Dynamics and Geometry of Moduli Spaces

Lecture 7. Train tracks. Integral measured laminations. Idea of the proof of Mirzakhani's count of simple closed geodesics

Anton Zorich
University Paris Cité

March 30, 2023

Space of multicurves
- Train tracks carrying simple closed curves
- Exercise on train-tracks
- Four basic train tracks on $S_{0,4}$
- Space of multicurves

Thurston's and Mirzakhani's measures on $\mathcal{ML}_{g,n}$

Proof of the main result

Teaser for next lectures

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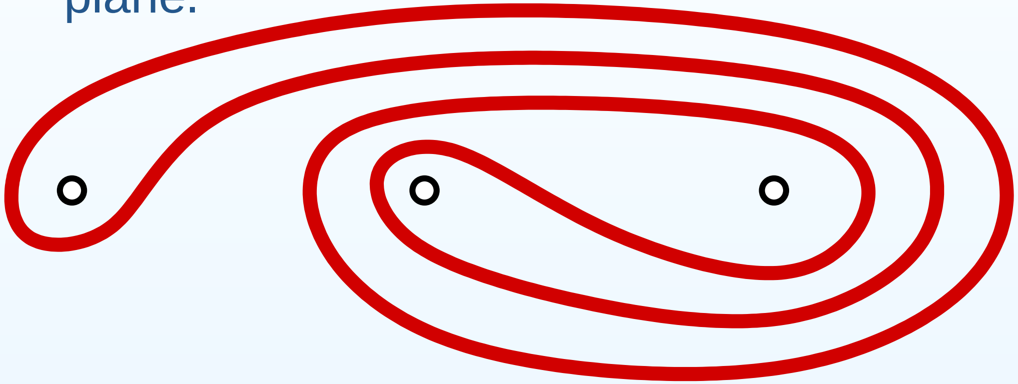

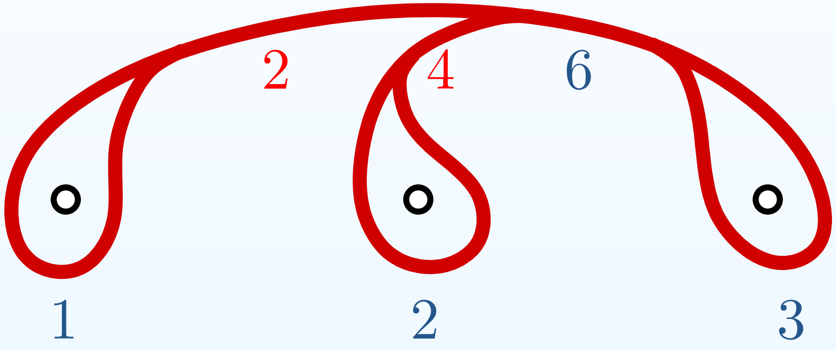

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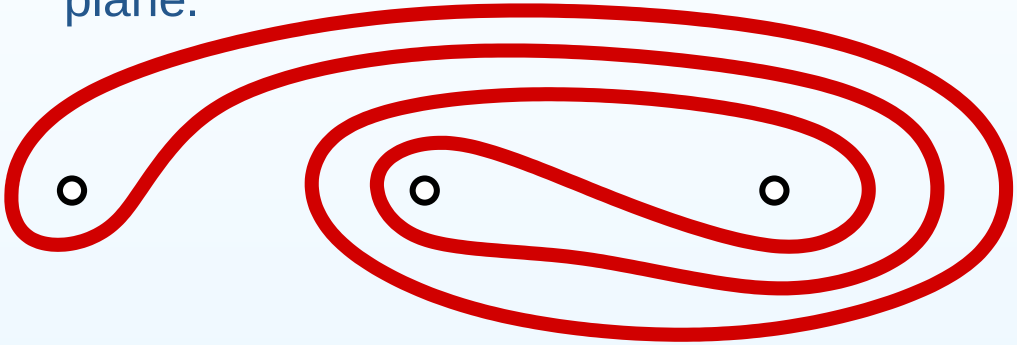

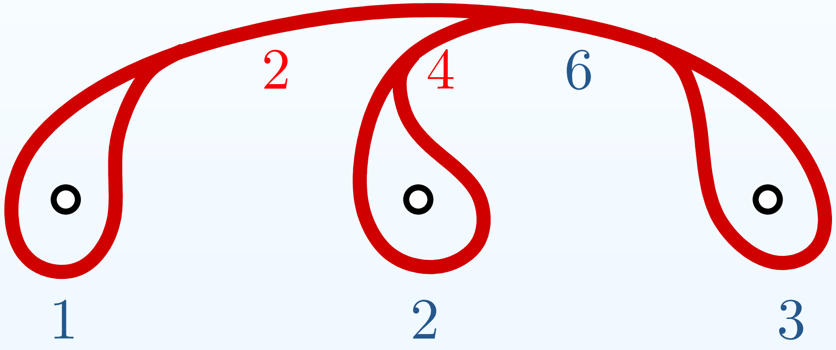

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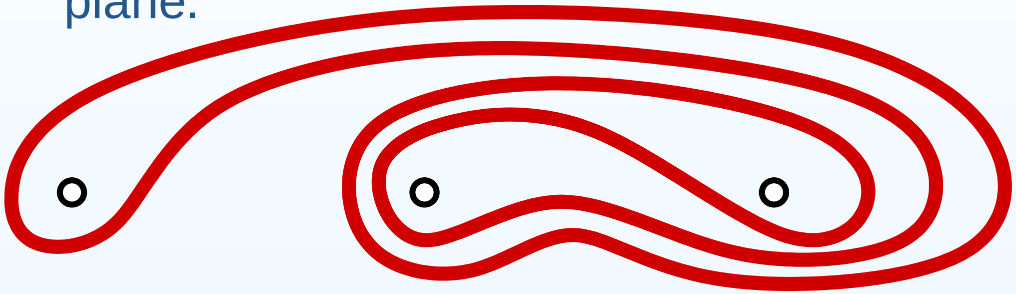

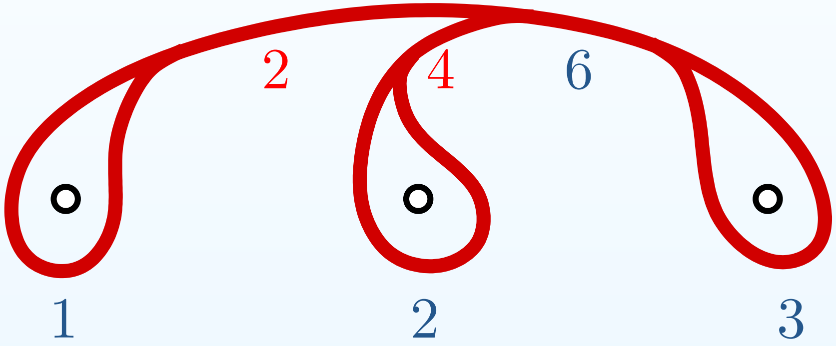

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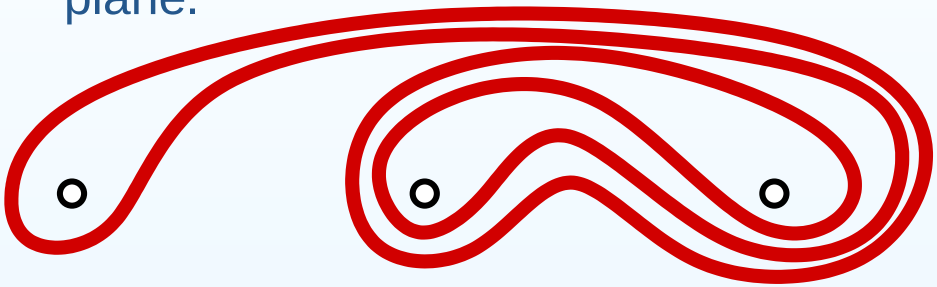

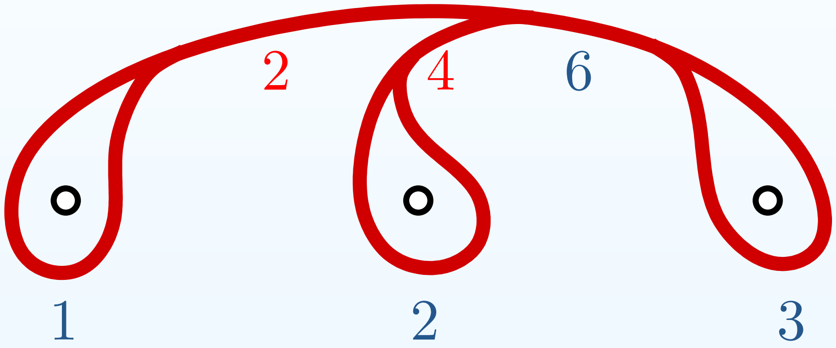

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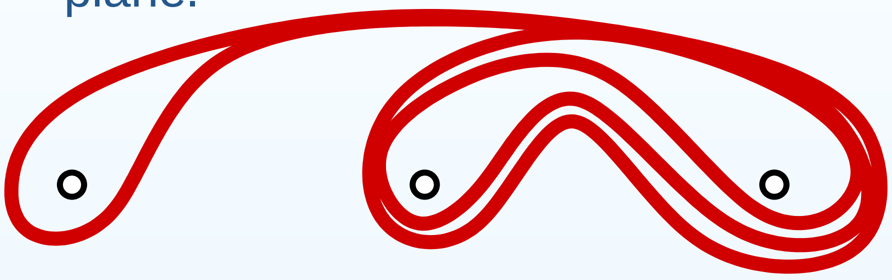

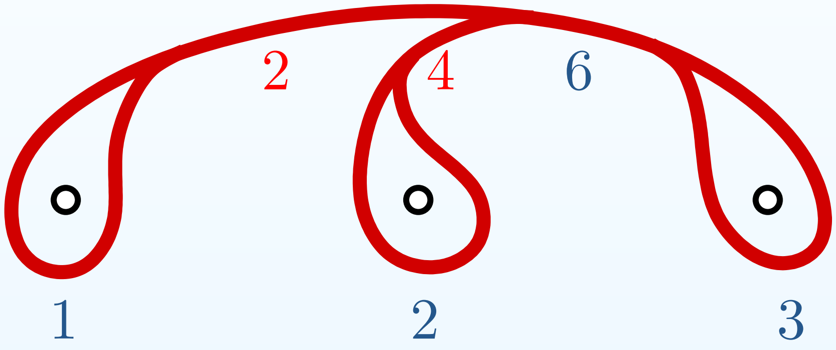

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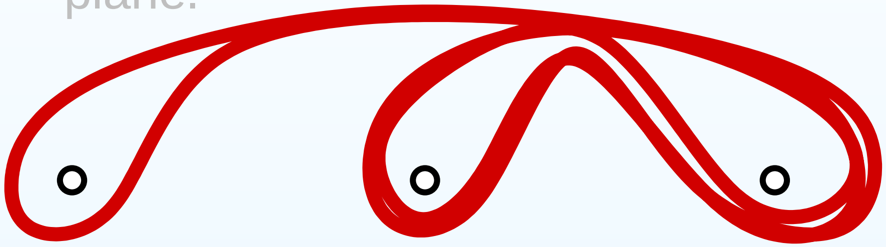

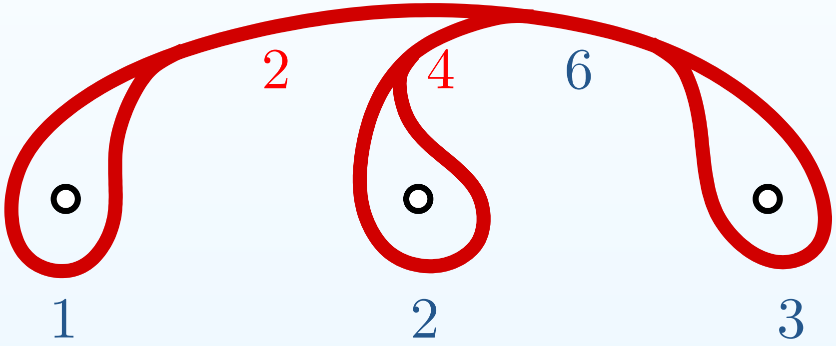

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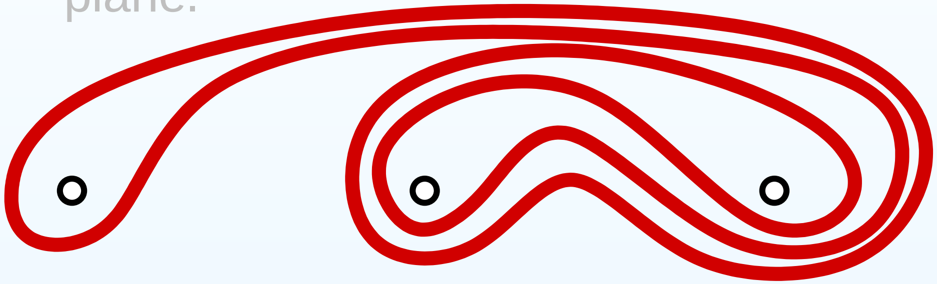

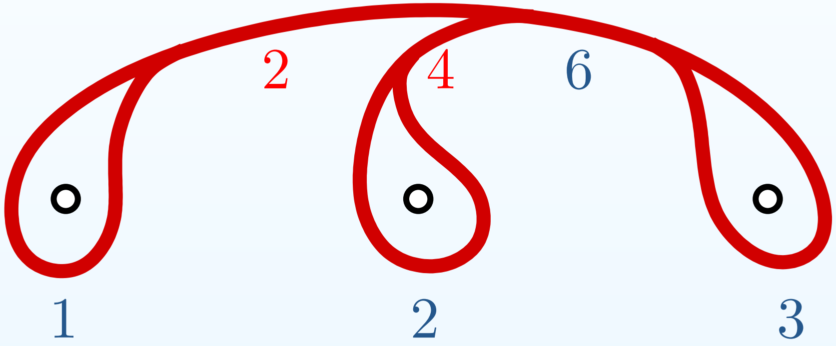

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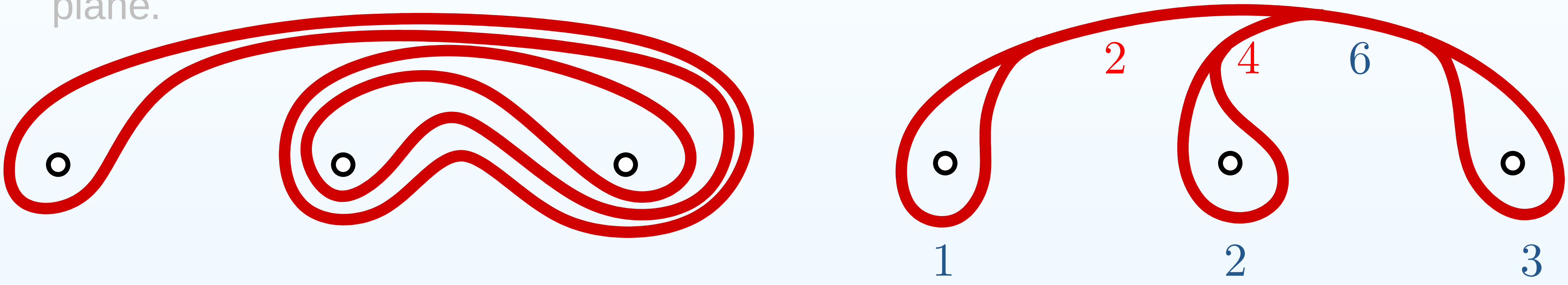

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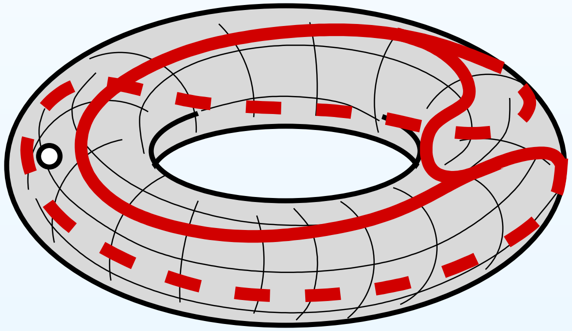
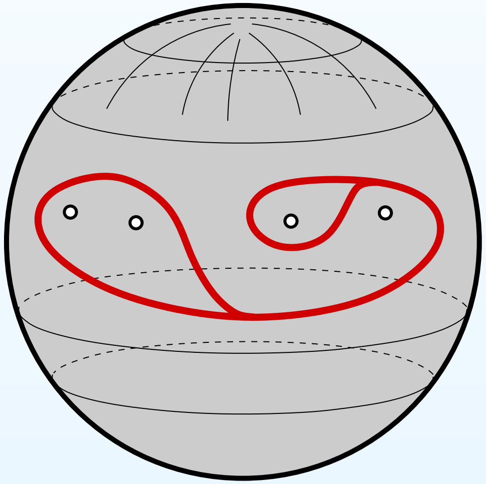
Exercise on train-tracks

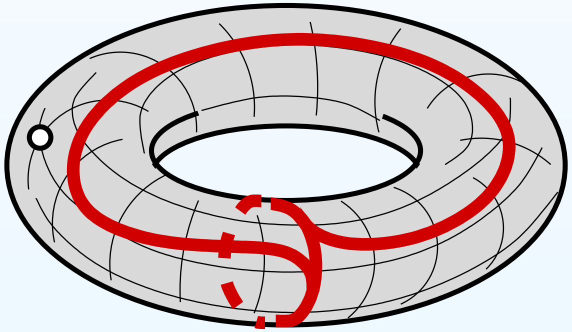
Which of the given train-tracks τ_1, τ_2, τ_3 might carry a simple closed hyperbolic geodesic? Indicate some legitimate weights if you claim that the train track carries a simple closed hyperbolic geodesic.



τ_1



τ_2



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Can any of the given train-tracks τ_1, τ_2, τ_3 carry *different* simple closed hyperbolic geodesic? Indicate the corresponding different legitimate collections of weights if you claim that.

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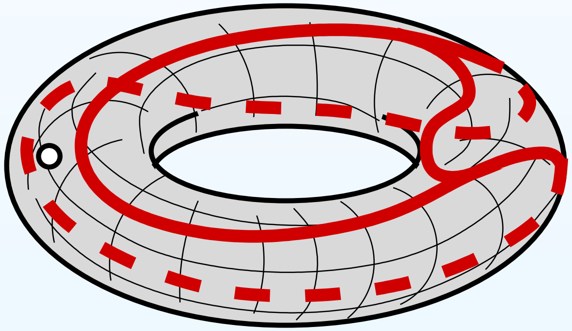

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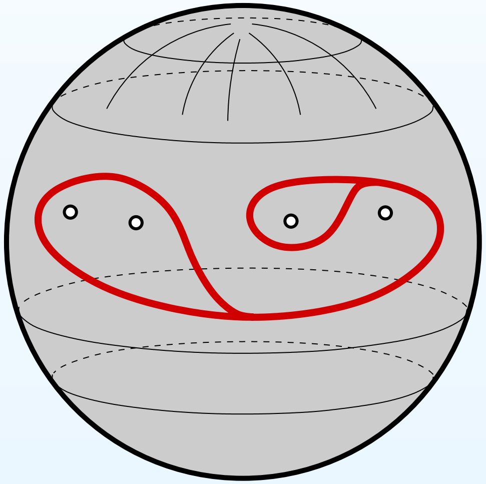

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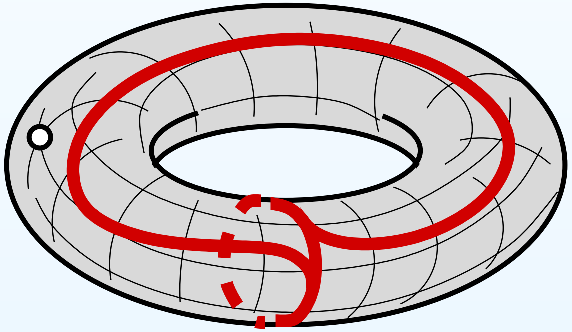

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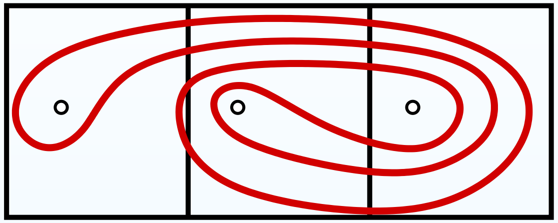
Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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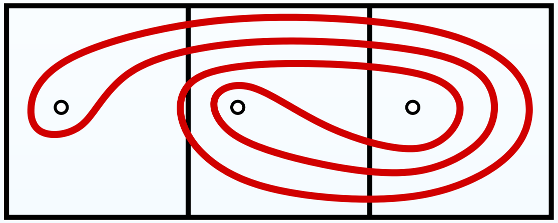

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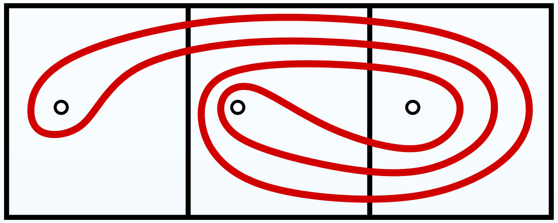

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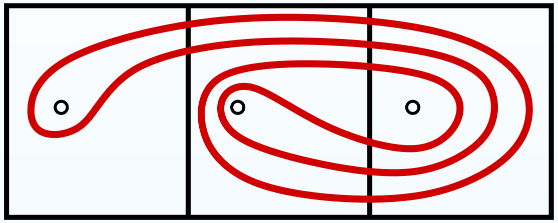

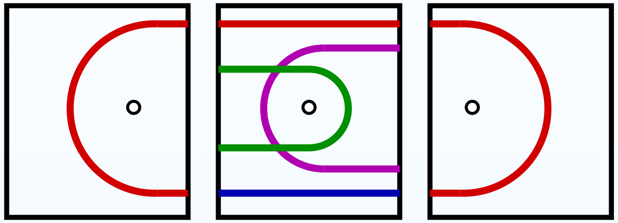

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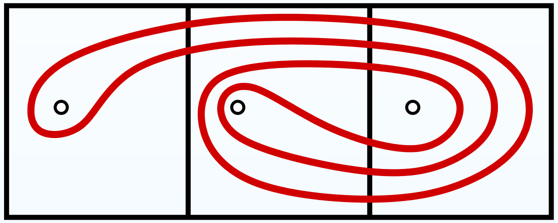

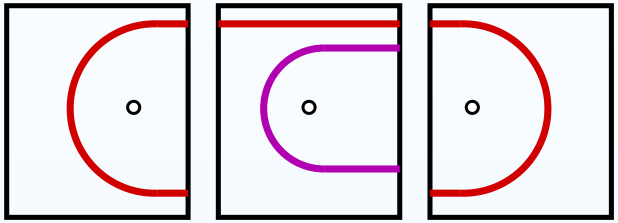

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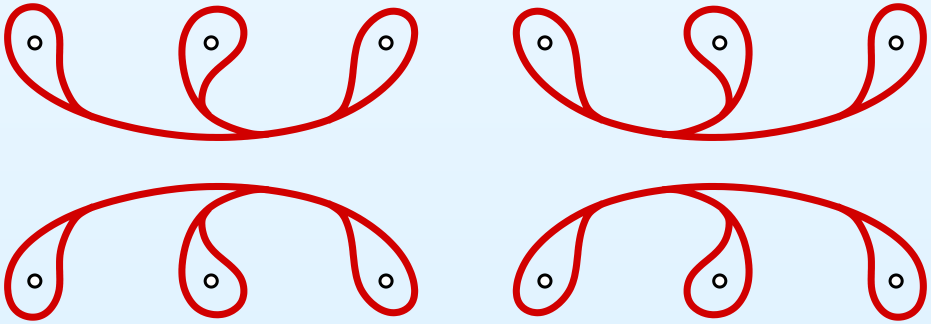
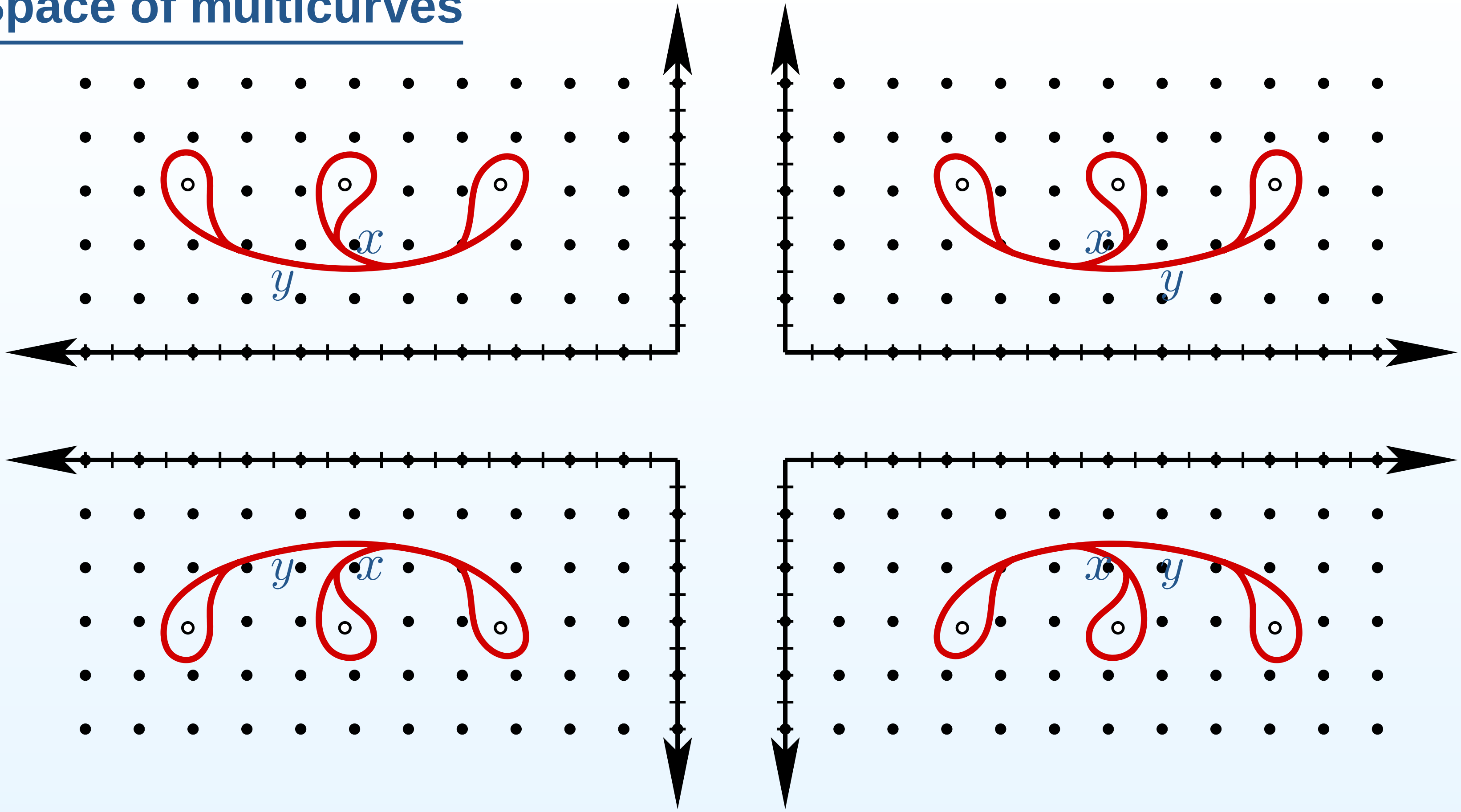
Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve is carried by one of the following four train tracks:



Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

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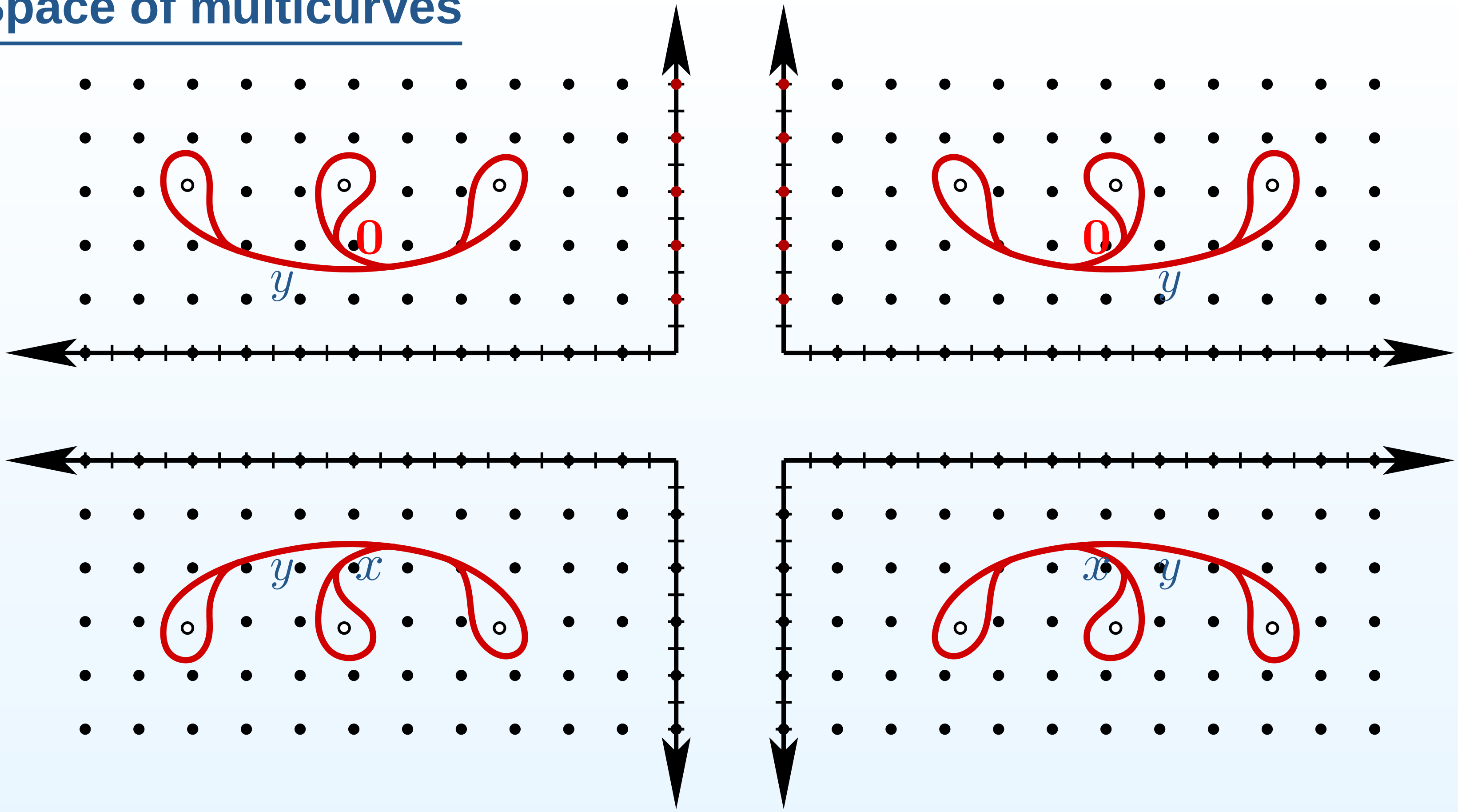

Weight zero on an edge of a train track tells that such edge can be deleted.
This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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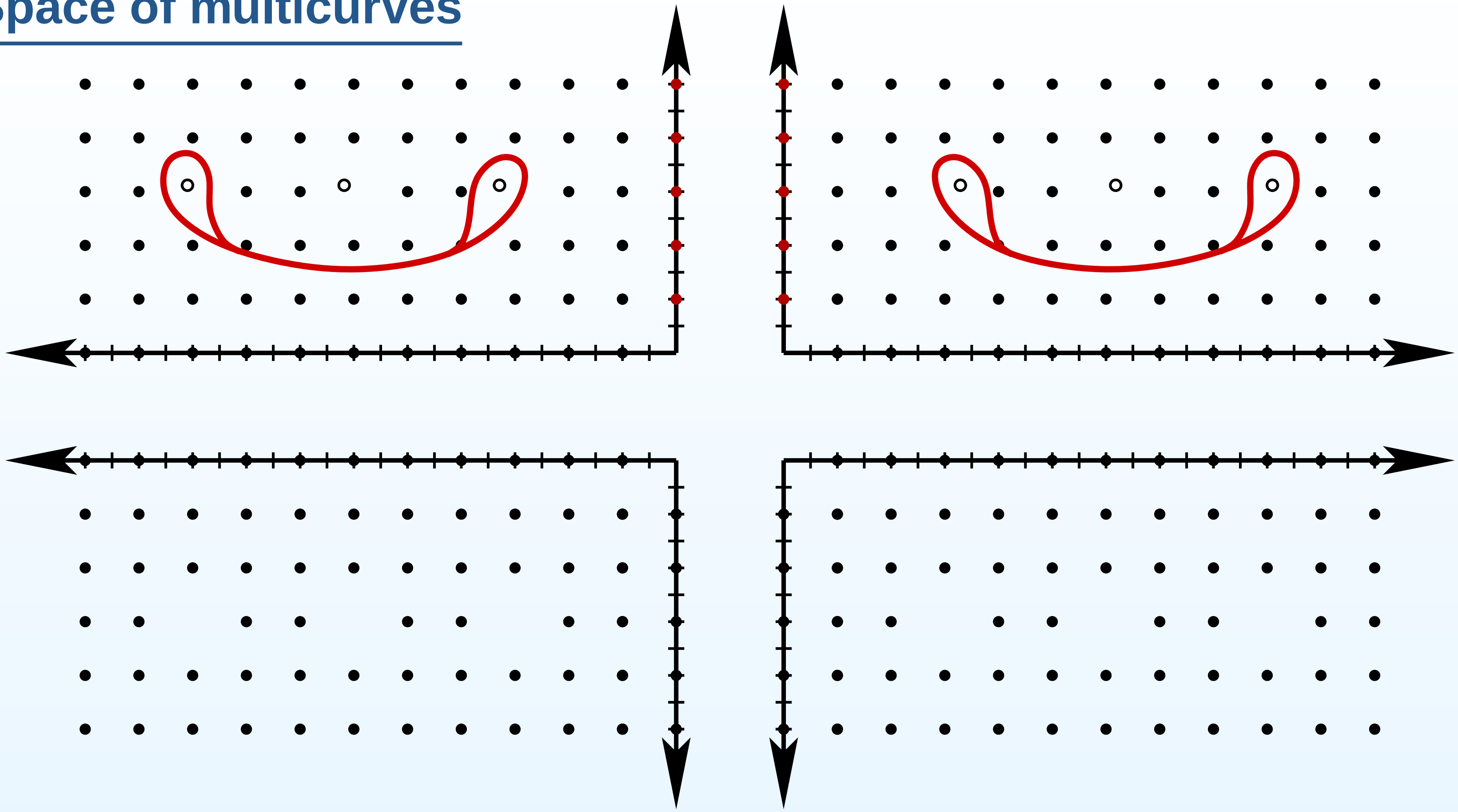

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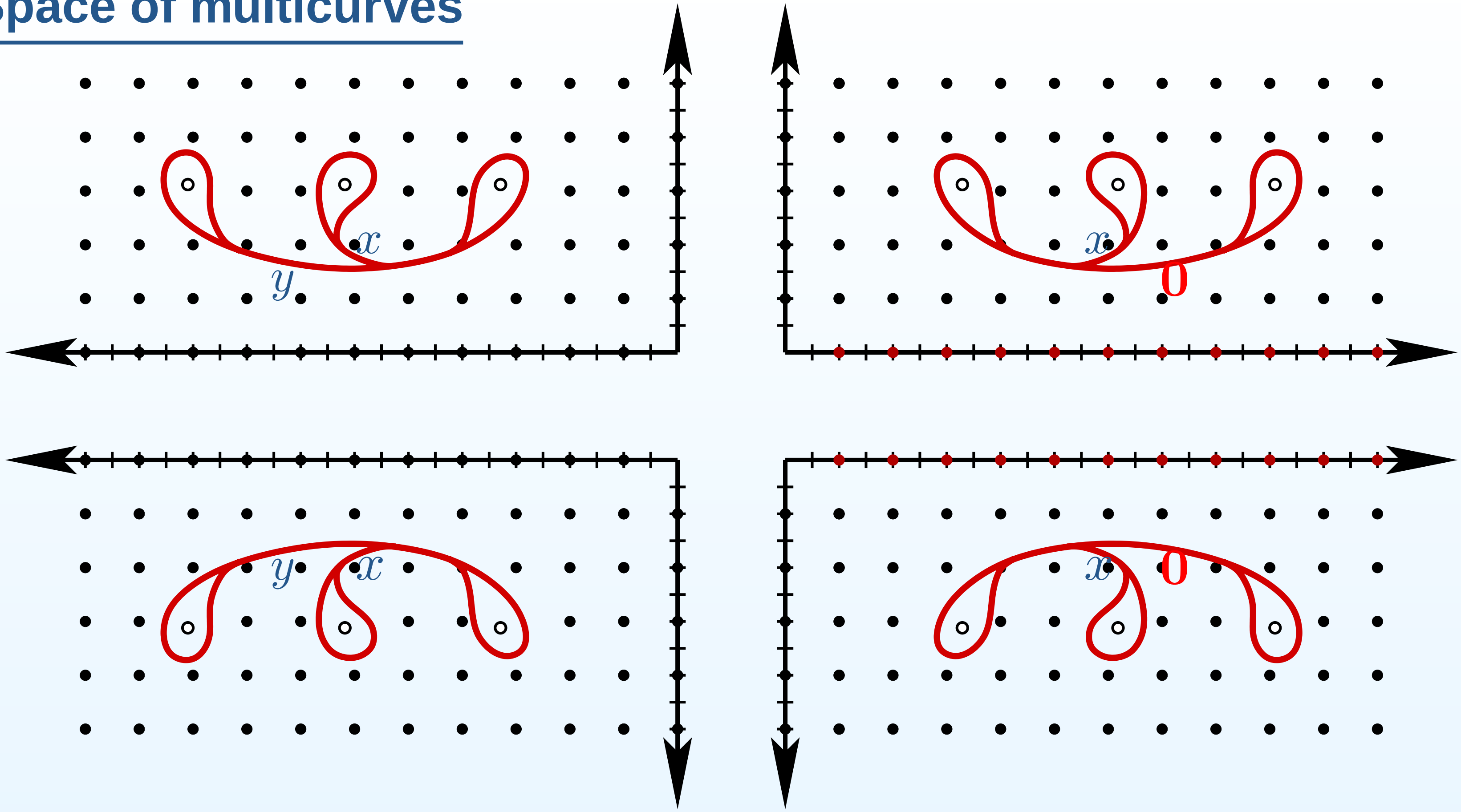

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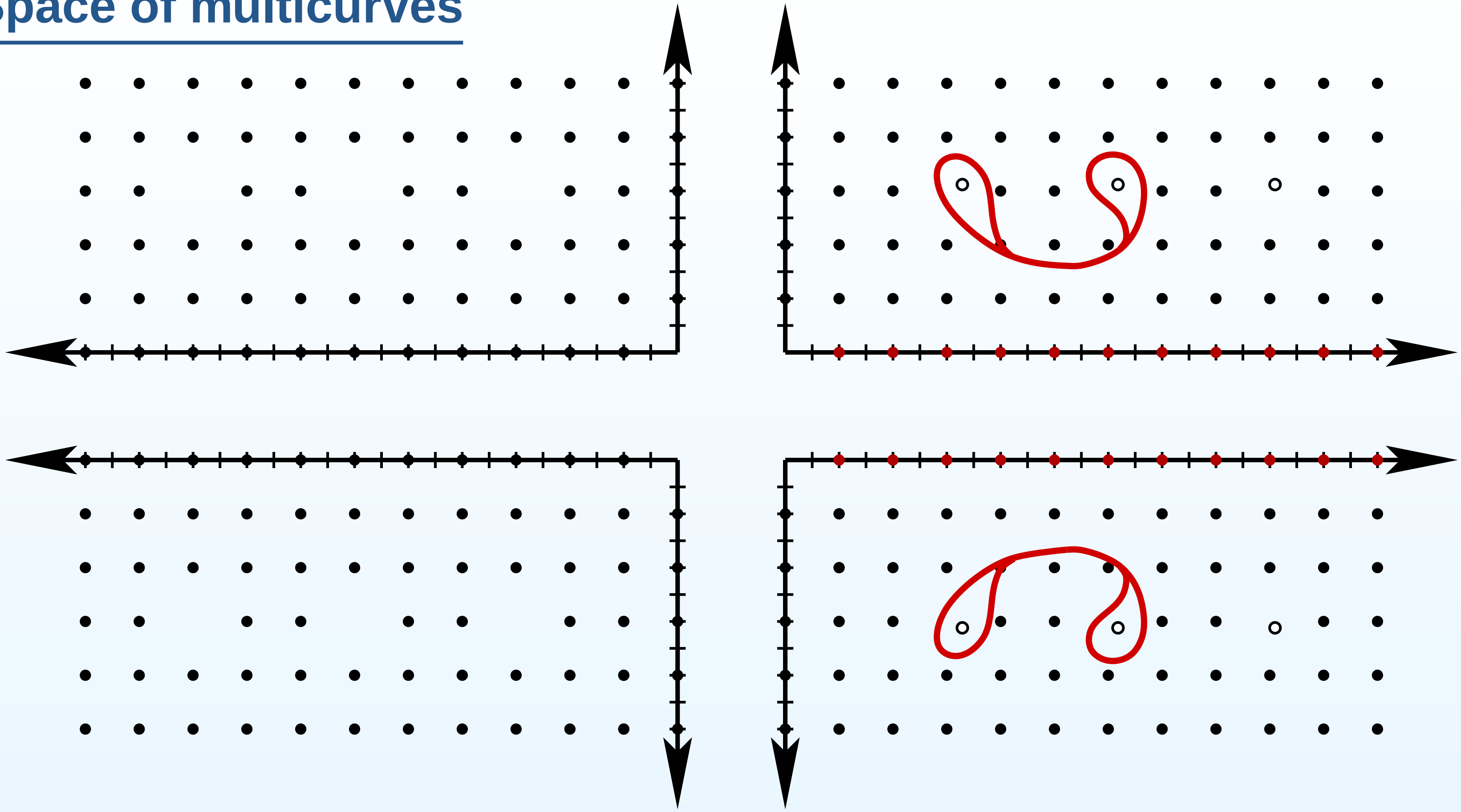

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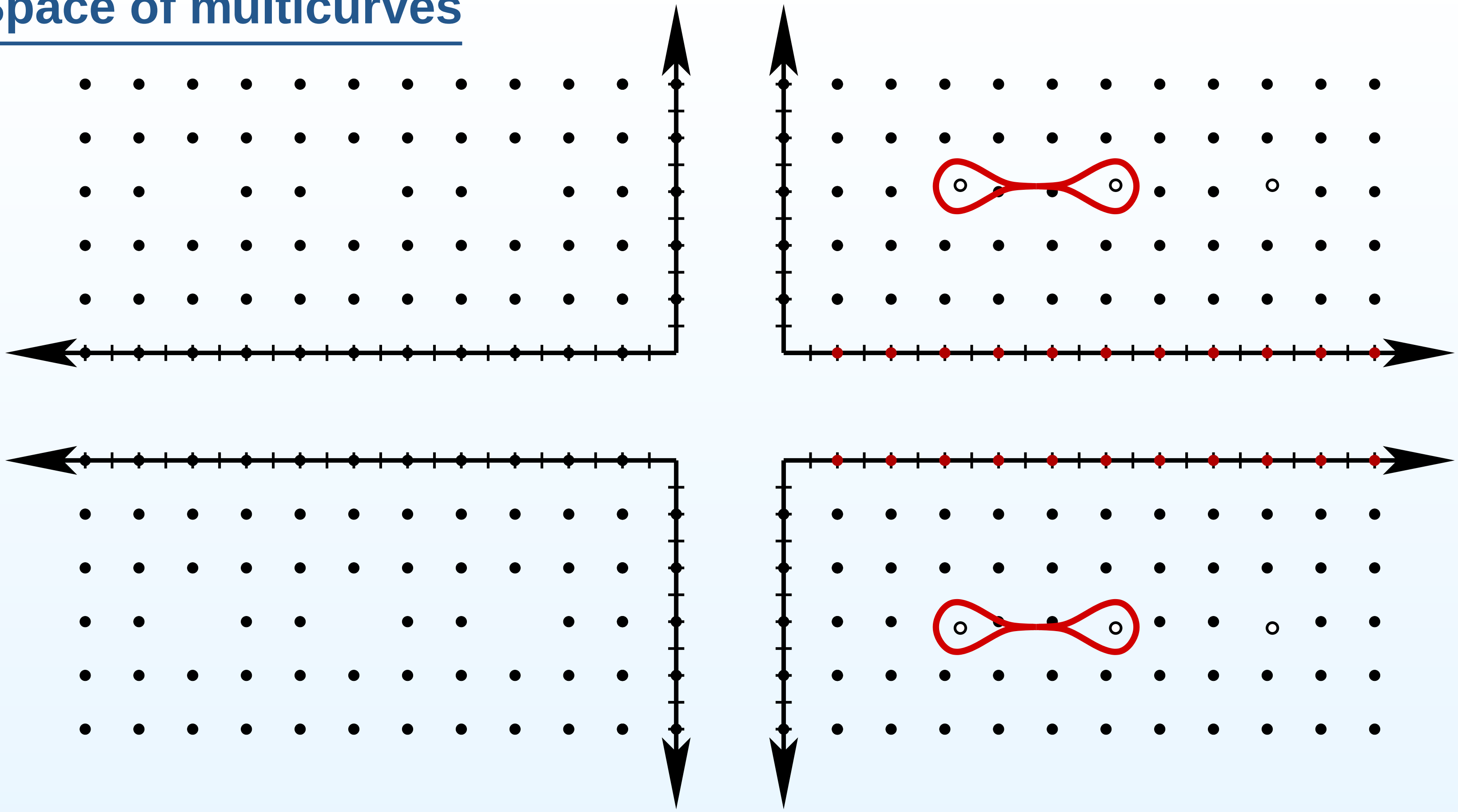

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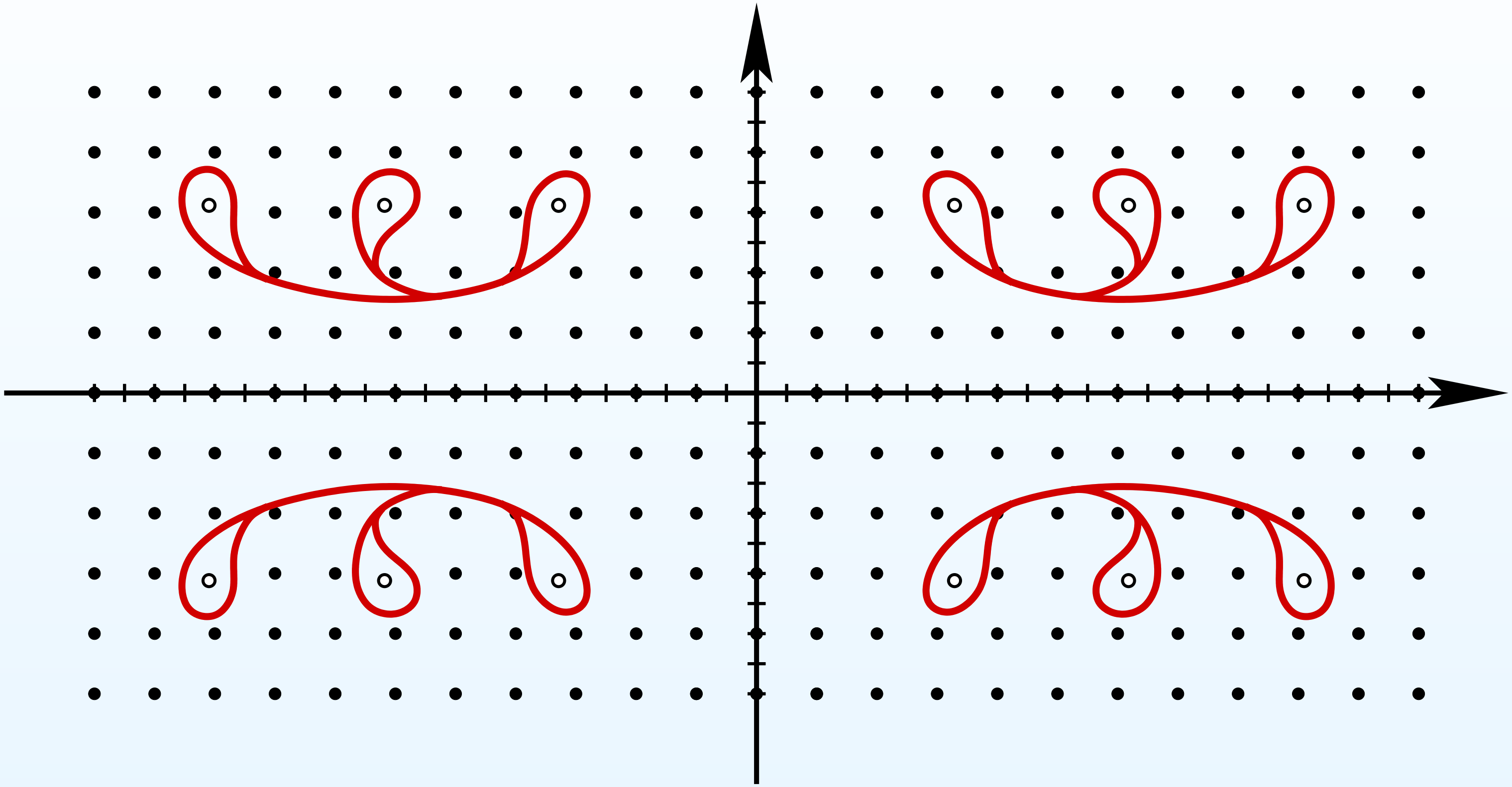

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Space of multicurves

Thurston’s and Mirzakhani’s measures on $\mathcal{ML}_{g,n}$

- Space of multicurves
- Space $\mathcal{ML}_{g,n}$ and the length function
- Thurston measure on $\mathcal{ML}_{g,n}$
- Counting the measure of a set
- Mirzakhani’s measures on $\mathcal{ML}_{g,n}$

Proof of the main result

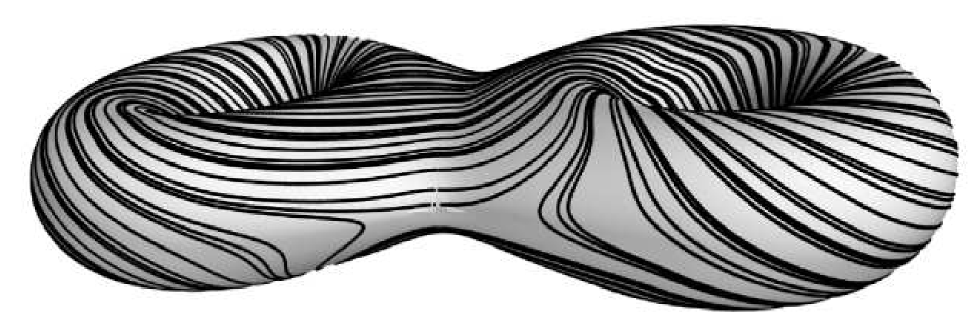
Teaser for next lectures

Thurston’s and Mirzakhani’s measures on $\mathcal{ML}_{g,n}$

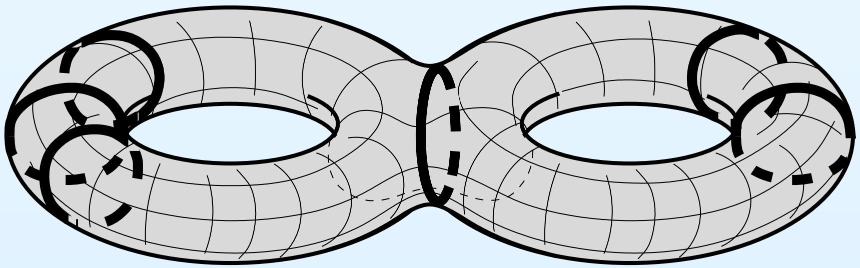
Orbits of multicurves

Thurston suggested to consider simple closed multicurves as integral points in the piecewise-linear space of measured laminations. All integral multicurves are partitioned in orbits under action of the mapping class group.

A general multicurve ρ:



the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.



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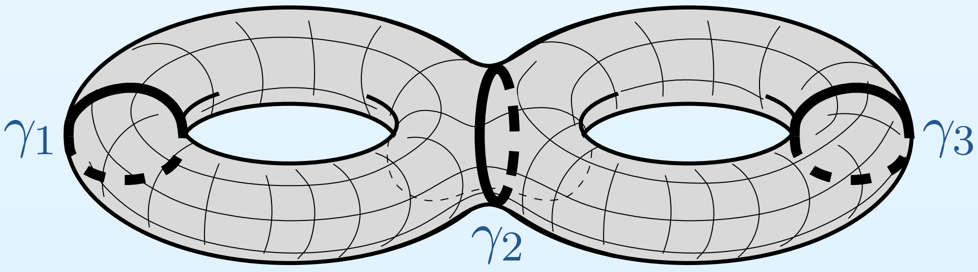
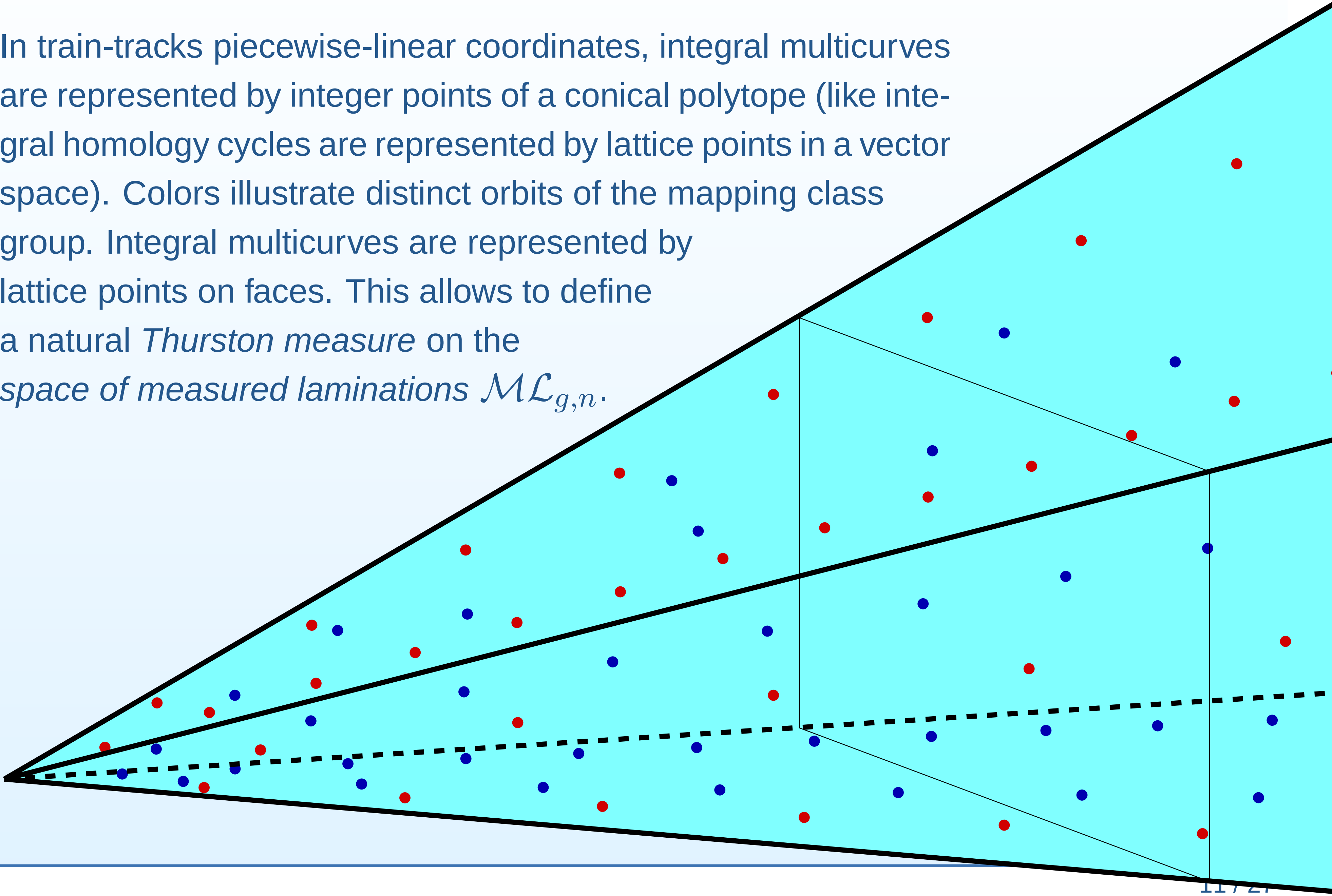


$\gamma_{reduced} = \gamma_1 + \gamma_2 + \gamma_3$

Space of multicurves

In train-tracks piecewise-linear coordinates, integral multicurves are represented by integer points of a conical polytope (like integral homology cycles are represented by lattice points in a vector space). Colors illustrate distinct orbits of the mapping class group. Integral multicurves are represented by lattice points on faces. This allows to define a natural *Thurston measure* on the *space of measured laminations* $\mathcal{ML}_{g,n}$.



Space $\mathcal{ML}_{g,n}$ and the length function

Similar considerations applied to a smooth surface $S_{g,n}$ lead to analogous space $\mathcal{ML}_{g,n}$ endowed with a **piecewise linear structure**.

Up to now we did not use hyperbolic metric on $S_{g,n}$. In the presence of a hyperbolic metric, integral points of $\mathcal{ML}_{g,n}$ can be interpreted as simple closed geodesic multicurves.

Moreover: all other points also get geometric realization as *measured geodesic laminations* — disjoint unions of non self-intersecting infinite geodesics.

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space.

One can extend the length function by linearity to simple closed multicurves:

$$\ell_{\sum a_i\gamma_i} := \sum a_i \ell_{\gamma_i}(X)\,.$$

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Space $\mathcal{ML}_{g,n}$ and the length function

Similar considerations applied to a smooth surface $S_{g,n}$ lead to analogous space $\mathcal{ML}_{g,n}$ endowed with a **piecewise linear structure**.

Up to now we did not use hyperbolic metric on $S_{g,n}$. In the presence of a hyperbolic metric, integral points of $\mathcal{ML}_{g,n}$ can be interpreted as simple closed geodesic multicurves.

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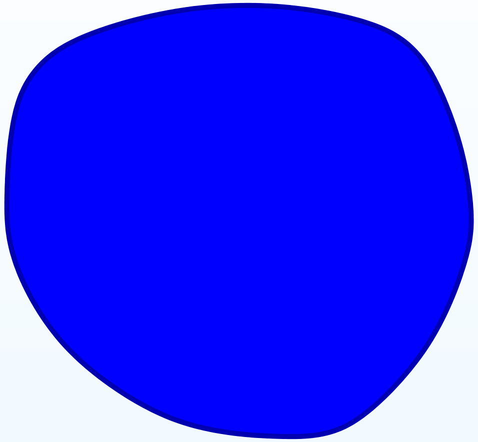
Counting the measure of a set



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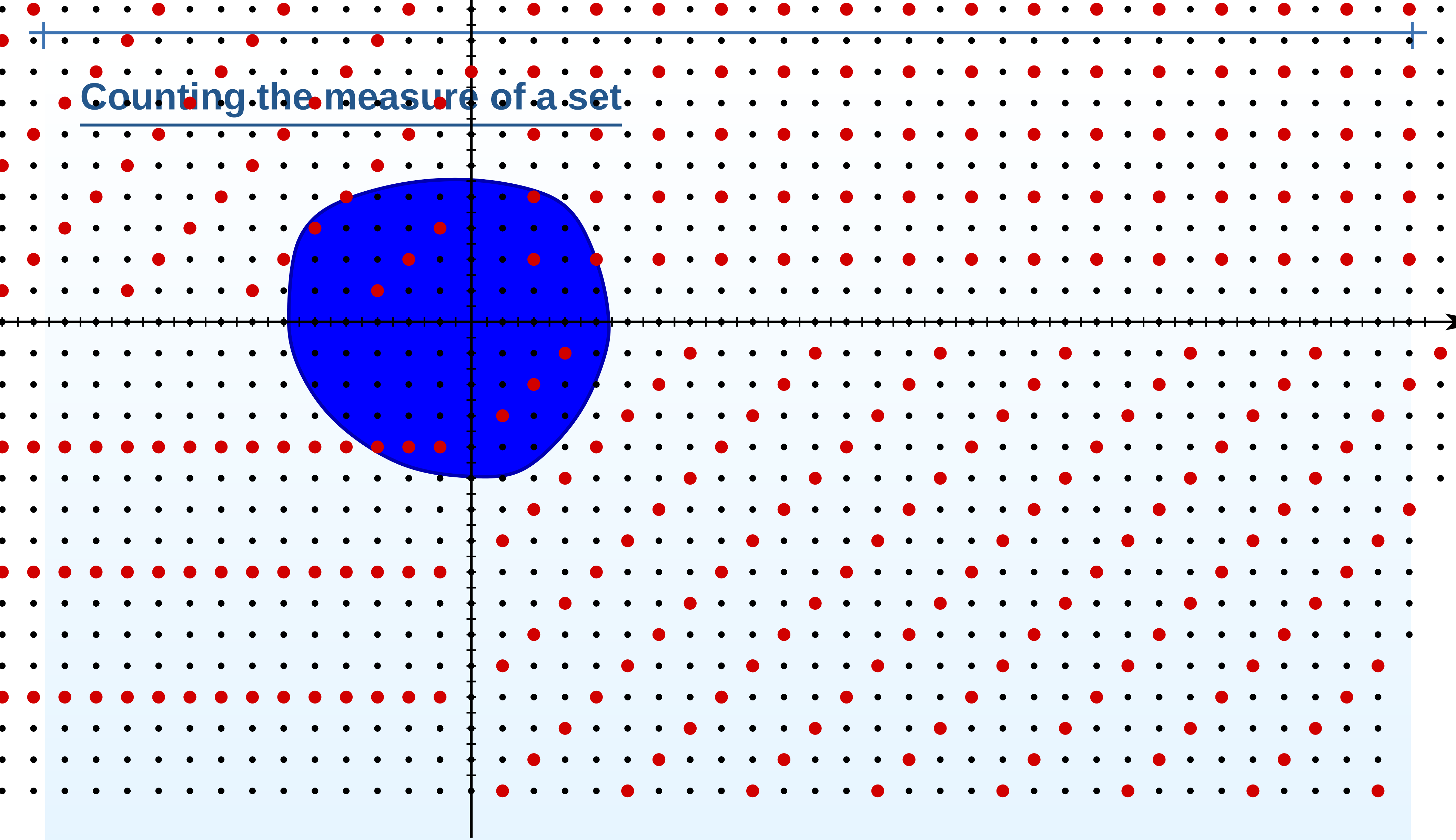

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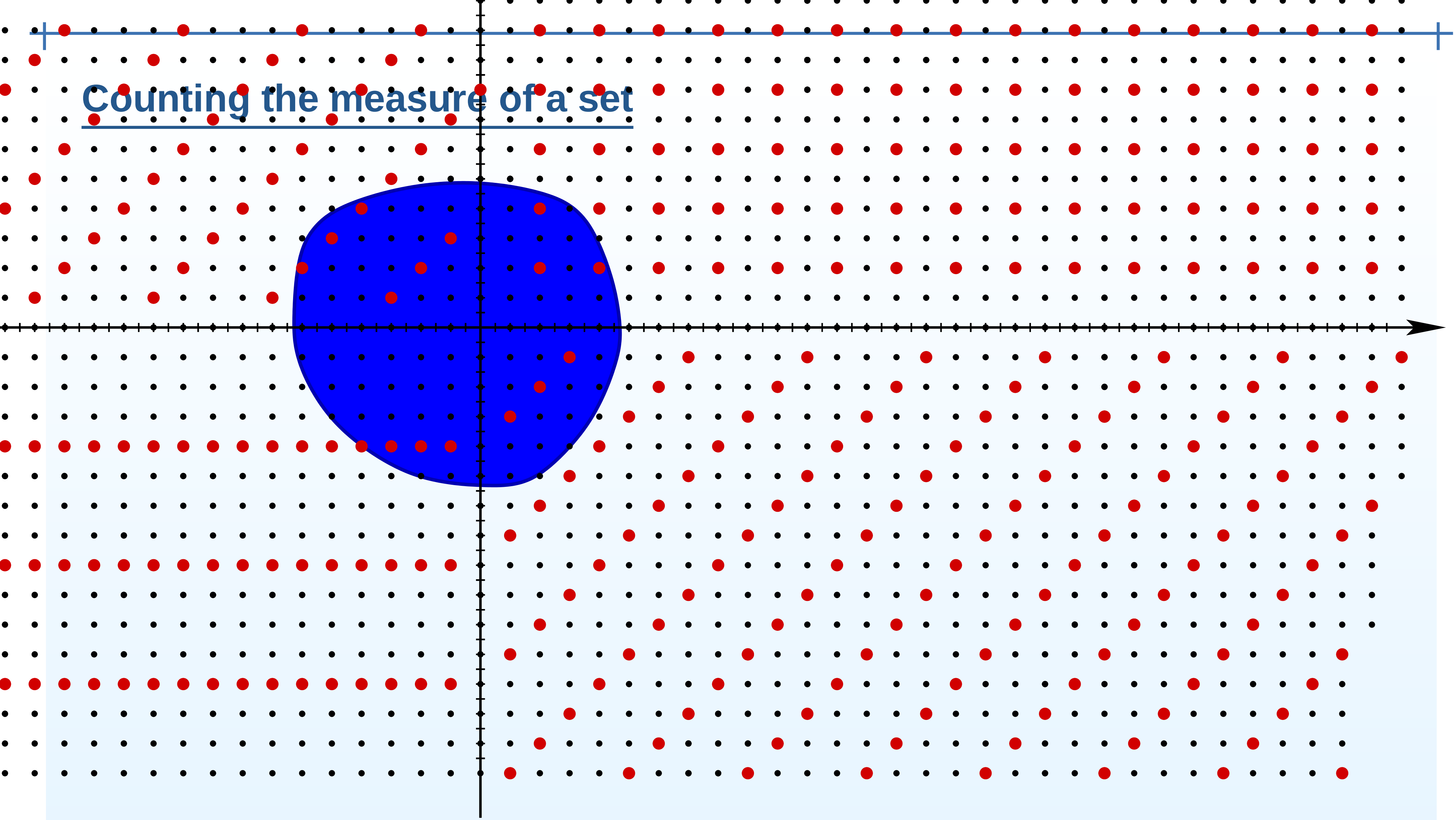

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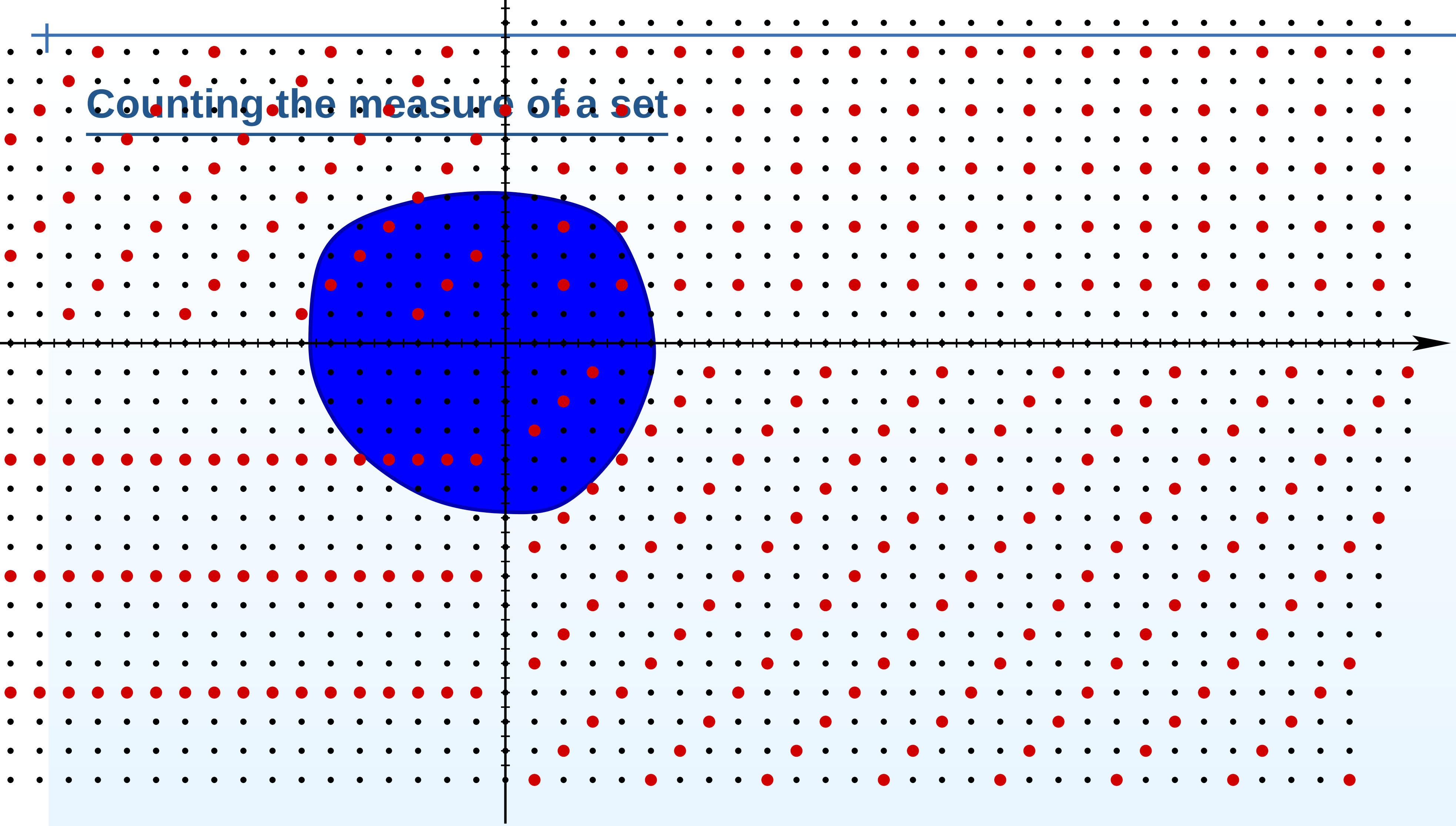

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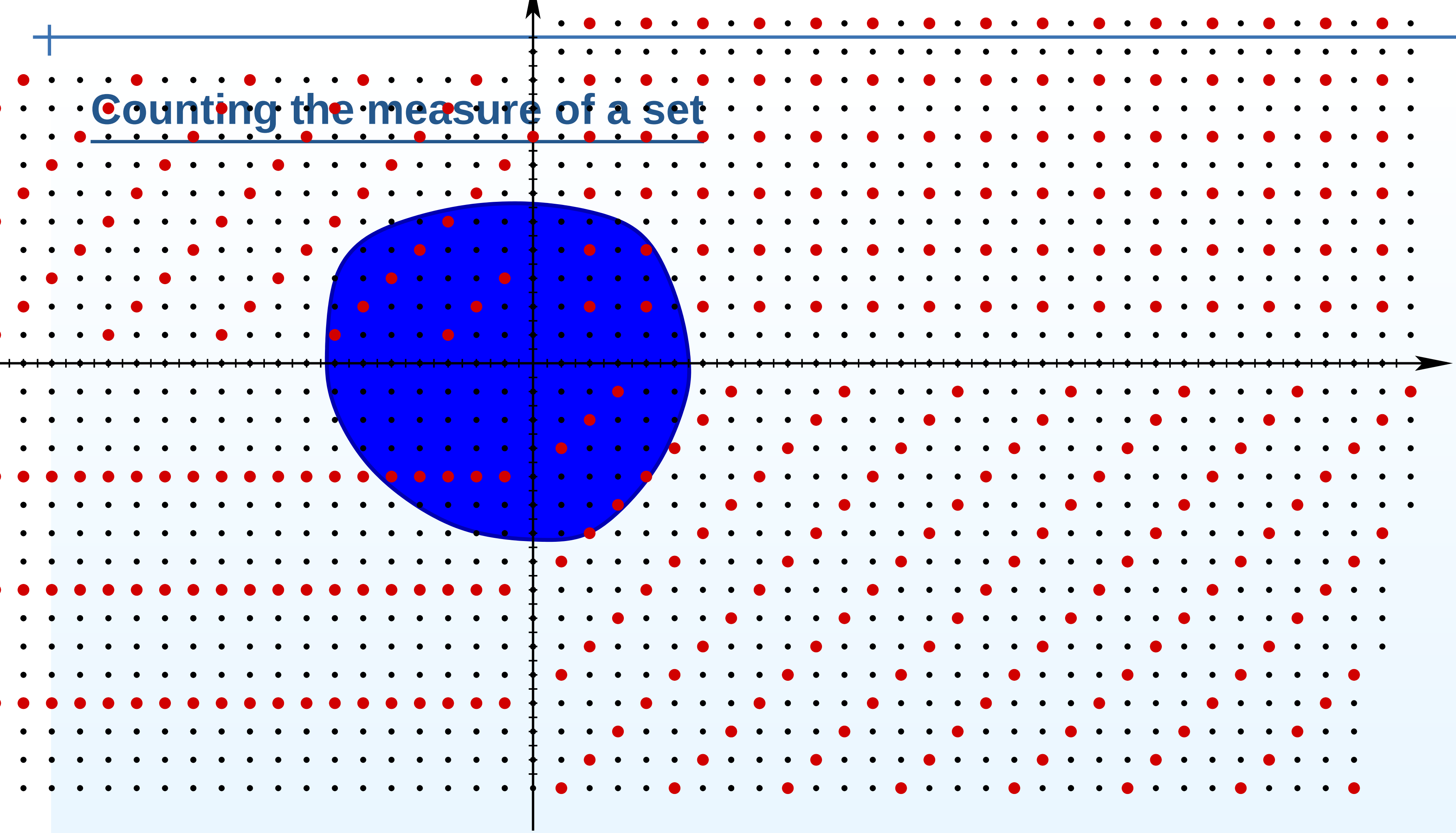

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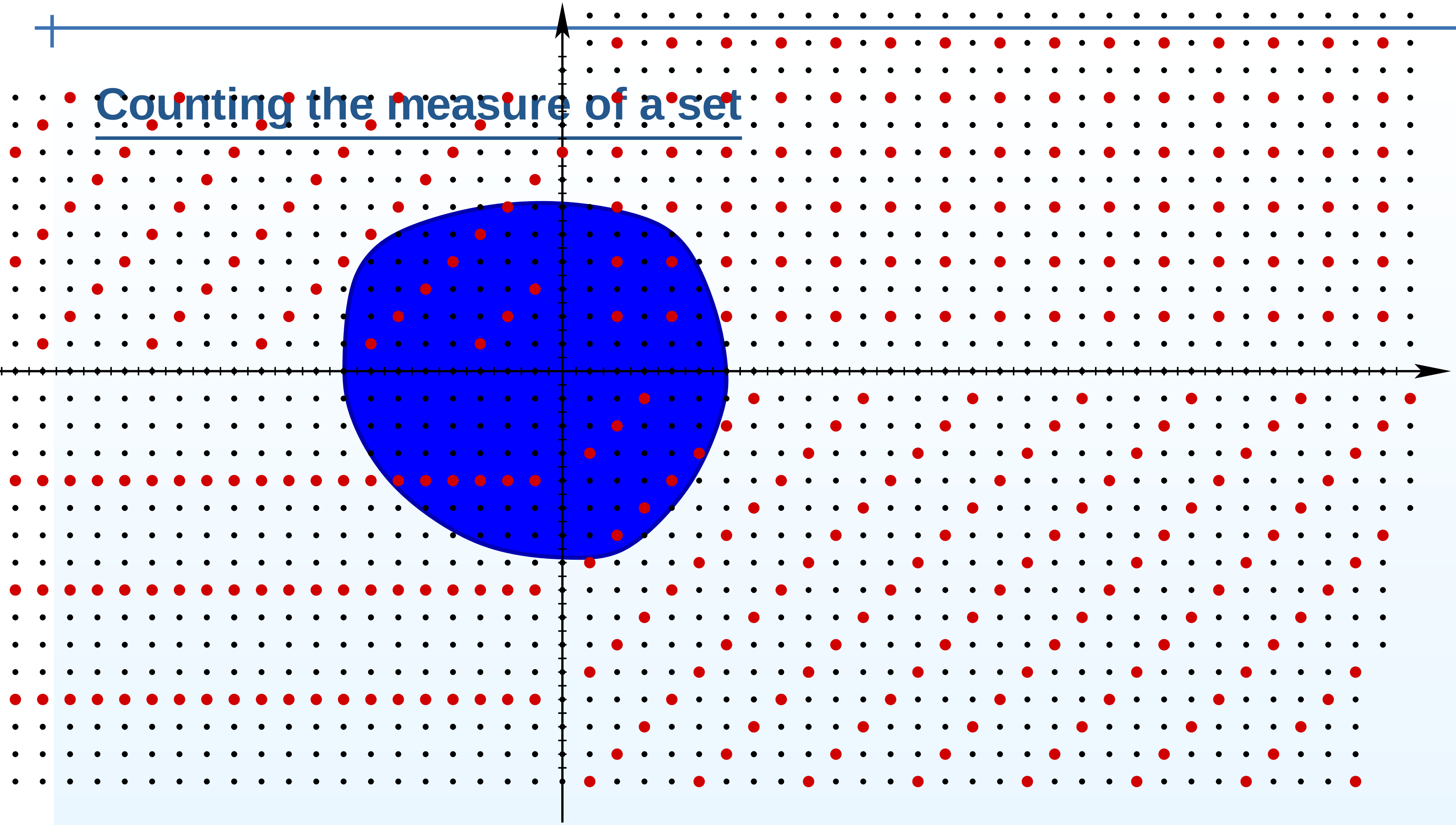

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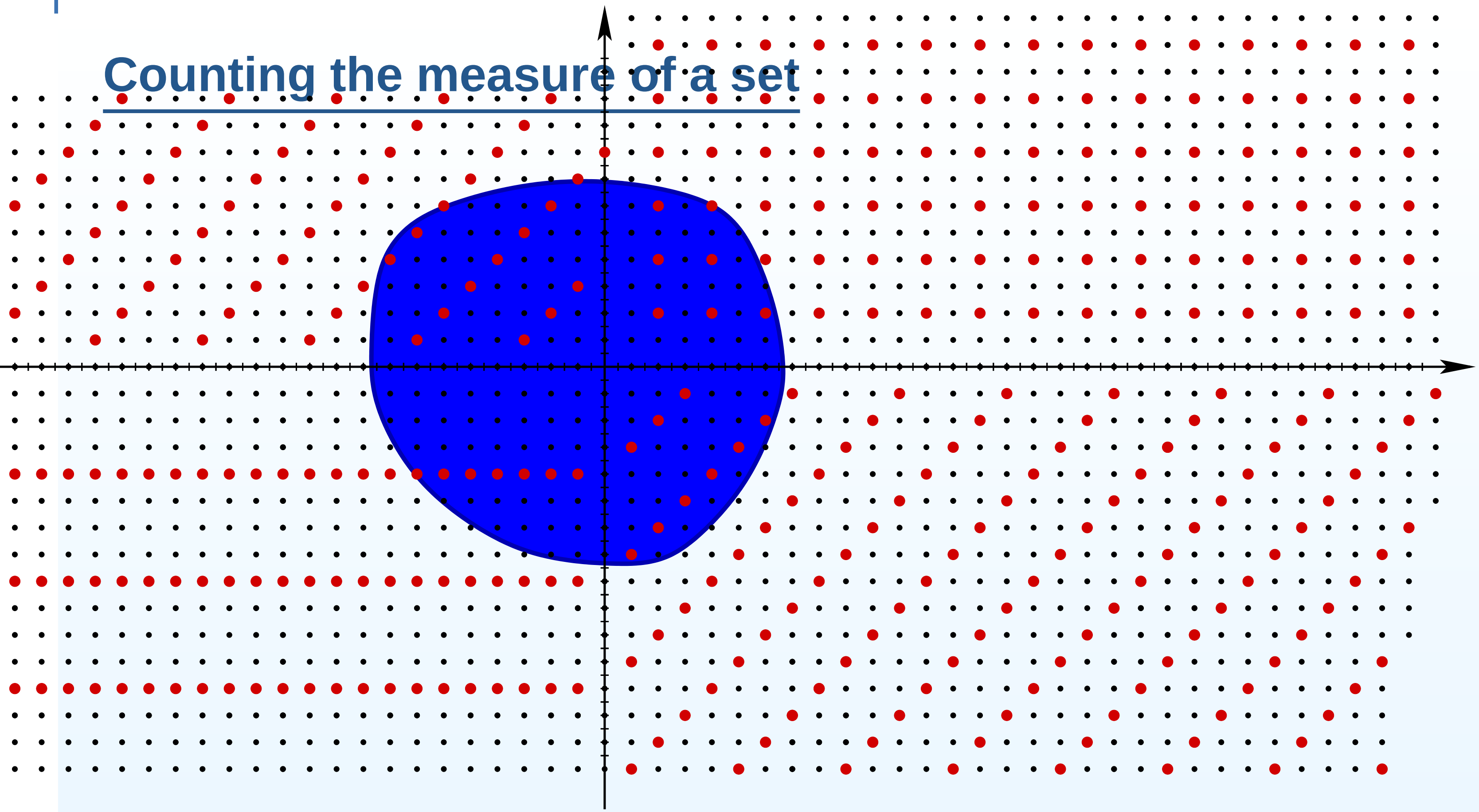

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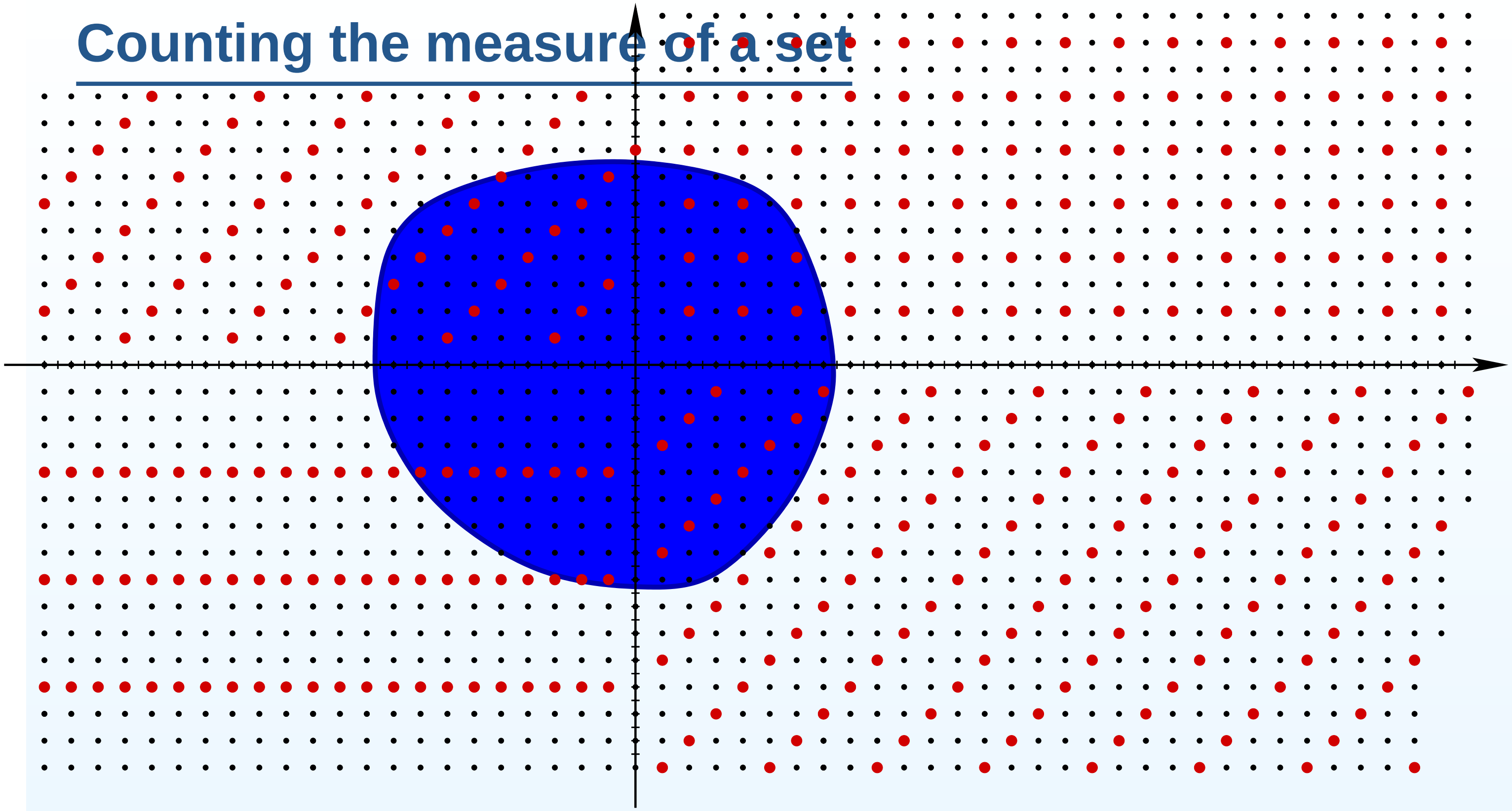

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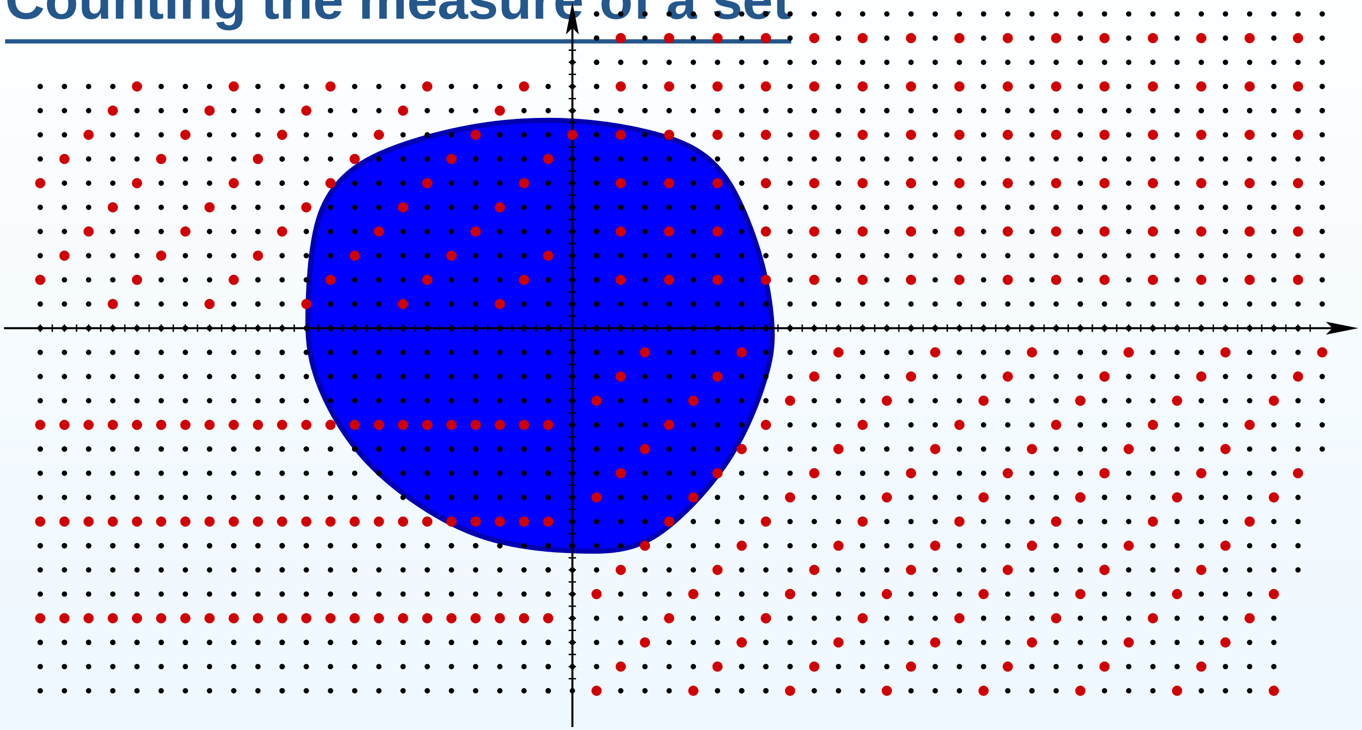

By definition, the Lebesgue measure $\mu(U)$ of a set $U \subset \mathbb{R}^n$ is defined as the limit of the normalized number of points of the ε-grid which get to U:

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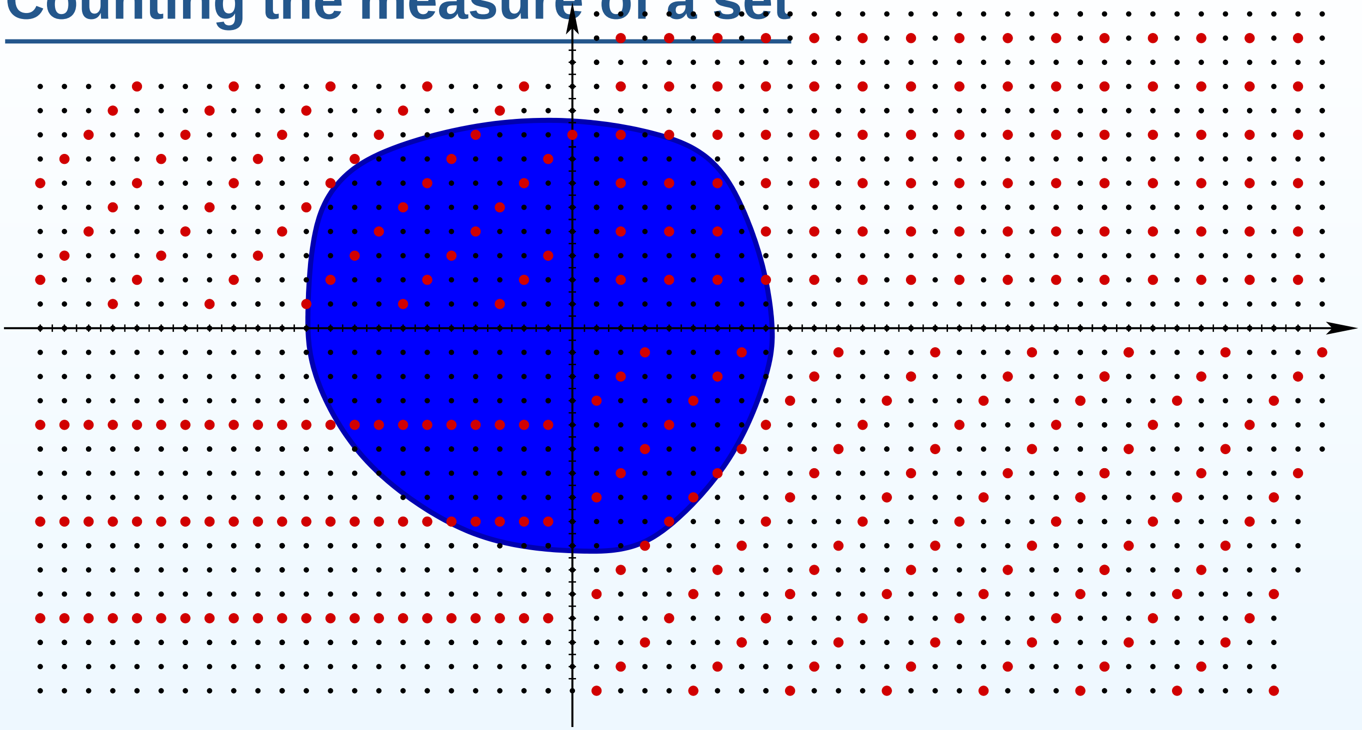

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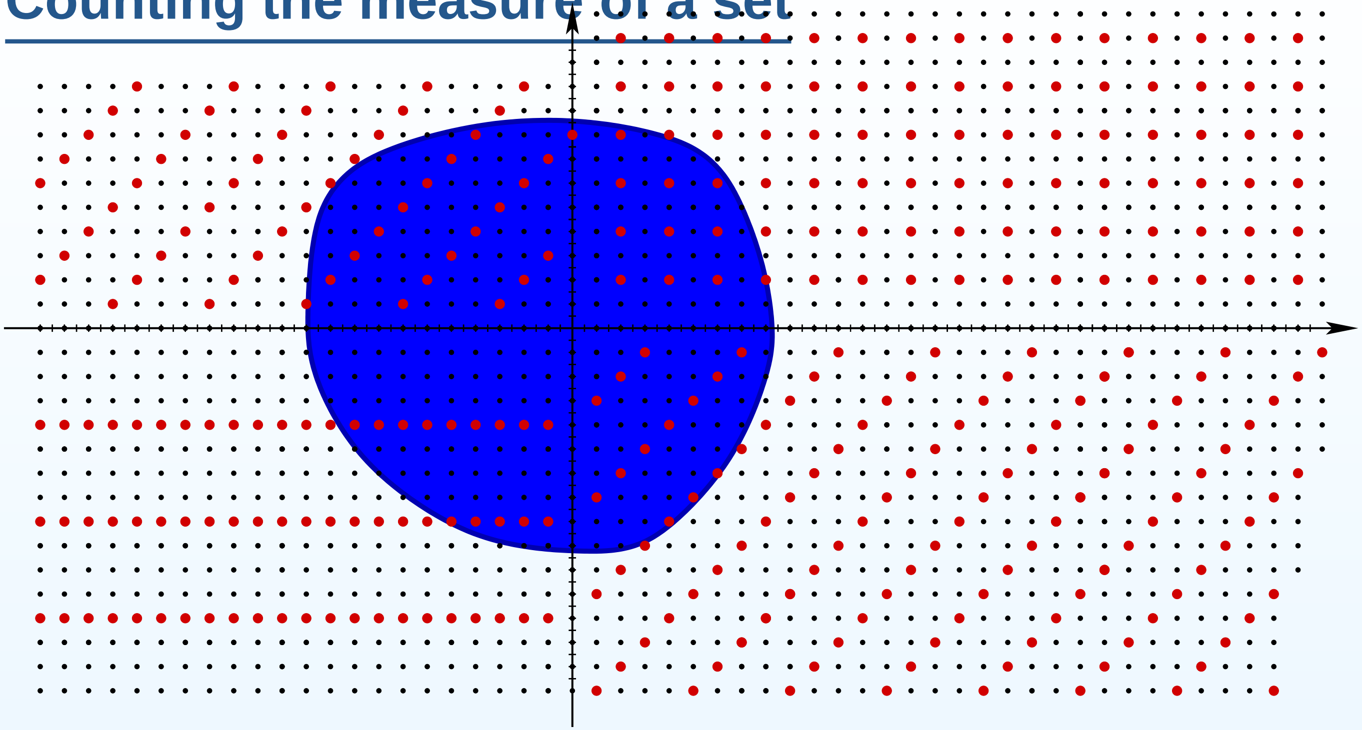
We can fix U and scale the lattice or can fix the lattice and scale U:

$$\operatorname{card}(U \cap \varepsilon\mathbb{Z}^n) = \operatorname{card}\left(\frac{1}{\varepsilon} U \cap \mathbb{Z}^n\right)$$

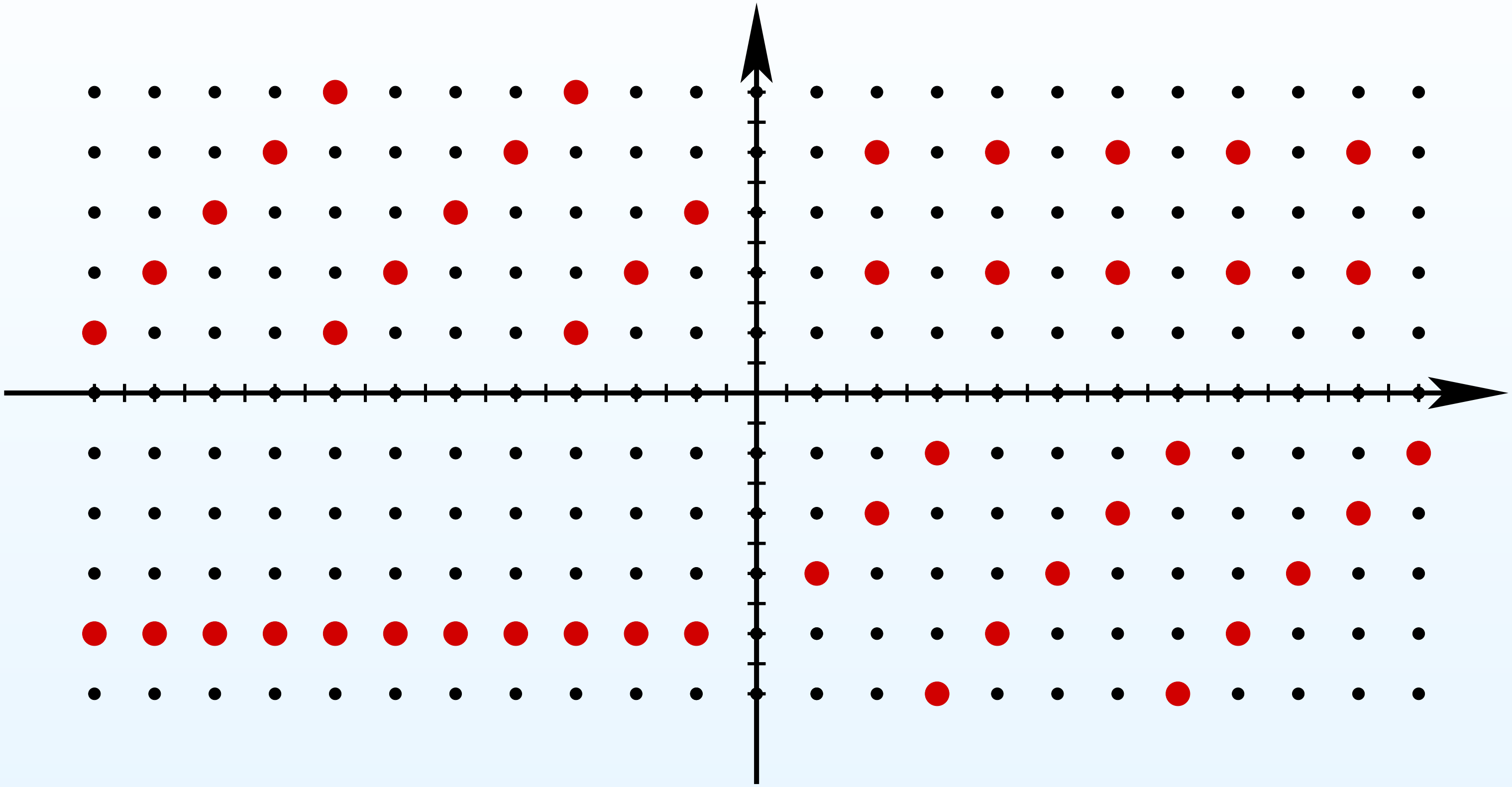
Counting the measure of a set



Finally, instead of using the entire lattice $\mathbb{Z}^n$ we can use any sublattice $\mathbb{L}^n \subset \mathbb{Z}^n$ having some nonzero *density* $k > 0$ in $\mathbb{Z}^n$.

For example, the set of coprime integral points in $\mathbb{Z}^2$ has density $k = \dfrac{6}{\pi^2}$ and can be also used to define the Lebesgue measure (scaled by the factor k) in any of the two ways discussed above.

Mirzakhani's measures on $\mathcal{ML}_{g,n}$



Choose some integral multicurve γ, say, a simple closed curve on $S_{g,n}$. The subset $\mathcal{O}_\gamma := \mathrm{Mod}_{g,n} \cdot \gamma$ can be seen as an analog of a "sublattice" in $\mathcal{ML}_{g,n}(\mathbb{Z})$. The insight of Mirzakhani was to realize that replacing the discrete set $\mathcal{ML}_{g,n}(\mathbb{Z})$ with the subset $\mathcal{O}_\gamma$ we get a new measure on $\mathcal{ML}_{g,n}$ which is proportional to the Thurston measure μ_{Th} with coefficient depending only on the homotopy type of γ.

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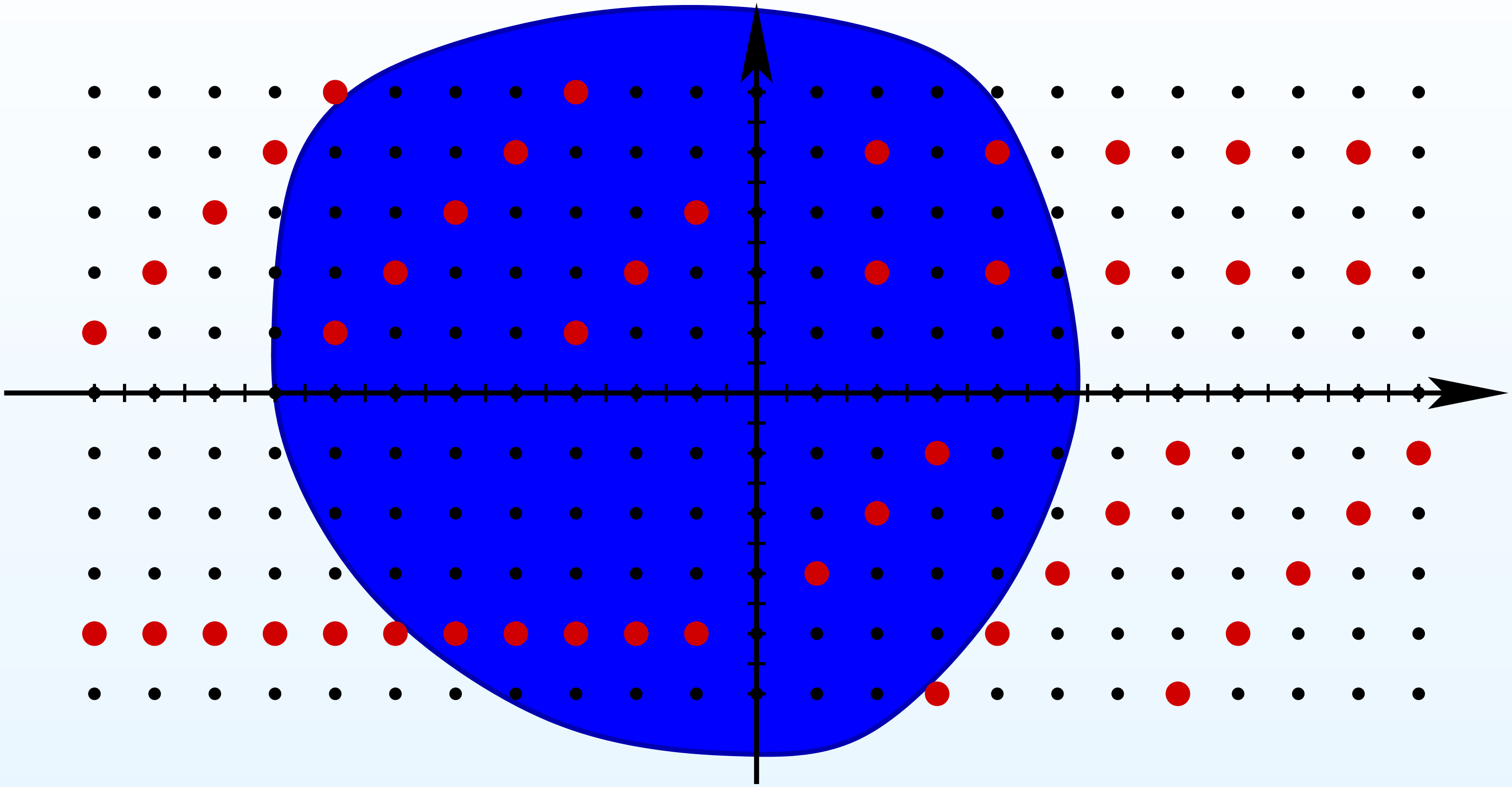

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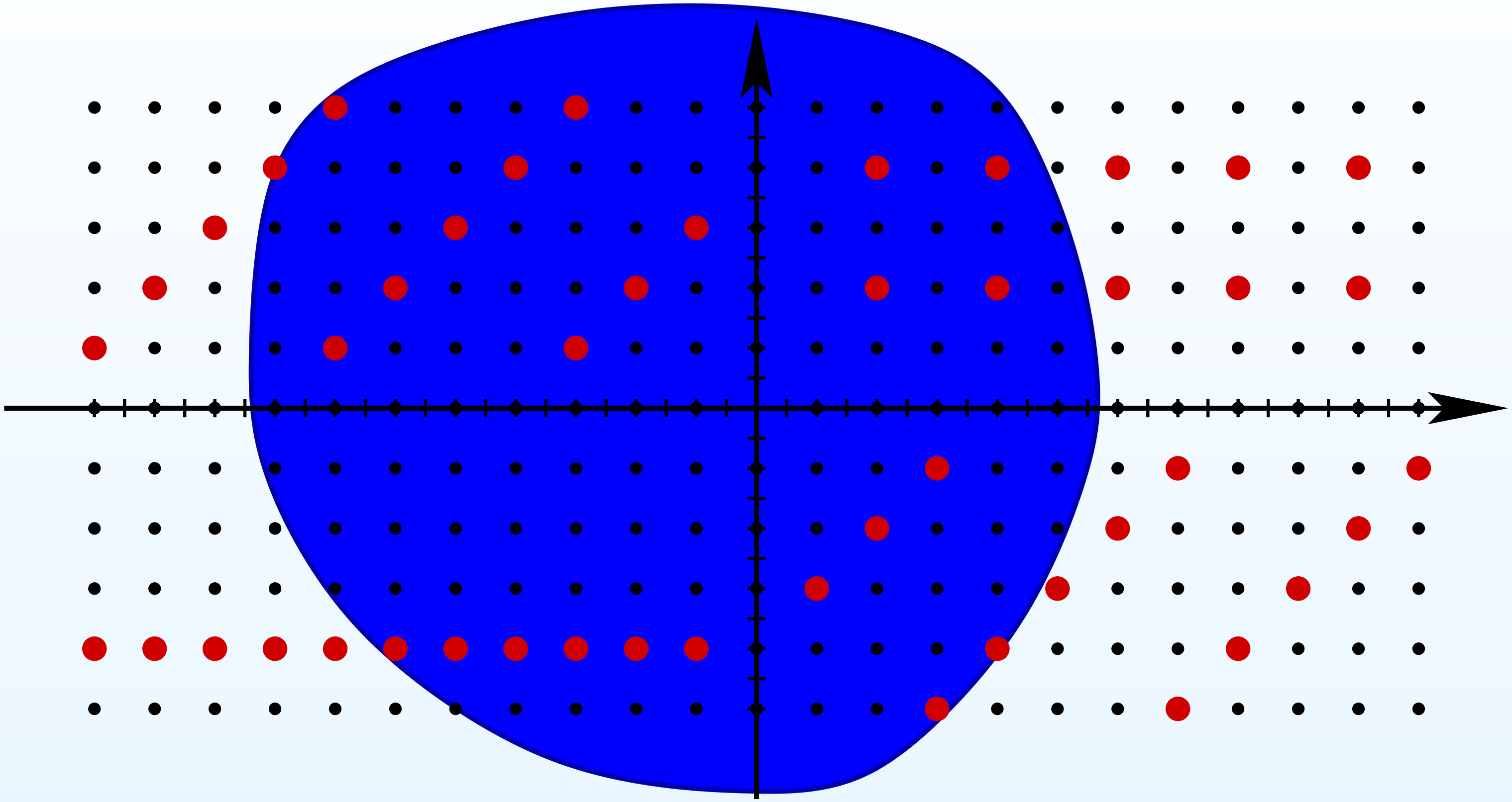

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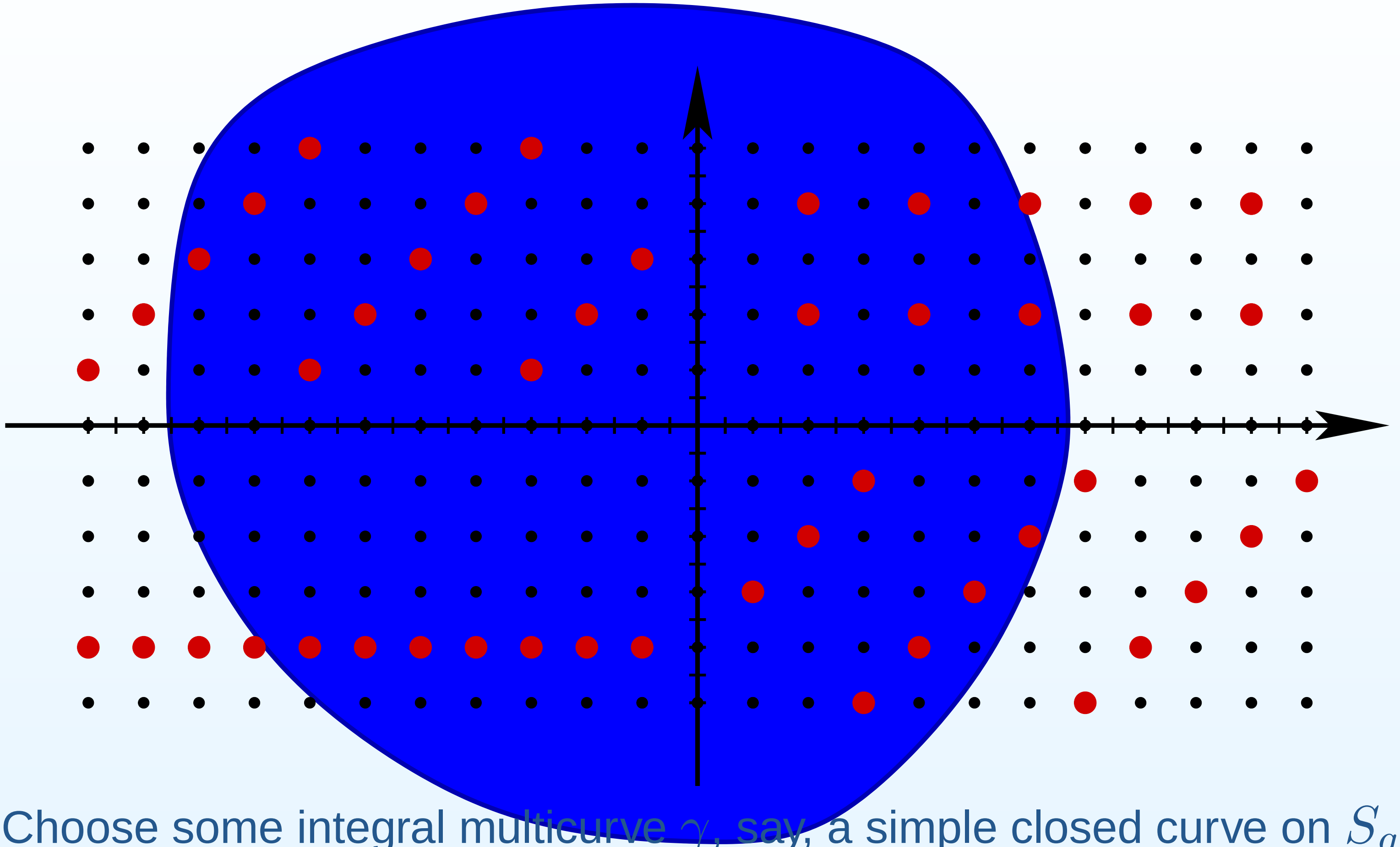

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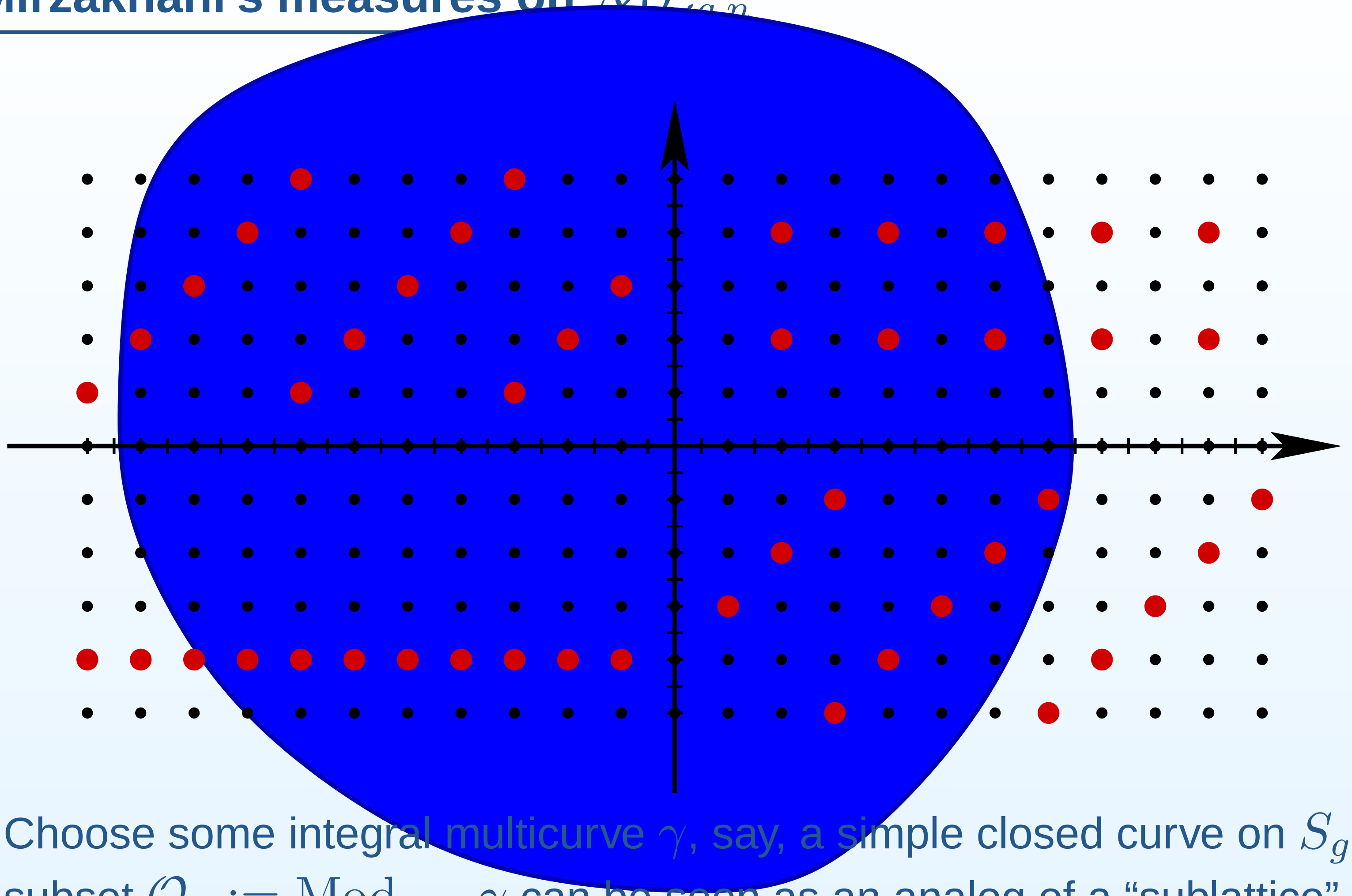

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Mirzakhani's measures on $\mathcal{ML}_{g,n}$

More formally: the Thurston measure of a subset $U \subset \mathcal{ML}_{g,n}$ is defined as

$$\mu_{\mathrm{Th}}(U) := \lim_{t \to +\infty} \frac{\operatorname{card}\{tU \cap \mathcal{ML}_{g,n}(\mathbb{Z})\}}{t^{6g-6+2n}}\,.$$

Mirzakhani defines a new measure μ_γ as

$$\mu_\gamma(U) := \lim_{t \to +\infty} \frac{\operatorname{card}\{tU \cap \mathcal{O}_\gamma\}}{t^{6g-6+2n}}\,.$$

Clearly, for any U we have $\mu_\gamma(U) \leq \mu_{\mathrm{Th}}(U)$ since $\mathcal{O}_\gamma \subset \mathcal{ML}_{g,n}(\mathbb{Z})$, so μ_γ belongs to the Lebesgue measure class. By construction μ_γ is $\mathrm{Mod}_{g,n}$-invariant. Ergodicity of μ_{Th} implies that $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ where $k_\gamma = const$.

Space of multicurves

Thurston's and Mirzakhani's measures on $\mathcal{ML}_{g,n}$

Proof of the main result
- Length function and unit ball
- Summary of notations
- Main counting results
- Example
- Idea of the proof and a notion of a "random multicurve"
- More honest idea of the proof

Teaser for next lectures

Proof of the main result

Length function and unit ball

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space. One can extend the length function to simple closed multicurves $\ell_{\sum a_i\gamma_i} = \sum a_i \ell_{\gamma_i}(X)$ by linearity. By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t\cdot\lambda}(X) = t \cdot \ell_\lambda(X)$.

Each hyperbolic metric X defines its own “unit ball” B_X in $\mathcal{ML}_{g,n}$:

$$B_X := \{\lambda \in \mathcal{ML}_{g,n} \,|\, \ell_\lambda(X) \le 1\}\,.$$

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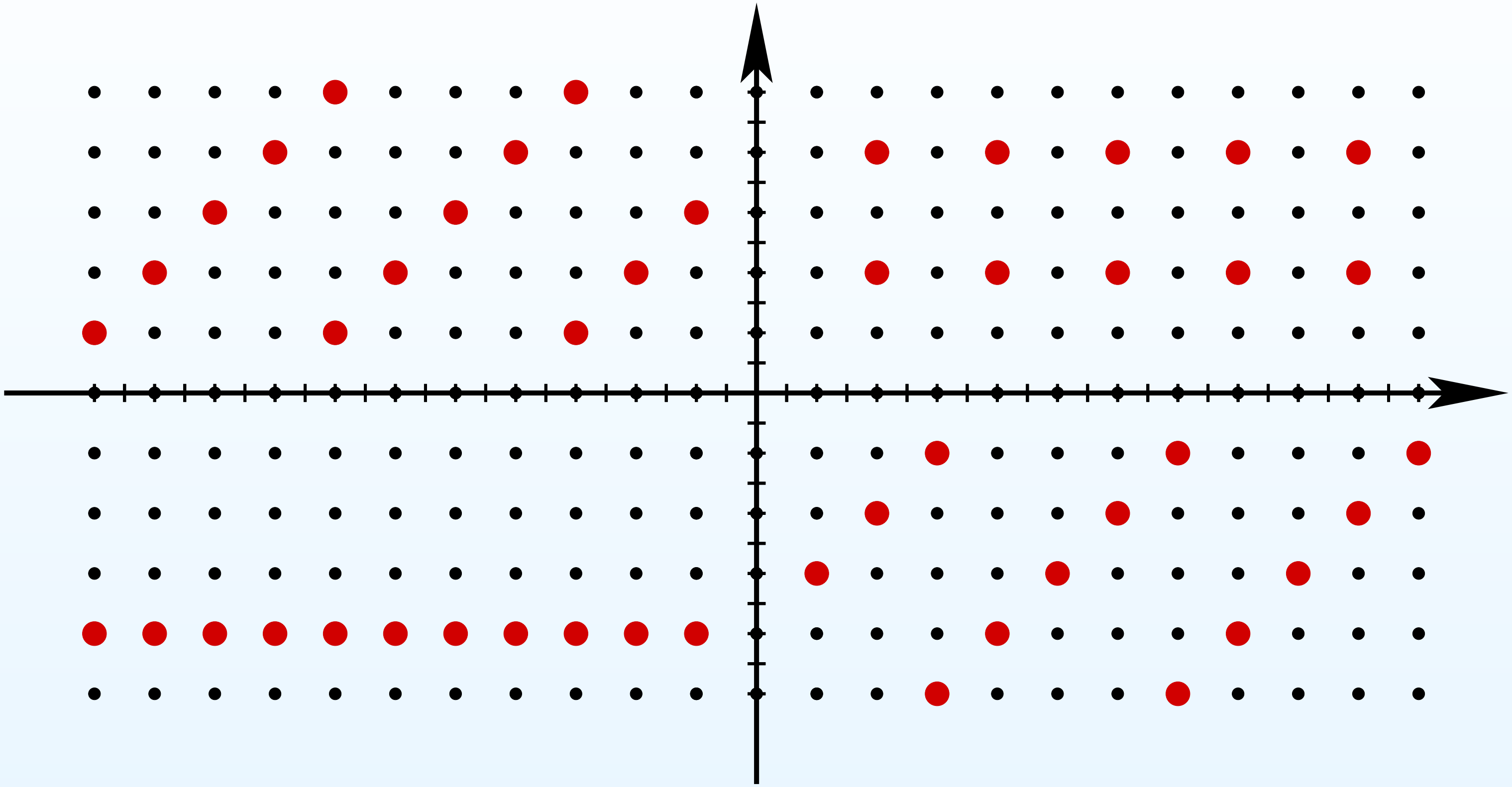

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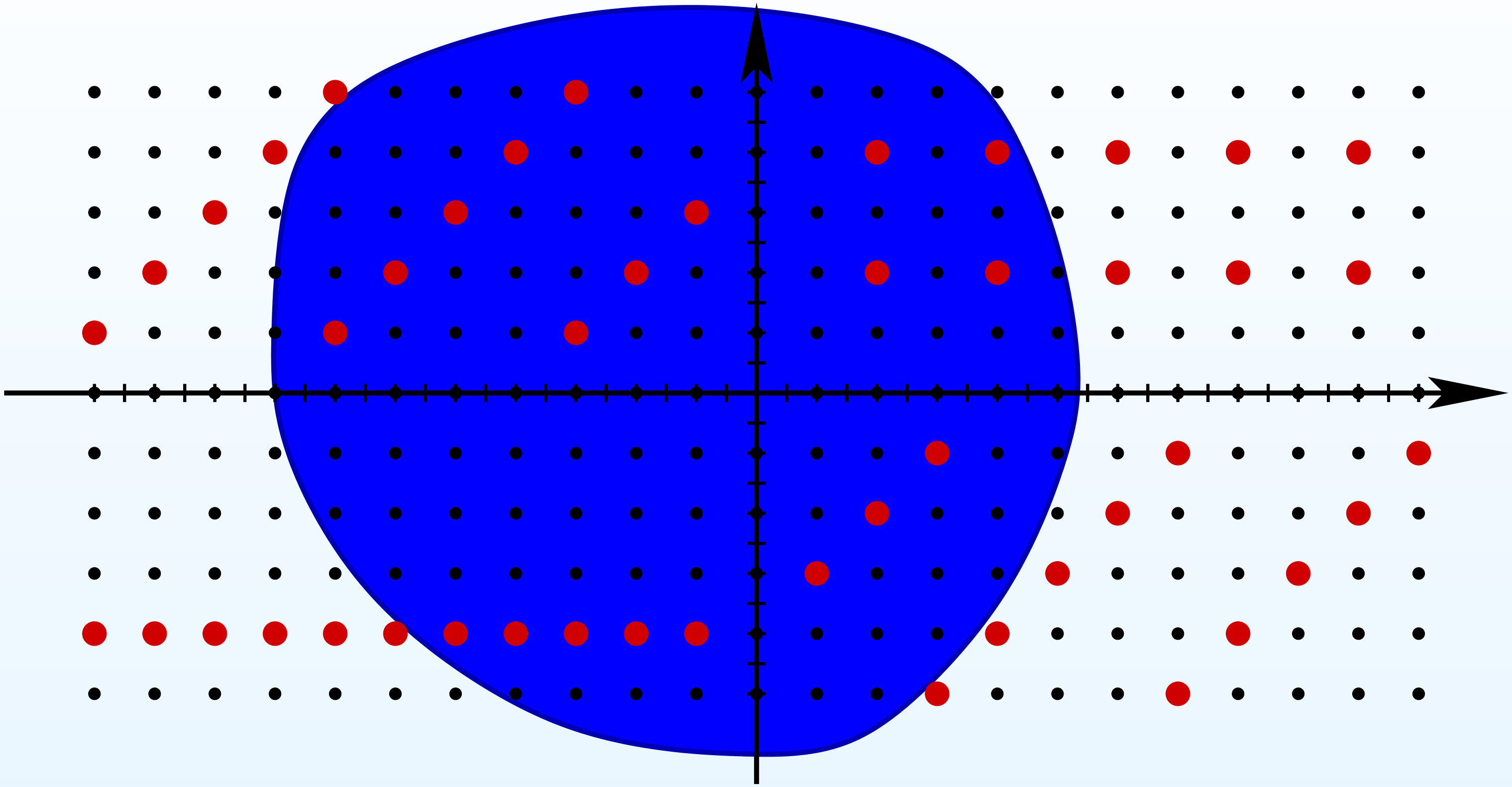

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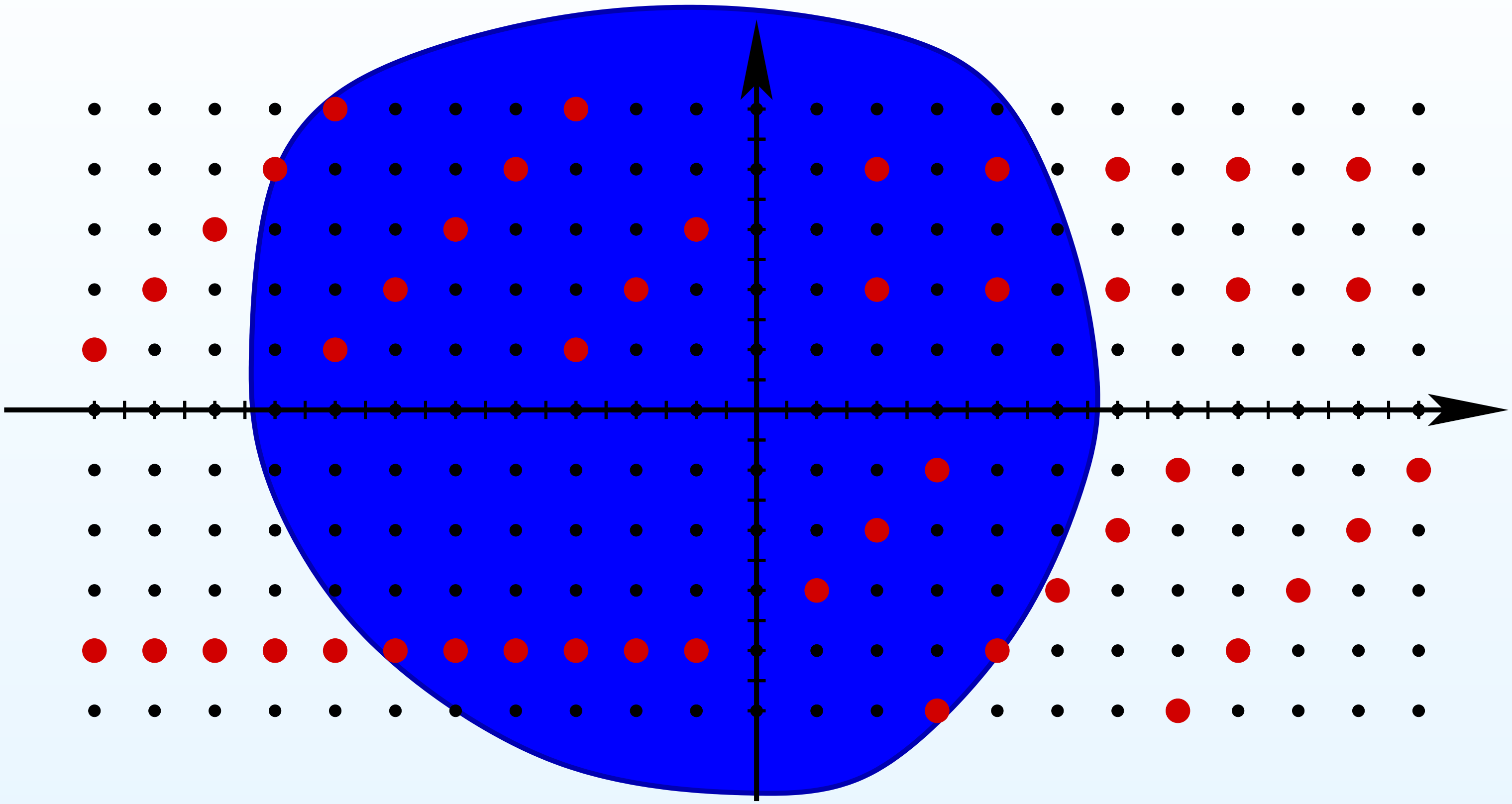

By definition of μ_{Th}, the Thurston volume of the unit ball is equal to the normalized number of integral points in a “ball of radius L” associated to X:

$$\mu_{\text{Th}}(B_X) = \lim_{L \to +\infty} \frac{\operatorname{card}\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell_\lambda(X) \leq L\}}{L^{6g-6+2n}} .$$

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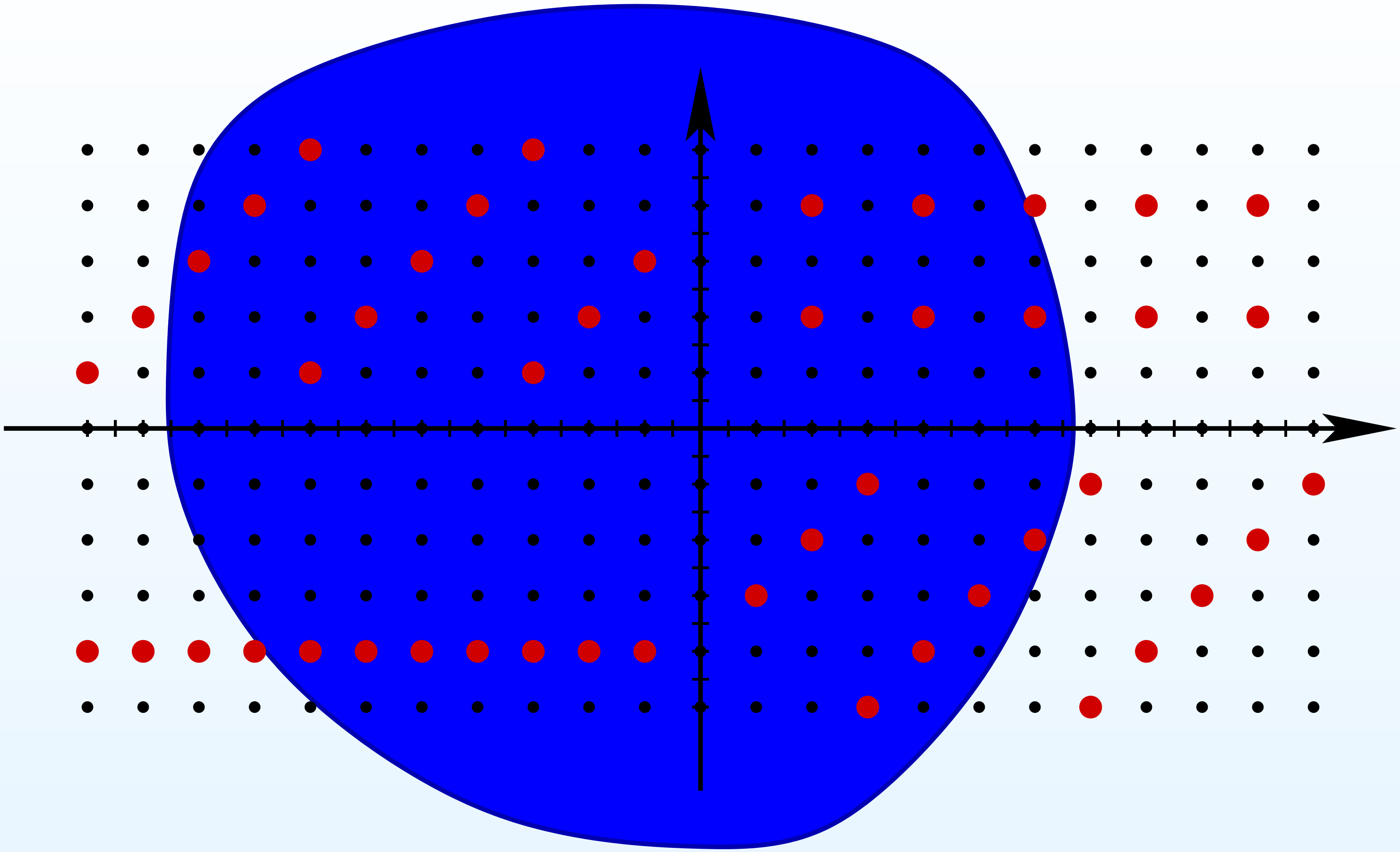

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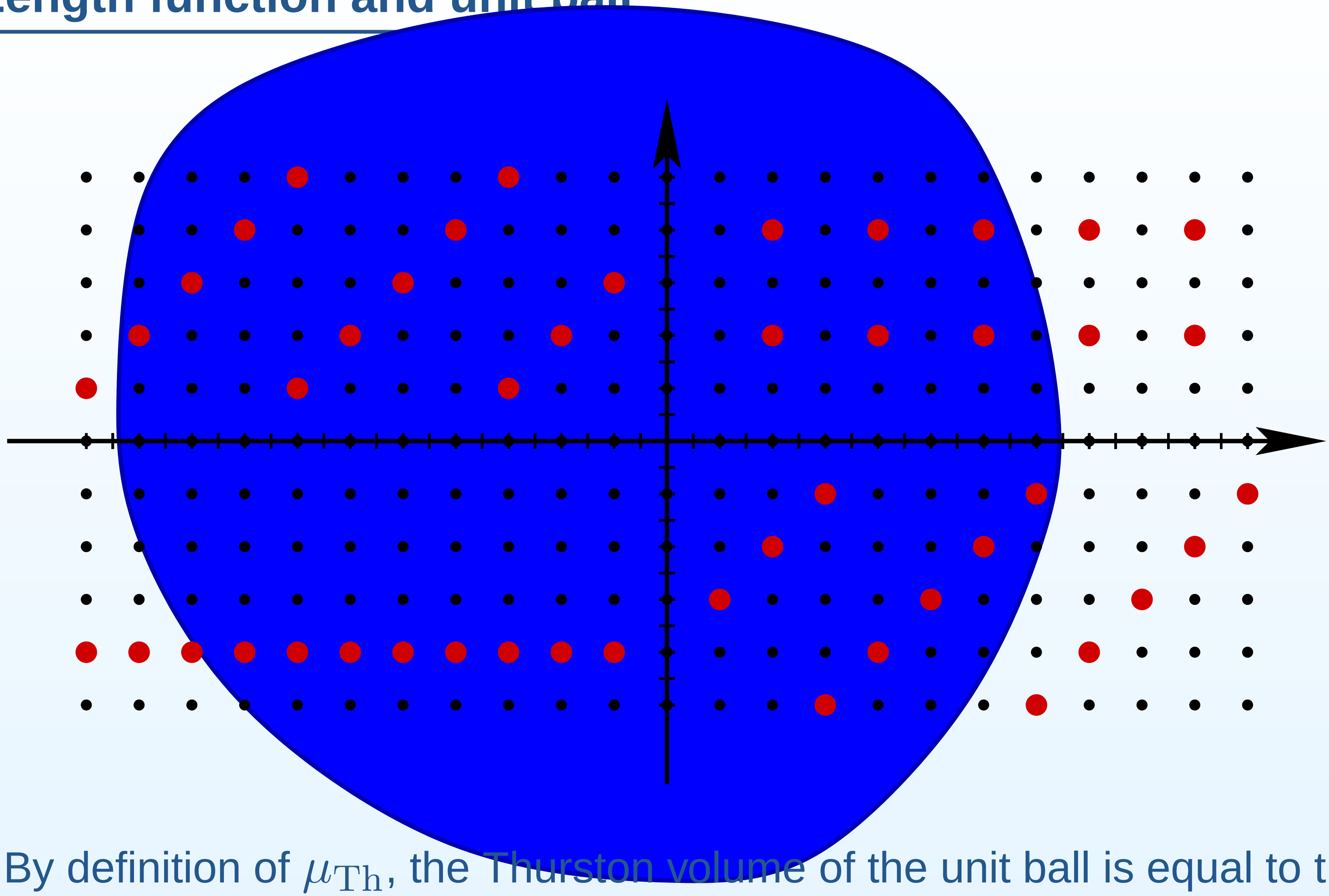

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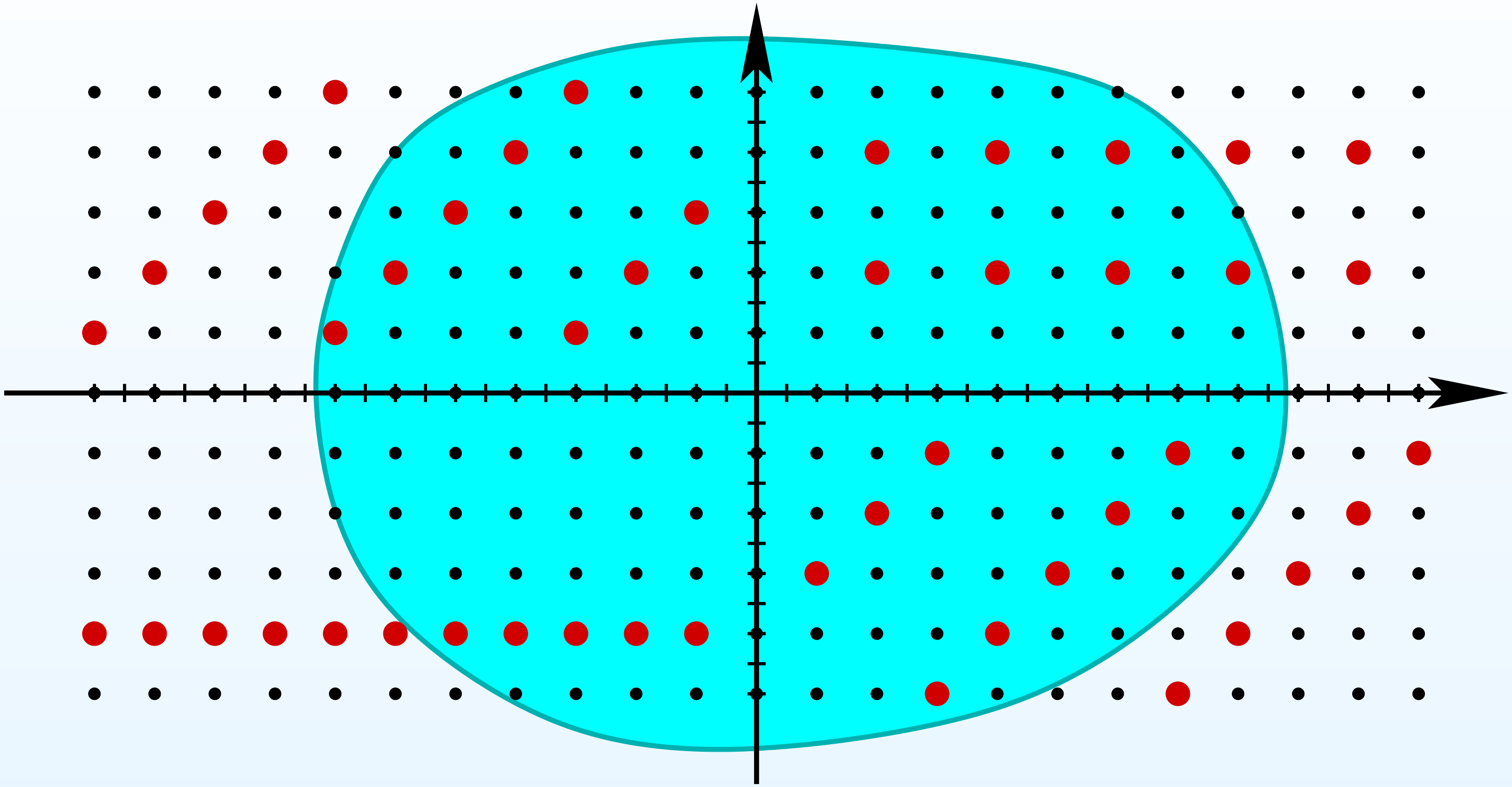

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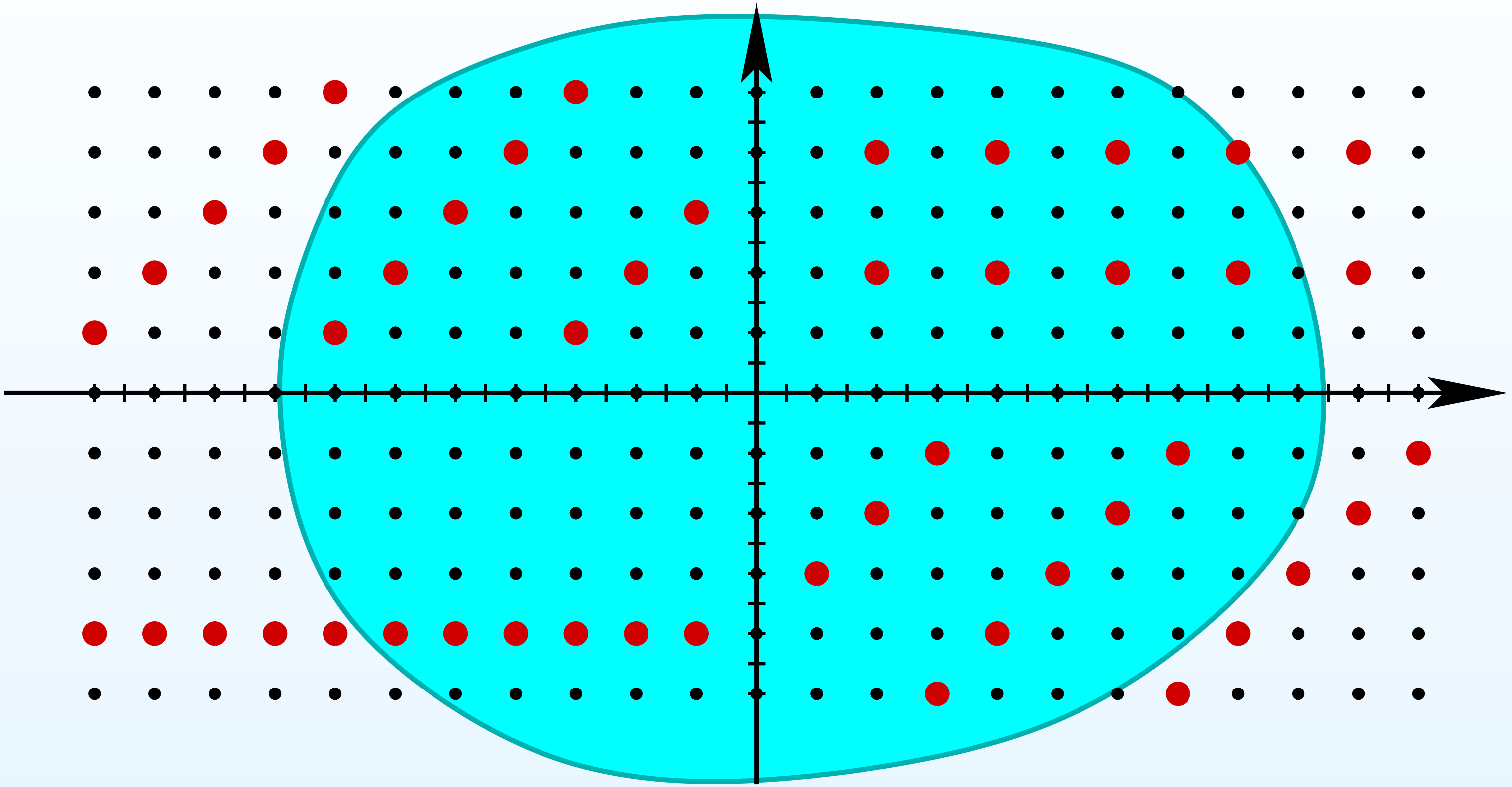

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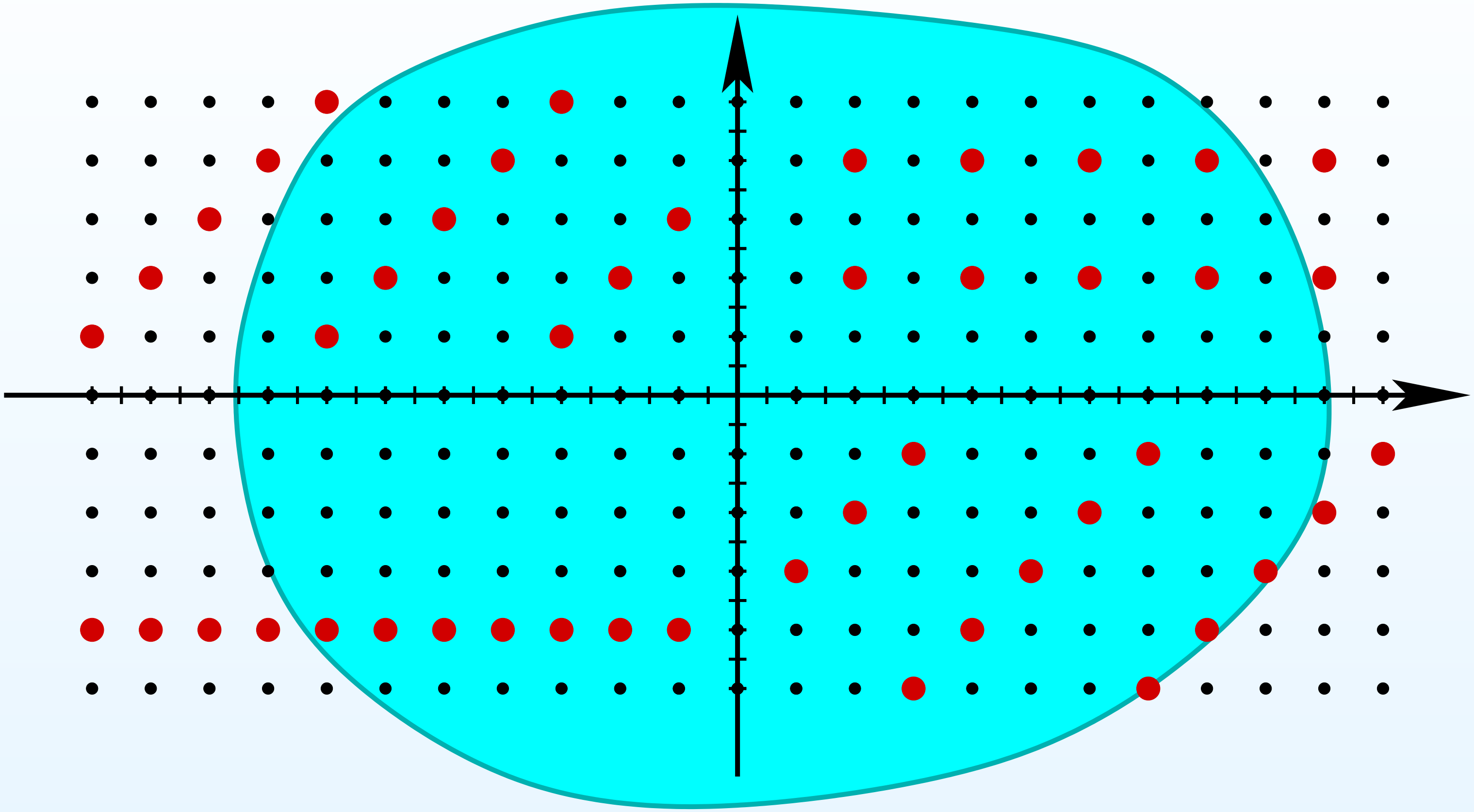

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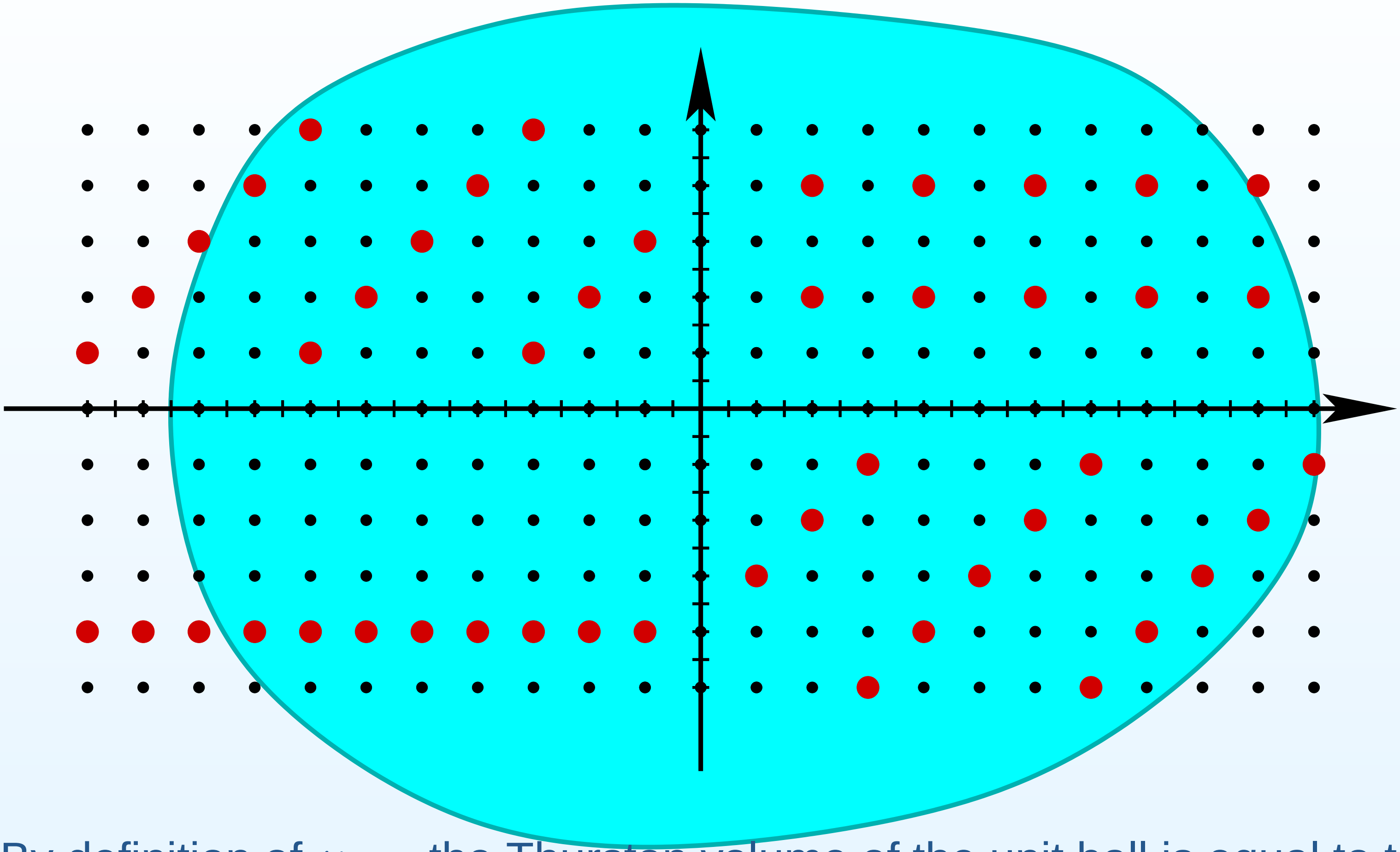

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Summary of notations

- X — a hyperbolic surface in $\mathcal{M}_{g,n}$.
- $s_X(L,\gamma)$ — the number of geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.
- $P(L\,,\gamma) := \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\,dX$ — the polynomial in L providing the *average* number of geodesic multicurves of topological type $[\gamma]$ and of hyperbolic length at most L over all hyperbolic surfaces $X \in \mathcal{M}_{g,n}$.
- $c(\gamma)$ — the coefficient of the leading term $L^{6g-6+2n}$ of the polynomial $P(L\,,\gamma)$.

- $B(X)$ — “Unit ball” in $\mathcal{ML}_{g,n}$ defined by means of the length function $\ell_X(\alpha)$, where $\alpha \in \mathcal{ML}_{g,n}$.
- $\mu_{\mathrm{Th}}(B(X)) := \lim_{L\to+\infty} \dfrac{\operatorname{card}\{L\cdot B_X \cap \mathcal{ML}(\mathbb{Z})\}}{L^{6g-6+2n}}$ is the Thurston measure of the unit ball $B(X)$
- $\mu_\gamma(B(X)) := \lim_{L\to+\infty} \dfrac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}}$ is the Mirzakhani measure of the unit ball $B(X)$ defined by the sublattice $\operatorname{Mod}_{g,n}\cdot\gamma \subset \mathcal{ML}(\mathbb{Z})$.

Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (which is the average value of $B(X)$ over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ (expressed in terms of the Witten–Kontsevich correlators).

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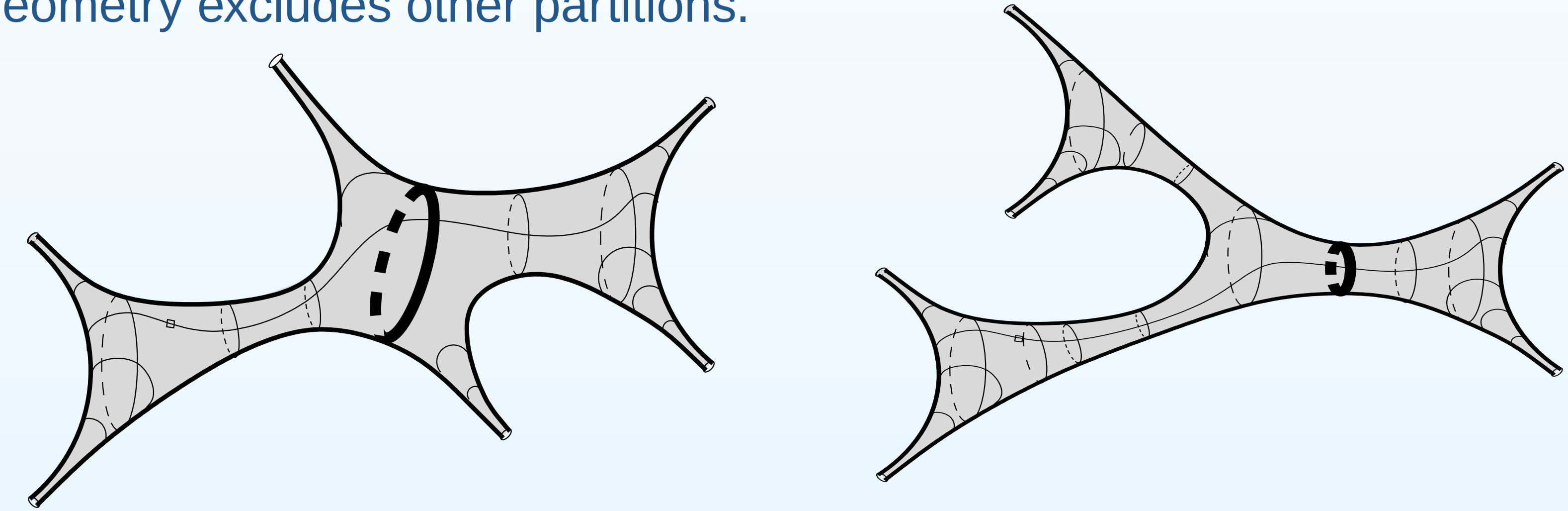
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$*, and any two rational multicurves* γ_1, γ_2 *on a smooth surface* $S_{g,n}$ *considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.



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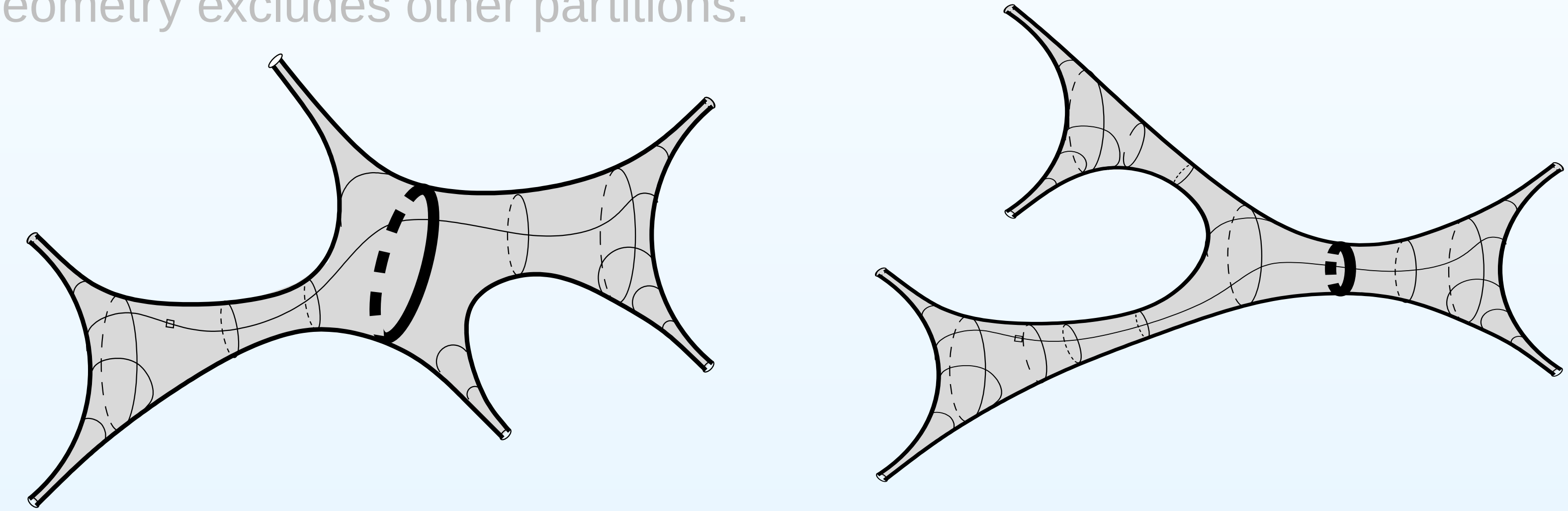
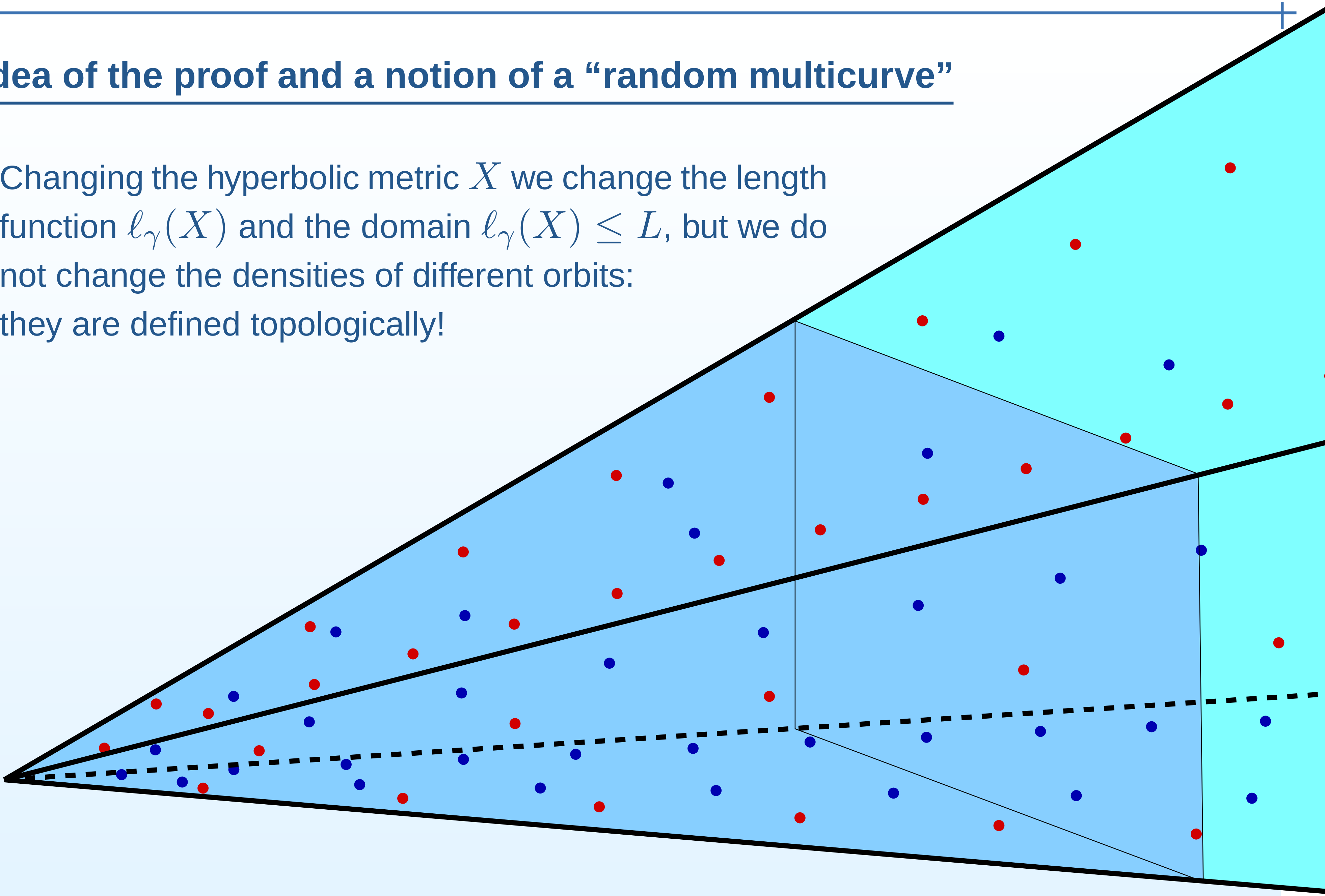


Example. (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by other means.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

Idea of the proof and a notion of a "random multicurve"

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \leq L$, but we do not change the densities of different orbits: they are defined topologically!



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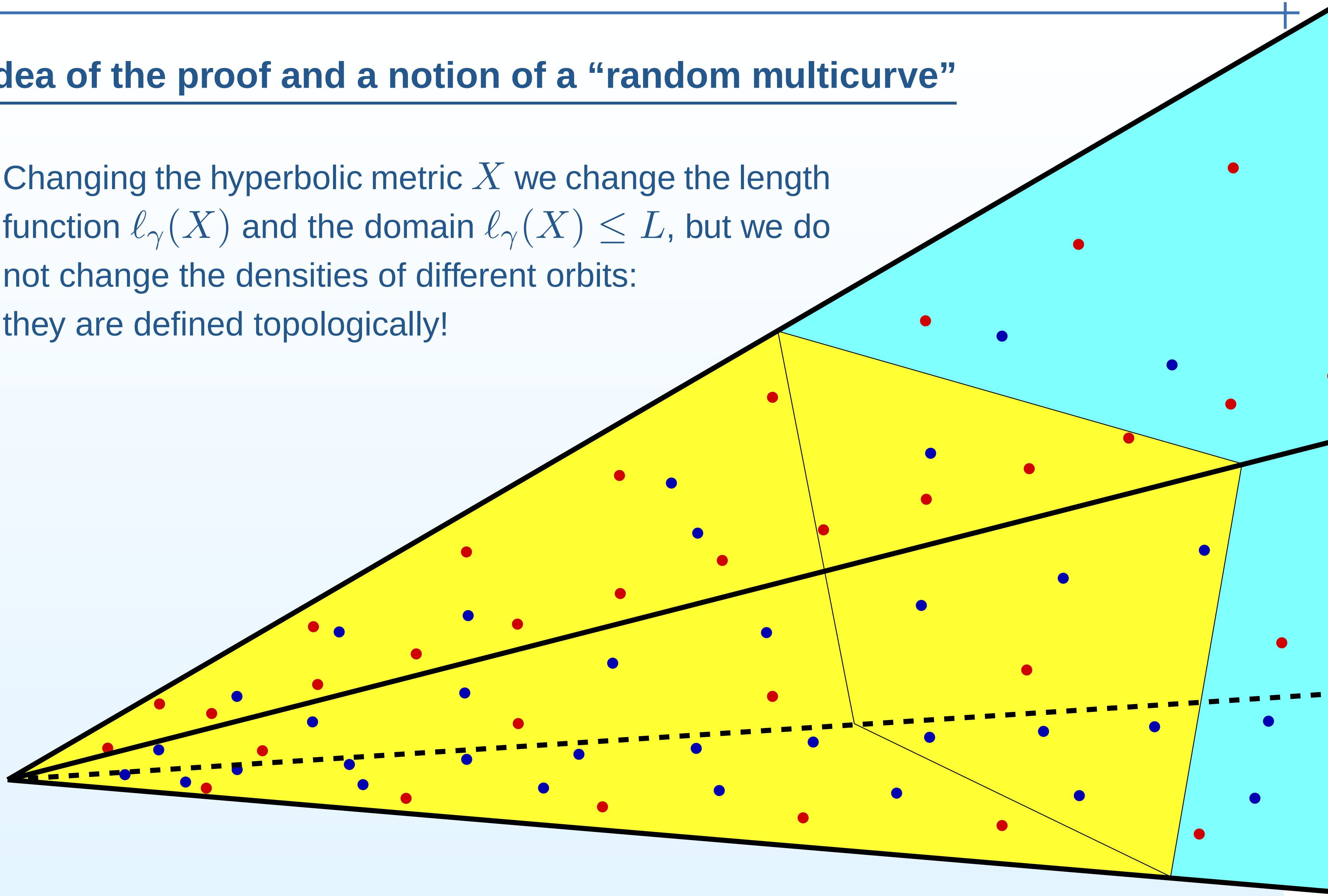

More honest idea of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the “unit ball” B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\operatorname{card}\{L \cdot B_X \cap \operatorname{Mod}_{g,n} \cdot \gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\, dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\, dX =$$

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so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzakhani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \leq F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

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Space of multicurves

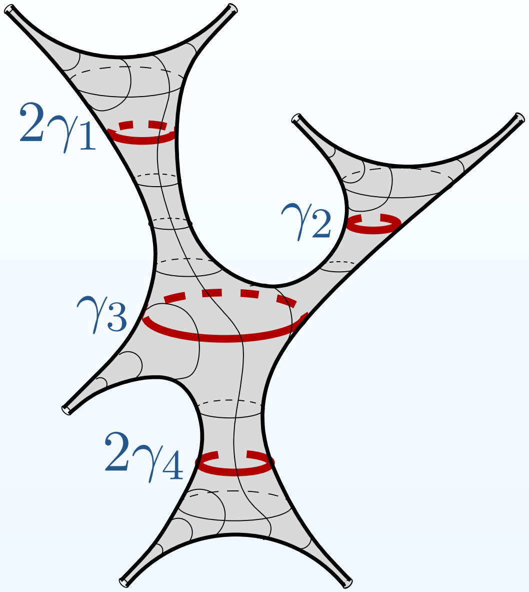
Thurston’s and Mirzakhani’s measures on $\mathcal{ML}_{g,n}$

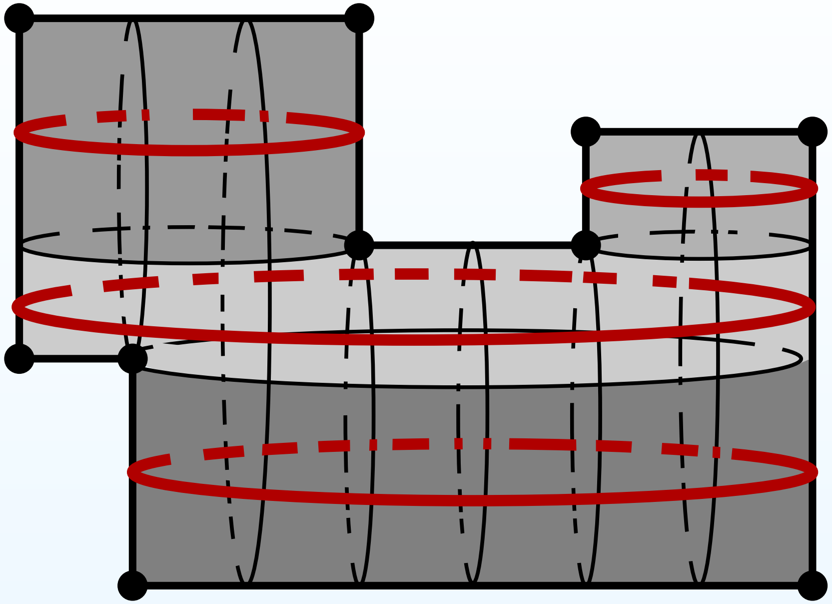
Proof of the main result

Teaser for next lectures
- Hyperbolic and flat geodesic multicurves
- Average volume of unit balls

Teaser for next lectures

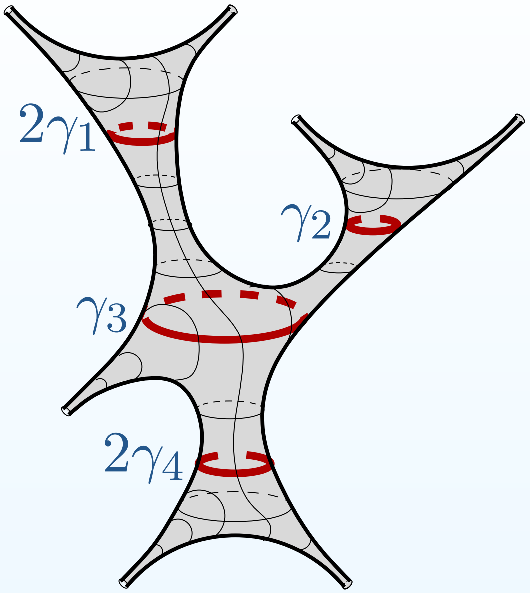
Hyperbolic and flat geodesic multicurves

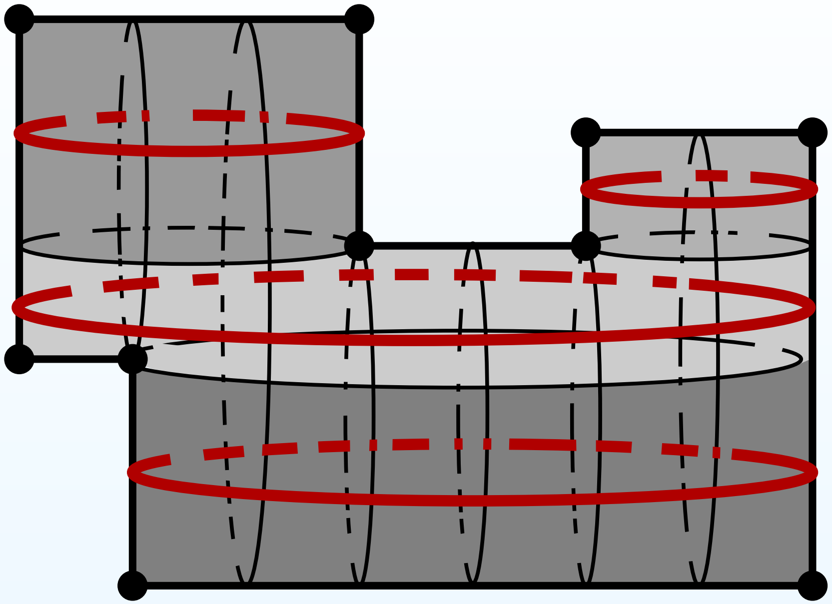




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves





Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of **hyperbolic** multicurves coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Average volume of unit balls

Recall that

$$b_{g,n} := \int_{\mathcal{M}_{g,n}} \mu_{\mathrm{Th}}(B(X))\, dX$$

denotes the average volume of “unit balls” measured in Thurston measure.

Theorem (M. Mirzakhani, 2008). *The quantity* $b_{g,n}$ *admits explicit expression as a weighted sum of all* $c(\gamma)$ *over (a finite collection) of all topological types* $[\gamma]$ *of multicurves.*

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Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z., 2017).

$$\mathrm{Vol}\, \mathcal{Q}(1^{4g-4+n}, -1^n) = 2 \cdot (6g-6+2n) \cdot (4g-4+n)! \cdot b_{g,n}\,.$$

The same constant was obtained by F. Arana-Herrera and by L. Monin and I. Telpukhovskiy by different methods.