Lecture 2. Bridging flat and hyperbolic enumerative geometry

Anton Zorich

(after a joint work with V. Delecroix, E. Goujard and P. Zograf;
partly published in *Duke Math. Jour.* **170:12** (2021), 2633–2718;
see `arXiv:2011.05306` and also `arXiv:2007.04740`.)

Counting Problems

Ventotene, September 8, 2021

Mirzakhani's count of simple closed geodesics
- Frequencies of multicurves
- Example
- Hyperbolic and flat geodesic multicurves

Masur–Veech volumes. Square-tiled surfaces

Masur–Veech versus Weil–Petersson volume

Shape of a random multicurve: genus two

Shape of a random multicurve: large genus

Mirzakhani's count of simple closed geodesics

Frequencies of multicurves

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *the number* $s_X(L,\gamma)$ *of simple closed geodesic multicurves on* X *of topological type* $[\gamma]$ *and of hyperbolic length at most* L *has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

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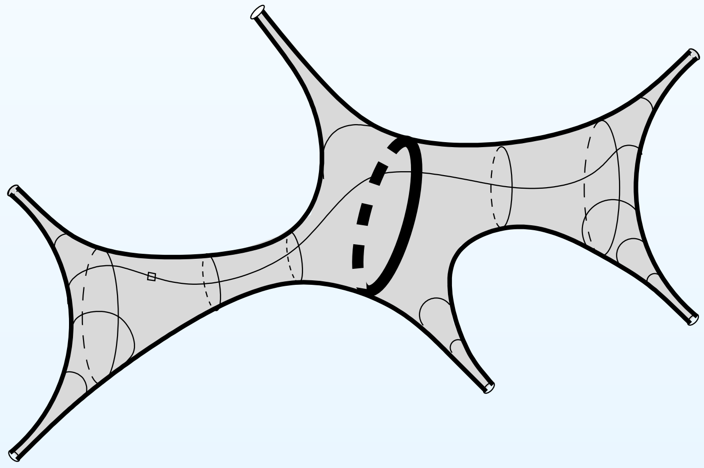
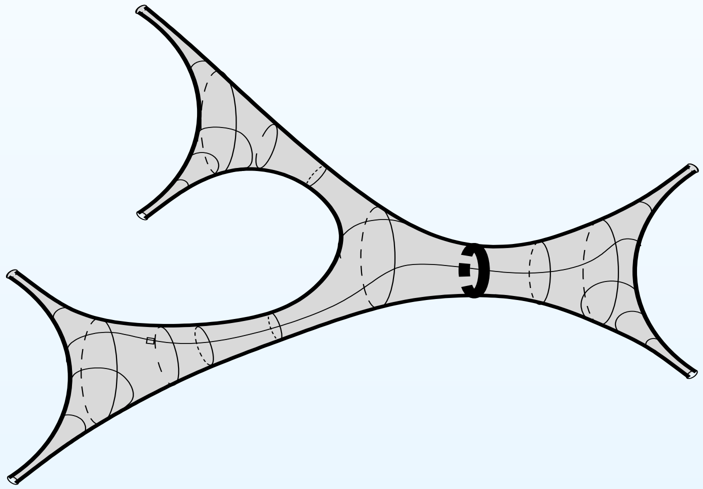
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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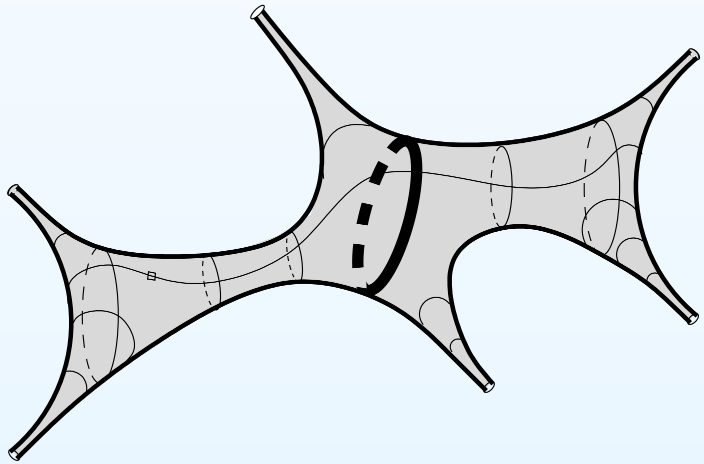
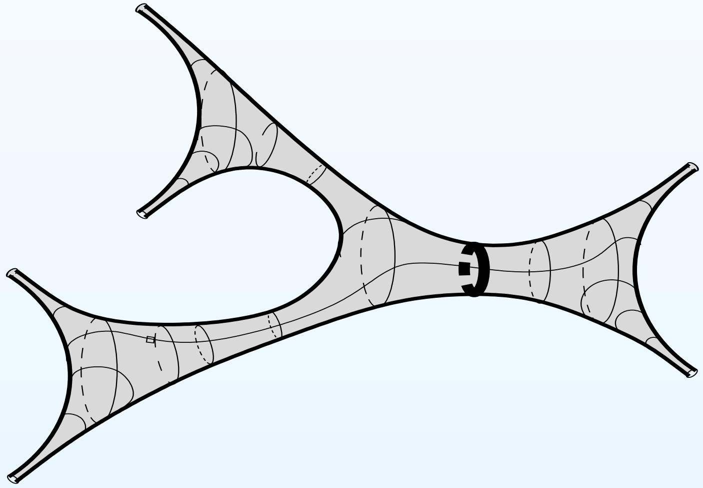
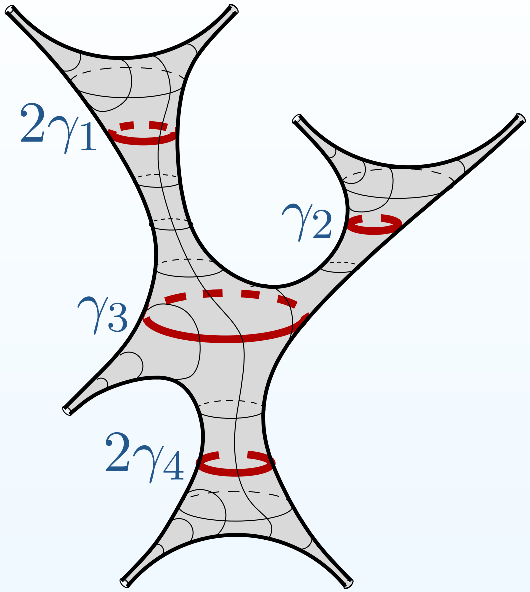


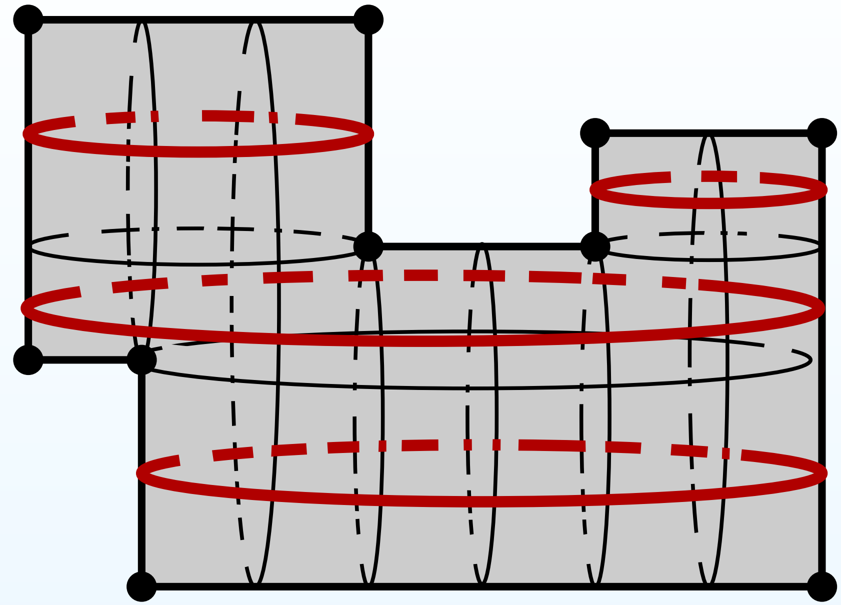


Example (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

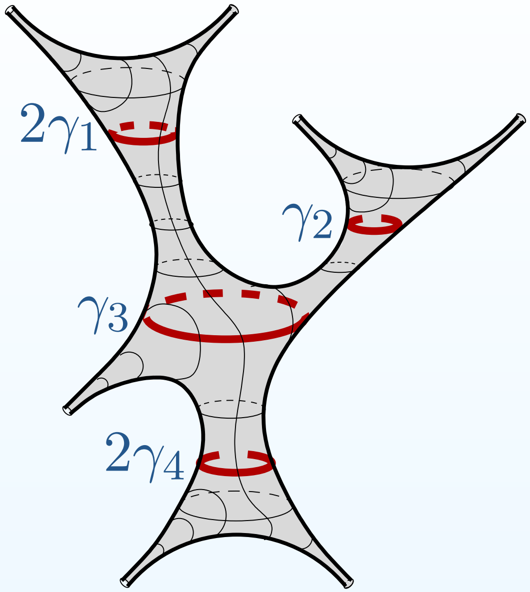
Hyperbolic and flat geodesic multicurves

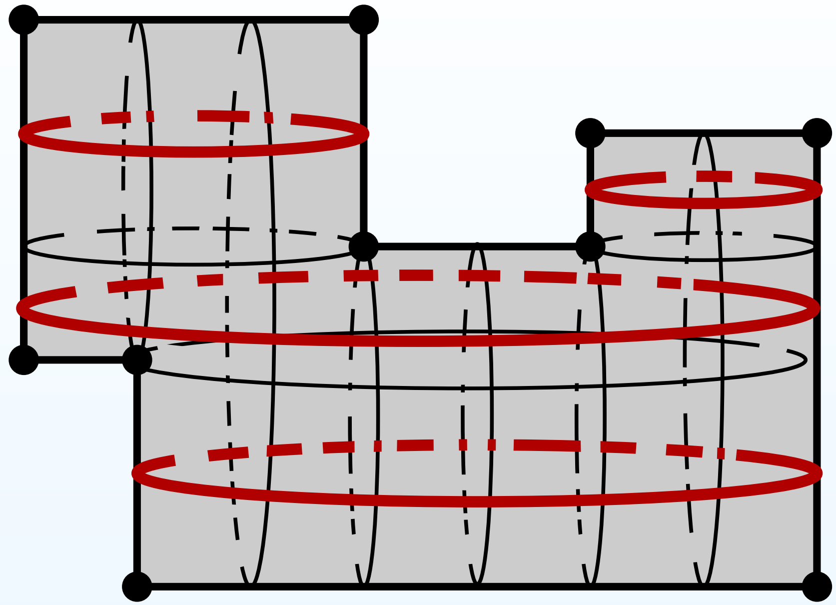




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves





Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of* ***hyperbolic*** *multicurves coincides with the asymptotic frequency of simple closed* ***flat*** *geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Mirzakhani's count of simple closed geodesics

Masur–Veech volumes. Square-tiled surfaces

- Very flat surface of genus two
- Period coordinates
- Masur–Veech volume
- Integer points as square-tiled surfaces
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- Counting volume by counting integer points
- Volume of the space of flat tori
- Methods of evaluation of Masur–Veech volumes

Masur–Veech versus Weil–Petersson volume

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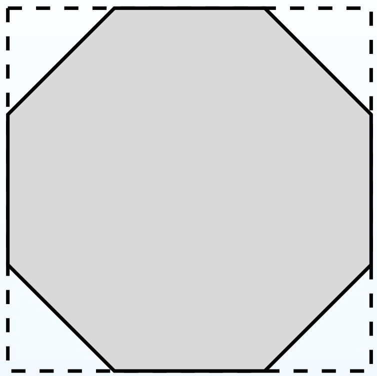
Shape of a random multicurve: large genus

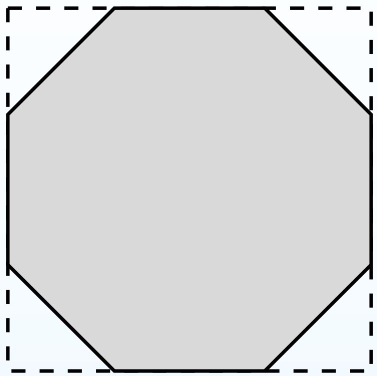
Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials. Square-tiled surfaces

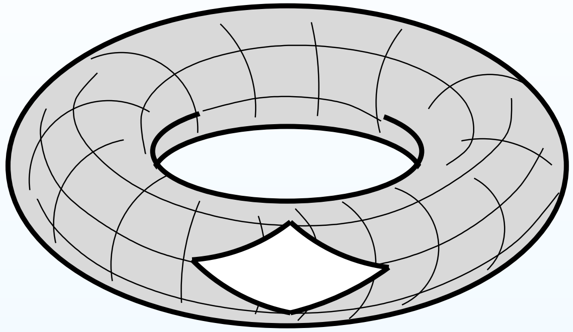
Very flat surface of genus two



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

Very flat surface of genus two





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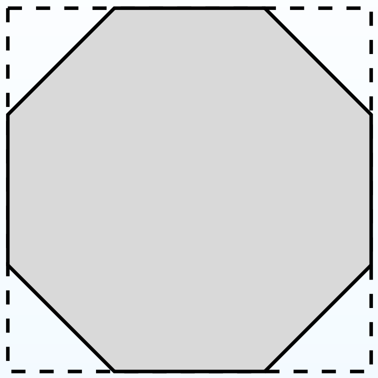

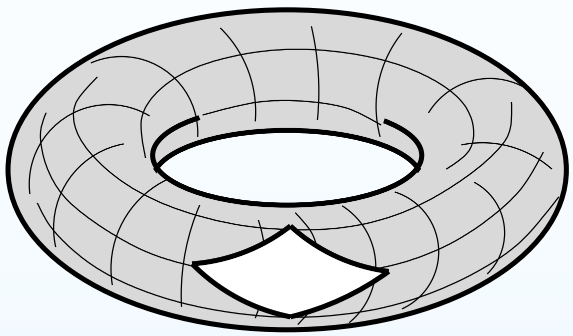

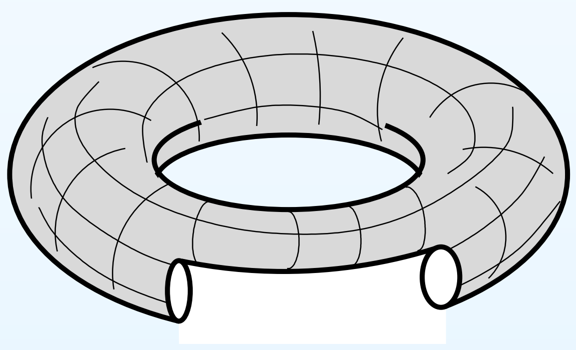

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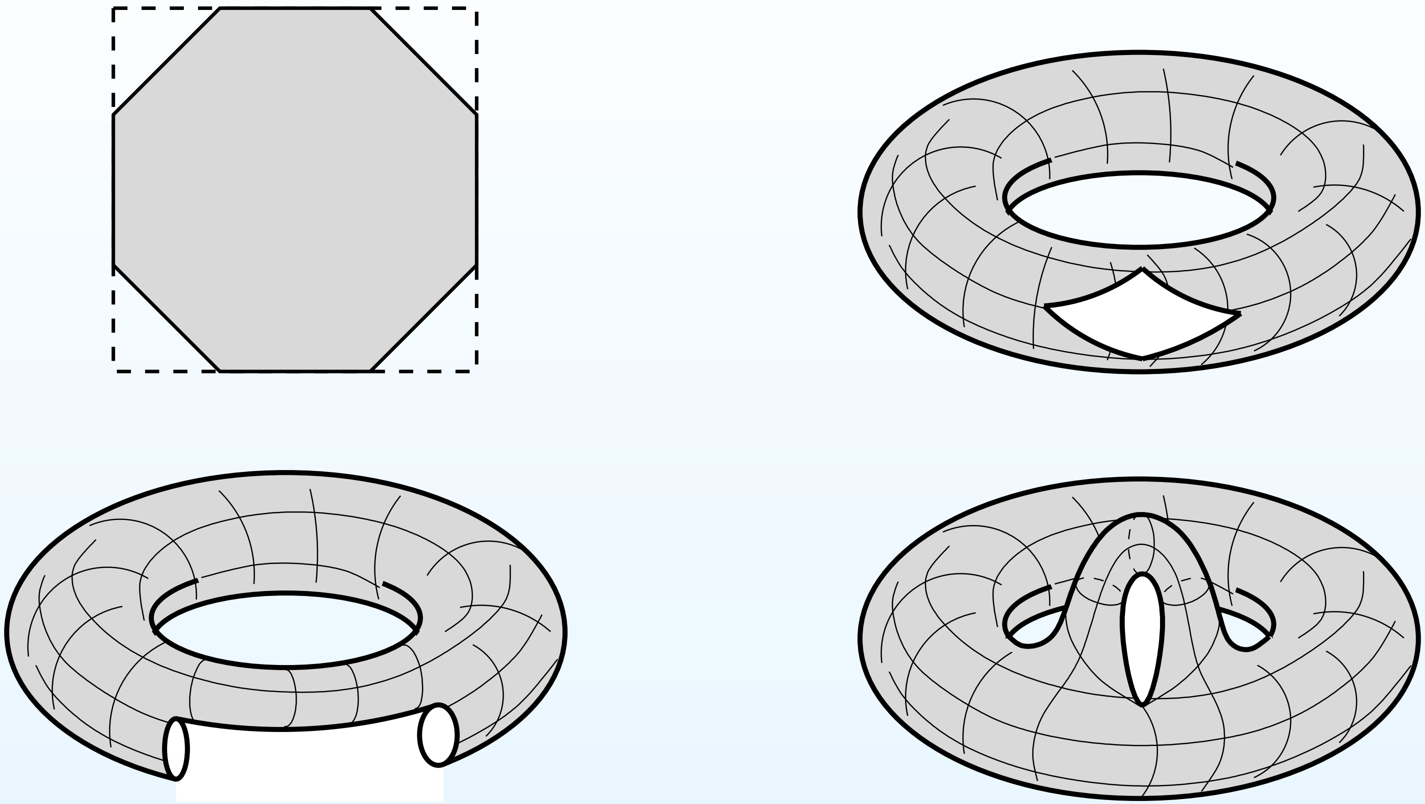

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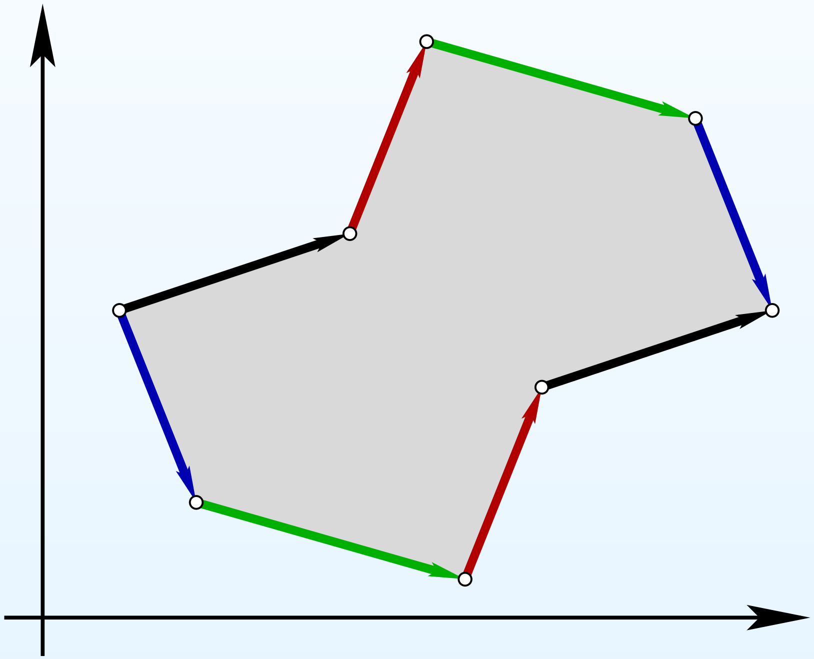
Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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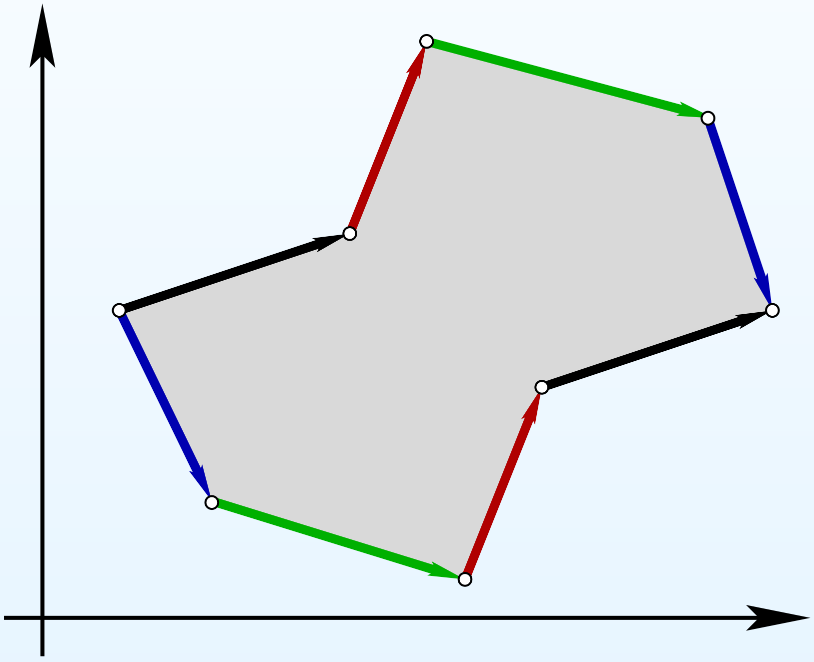

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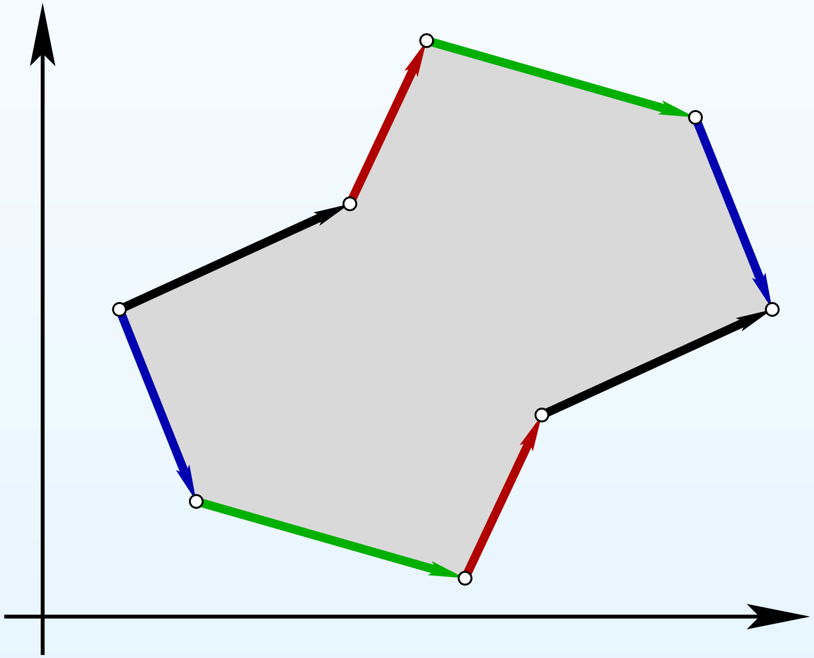

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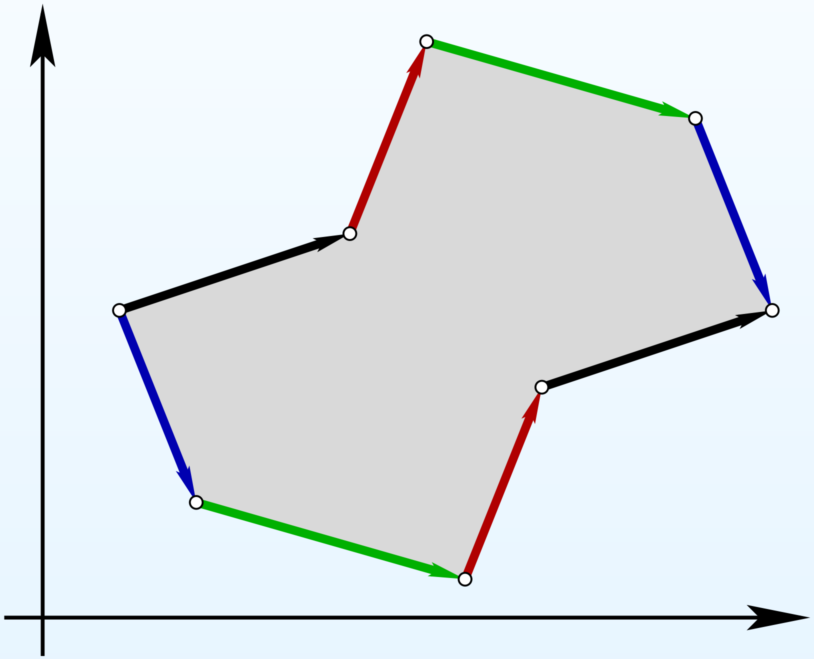
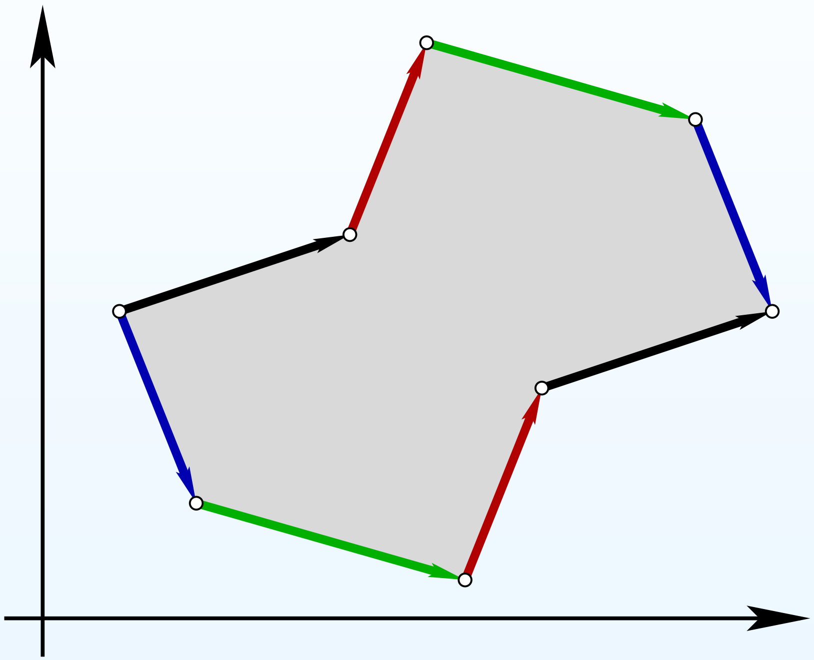


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

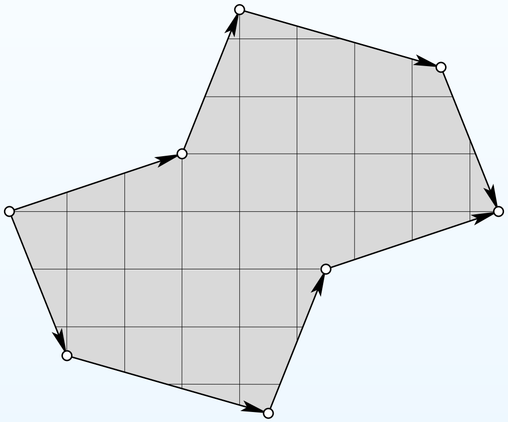
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs (C,ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modeled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any moduli space $\mathcal{H}(m_1, \ldots, m_n)$: given a positive integer $r > 0$ we can rescale a flat surface by factor r. The flat area of the surface gets rescaled by the factor r^2.



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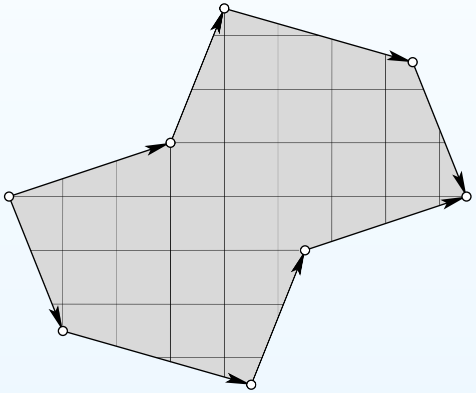

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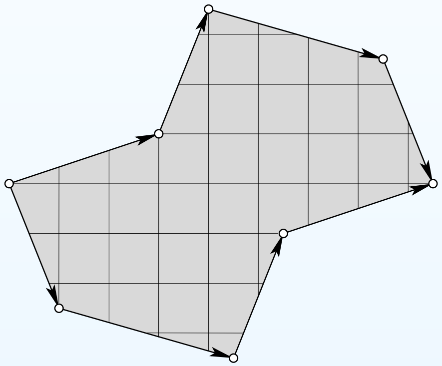

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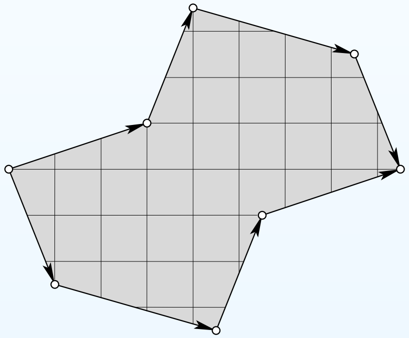

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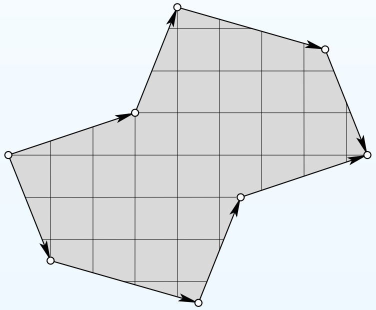

Flat area of the surface as a positive homogeneous function

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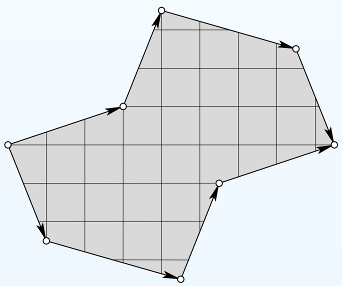


Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1=\mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1=\text{area}(S)=\frac{i}{2}\int_C \omega\wedge\bar\omega=\sum_{i=1}^{g}(A_i\bar B_i-\bar A_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu=r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

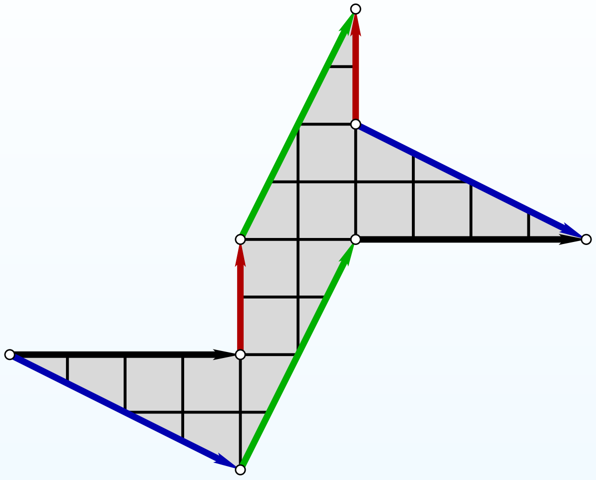
Masur–Veech volume

Summary. Every stratum of Abelian differentials admits

- A local structure of a vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$;
- An integer lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1, \dots, m_n)$ or $\mathcal{Q}_1(m_1, \dots, m_n)$ of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

Integer points as square-tiled surfaces



Integer points in period coordinates are represented by *square-tiled surfaces*.

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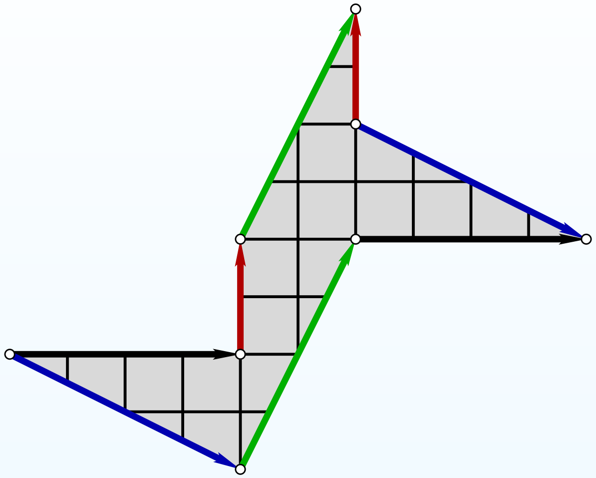

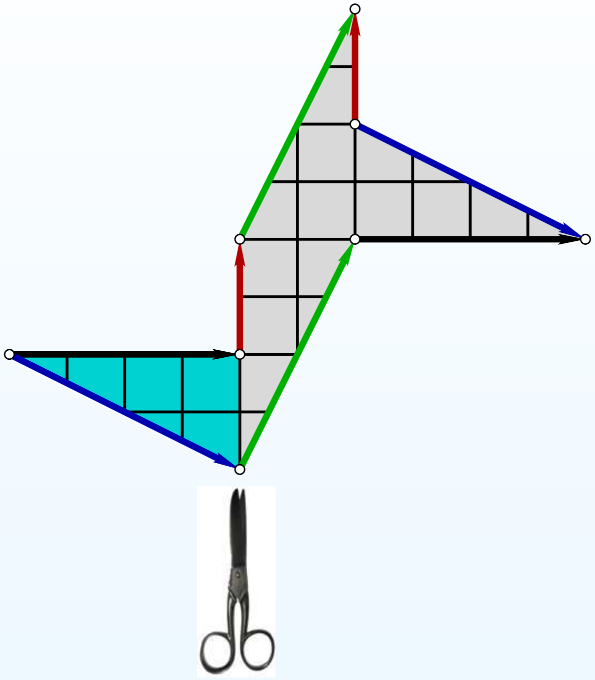

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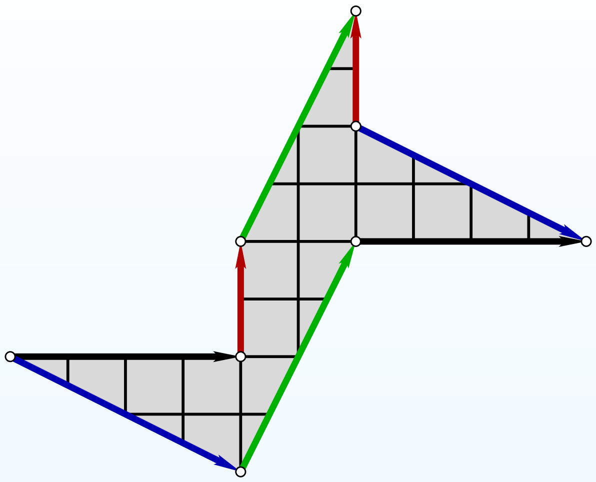

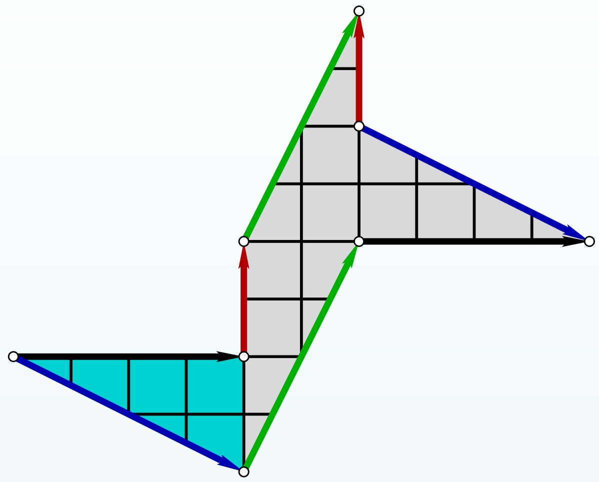

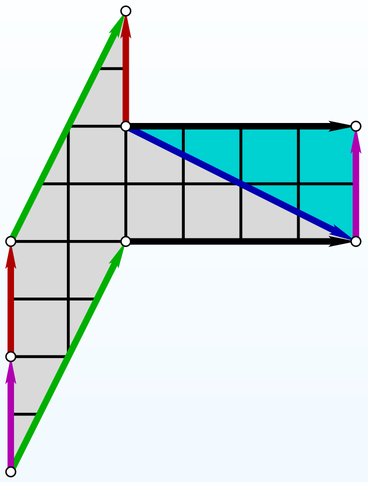

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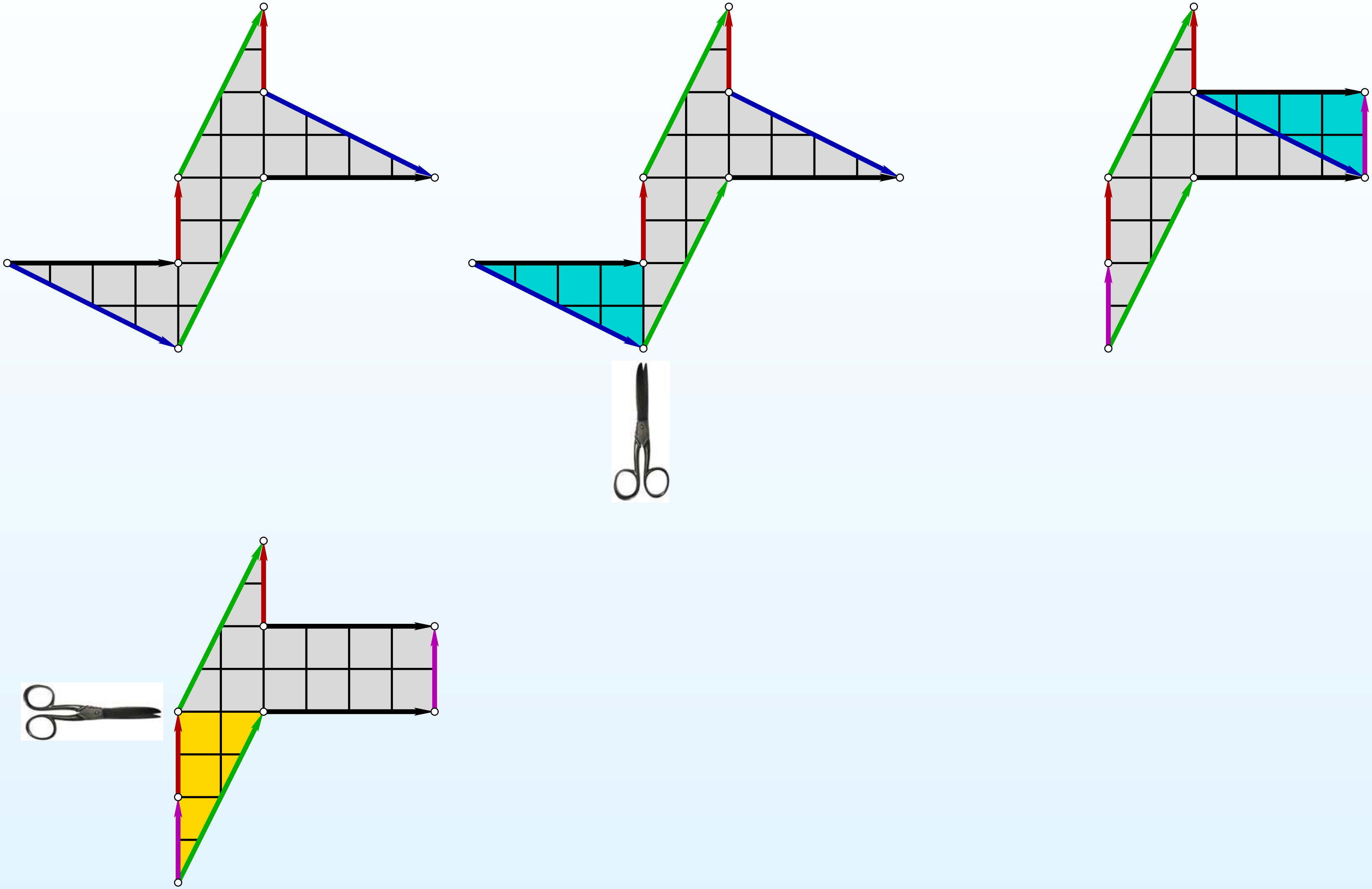

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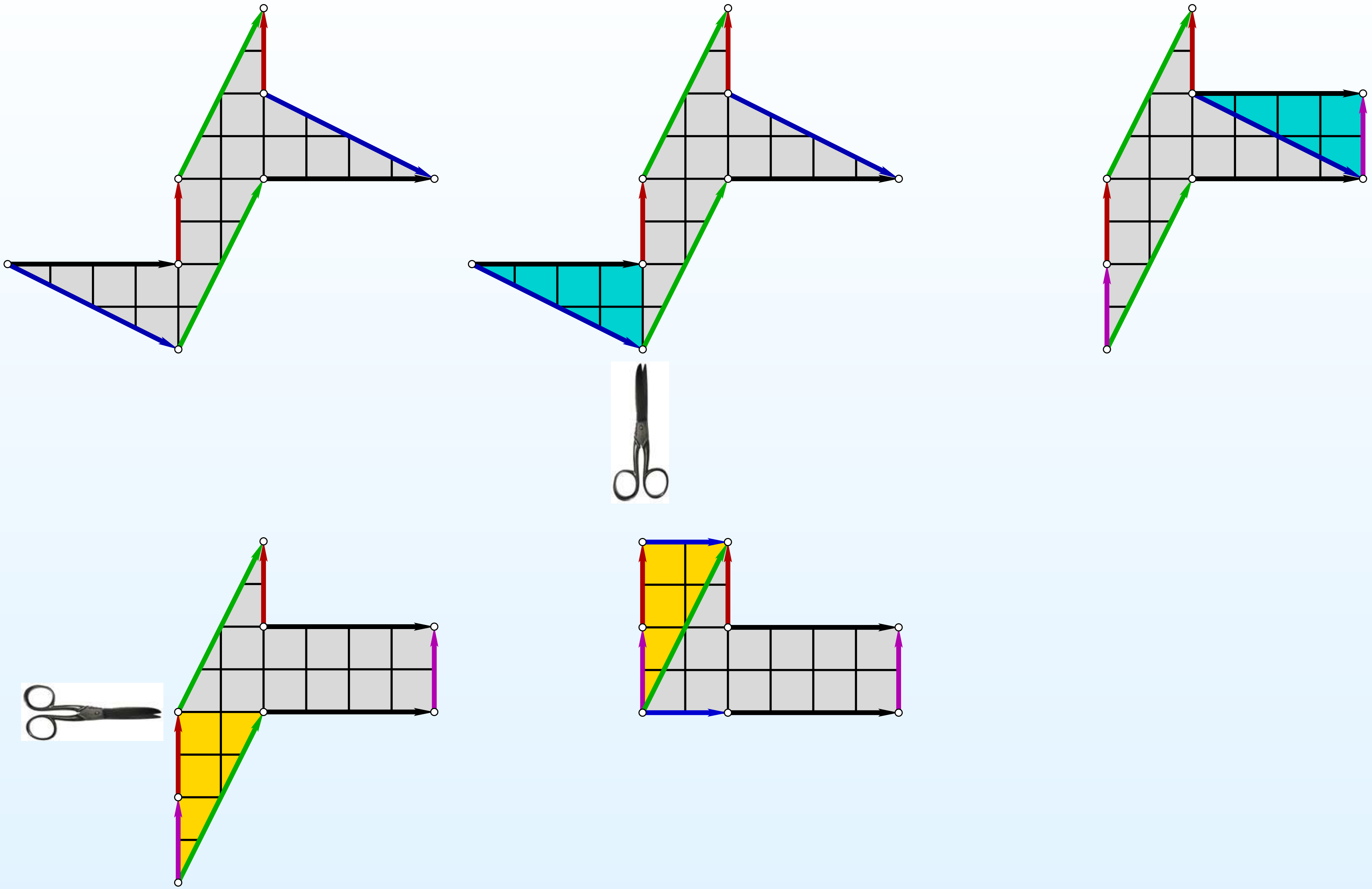

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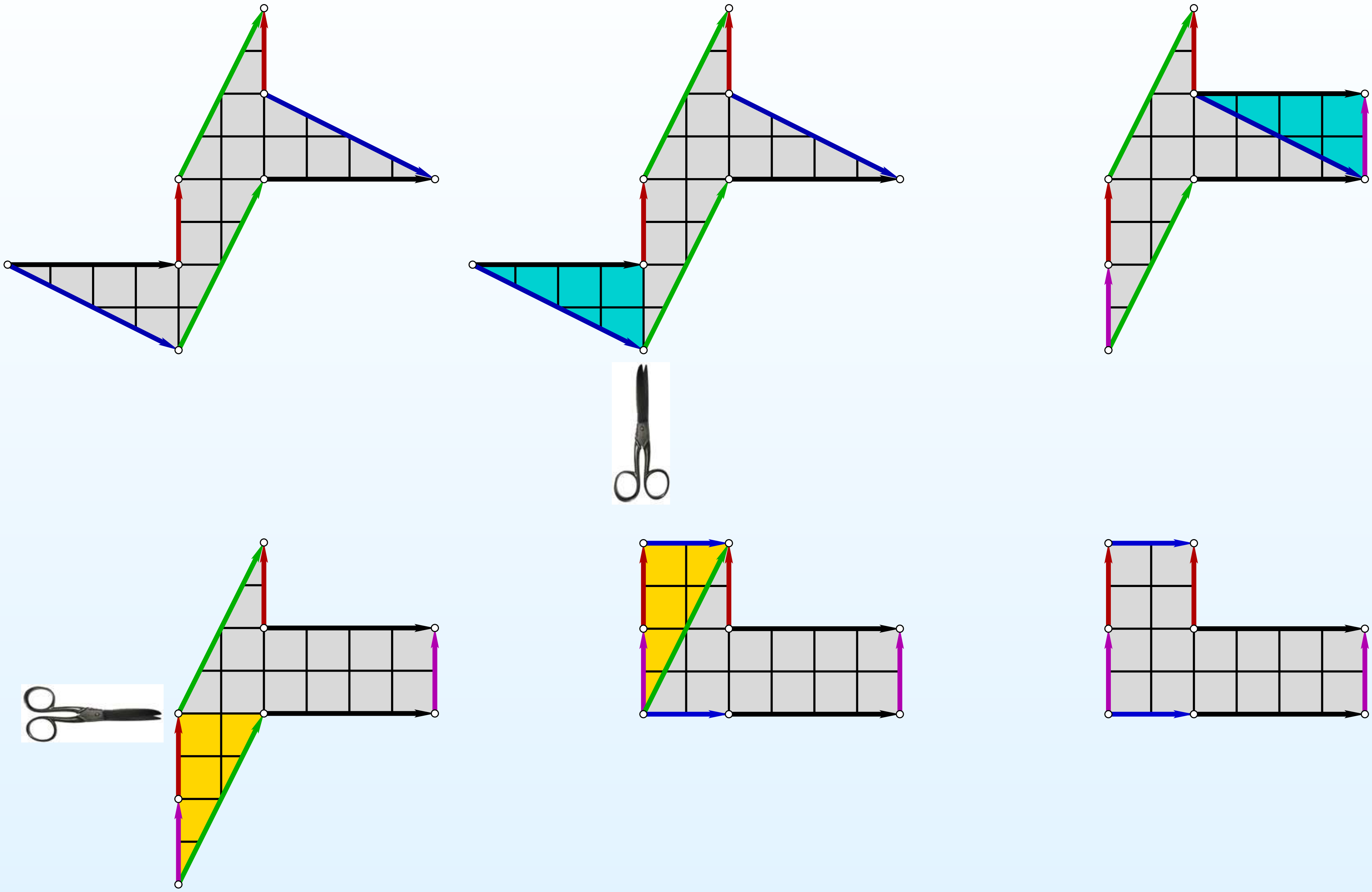

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Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}, \text{ where } P_1 \text{ is a zero of } \omega \,.$$

The ramification points of the cover are exactly the zeroes of ω.

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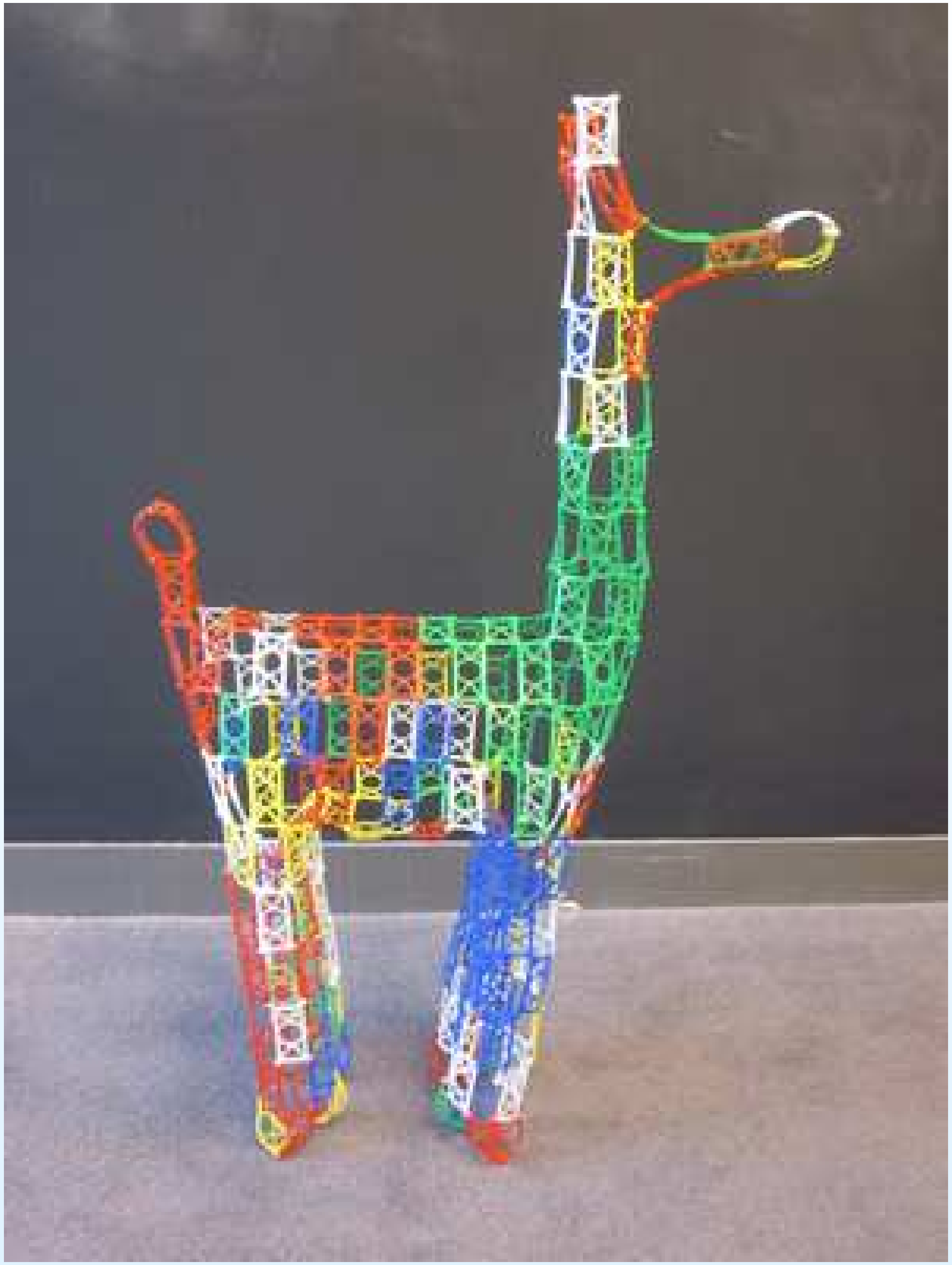
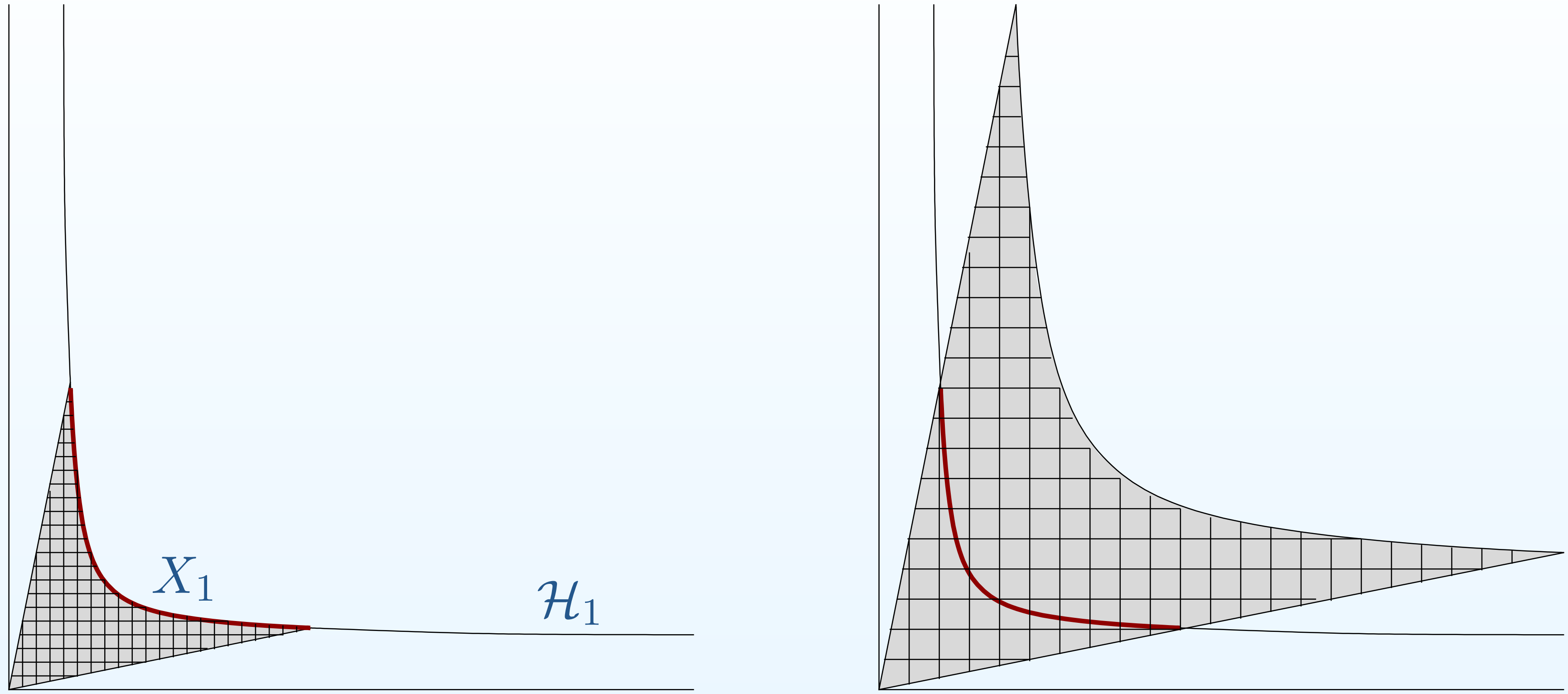
Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous "pillowcase covers" over $\mathbb{C}\mathrm{P}^1$ branched at four points. Thus, counting volumes of the strata is similar to counting analogs of Hurwitz numbers.

An example of a square-tiled surface

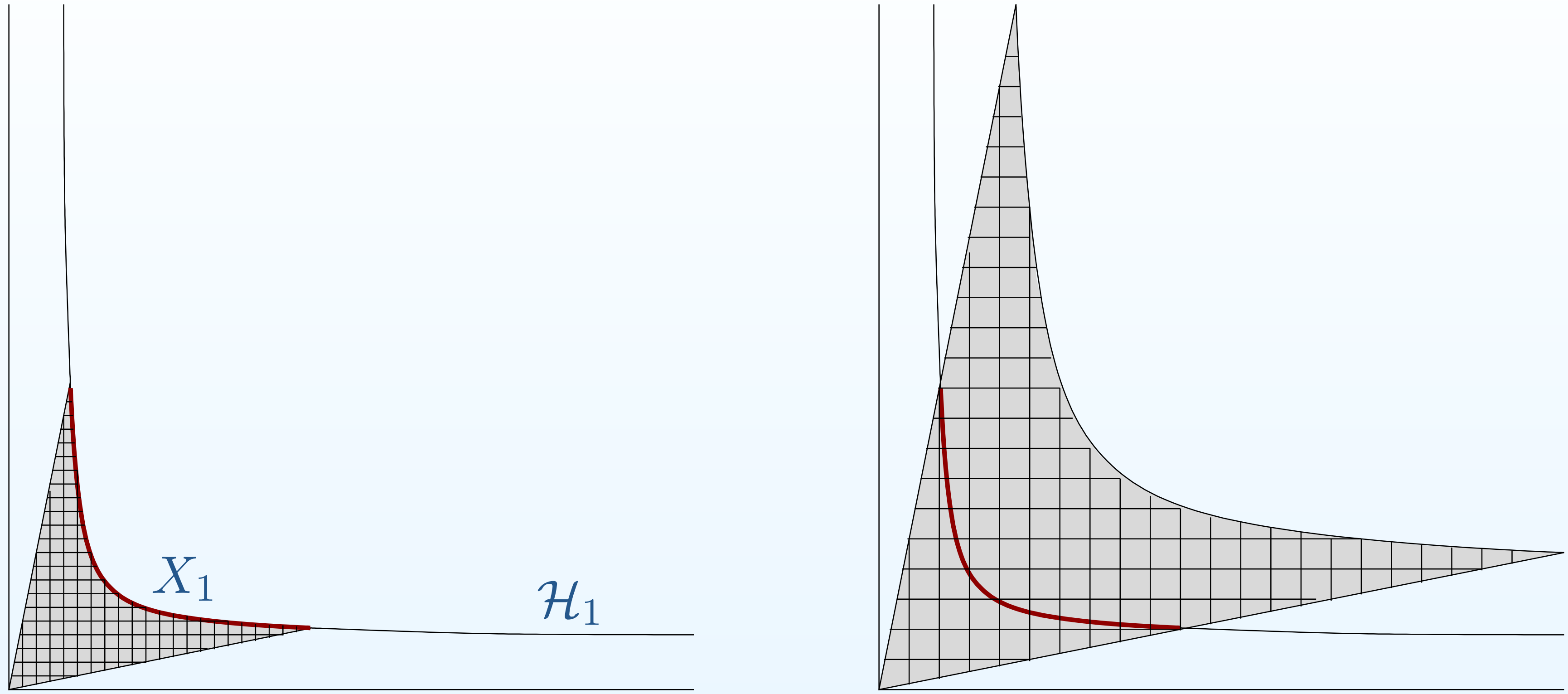


Counting volume by counting integer points in a large cone



To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it. Counting points of the $\frac{1}{N}$-grid in the cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ is the same as counting integer points in the larger proportionally rescaled cone $C_N(X_1) = \{r \cdot S | S \in X_1, r \leq N\}$.

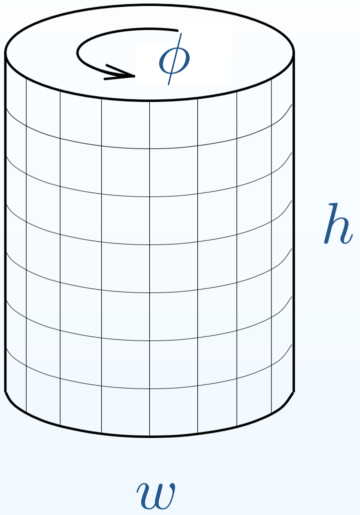
Counting volume by counting integer points in a large cone

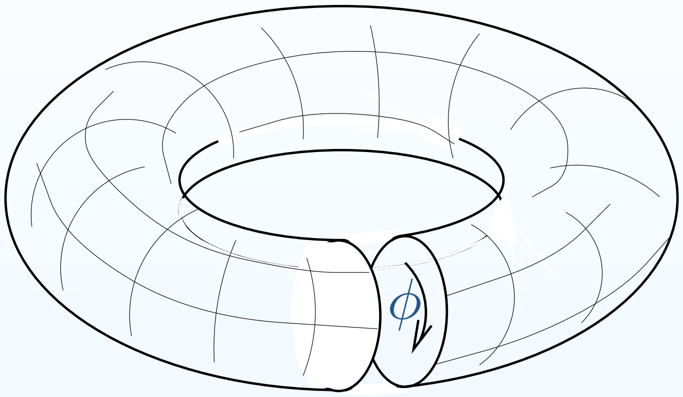


Let $\mathcal{H} = \mathcal{H}(m_1, \dots, m_n)$; let $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n) = 2g + n - 1$. We get:

$$\operatorname{Vol} \mathcal{H}_1 = 2d \cdot \lim_{N \to +\infty} \frac{\begin{pmatrix} \text{number of square-tiled surfaces in } \mathcal{H} \\ \text{tiled with at most } N \text{ identical squares} \end{pmatrix}}{N^d} .$$

Volume of the space of flat tori





The number of square-tiled tori tiled with at most N squares has asymptotics

$$\sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w = \sum_{\substack{w,h\in\mathbb{N}\\ w\le \frac{N}{h}}} w \approx \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\zeta(2) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}\,.$$

$$\operatorname{Vol}\mathcal{H}_1(0) = 2\cdot 2\cdot \lim_{N\to+\infty}\frac{\begin{pmatrix}\text{number of square-tiled surfaces in } \mathcal{H}\\ \text{tiled with at most } N \text{ identical squares}\end{pmatrix}}{N^2} = \frac{\pi^2}{3}\,.$$

Methods of evaluation of Masur–Veech volumes

- M. Kontsevich–A. Zorich (1998). Straightforward calculation of square-tiled surfaces.
- (A. Eskin–A. Okounkov–R. Pandharipande; D. Chen–M. Moëller–D. Zagier; E. Goujard) A. Eskin and A. Okounkov observed in 2000 that the generating function for the count of square-tiled surfaces is a quasimodular form.
- D. Chen–M. Möeller–A. Sauvaget; M. Kazarian; Di Yang–D. Zagier–Y. Zhang (2018–) Using recent BCGGM smooth compactification of the moduli space, one can work with the volume element as with the cohomology class. Intersection theory.
- V. Delecroix–E. Goujard–P. Zograf–A. Zorich (2018) (F. Arana–Herrera): volume of the principal stratum of quadratic differentials through Kontsevich's count of metric ribbon graphs in terms of Witten–Kontsevich correlators.
- D. Chen–M. Möeller–A. Sauvaget–D. Zagier; A. Aggarwal (2018–) Large genus asymptotics for any stratum of Abelian differentials (proving conjectures of Eskin–Zorich and of Delecroix–Goujard–Zograf–Zorich).
- Andersen–Borot–Charbonnier–Delecroix–Giacchetto–Lewanski–Wheeler, 2020 (inspired by the formula of Delecroix-Goujard–Zograf–Zorich): topological recursion.

Mirzakhani's count of simple closed geodesics

Masur–Veech volumes. Square-tiled surfaces

Masur–Veech versus Weil–Petersson volume
- Intersection numbers (correlators)
- Volume polynomials
- Trivalent ribbon graphs
- Kontsevich's count of metric ribbon graphs
- Surface decompositions
- Associated polynomials
- Volume of $\mathcal{Q}_2$
- Volume of $\mathcal{Q}_{g,n}$

Shape of a random multicurve: genus two

Shape of a random multicurve: large genus

Masur–Veech versus Weil–Petersson volume

Intersection numbers (correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \dots, P_n \in C$ is a complex orbifold of complex dimension $3g - 3 + n$.
Choose index i in $\{1, \dots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g - 3 + n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} .$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in* 2*-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; one of alternative proofs belongs to M. Mirzakhani.

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Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\, \mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

Volume polynomials

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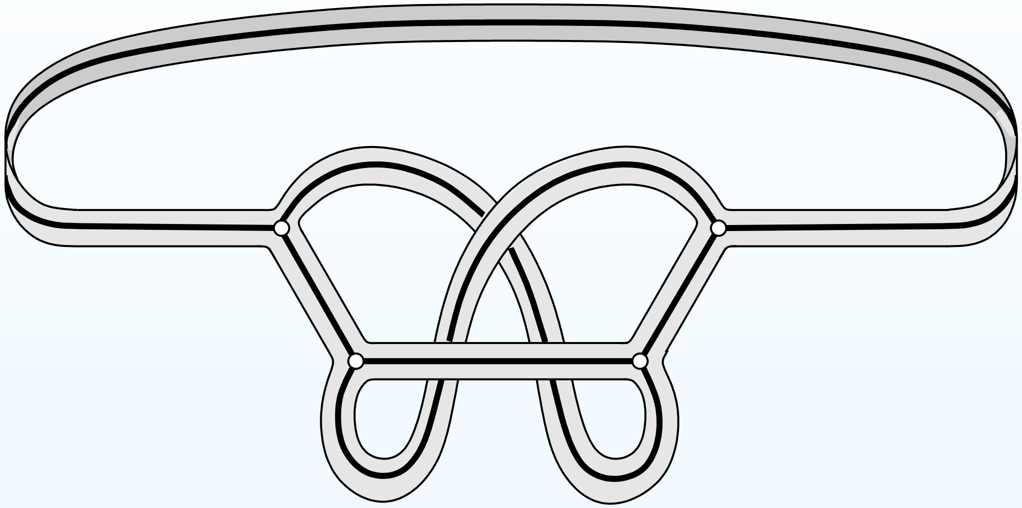
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 2$ with $n = 2$ boundary components. If we assigned lengths to all edges of the core graph, each boundary component gets induced length, namely, the sum of the lengths of the edges which it follow.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorem of Kontsevich counts them.

Kontsevich’s count of metric ribbon graphs

Each horizontal layer containing zeroes or poles of a square-tiled surface can be seen as a metric ribbon graph. When the associate quadratic differential has only simple zeroes, the metric ribbon graph is trivalent.

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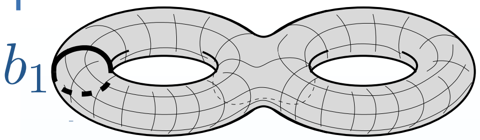
Theorem (M. Kontsevich; in this form — P. Norbury). *Consider a collection of positive integers $b_1, \dots, b_n$ such that $\sum_{i=1}^{n} b_i$ is even. The weighted count of genus g connected trivalent metric ribbon graphs Γ with integer edges and with n labeled boundary components of lengths $b_1, \dots, b_n$ is equal to $N_{g,n}(b_1, \dots, b_n)$ up to the lower order terms:*

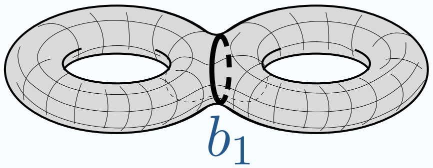
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \dots, b_n) = N_{g,n}(b_1, \dots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ denote the set of (nonisomorphic) trivalent ribbon graphs Γ of genus g and with n boundary components.

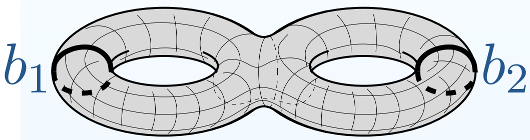
This Theorem is an important part of Kontsevich's proof of Witten's conjecture.



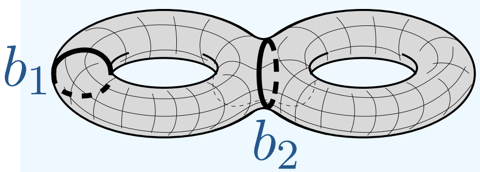
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



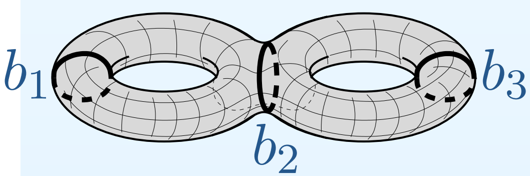
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



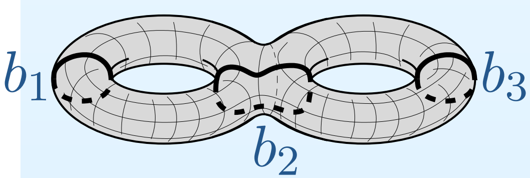
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



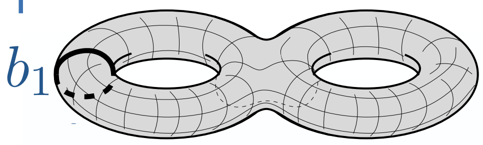
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



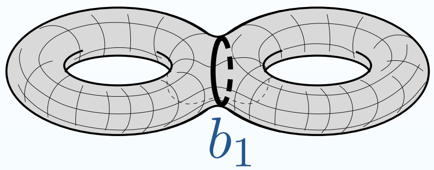
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



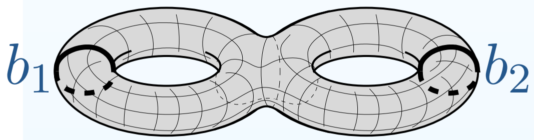
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$



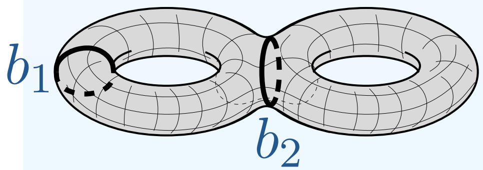
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) \qquad = \frac{1}{2} \cdot b_1 \left(\tfrac{1}{384} (2b_1^2)(2b_1^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) \qquad = \frac{1}{4} \cdot b_1 \left(\tfrac{1}{48} b_1^2 \right) \left(\tfrac{1}{48} b_1^2 \right)$$

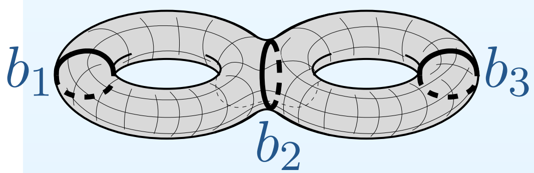


$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) \qquad = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\tfrac{1}{4} (2b_1^2 + 2b_2^2) \right)$$



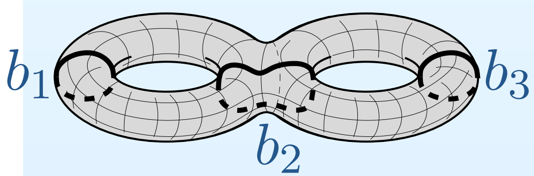
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

$$\cdot N_{1,1}(b_2) \qquad = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\tfrac{1}{48} b_2^2 \right)$$



$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

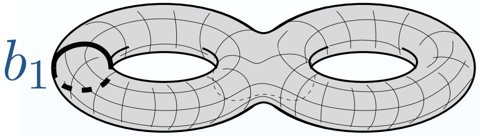
$$\cdot N_{0,3}(b_2, b_3, b_3) \qquad = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$



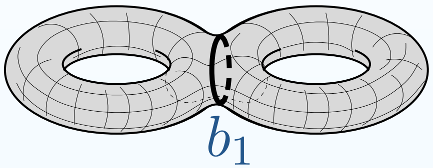
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$

$$\cdot N_{0,3}(b_1, b_2, b_3) \qquad = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

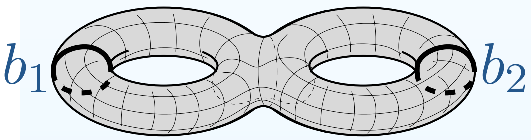
Volume of $\mathcal{Q}_2$



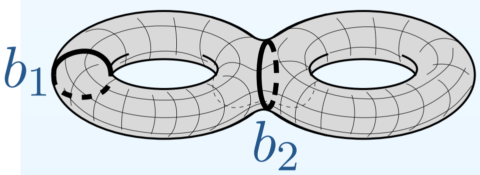
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



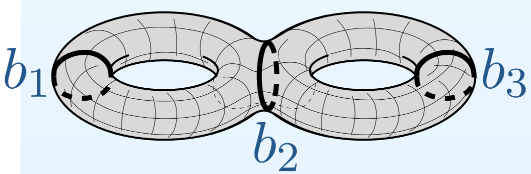
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



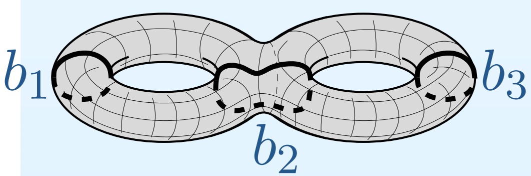
$$\frac{1}{16}(b_1^3 b_2 + + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



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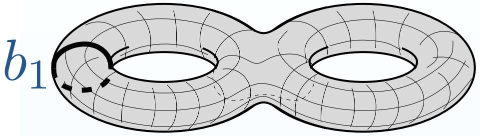
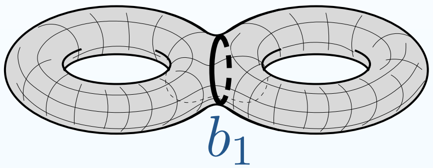
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

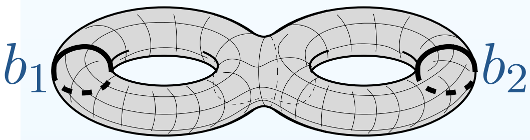
Volume of $\mathcal{Q}_2$



$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



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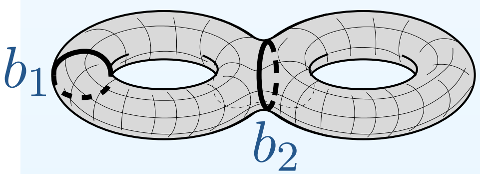
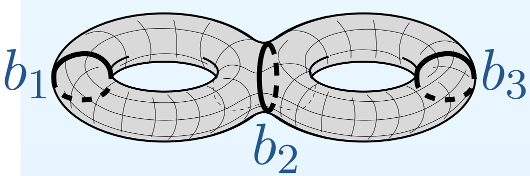
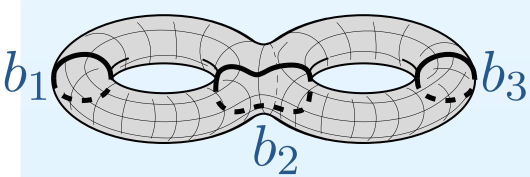


$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix, Goujard, Zograf, Zorich). *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs } \Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2, \underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k *of edges gives the contribution of* k*-cylinder square-tiled surfaces.*

Volume of $\mathcal{Q}_{g,n}$

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$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

Mirzakhani's count of simple closed geodesics

Masur–Veech volumes. Square-tiled surfaces

Masur–Veech versus Weil–Petersson volume

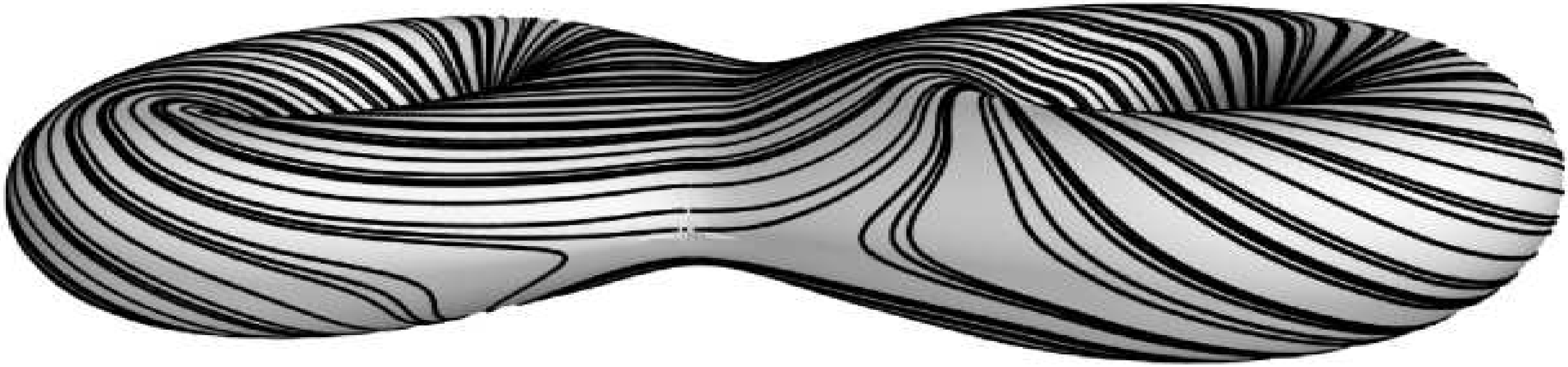
Shape of a random multicurve: genus two

● Separating versus non-separating

Shape of a random multicurve: large genus

Shape of a random multicurve on a surface of genus two

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

• *Which simple closed geodesics are more frequent: separating or non-separating?*

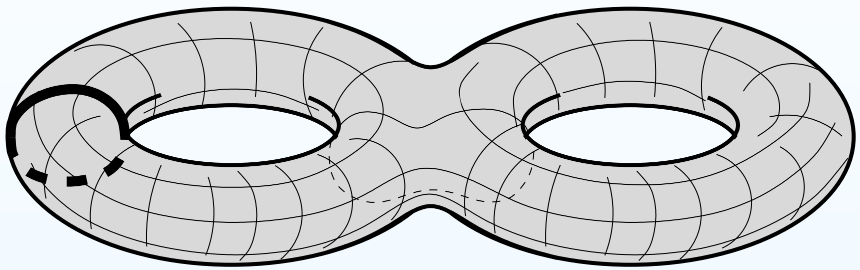
Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

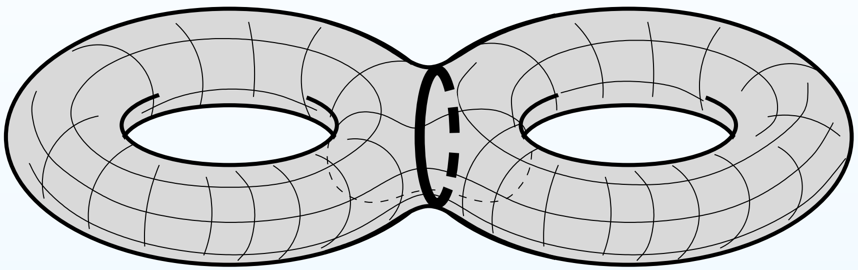
• *With what probability that $\gamma_{reduced}$ slices the surface into $1, ..., 2g-2$ connected components?*

• *With what probability $\gamma_{reduced}$ has $k = 1, 2, \ldots, 3g-3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$

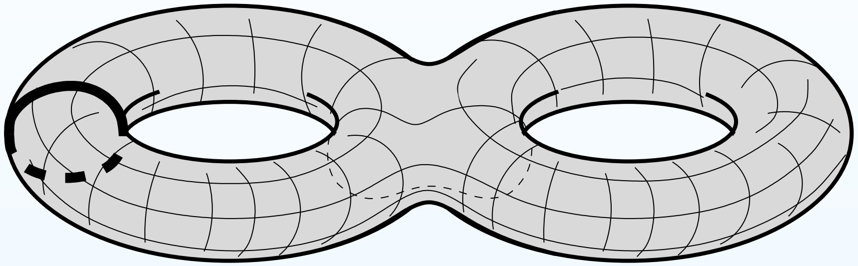


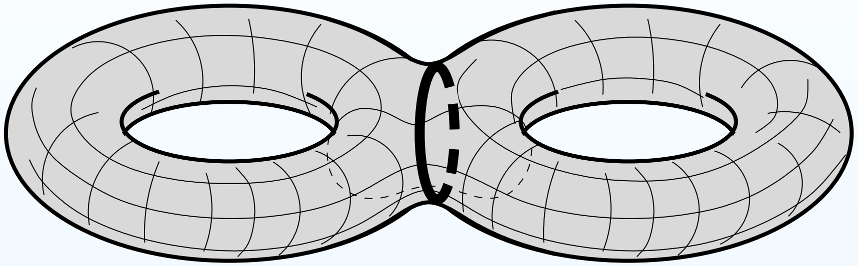


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$



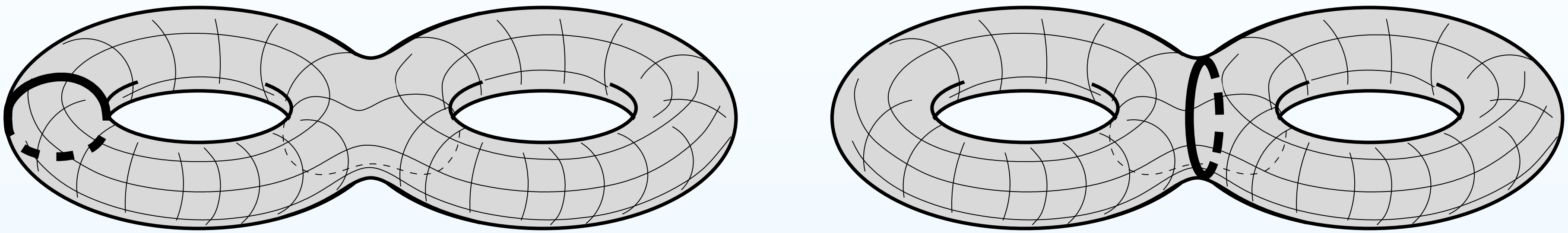


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Mirzakhani's calculation.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$

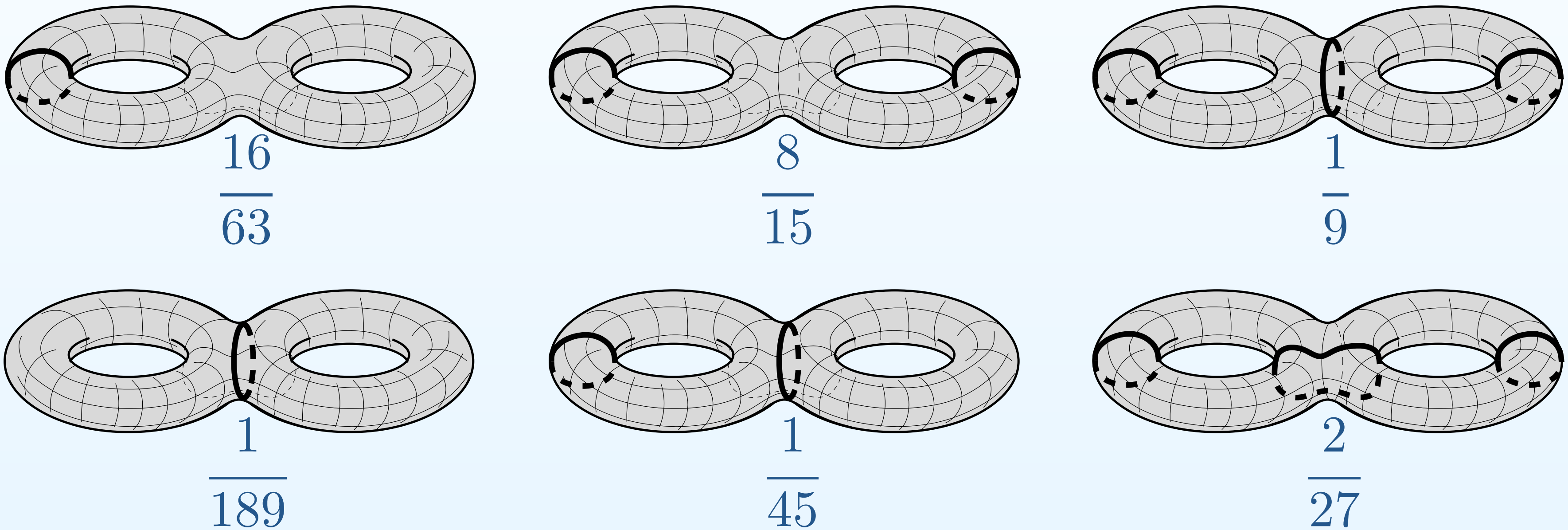


$$\lim_{L\to+\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell. Most recently it was independently confirmed by V. Erlandsson, K. Rafi, J. Souto and by A. Wright by methods independent of ours.

Multicurves on a surface of genus two and their frequencies

The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Mirzakhani's count of simple closed geodesics

Masur–Veech volumes. Square-tiled surfaces

Masur–Veech versus Weil–Petersson volume

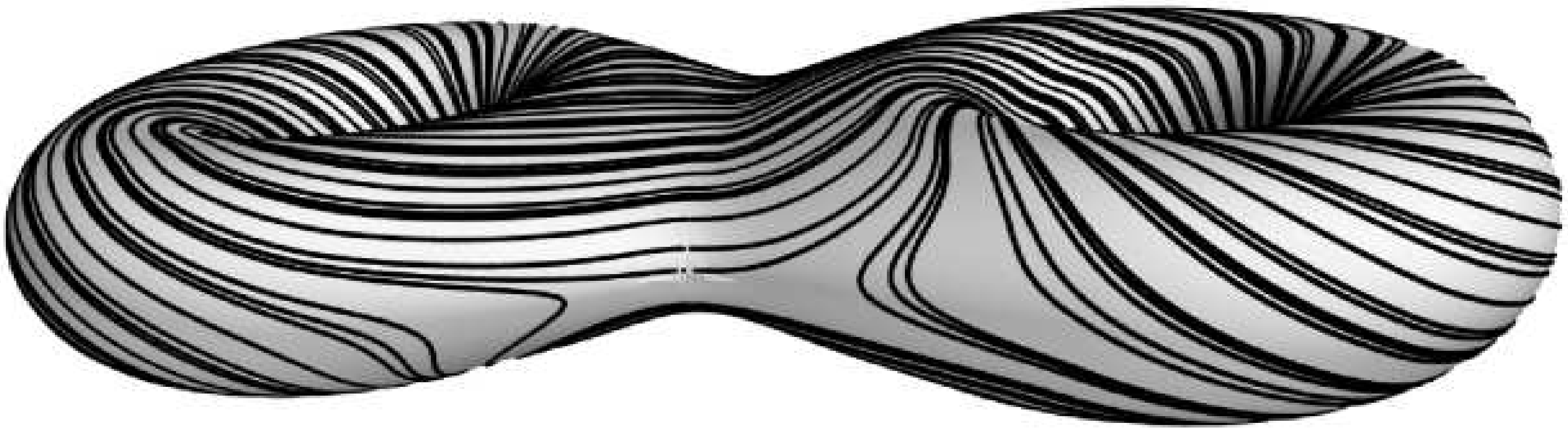
Shape of a random multicurve: genus two

Shape of a random multicurve: large genus

- Random multicurves and random square-tiled surfaces
- Shape of a random multicurve
- Weights of a random multicurve
- Main Theorem (informally)
- Keystone underlying results
- Another Keystone result

Shape of a random multicurve on a surface of large genus

What shape has a random simple closed multicurve on a surface of large genus?



Picture from a book of Danny Calegari

Questions.

- *With what probability a random primitive multicurve on a surface of genus g slices the surface into $1, 2, 3, \ldots$ connected components?*
- *With what probability a random multicurve $m_1\gamma_1 + m_2\gamma_2 + \cdots + m_k\gamma_k$ has $k = 1, 2, \ldots, 3g-3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*
- *What are the typical weights $m_1, \ldots, m_k$?*
- *What is the shape of a random multicurve on a surface of large genus?*

Partly suggested to Leonid Monin and to Peter Pushkar Jr. about ten years ago as a 3-weeks student's project

Random multicurves and random square-tiled surfaces

Denote by $K_g(\gamma)$ the number of components k of the multicurve $\gamma = \sum_{i=1}^{k} m_i \gamma_i$ on a surface of genus g counted without multiplicities.

Denote by $K_g(S)$ the number of maximal horizontal cylinders in the cylinder decomposition of a square-tiled surface S of genus g. We will always consider square-tiled surfaces without cone-angles π, i.e. the ones corresponding to holomorphic quadratic differentials.

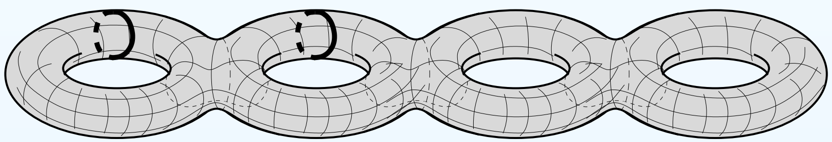
Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *For any genus $g \geq 2$ and for any $k \in \mathbb{N}$, the probability $p_g(k)$ that a random multicurve γ on a surface of genus g has exactly k components counted without multiplicities coincides with the probability that a random square-tiled surface S of genus g has exactly k maximal horizontal cylinders:*

$$p_g(k) = \mathbb{P}\big(K_g(\gamma) = k\big) = \mathbb{P}\big(K_g(S) = k\big)\,.$$

In other words, $K_g(\gamma)$ and $K_g(S)$, considered as random variables, determine the same probability distribution $p_g(k)$, where $k = 1, 2, \ldots, 3g-3$.

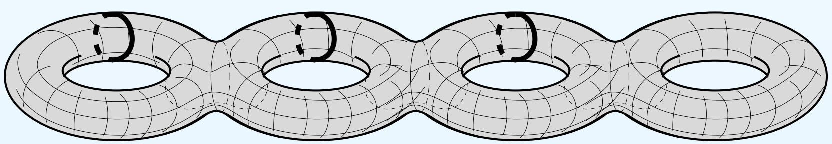
Shape of a random multicurve (random square-tiled surface) on a surface of large genus in simple words

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *The reduced multicurve* $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ *associated to a random integral multicurve* $m_1\gamma_1 + \ldots m_k\gamma_k$ *separates the surface with probability which tends to zero as genus* g *grows. For large* g, $\gamma_{reduced}$ *has about* $(\log g)/2$ *components and has one of the following topological types*



$0.09 \log(g)$ components

$\ldots$ $\ldots$ $\ldots$ $\ldots$



$0.62 \log(g)$ components

$$\mathbb{P}\Big(0.09 \log g < K_g(\gamma) < 0.62 \log g\Big) = 1 - O\left((\log g)^{24} g^{-1/4}\right) .$$

A random square-tiled surface (without conical points of angle π*) of large genus has about* $\frac{\log(g)}{2}$ *cylinders, and all its conical points sit at the same level.*

Weights of a random multicurve (heights of cylinders of a ransom square-tiled surface)

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *A random integer multicurve $m_1\gamma_1 + \dots + m_k\gamma_k$ with bounded number k of primitive components is reduced (i.e., $m_1 = \dots = m_k = 1$) with probability which tends to 1 as $g \to +\infty$. In other terms, if we consider a random square-tiled surface with at most K cylinders, the heights of all cylinders would very likely be equal to 1 for $g \gg 1$.*

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *A general random integer multicurve $m_1\gamma_1 + \dots + m_k\gamma_k$ type is reduced (i.e., $m_1 = \dots = m_k = 1$) with probability which tends to $\frac{\sqrt{2}}{2}$ as genus grows. More generally, all weights $m_1, \dots m_k$ of a random multicurve are bounded from above by an integer m with probability which tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

(In other words, for more 70% of square-tiled surfaces of large genus, the heights of all cylinders are equal to 1.)

Main Theorem (informally)

Main Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *As g grows, the probability distribution p_g rapidly becomes, basically, indistinguishable from the distribution of the number of cycles in a (very explicitly nonuniform) random permutation. In particular, for any $k \in \mathbb{N}$ the difference of the k-th moments of the two distributions is of the order $O(g^{-1})$.*

Actually, we have an explicit asymptotic formula for all cumulants. For example

$$\mathbb{E}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 + o(1)\,,$$
$$\mathbb{V}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1)\,,$$

where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

In practice, already for $g = 12$ the match of the graphs of the distributions is such that they are visually indistinguishable.

Mod-Poisson convergence: the distribution of the number of cycles of a usual random permutation of n elements is uniformly well-approximated in a neighborhood of $x \log n$ by the Poisson distribution with parameter $\log n + a(x)$ with an explicit correctional term $a(x)$.

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where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

Let $\lambda_{3g-3} = \log(6g-6)/2$. We have uniformly in $0 \le k \le 1.233 \cdot \lambda_{3g-3}$

$$\mathbb{P}\big(K_g(\gamma) = k+1\big) = e^{-\lambda_{3g-3}} \cdot \frac{\lambda_{3g-3}^k}{k!} \cdot \left(\frac{\sqrt{\pi}}{2\Gamma\left(1 + \frac{k}{2\lambda_{3g-3}}\right)} + O\left(\frac{k}{(\log g)^2}\right)\right)\,.$$

Keystone underlying results

Our results are strongly based on the following conjecture which we stated in August 2019, and which Amol Aggarwal proved in April 2020.

Theorem (Aggarwal). *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol}\mathcal{Q}_g \sim \frac{4}{\pi}\cdot\left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g \to +\infty\,.$$

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Conjecture. *The Masur–Veech volume of any stratum of meromorphic quadratic differentials with at most simple poles has the following large genus asymptotics (with the error term uniformly small for all partitions $\boldsymbol{d}$):*

$$\operatorname{Vol}\mathcal{Q}(d_1,\dots,d_n) \overset{?}{\sim} \frac{4}{\pi}\cdot\prod_{i=1}^{n}\frac{2^{d_i+2}}{d_i+2} \quad \text{as } g \to +\infty\,,$$

under assumption that the number of simple poles is bounded or grows much slower than the genus.

Another Keystone result

Another Conjecture which we stated in August 2019 was also proved by Amol Aggarwal in April 2020.

Theorem (Aggarwal). *The following **uniform** asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1+\frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1+\cdots+d_n = 3g-3+n$*, as* $g \to +\infty$*.*