ *is ergodic with respect to the Lebesgue measure class (i.e. any measurable subset of* $\mathcal{ML}_{g,n}$ *invariant under* $\mathrm{Mod}_{g,n}$ *has measure zero or its complement has measure zero). Any* $\mathrm{Mod}_{g,n}$*-invariant measure in the Lebesgue measure class is just Thurston measure rescaled by some constant factor.*

Mirzakhani’s count of simple closed geodesics: statement of results

Picture by François Labourie taken at CIRM

Geodesic representatives of multicurves

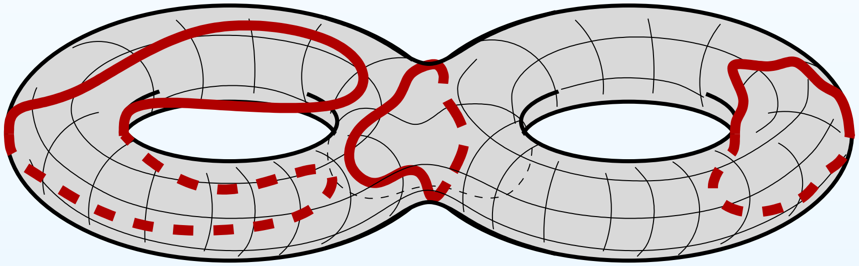
Consider now several pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. We have seen that in the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves become simple closed geodesics.

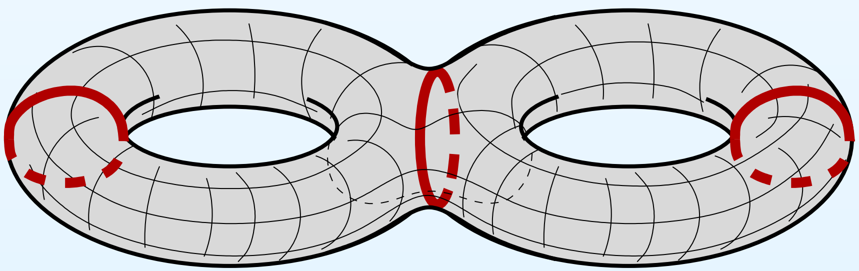


Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

Geodesic representatives of multicurves

Consider now several pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. We have seen that in the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves become simple closed geodesics.





Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

Hyperbolic length of a multicurve

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

Hyperbolic length of a multicurve

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (which is the average value of $B(X)$ over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ (expressed in terms of the Witten–Kontsevich correlators).

Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

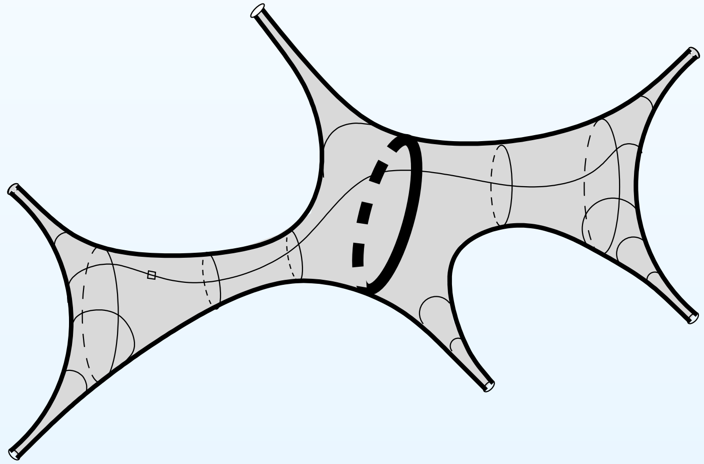
Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (which is the average value of $B(X)$ over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ (expressed in terms of the Witten–Kontsevich correlators).

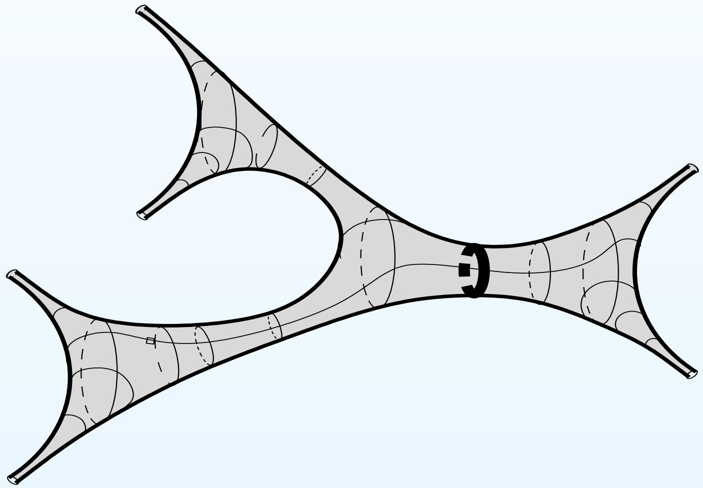
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

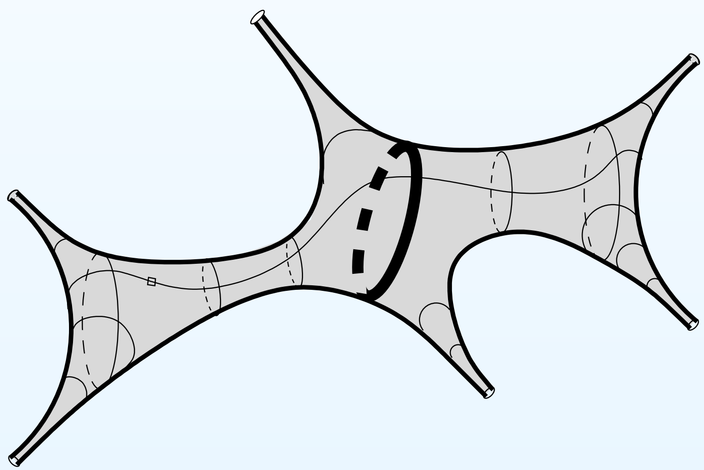
A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.

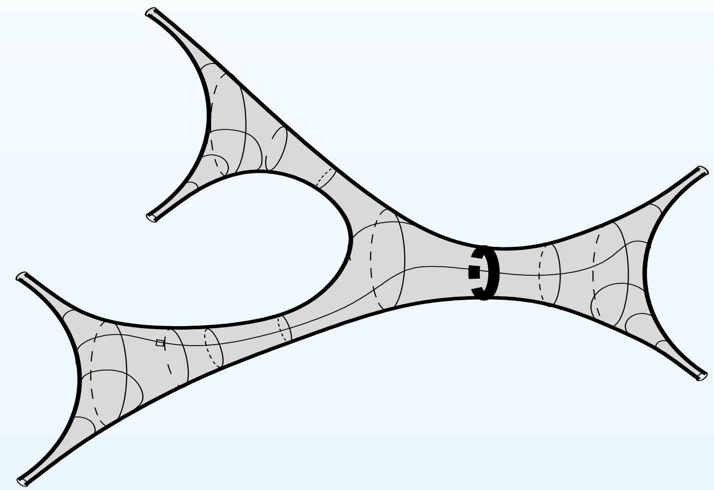




Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.



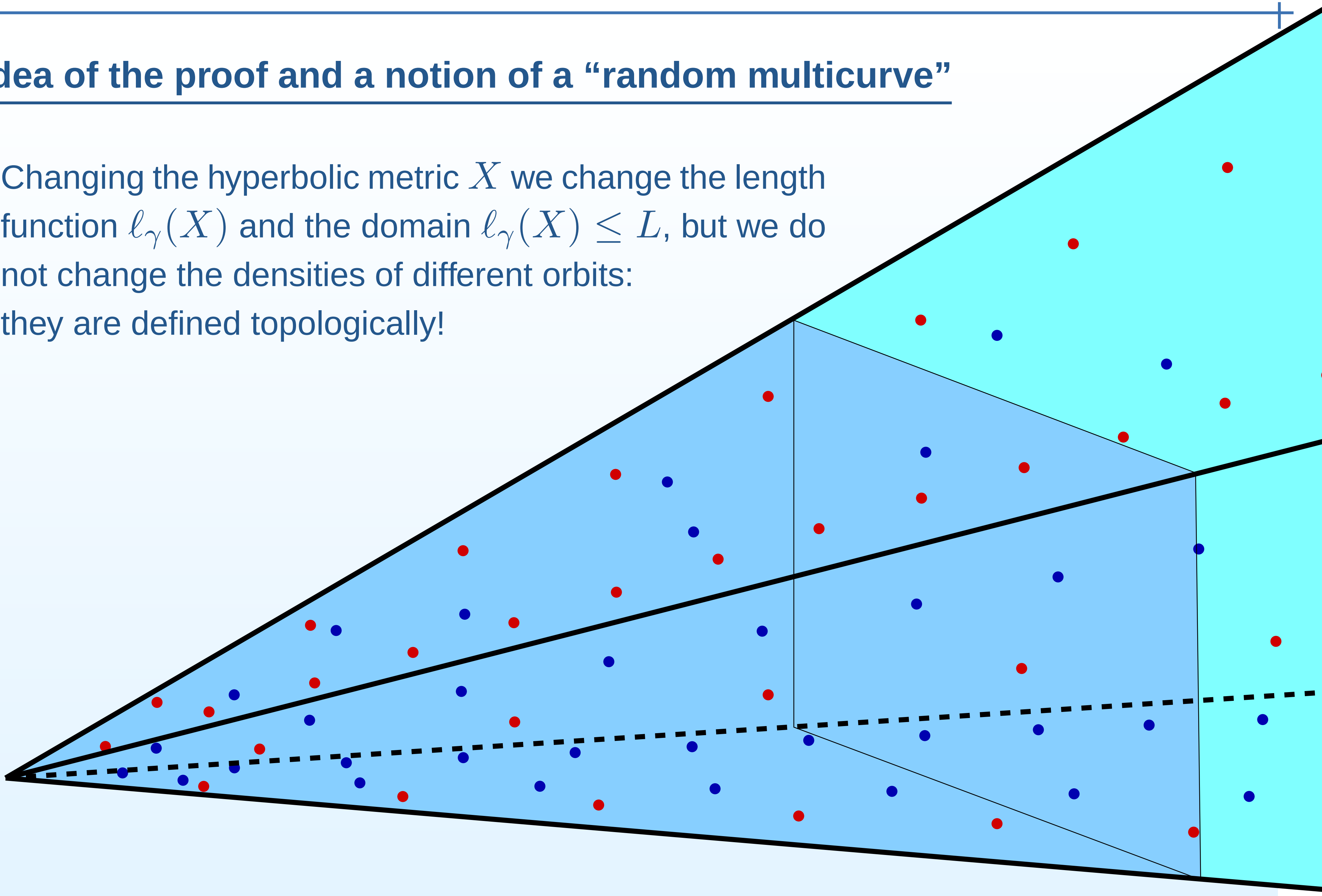


Example. (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by other means.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

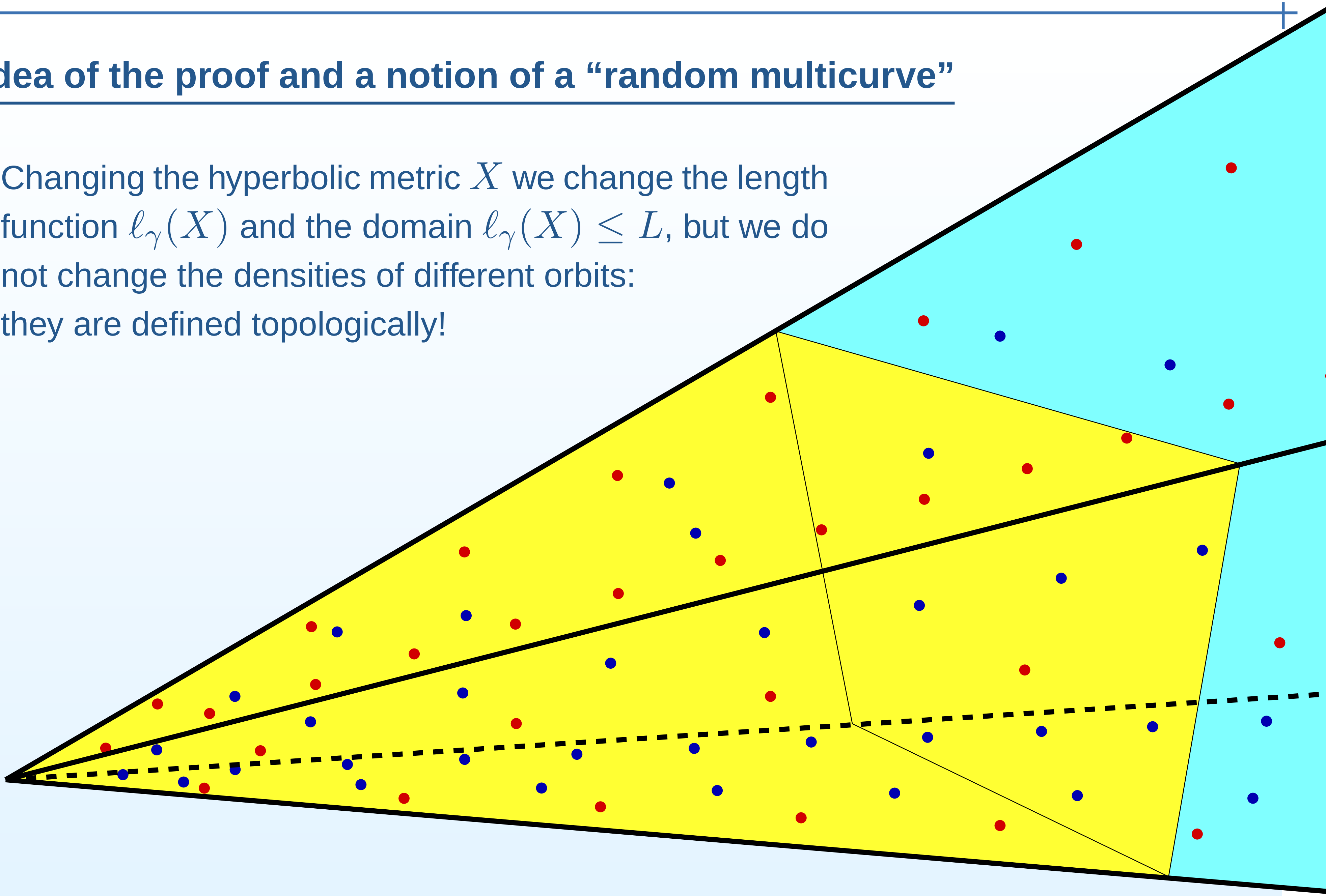
Idea of the proof and a notion of a “random multicurve”

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \le L$, but we do not change the densities of different orbits: they are defined topologically!



Idea of the proof and a notion of a “random multicurve”

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \leq L$, but we do not change the densities of different orbits: they are defined topologically!



More honest idea of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the “unit ball” B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\, dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\, dX =$$

$$= \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}}\, dX = \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}}\, dX =$$

$$= \lim_{L\to+\infty} \frac{1}{L^{6g-6+2n}} \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX = \lim_{L\to+\infty} \frac{P(L,\gamma)}{L^{6g-6+2n}}\, dX = c(\gamma)\,,$$

so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzahani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

More honest idea of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the "unit ball" B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n} \cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\, dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\, dX =$$

$$= \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}}\, dX = \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}}\, dX =$$

$$= \lim_{L\to+\infty} \frac{1}{L^{6g-6+2n}} \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX = \lim_{L\to+\infty} \frac{P(L,\gamma)}{L^{6g-6+2n}}\, dX = c(\gamma)\,,$$

so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzahani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

Witten–Kontsevich correlators. Mirzakhani's formula for frequencies.

Intersection numbers (correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \dots, P_n \in C$ is a complex orbifold of complex dimension $3g - 3 + n$.
Choose index i in $\{1, \dots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \dots + d_n = 3g - 3 + n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in* 2*-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; one of alternative proofs belongs to M. Mirzakhani.

Intersection numbers (correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g - 3 + n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$. The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g - 3 + n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in* 2*-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; one of alternative proofs belongs to M. Mirzakhani.

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.

Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\, \mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

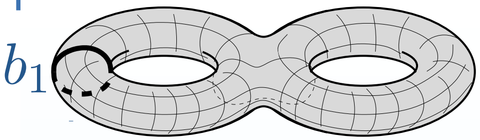
where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

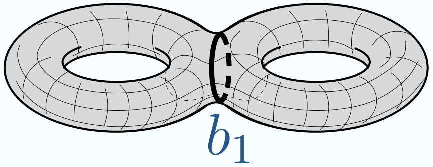
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

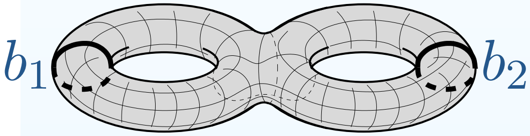
and extend it to symmetric polynomials in b_i by linearity.



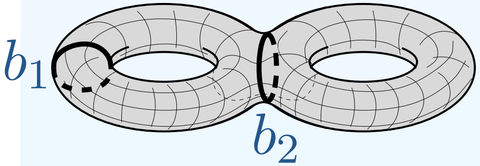
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



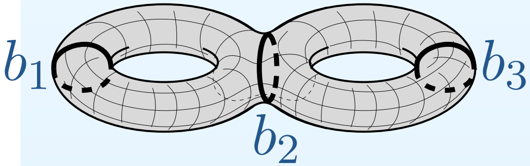
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



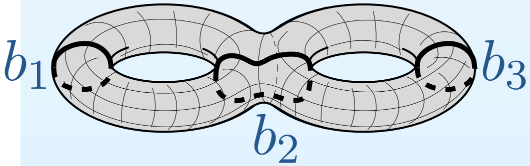
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



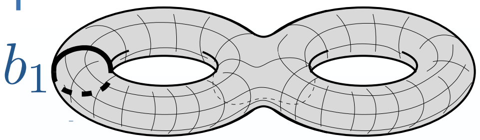
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



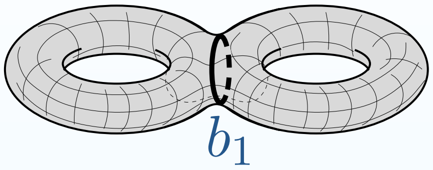
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



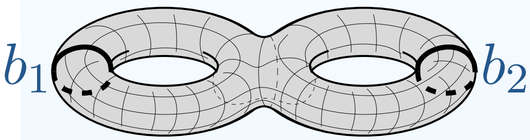
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$



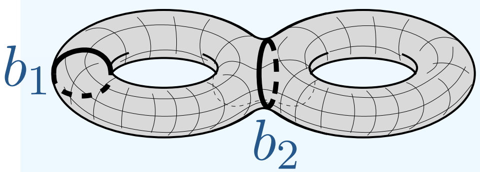
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) \qquad = \frac{1}{2} \cdot b_1 \left(\frac{1}{384} (2b_1^2)(2b_1^2) \right)$$



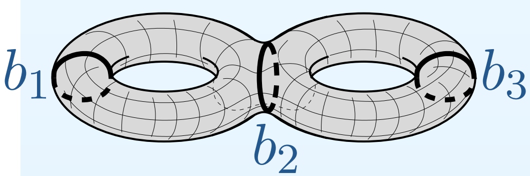
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) \qquad = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$



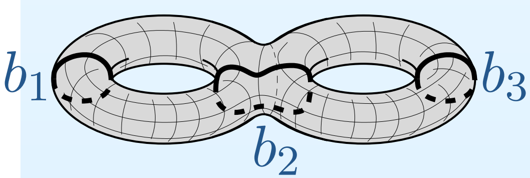
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) \qquad = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4} (2b_1^2 + 2b_2^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) \qquad = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$

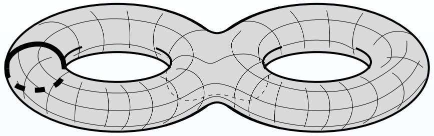


$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) \qquad = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

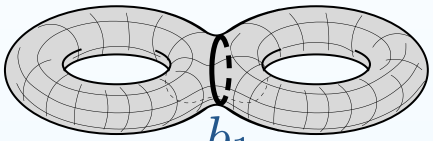


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) \qquad = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

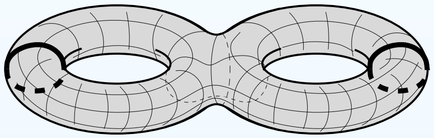
Computation of Mirzakhani's frequencies



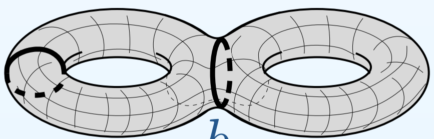
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



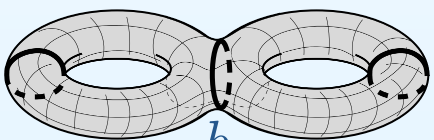
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



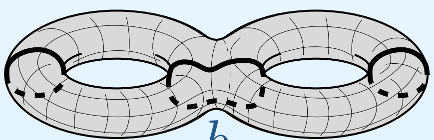
$$\frac{1}{16}(b_1^3 b_2 + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



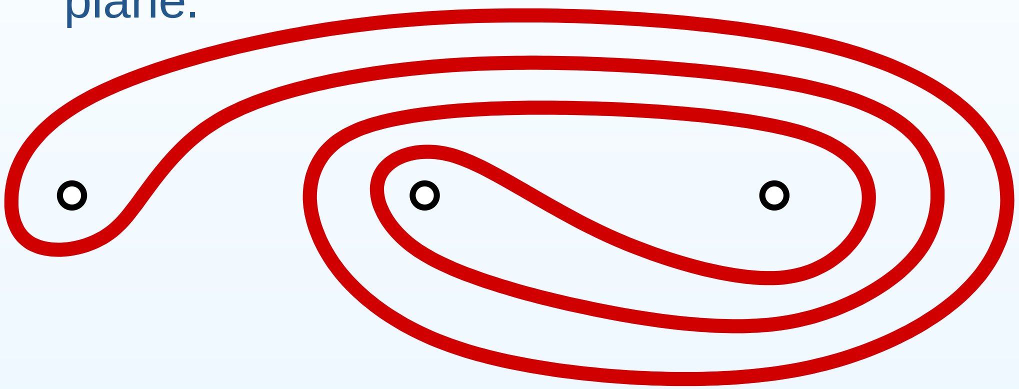
$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

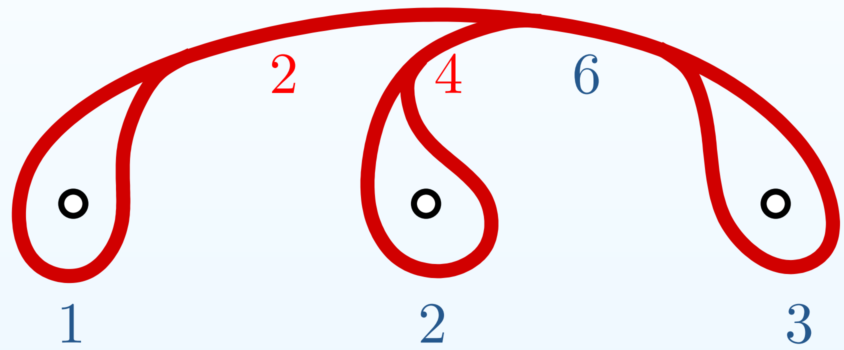
The resulting numbers are proportional to frequencies of corresponding multicurves.

Train track coordinates (after section 15.1 of the book of B. Farb and D. Margalit "A Primer on Mapping Class Groups")

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



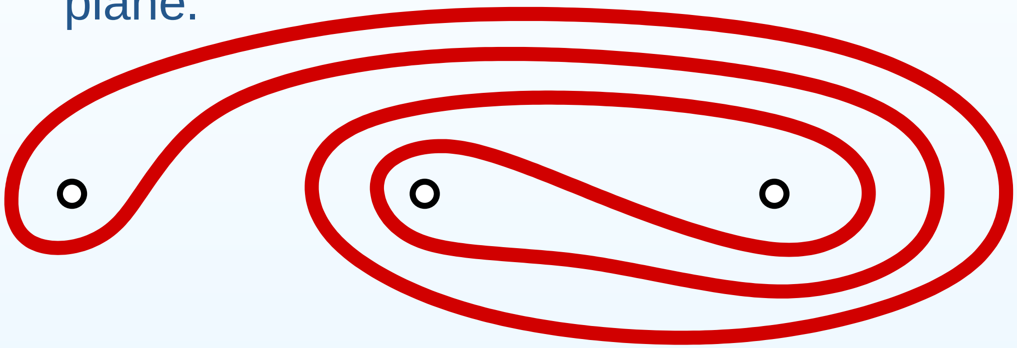


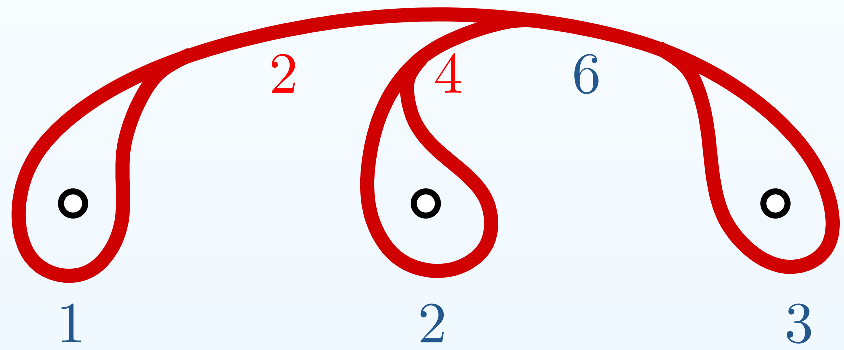
We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



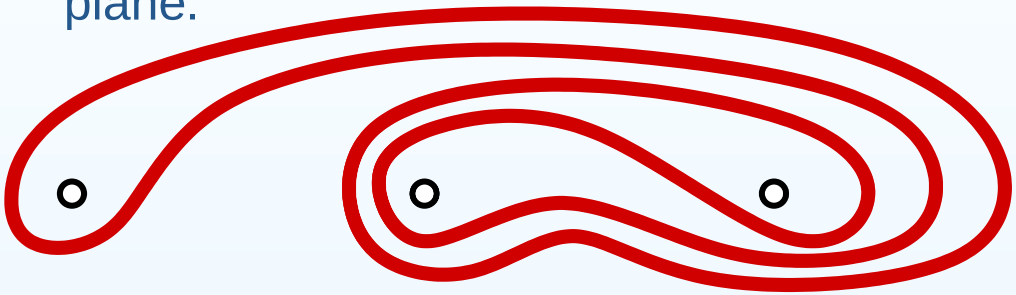


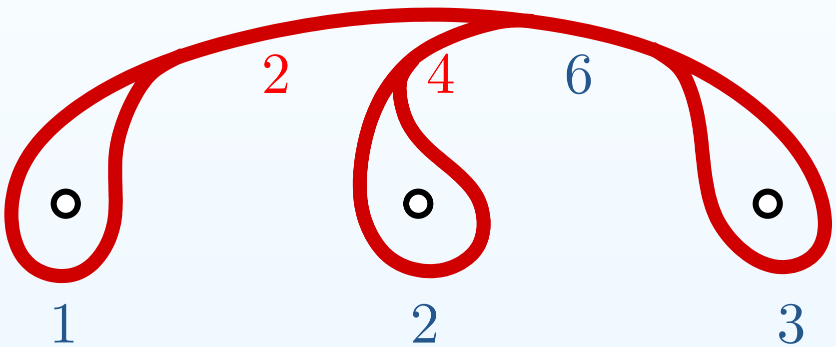
We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



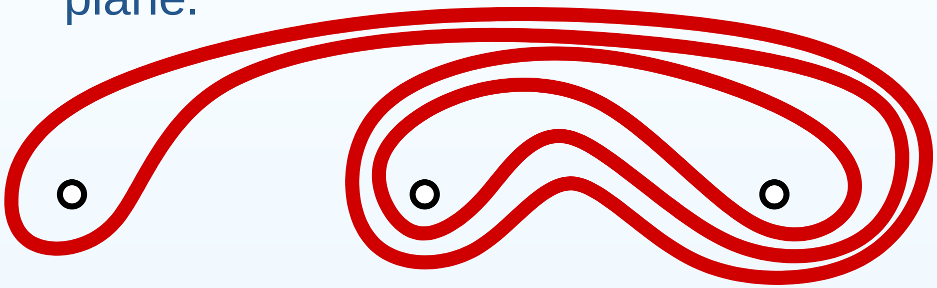


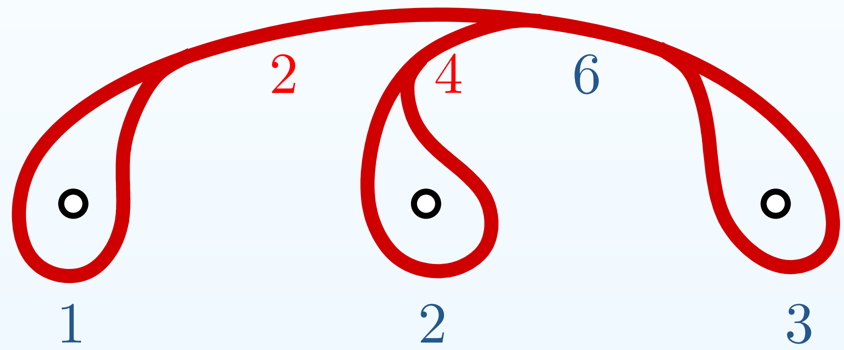
We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



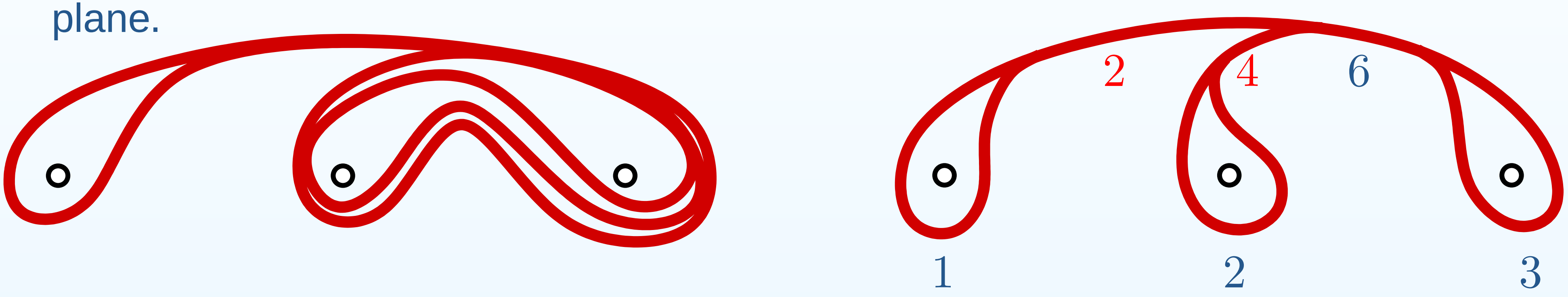


We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.

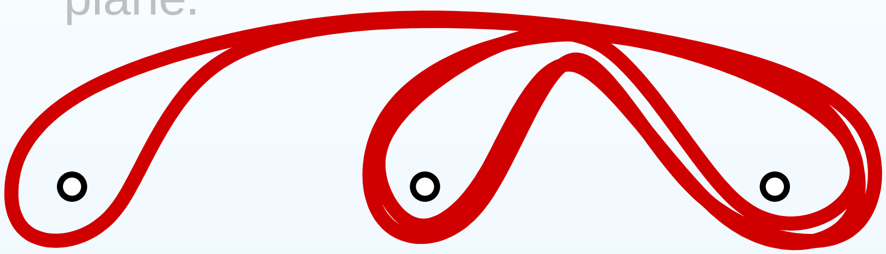


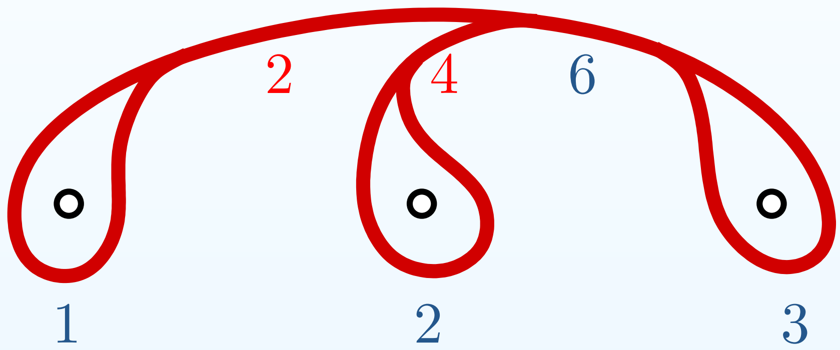
We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



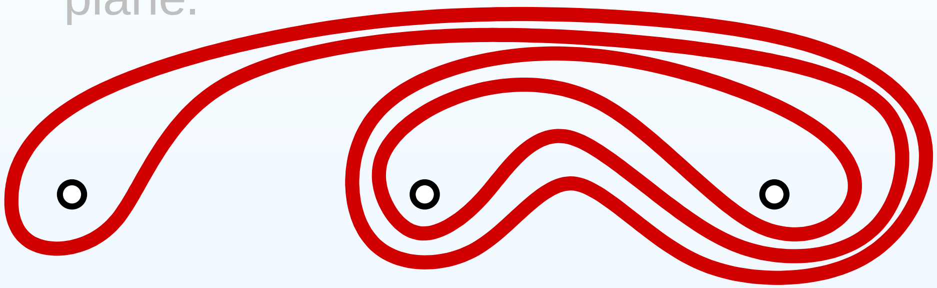


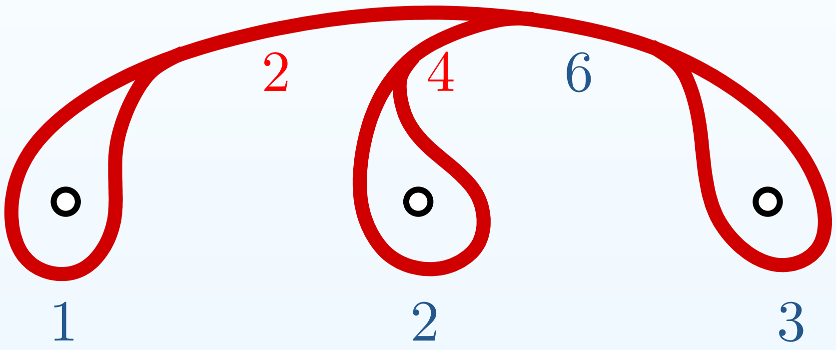
We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



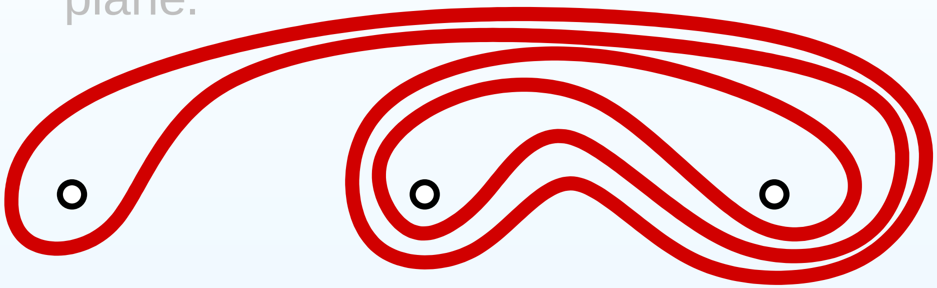


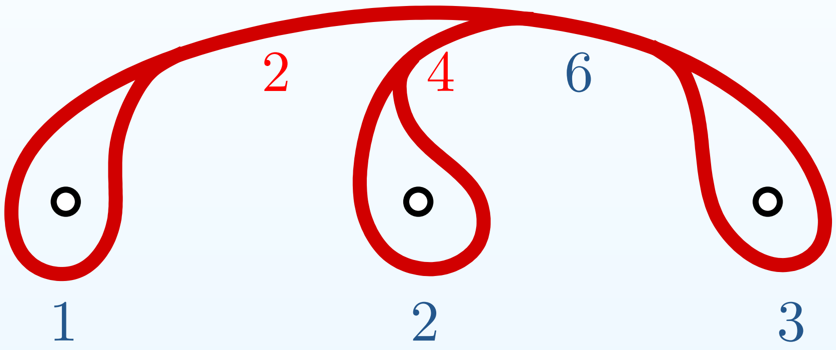
We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



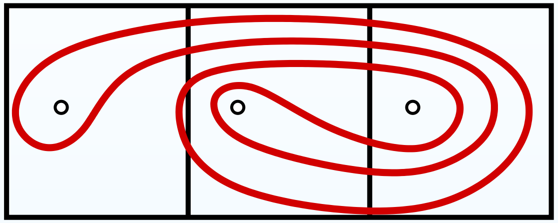


We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other. Note that the two weights in red uniquely determine all other weights.

Four basic train tracks on $S_{0,4}$

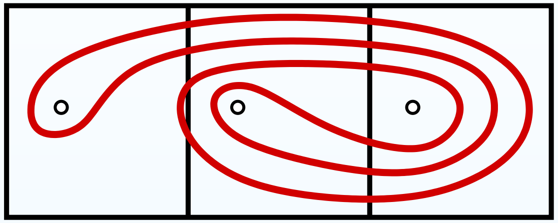
Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:

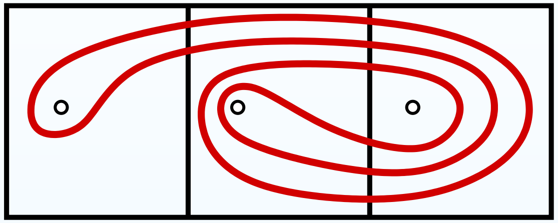


By further isotopy, we eliminate bigons with the vertical edges of the three squares.

Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:

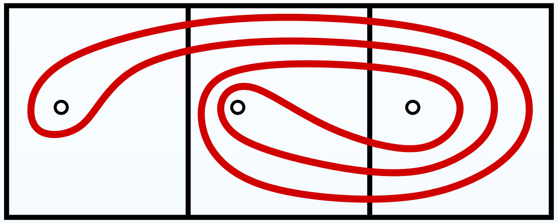


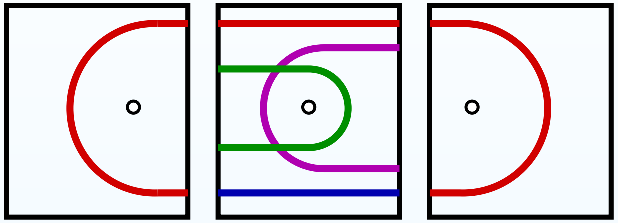
By further isotopy, we eliminate bigons with the vertical edges of the three squares.

Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



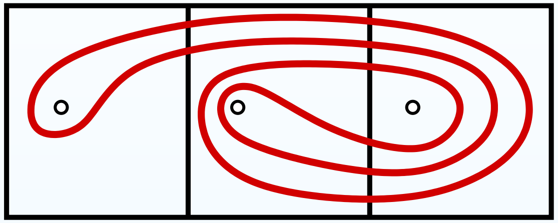


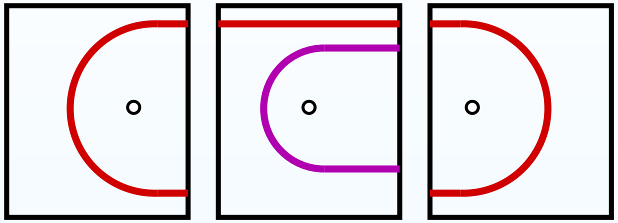
By further isotopy, we eliminate bigons with the vertical edges of the three squares.

Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

Four basic train tracks on $S_{0,4}$

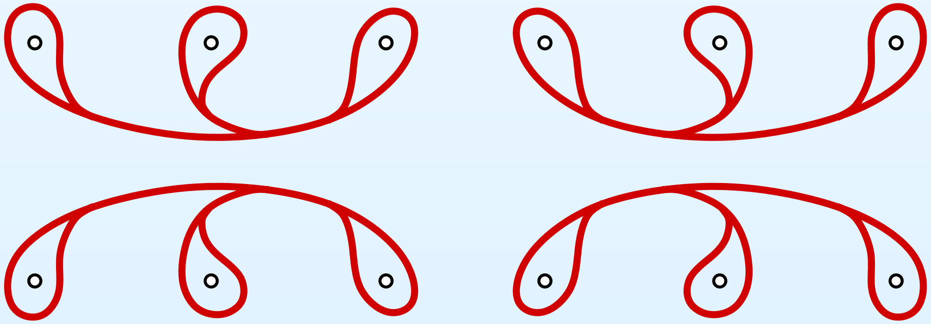
Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



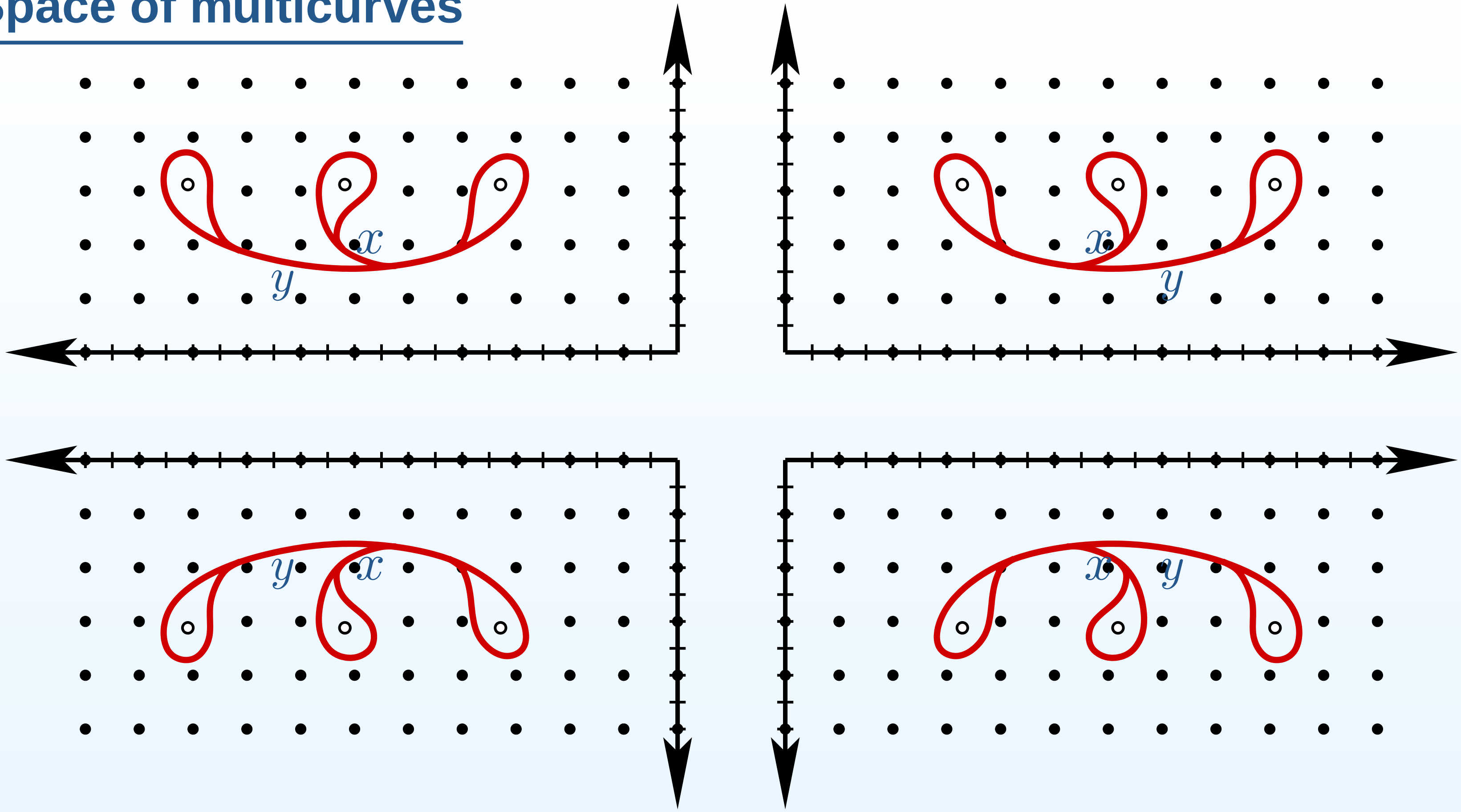


By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve in is carried by one of the following four train tracks:

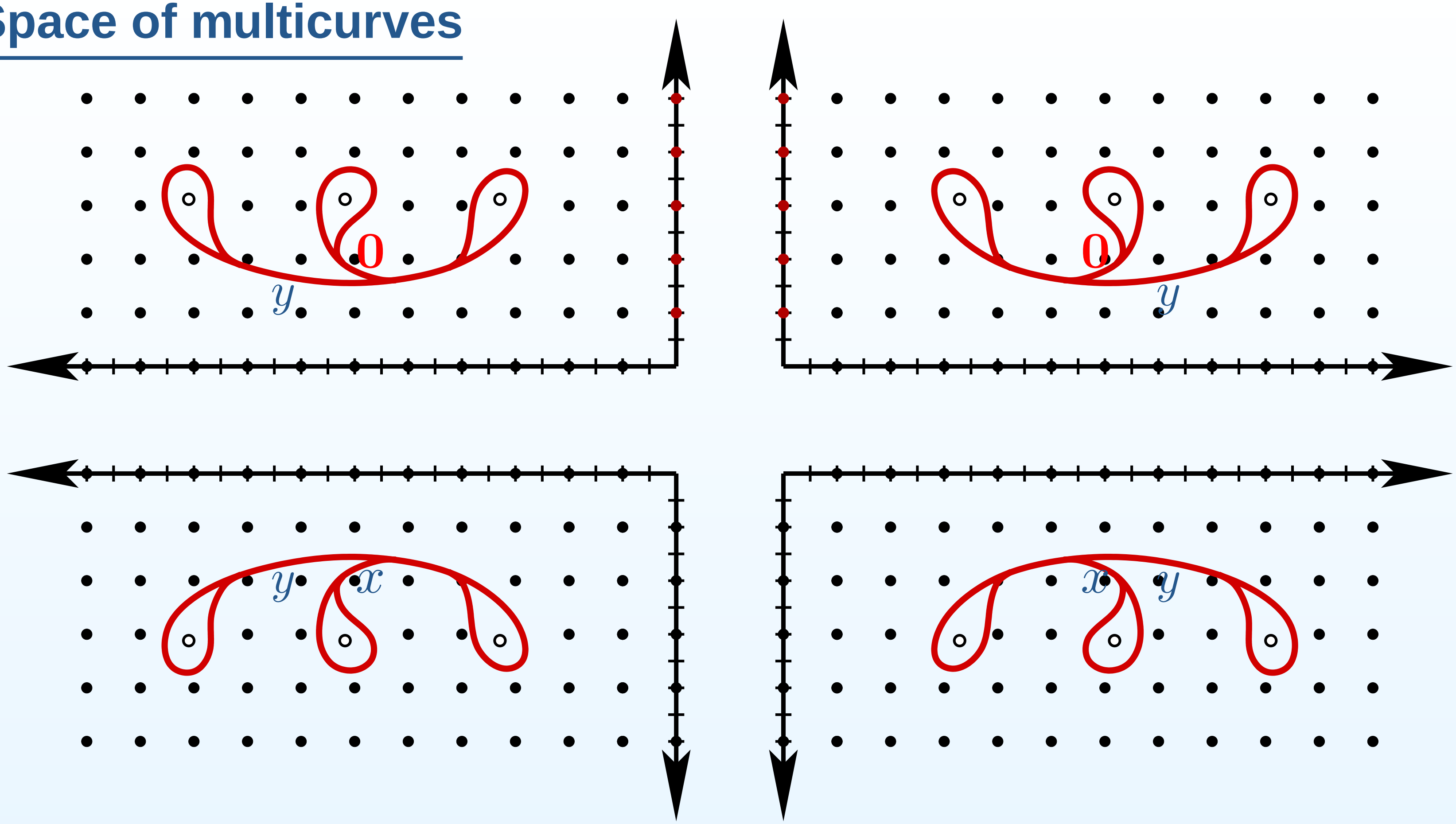


Space of multicurves



The four train tracks $\tau_1, \tau_2, \tau_3, \tau_4$ give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

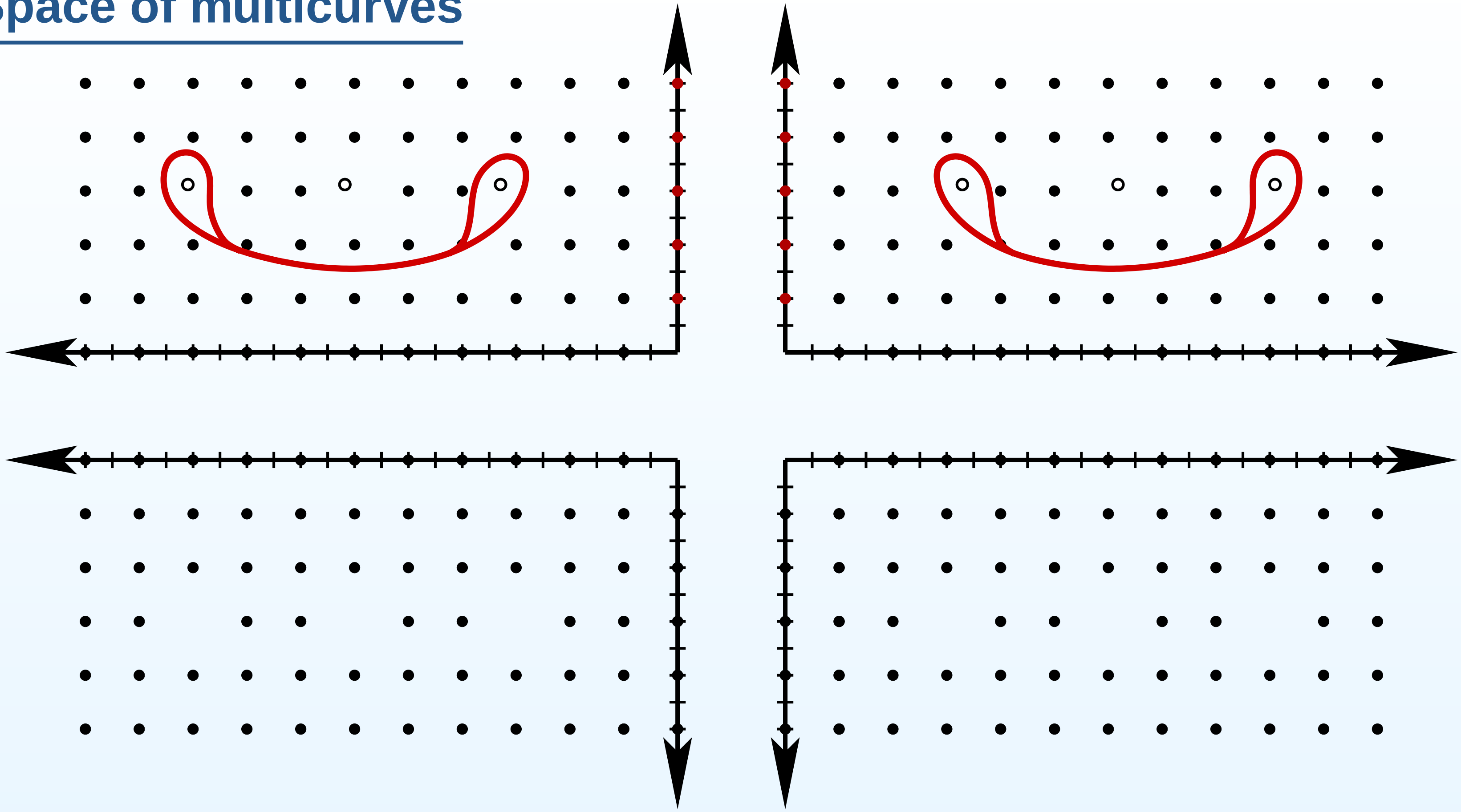
Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

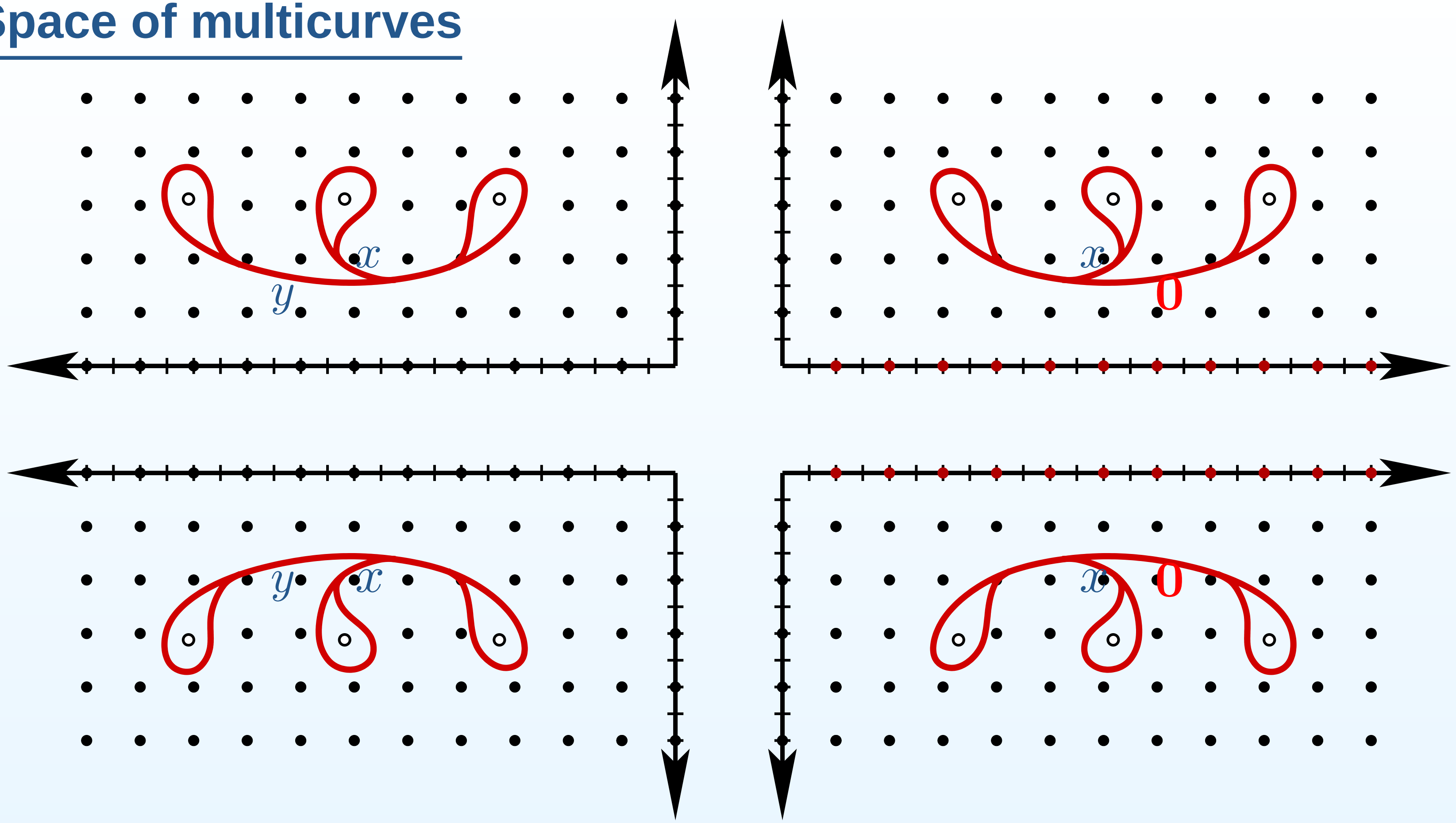
Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

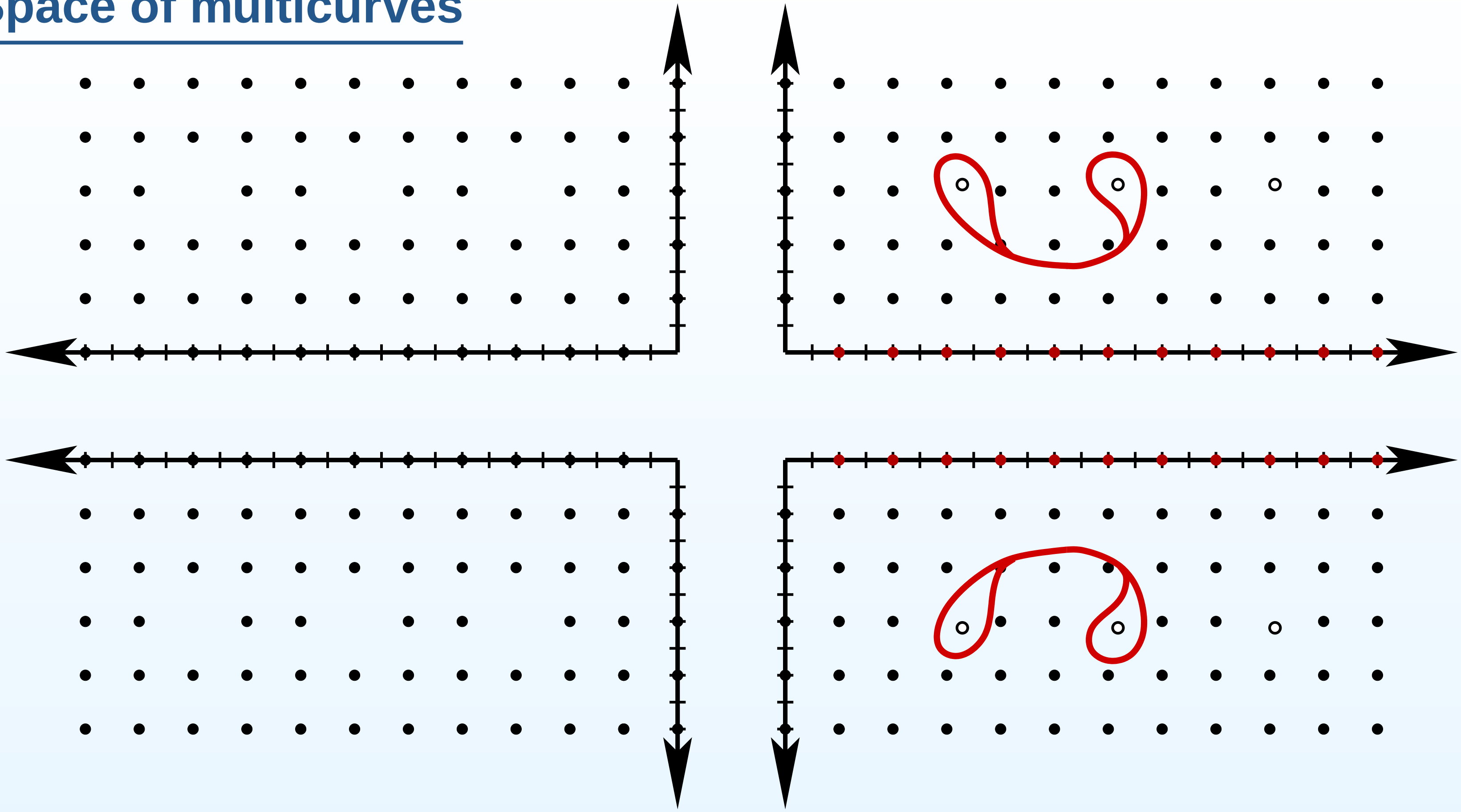
Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted.
This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

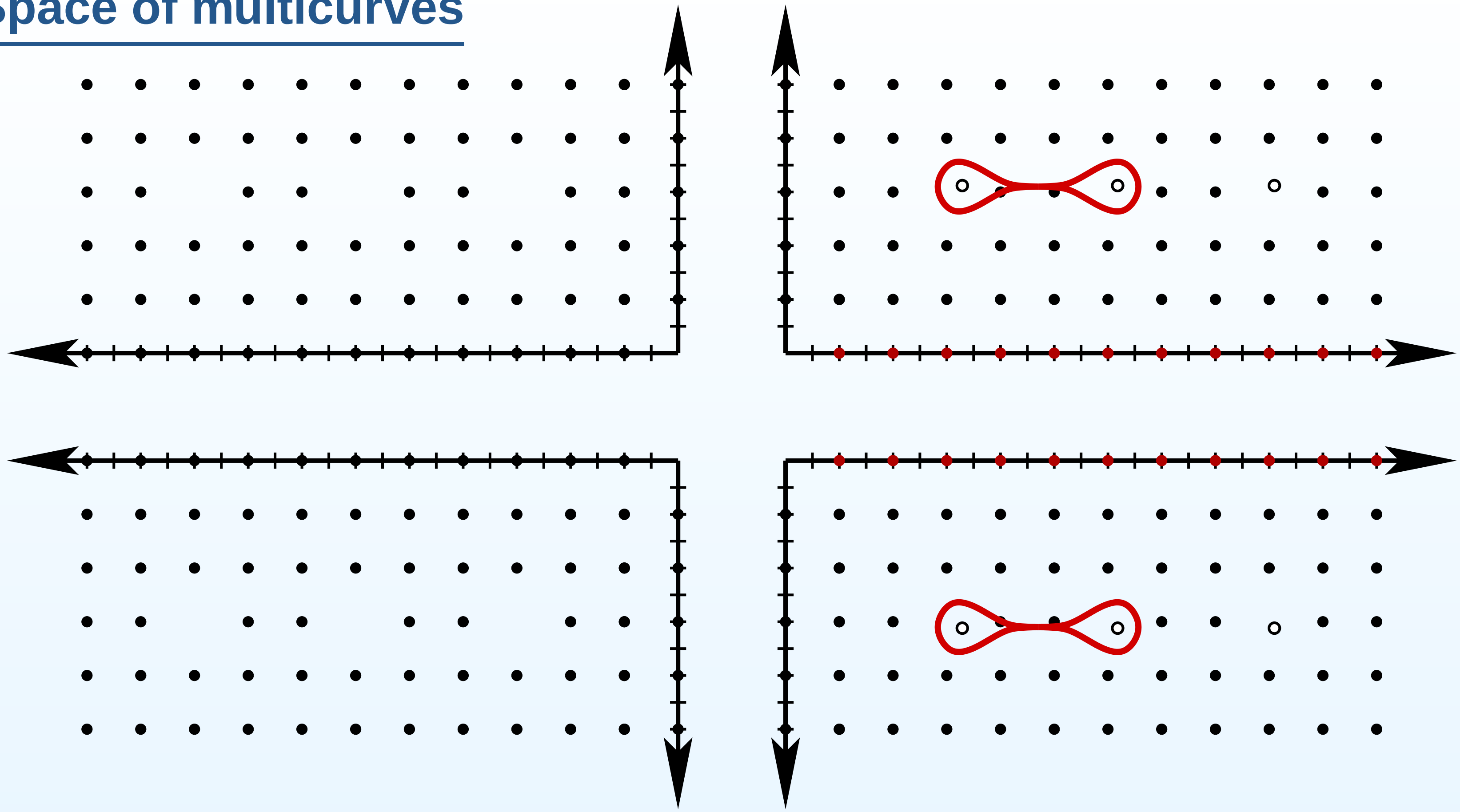
Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted.
This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

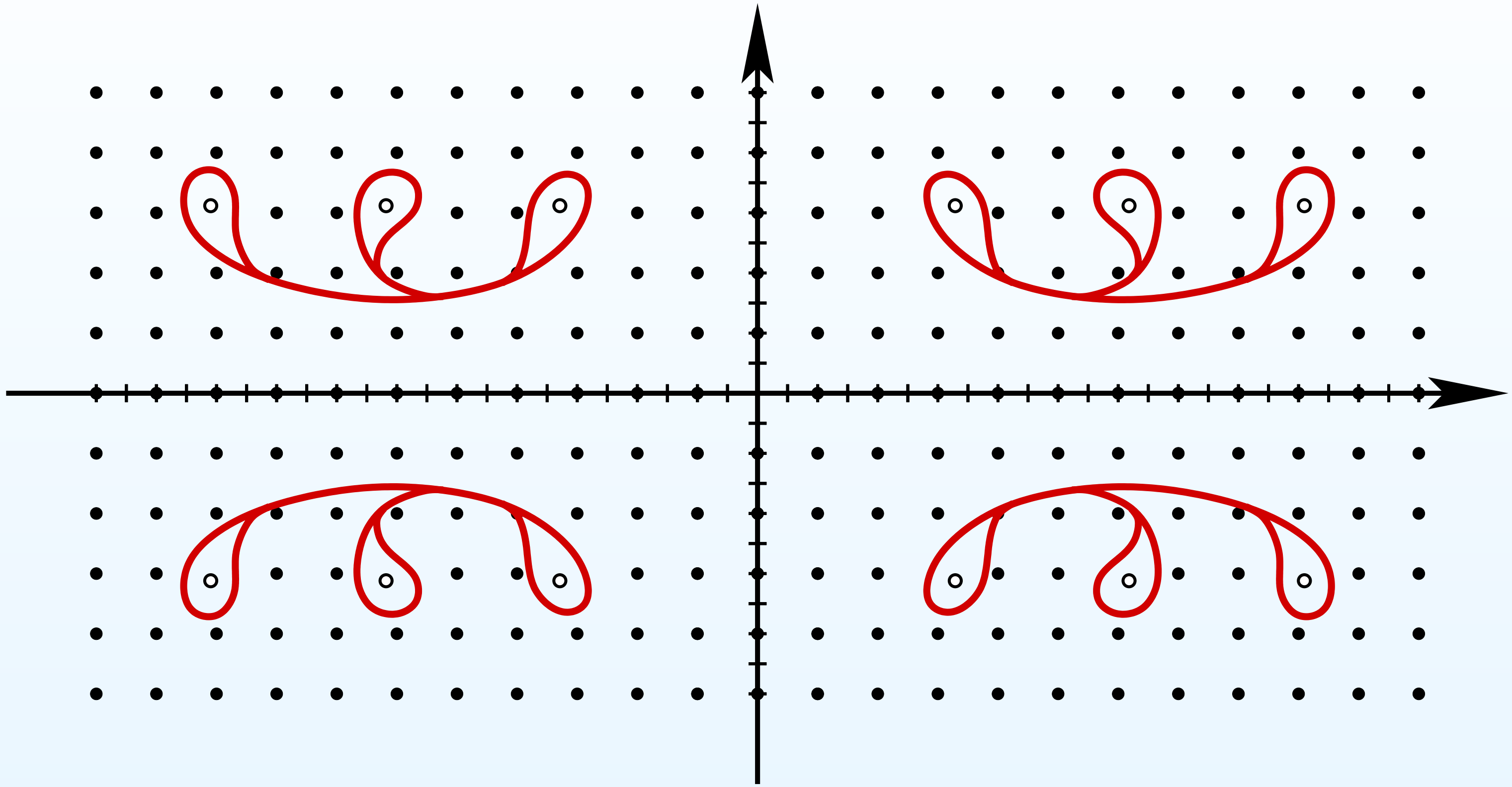
Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.