Dynamics and Geometry of Moduli Spaces

Lecture 5. Solutions of problems from the homework assignment based on Lectures 1–4

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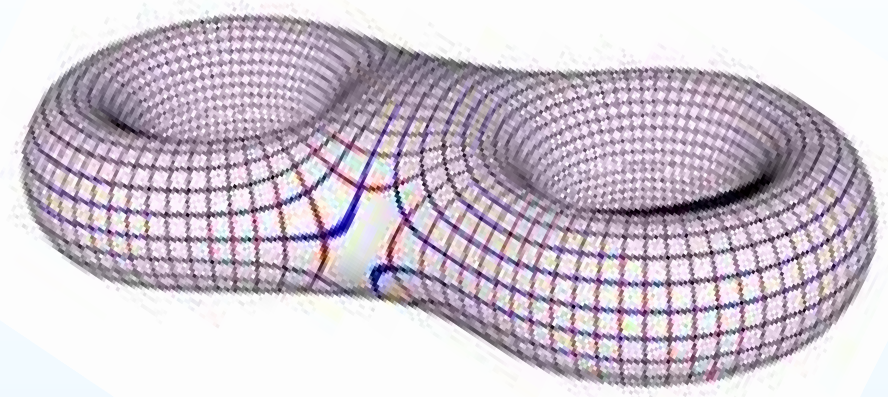
Separatrix diagrams

Problem 1: Which separatrix diagram?

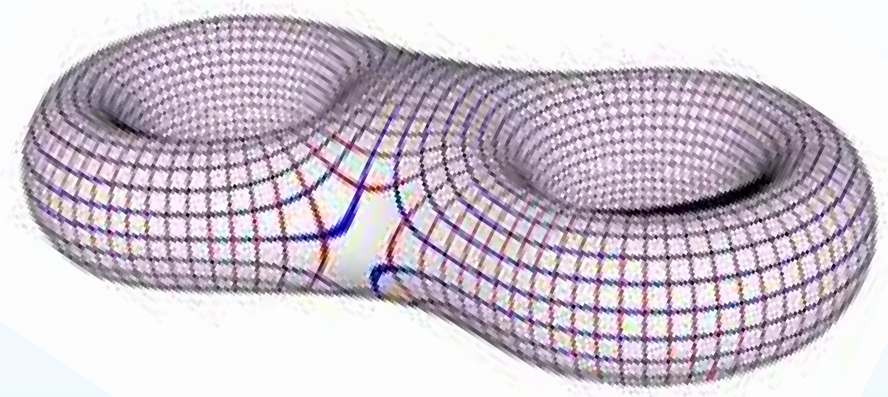


Picture created by Jian Jiang

Questions.

- *To what stratum belongs this square-tiled surface?*
- *Find all realizabe separatrix diagrams for this stratum.*
- *To which of the found diagrams corresponds the square-tiled surface from the picture?*

Which stratum?



Picture created by Jian Jiang

Question.

- *To what stratum belongs this square-tiled surface?*

Answer.

There are two strata in genus two: $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$. The surface in the picture has two symmetric conical singularities, so the ambient stratum is $\mathcal{H}(1,1)$.

One can also honestly count the cone angle at the visible conical singularity. The neighborhood is an octagon composed of four horizontal (blue) sides of the squares and of four vertical (red) sides. Thus, the cone angle is 4π, which excludes stratum $\mathcal{H}(2)$.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

We have two zeroes. Each has two outgoing and two incoming horizontal separatrices.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

Let us start with critical graphs (separatrix diagram) having no closed loops. Let us draw one saddle connection and discuss how we can complete it.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

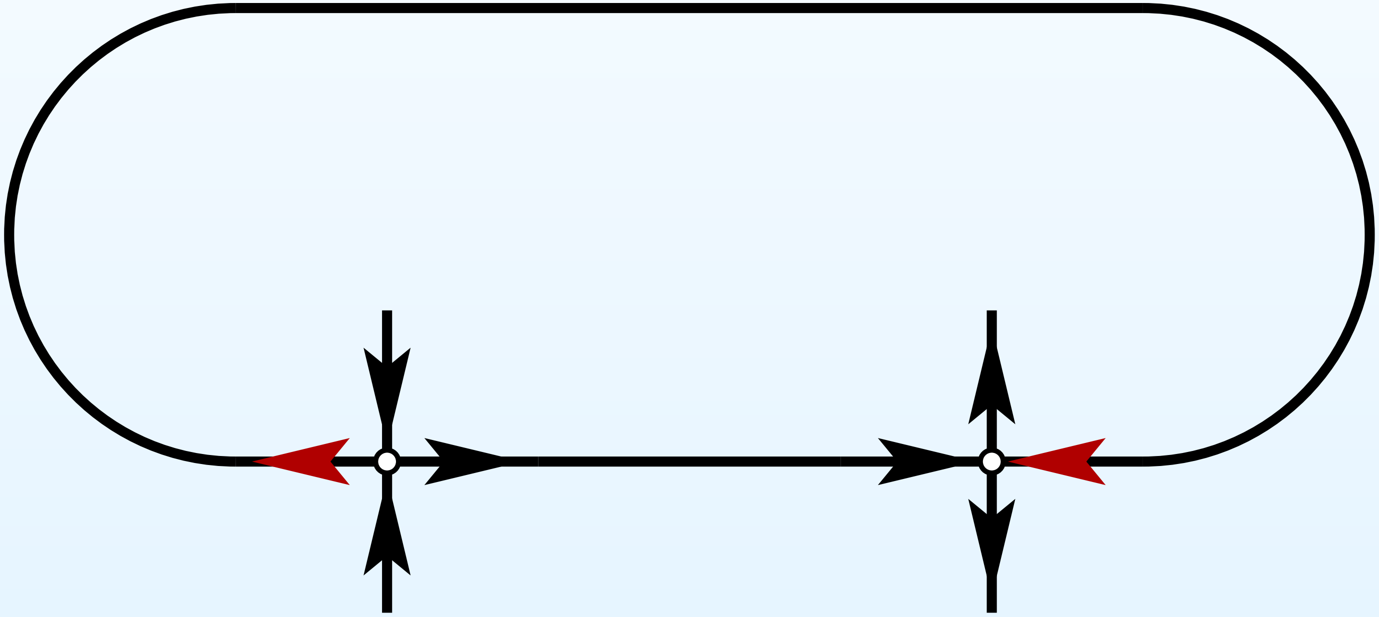
- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

On the left there is a single outgoing separatrix and on the right — only one incoming. We are forced to join them.

Admissible diagrams in $\mathcal{H}(1,1)$

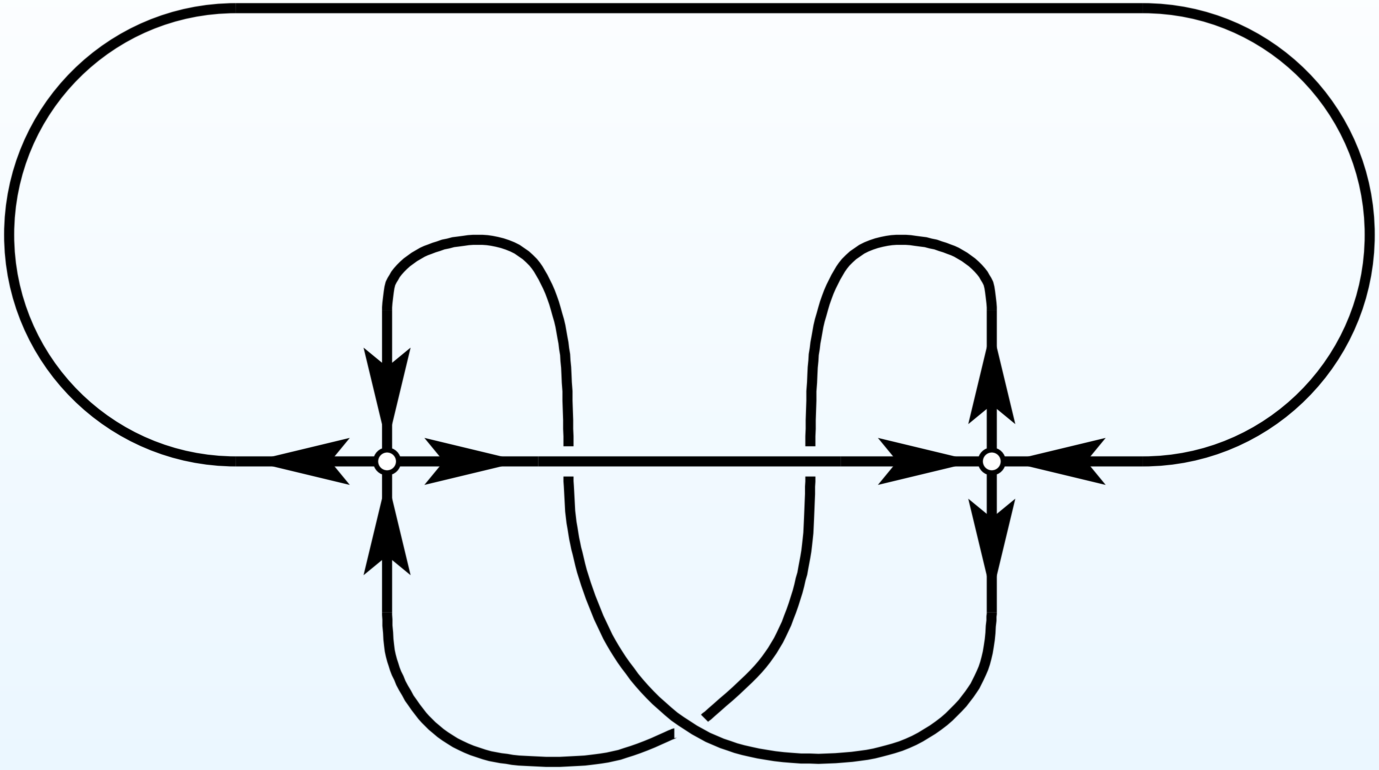
Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*



On the left there is a single outgoing separatrix and on the right — only one incoming. We are forced to join them.

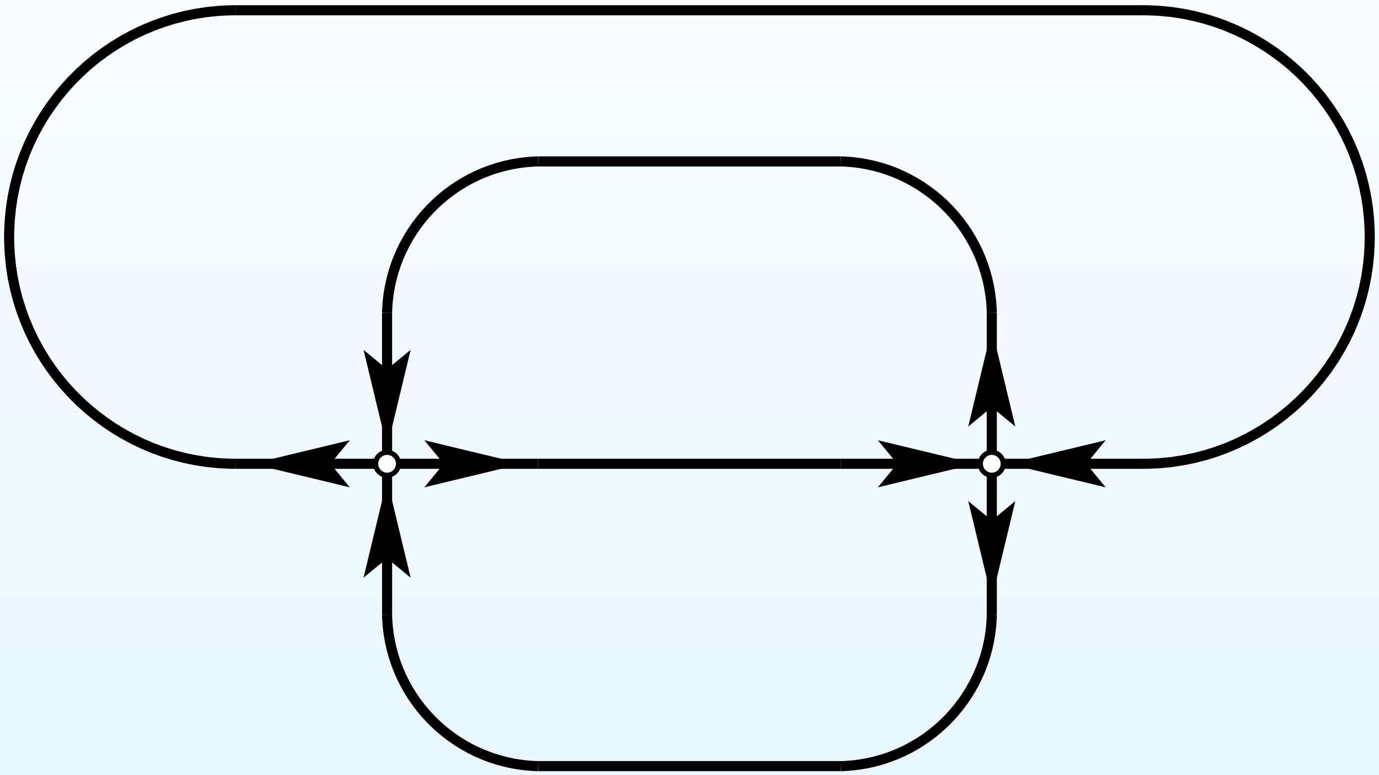
1-cylinder diagram in $\mathcal{H}(1,1)$



This is the first of the two ways of joining the remaining two pairs of separatrix rays.

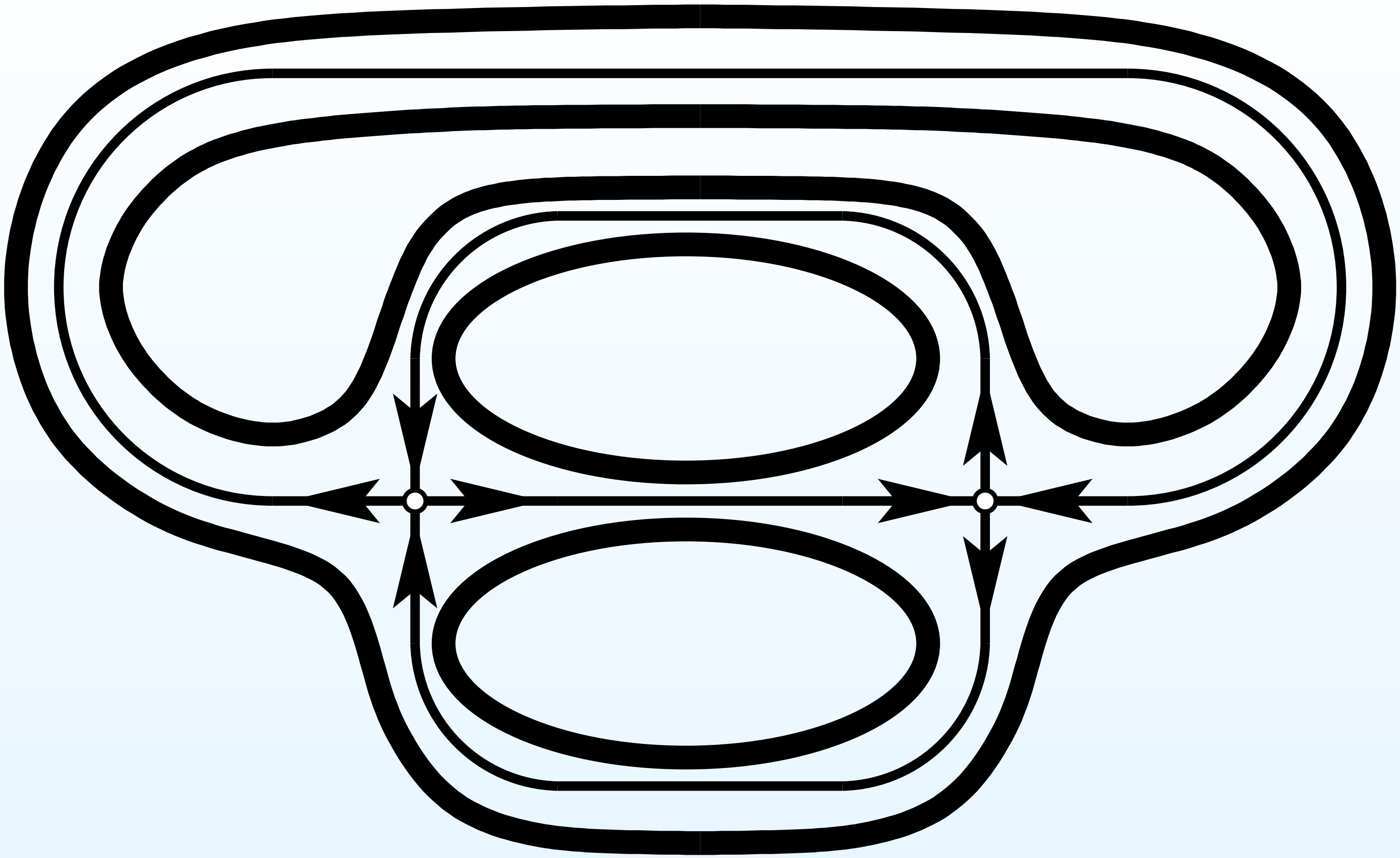
Mandatory Exercise. Check all of the following: The corresponding ribbon graph has two boundary components. Each component follows once each of the four saddle connection, so that the length of each of the two saddle connections is $\ell_1 + \ell_2 + \ell_3 + \ell_4$. There are no realtions on ℓ_i: this diagram is realizable for any choice of the lengths ℓ_i, where $i = 1, \ldots, 4$.

2-cylinder diagram in $\mathcal{H}(1,1)$



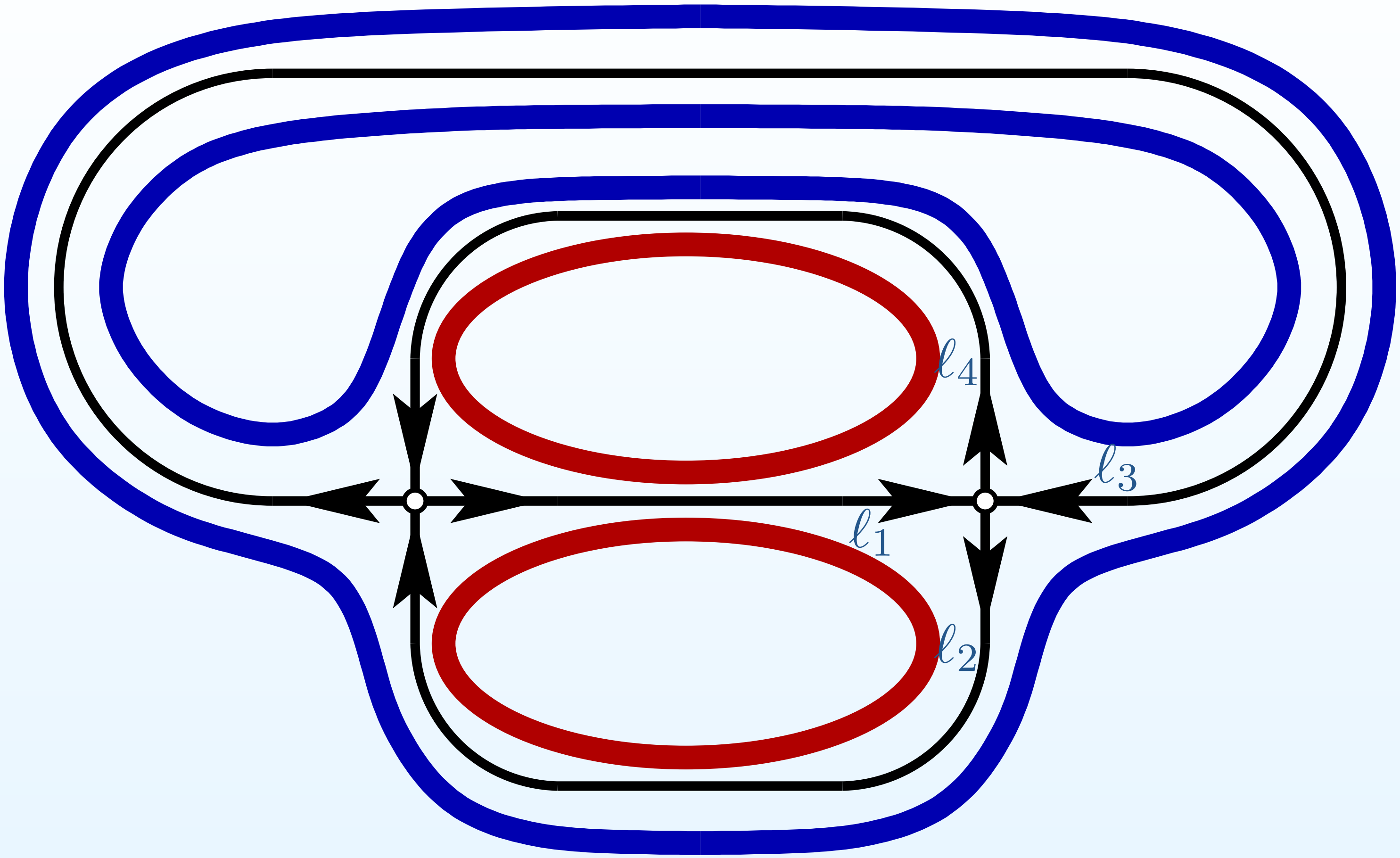
This is the other way to join the remaining two pairs of separatrix rays. Note that every maximal horizontal cylinder has one top and one boundary component. Thus, for every pair of boundary components to which we glue a cylinder, one component has the critical graph on the left and the other component has it on the right.

2-cylinder diagram in $\mathcal{H}(1,1)$



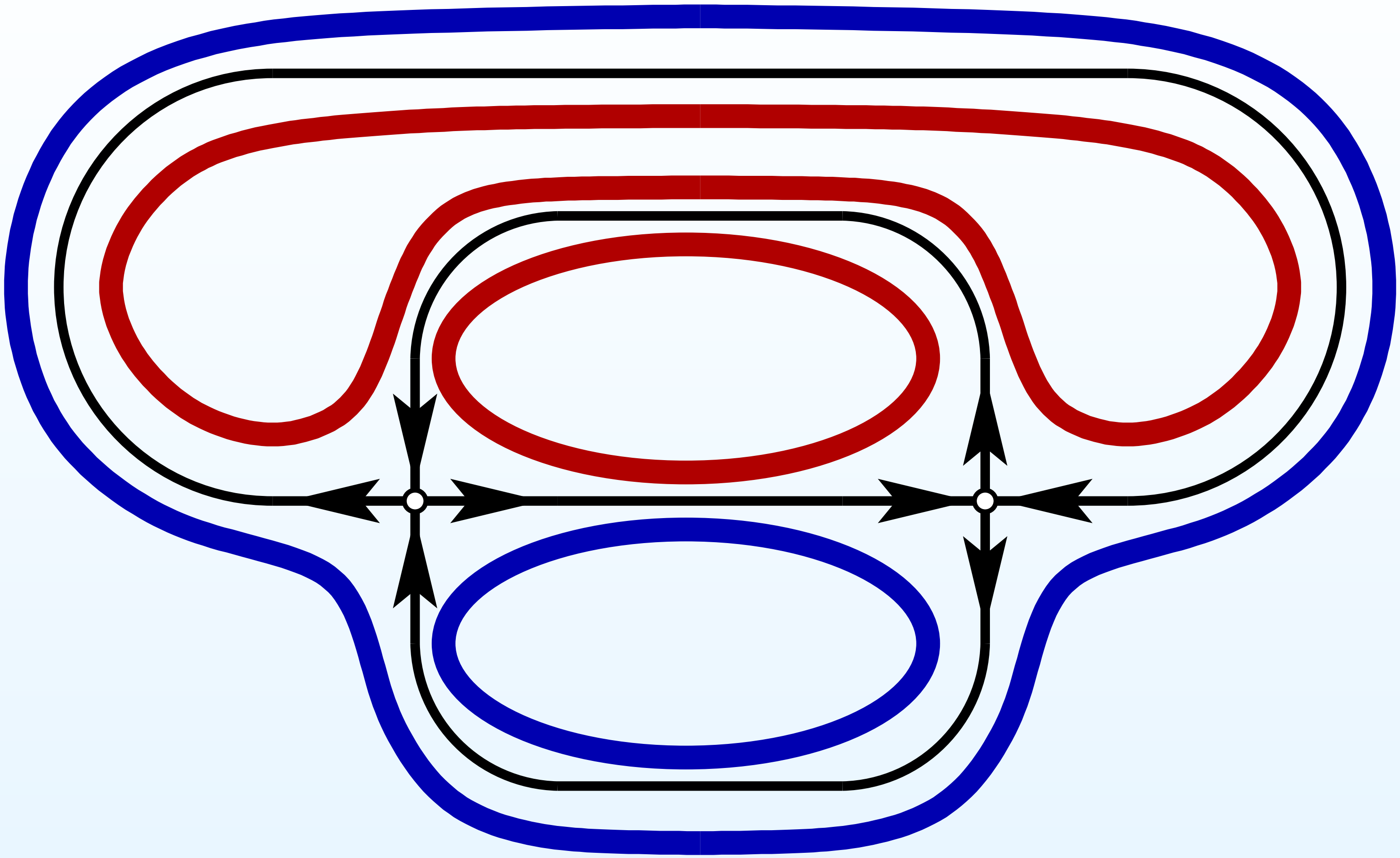
It gives us two ways in which we can organize the four boundary components into two pairs.

2-cylinder diagram in $\mathcal{H}(1,1)$



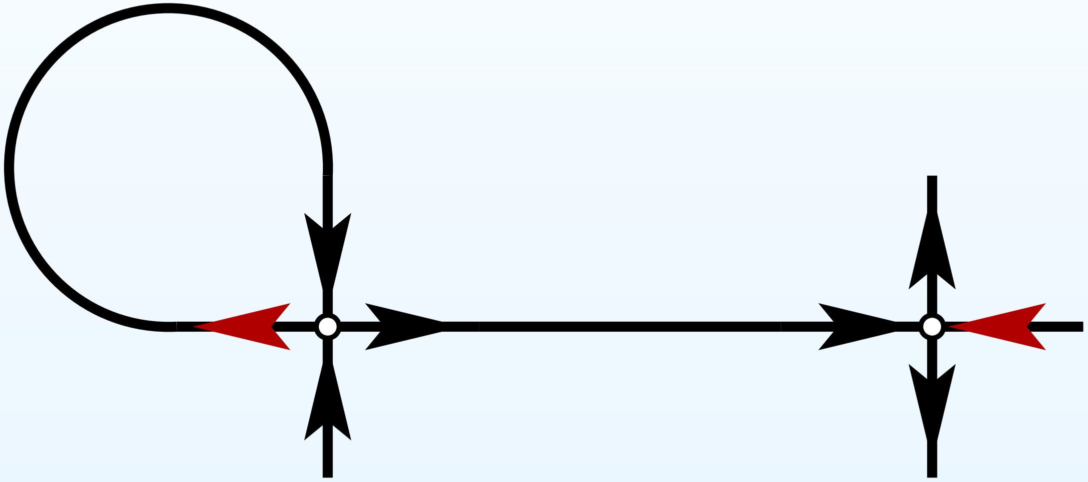
If we choose this way, we see that we have to impose the following conditions on the lengths of saddle connections: $\ell_2 = \ell_4$. Then the red cylinder has the waist curve of length $\ell_1 + \ell_2$ and the blue cylinder has the waist curve of length $\ell_3 + \ell_2$. We get an admissible diagram.

2-cylinder diagram in $\mathcal{H}(1,1)$



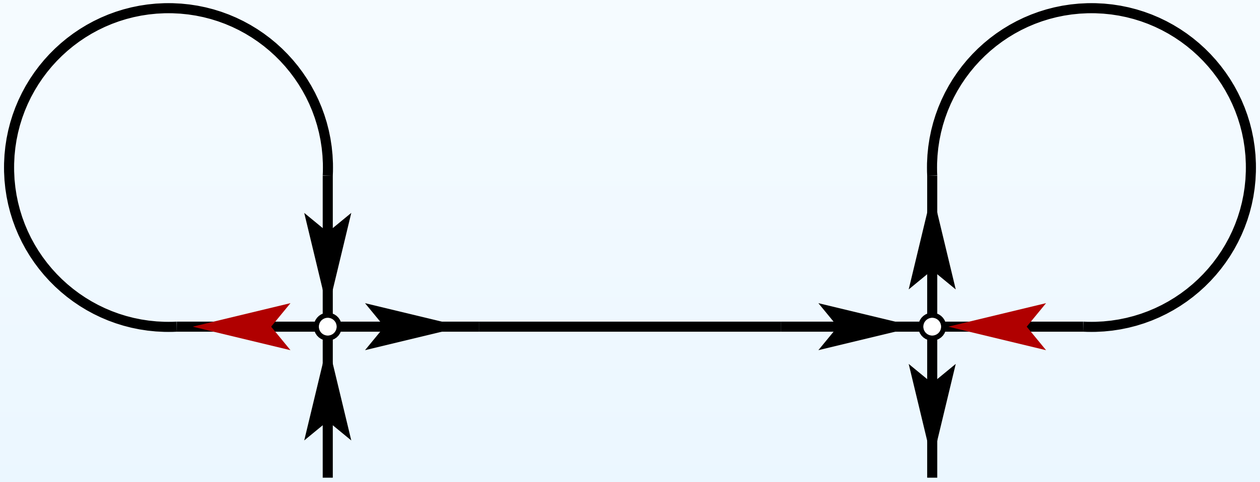
Exercise. Verify that the second way to arrange boundary components into pairs (as in the picture) is symmetric to the first one under interchanging the labels of the two singularities.

Diagrams with two loops in $\mathcal{H}(1,1)$



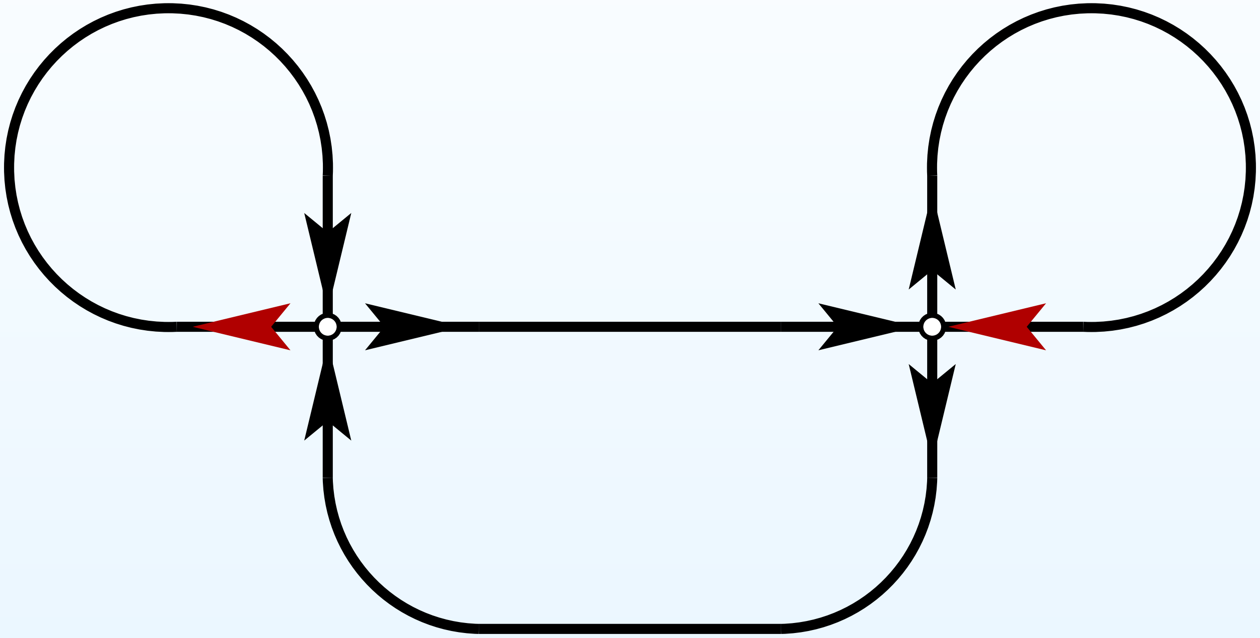
Now we have to consider diagrams having at least one loop. It is clear, that if a diagram has a loop and a saddle connection joining the two zeroes, it has to have another loop at the other zero.

Diagrams with two loops in $\mathcal{H}(1,1)$



There are two choices for the second loop. This is the first possible choice.

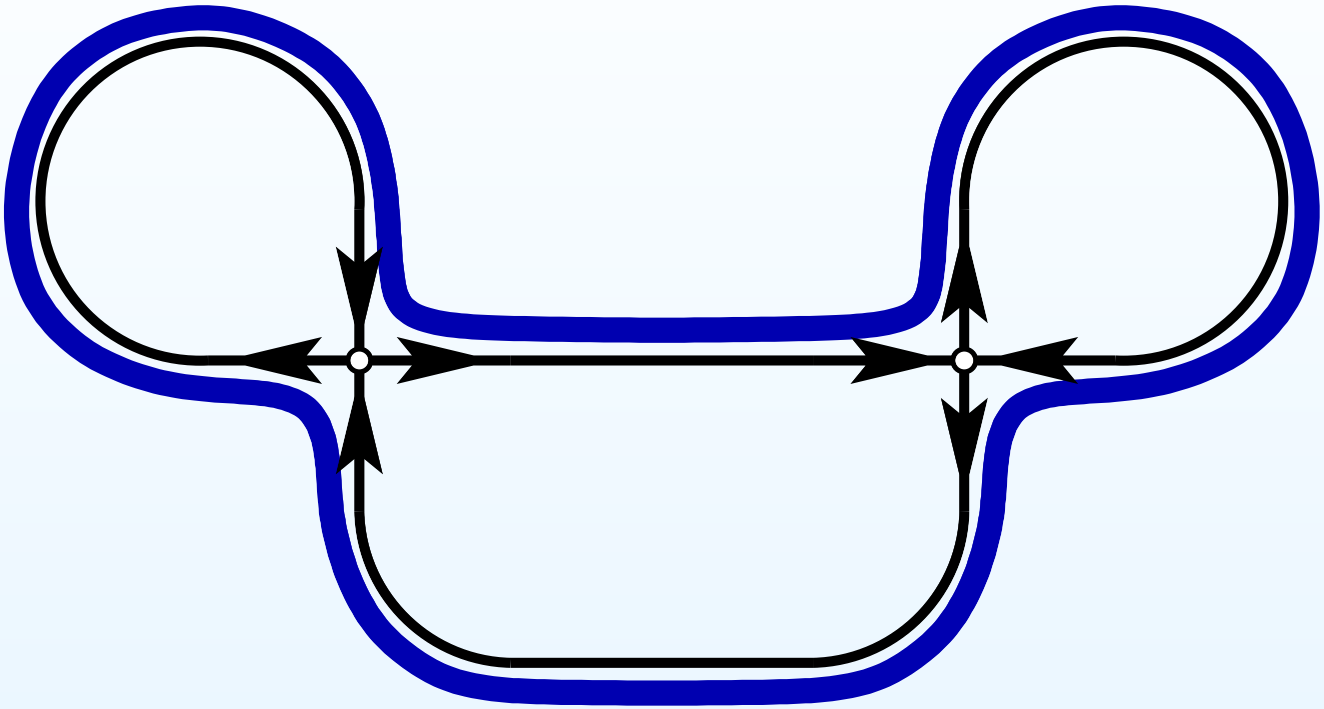
Diagrams with two loops in $\mathcal{H}(1,1)$



There are two choices for the second loop. This is the first possible choice.

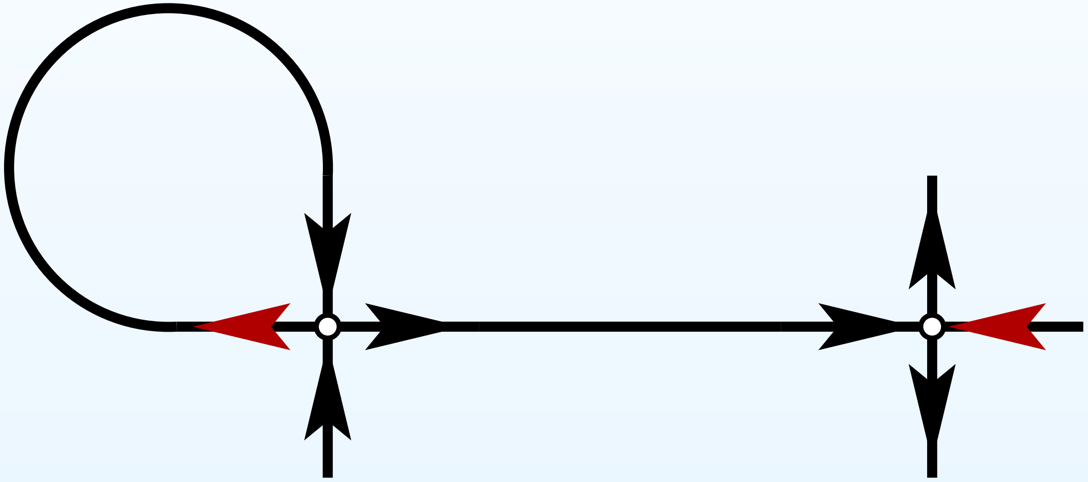
This is the unique way to join the remaining pair of separatrix rays.

Diagrams with two loops in $\mathcal{H}(1,1)$



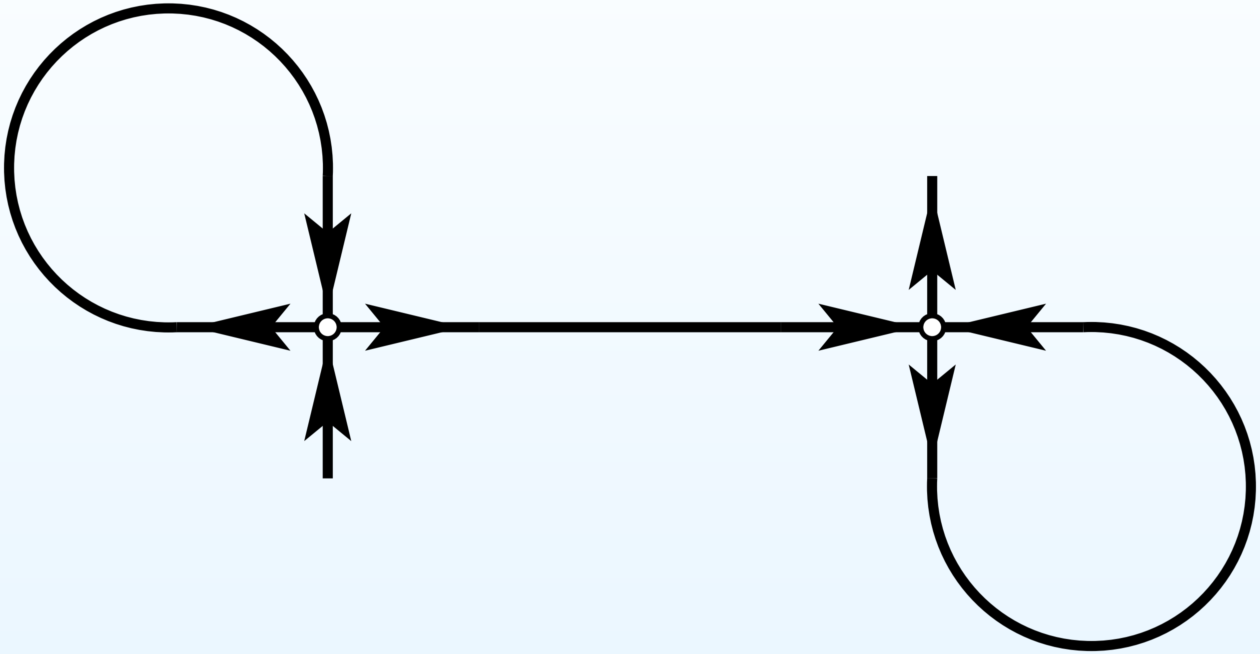
The boundary component of the resulting ribbon graph is longer than any other component for any choice of lengths of saddle connections (edges of the graph). This diagram is not realizable.

Diagrams with two loops in $\mathcal{H}(1,1)$



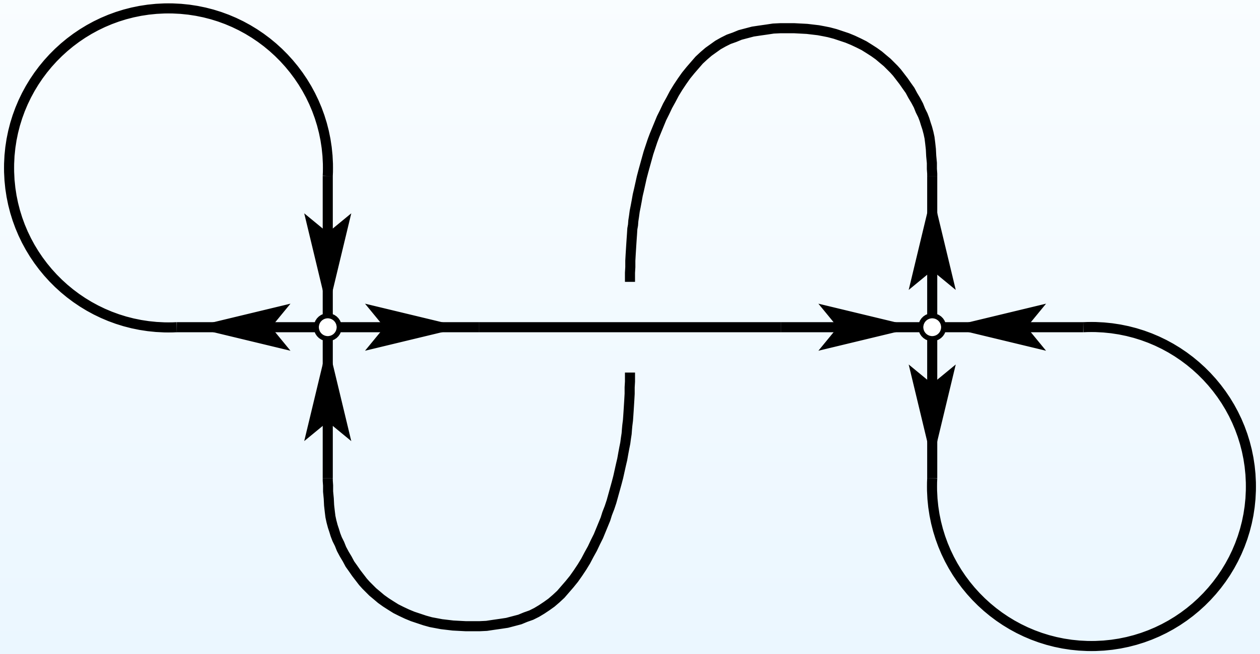
Recall that we are considering diagrams having at least one loop and a saddle connection joining the two zeroes.

Diagrams with two loops in $\mathcal{H}(1,1)$



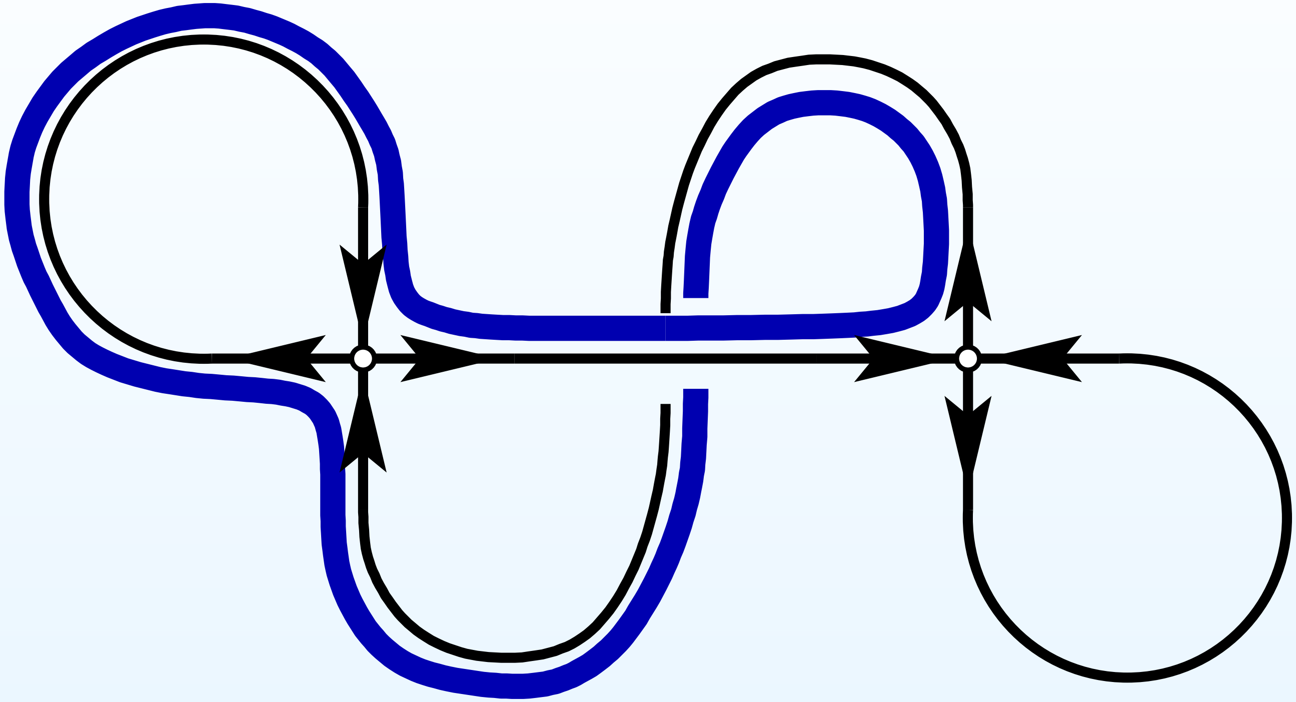
The second choice for the second loop as in the picture.

Diagrams with two loops in $\mathcal{H}(1,1)$



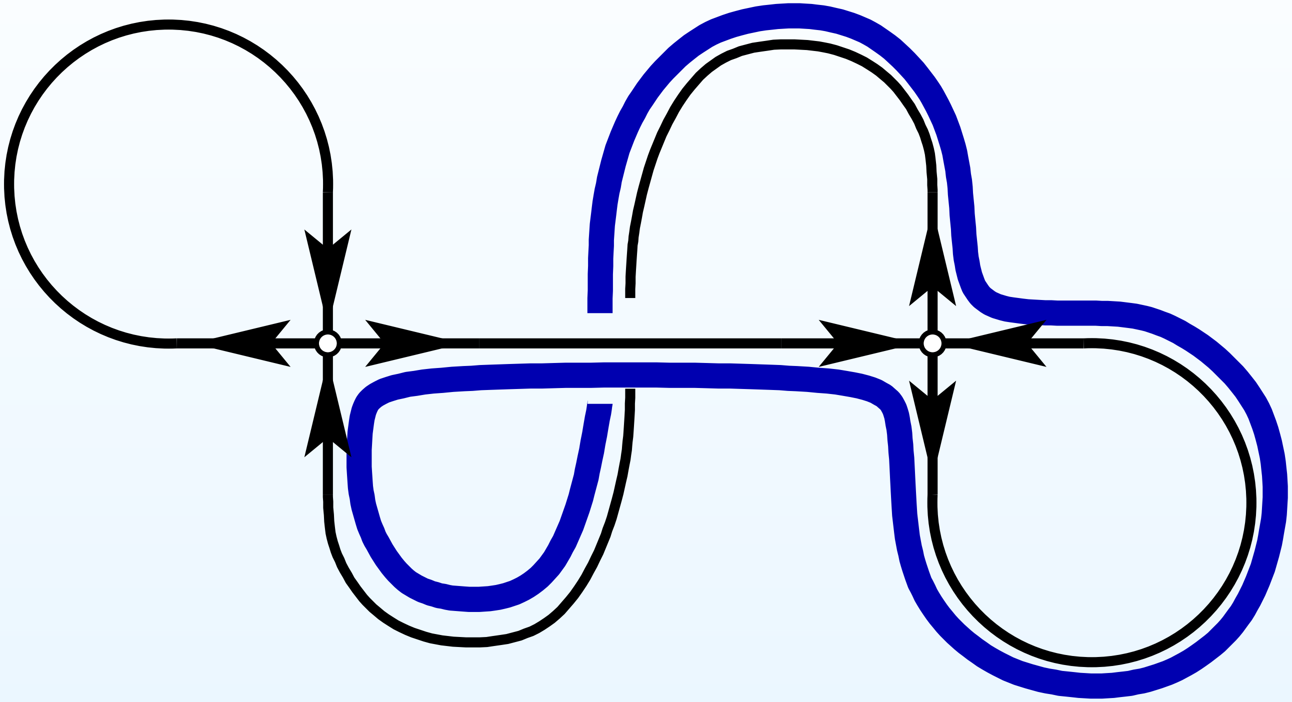
The second choice for the second loop as in the picture. This is the unique way to join the remaining pair of separatrix rays.

Diagrams with two loops in $\mathcal{H}(1,1)$



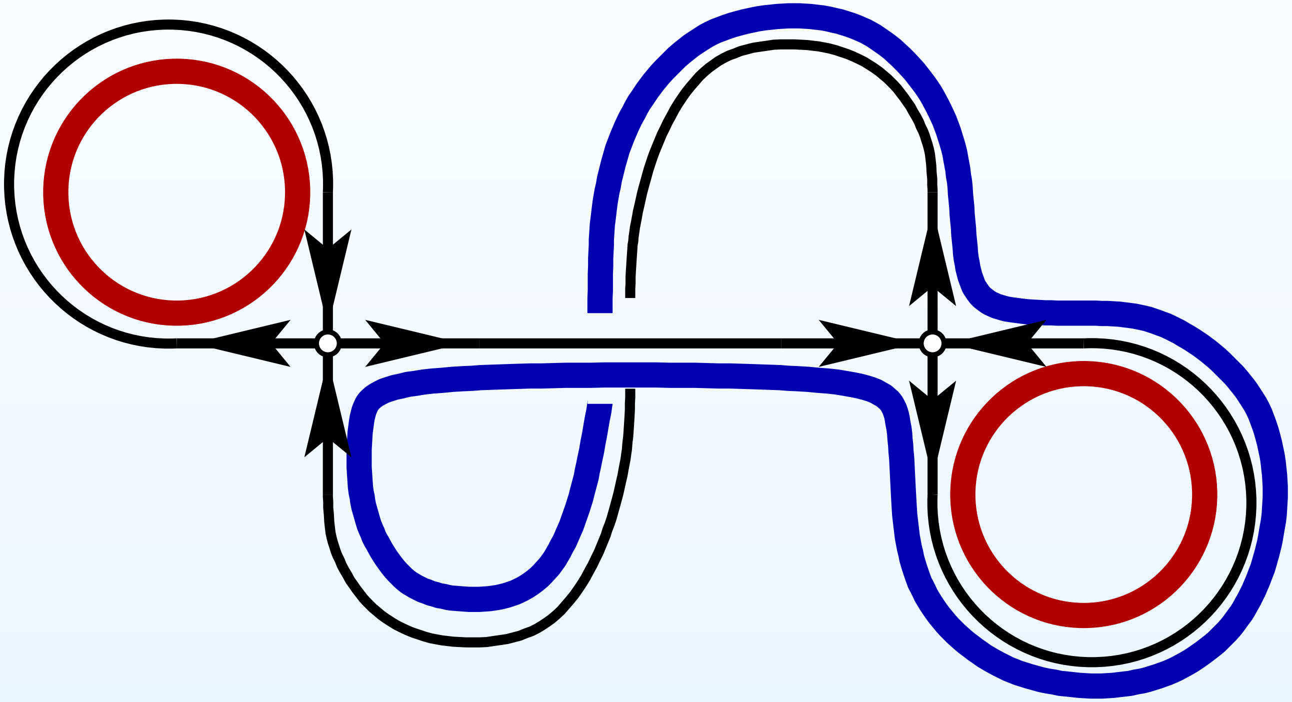
This is one of the four boundary components of the resulting ribbon graph.

Diagrams with two loops in $\mathcal{H}(1,1)$



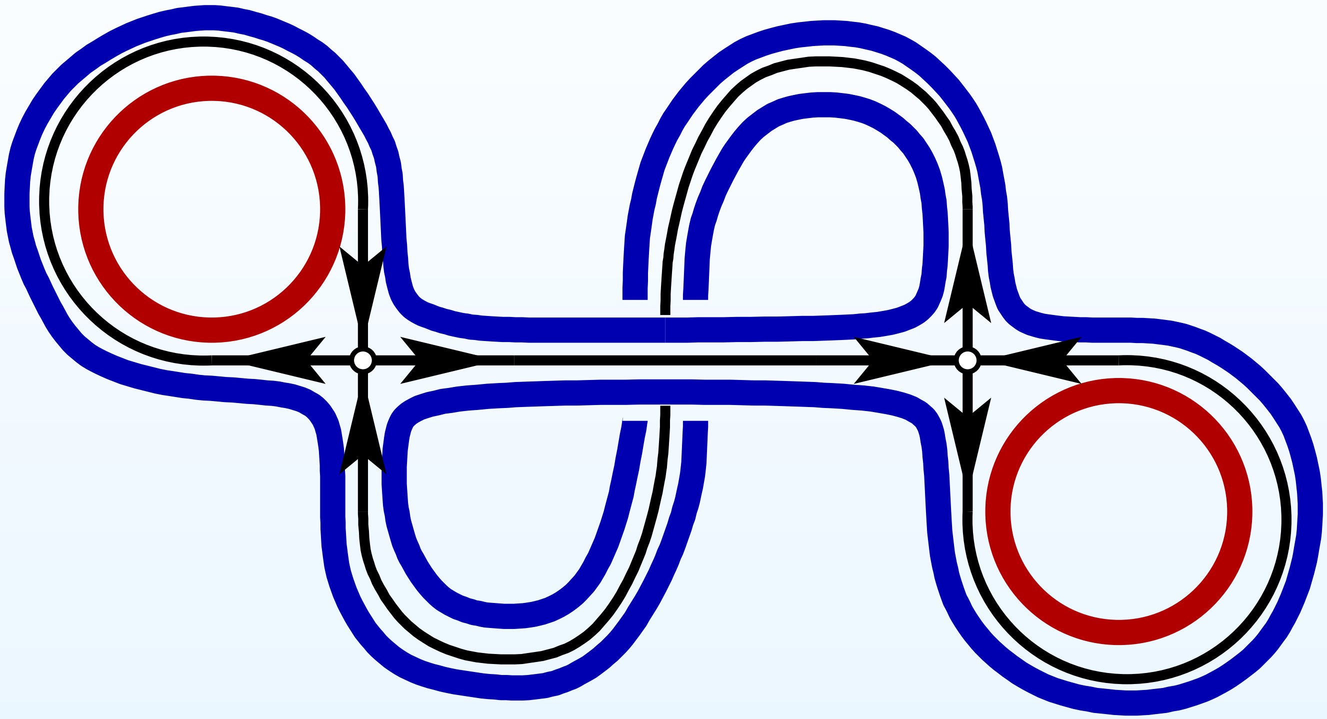
This is one more boundary component.

Diagrams with two loops in $\mathcal{H}(1,1)$



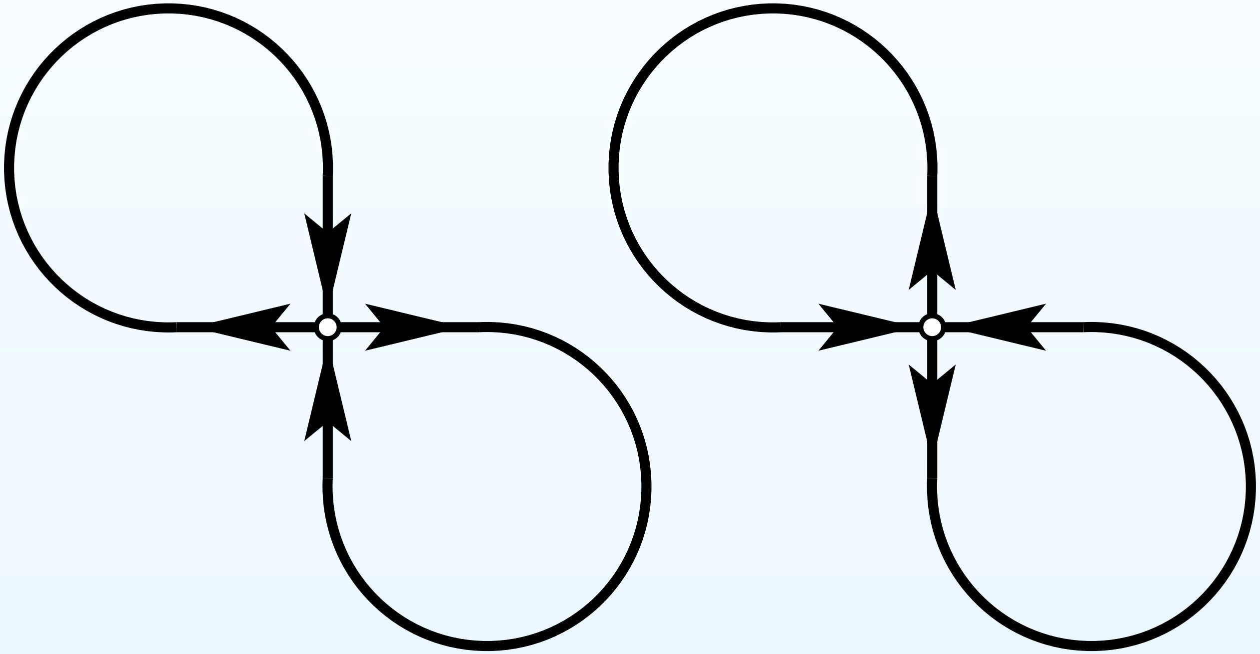
It is really easy to check that the only choice is to paste a cylinder to the pair of red boundary components. This implies a condition that the lengths of the corresponding loops are the same.

Diagrams with two loops in $\mathcal{H}(1,1)$



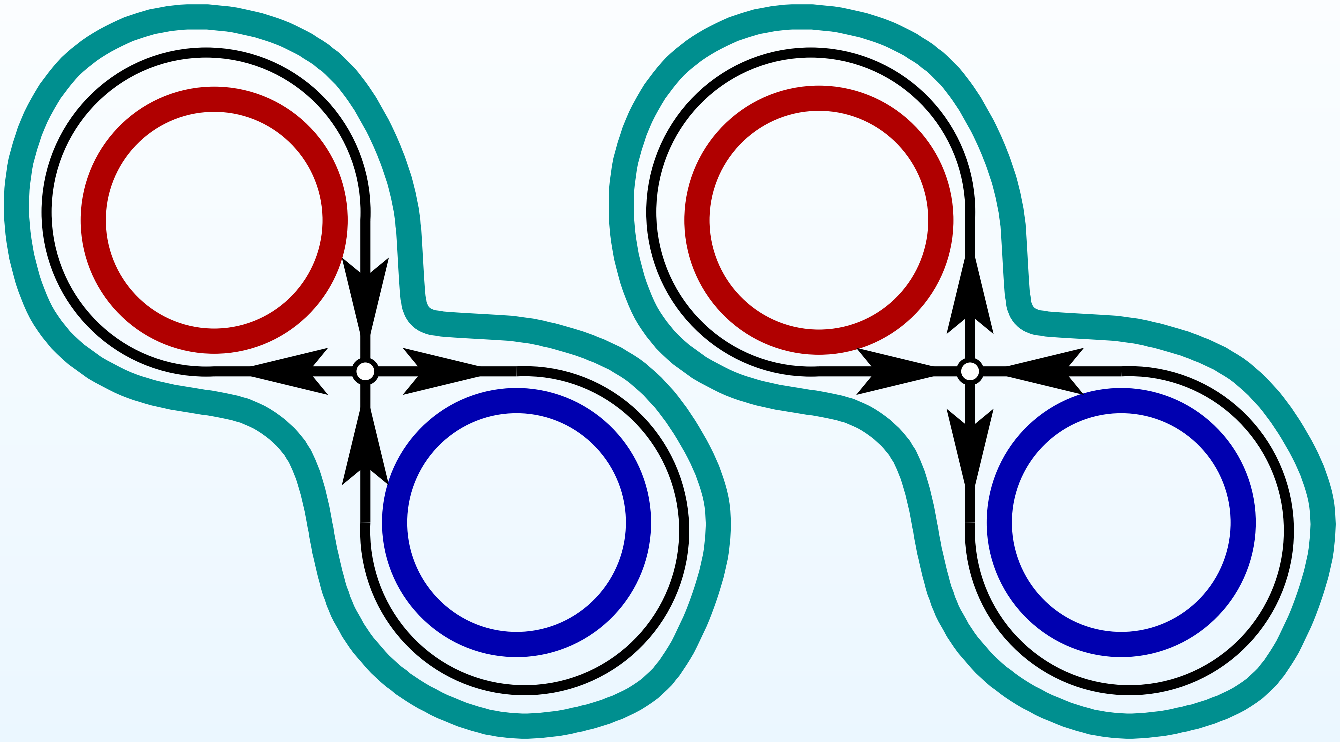
It is really easy to check that the only choice is to paste a cylinder to the pair of red boundary components. This implies a condition that the lengths of the corresponding loops are the same. This automatically implies that the lengths of the blue boundary components are the same. We get one more realizable diagram with two cylinders.

Diagrams with four loops in $\mathcal{H}(1,1)$



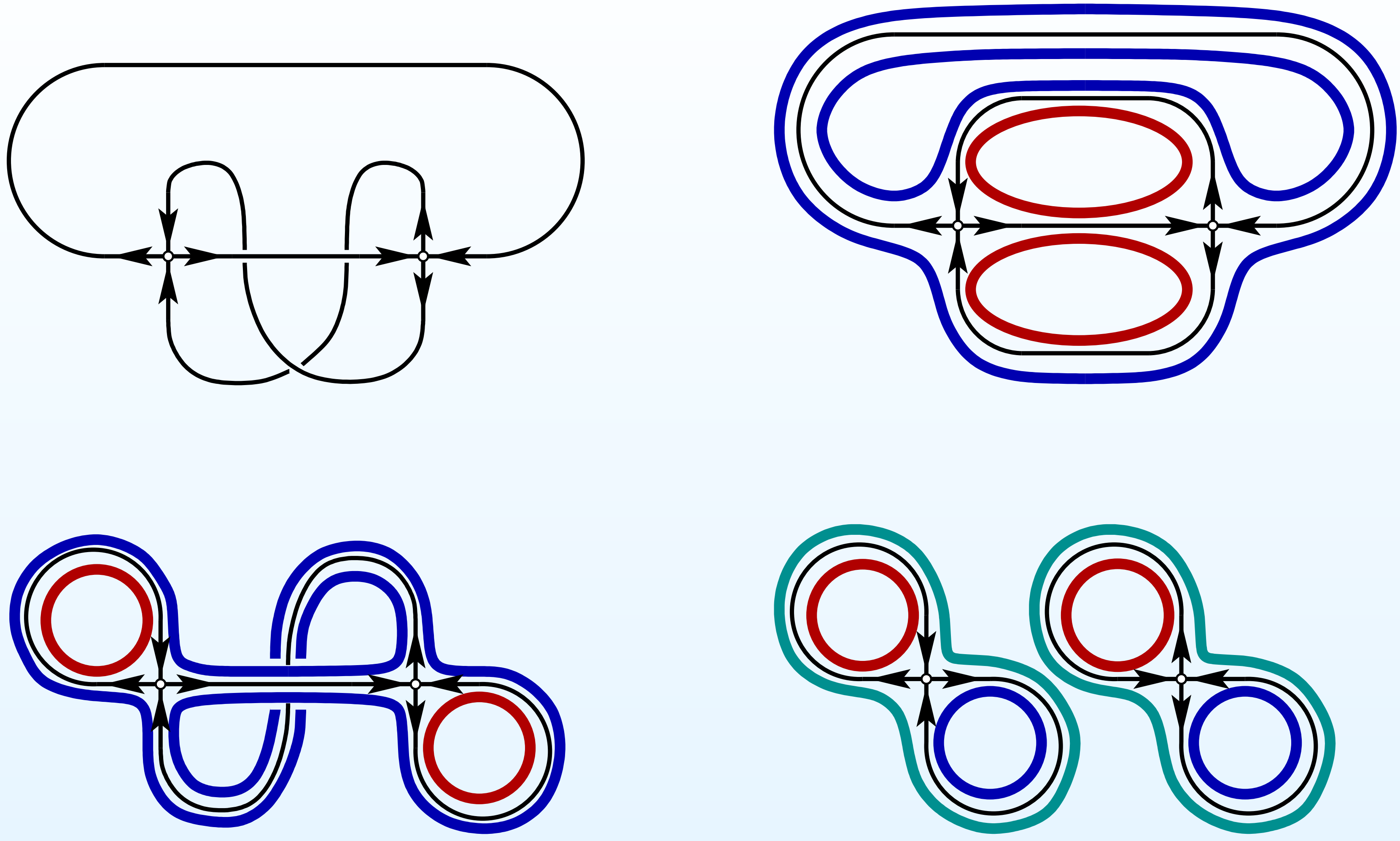
In the remaining case all the edges are loops.

Diagrams with four loops in $\mathcal{H}(1,1)$



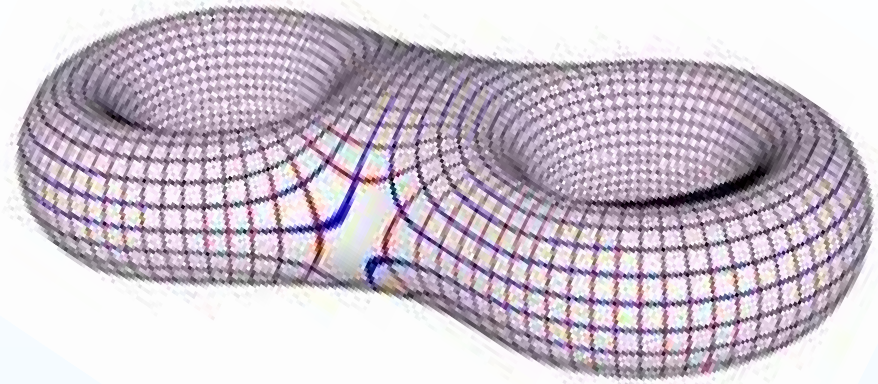
In the remaining case all the edges are loops. There is, clearly only one way to arrange boundary components into pairs. We get the last admissible (realizable) diagram in the stratum $\mathcal{H}(1,1)$.

Admissible diagrams in $\mathcal{H}(1,1)$



These four separatrix diagrams are admissible (realizable) diagrams in the stratum $\mathcal{H}(1,1)$ and there are no other ones (up to interchange of the labelling of the two zeroes).

Which diagram?



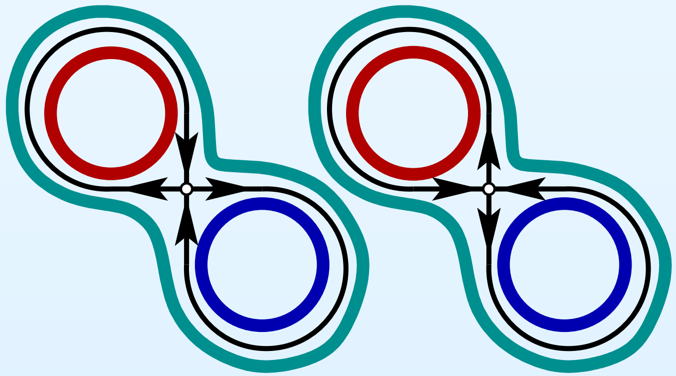
Picture created by Jian Jiang

Question.

• *To which of the found diagrams corresponds the red foliation of the square-tiled surface from the picture?*

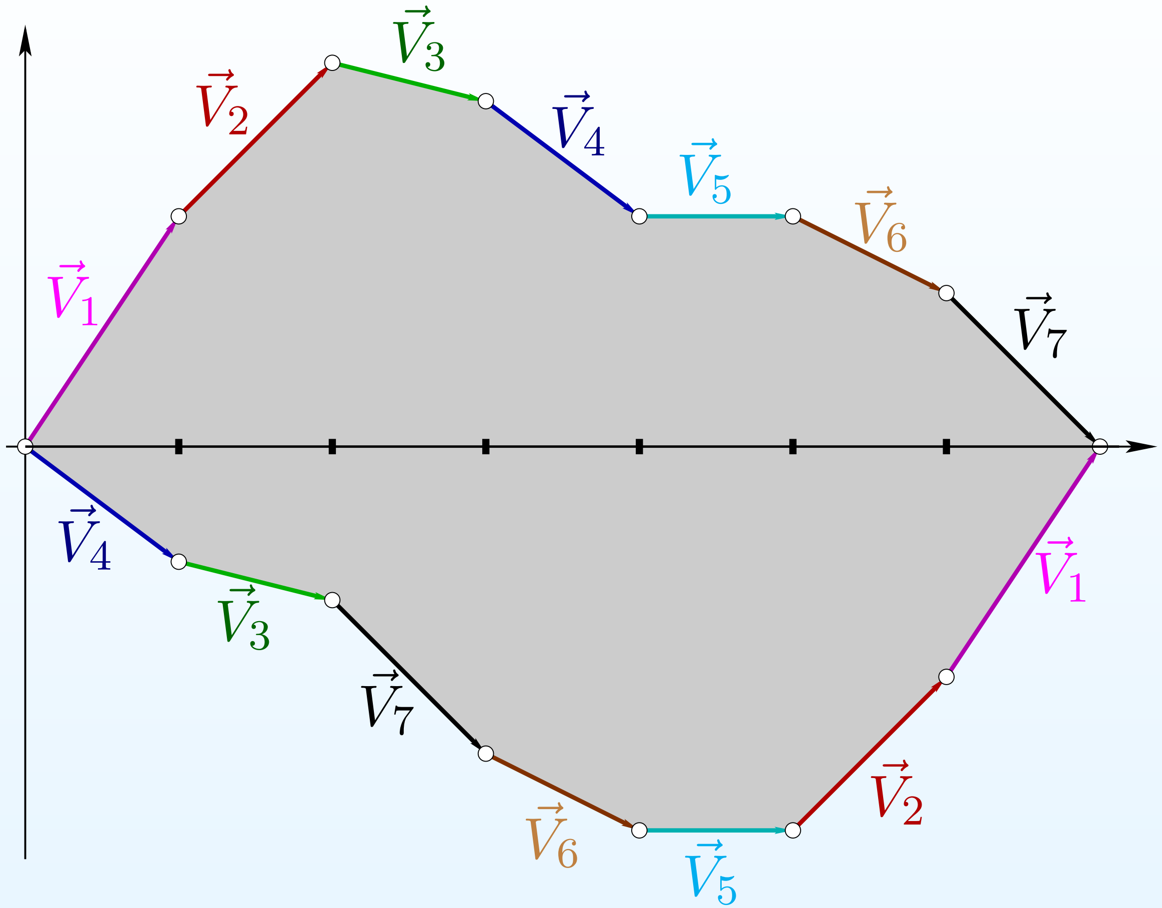
Answer.

There are, clearly, three distinct cylinders. There only one 3-cylinder diagram in the stratum $\mathcal{H}(1,1)$:



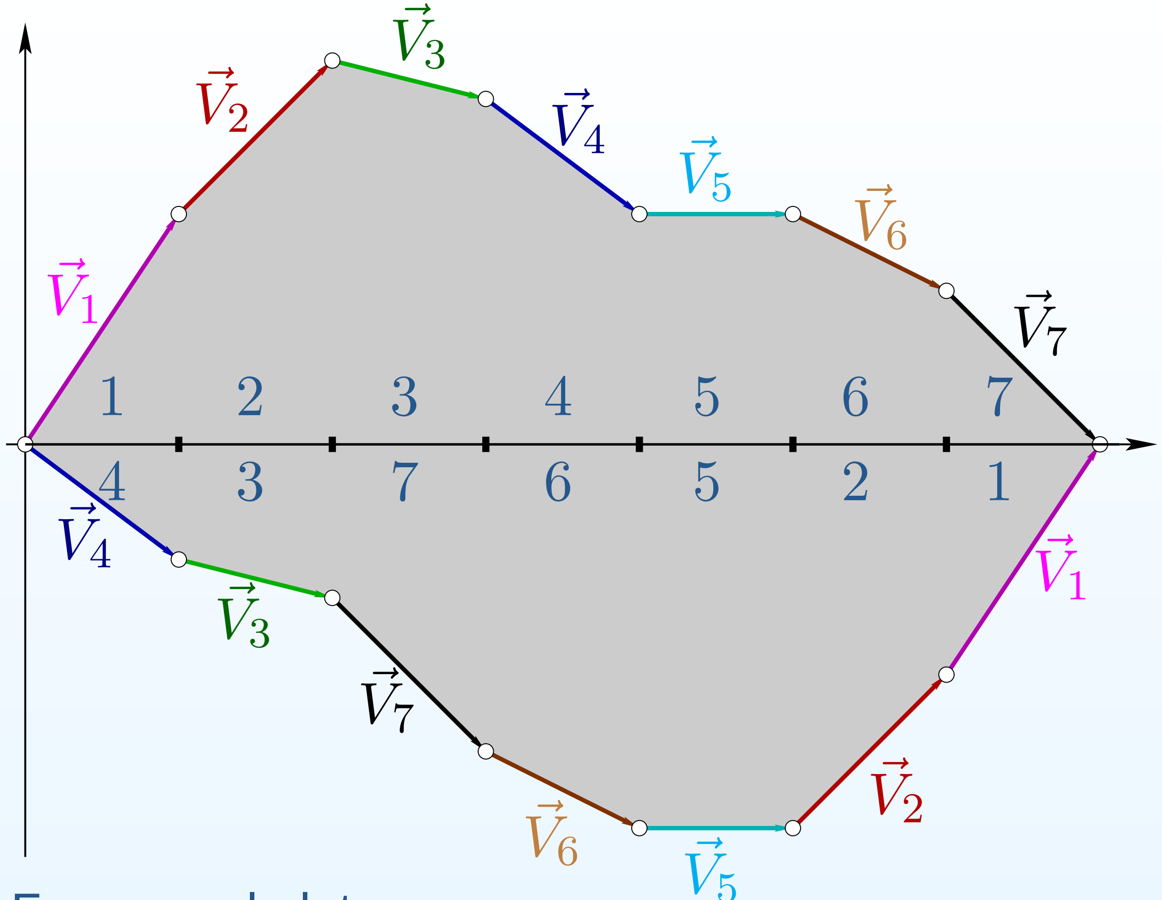
Detecting the stratum associated to an interval exchange transformation

Problem 2. What is the ambient stratum?



What is the ambient stratum $\mathcal{H}(d_1, \dots, d_n)$ for the translation surface obtained by identifying the pairs of sides corresponding to the same vectors $\vec{V}_1, \dots, \vec{V}_7$ by parallel translations?

Canonical suspension

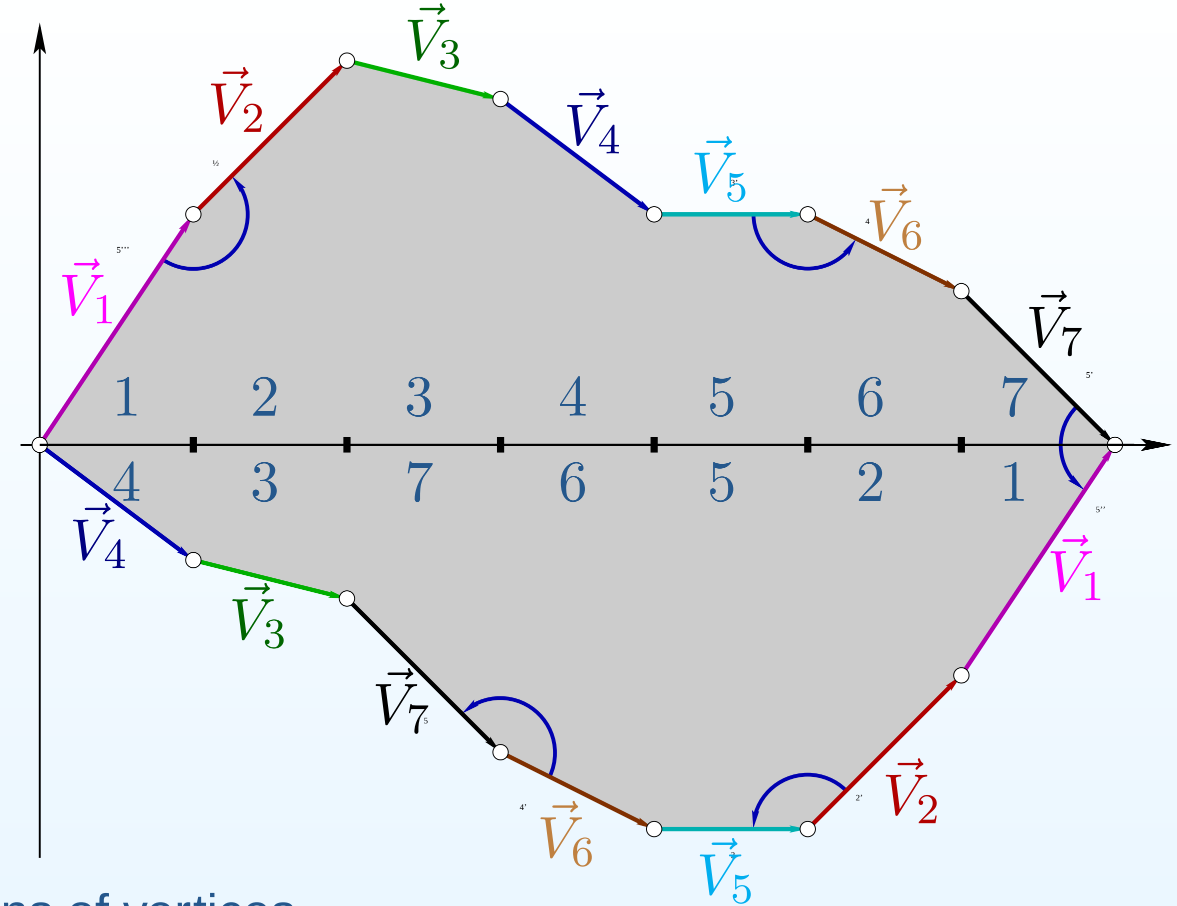


Consider some permutation. For example let

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 7 & 6 & 5 & 2 & 1 \end{pmatrix} .$$

Ambient stratum depends on the permutation but not on a particular choice of coordinates of vectors of the suspension. We can choose $\vec{V}_j = (1, \pi(j) - j)$.

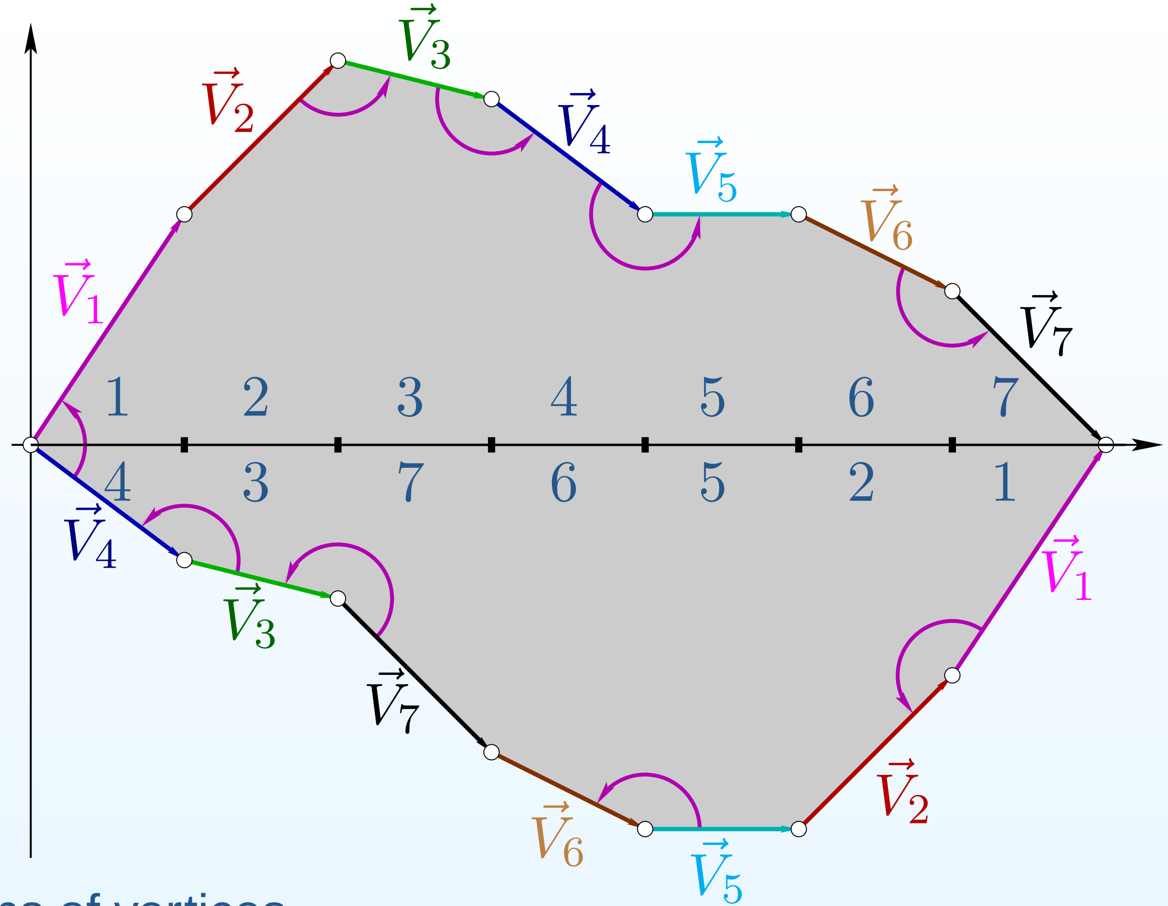
Canonical suspension



Now let us trace identifications of vertices.

We turn around a conical point on the surface before we complete a loop.

Canonical suspension

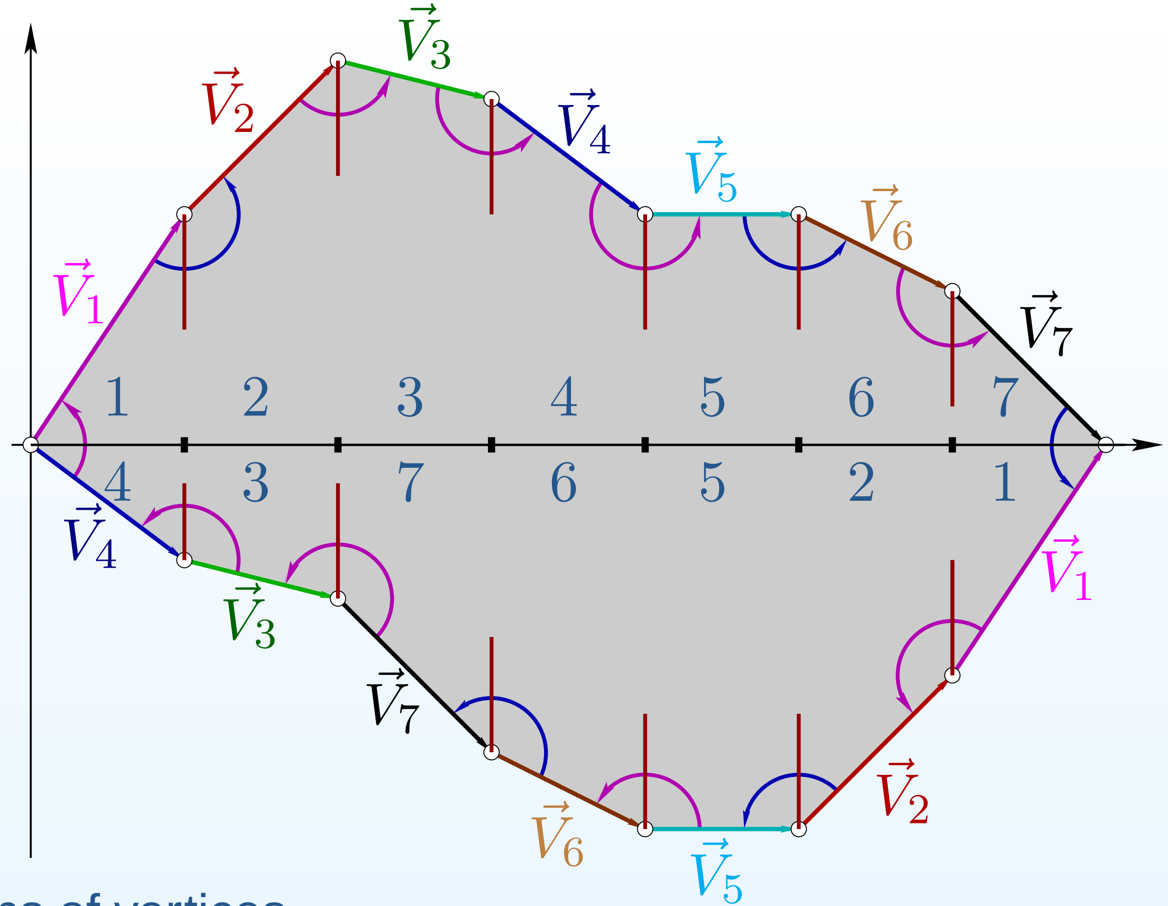


Now let us trace identifications of vertices.

We turn around a conical point on the surface before we complete a loop.

Then we proceed with the next loop.

Canonical suspension



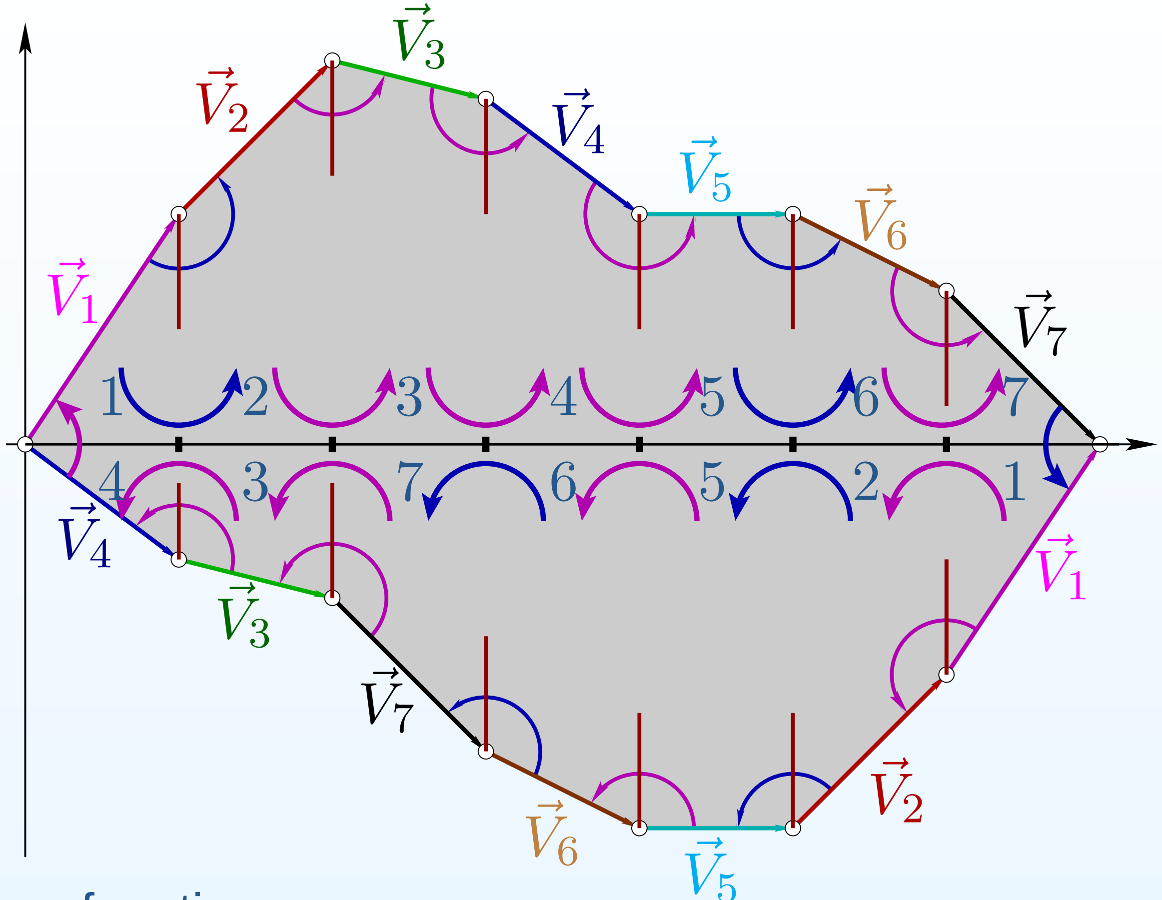
Now let us trace identifications of vertices.

We turn around a conical point on the surface before we complete a loop.

Then we proceed with the next loop.

To compute cone angles we count how many times our loops cross the vertical direction: 4 times the blue cycle and 8 times the purple one. We are in $\mathcal{H}(3,1)$.

Canonical suspension



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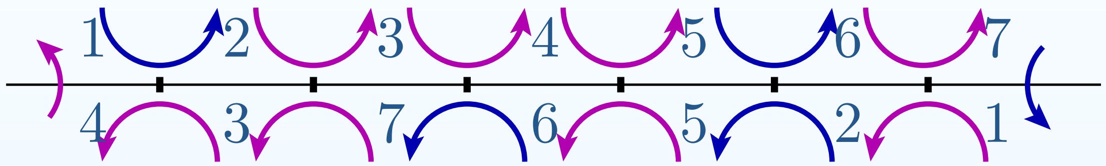
Canonical suspension



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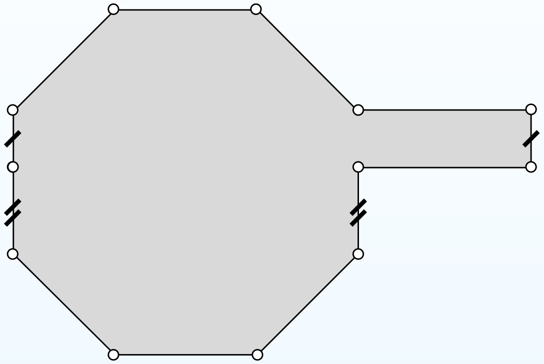
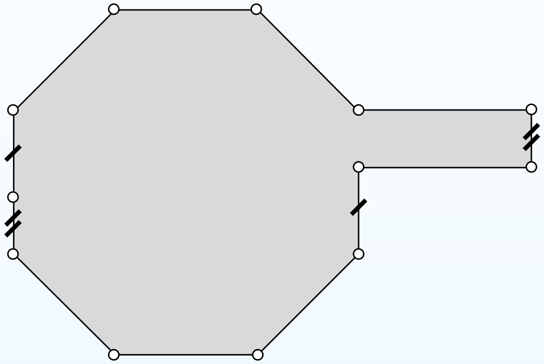
Connected components of the stratum $\mathcal{H}(4)$

Problem 3. Which connected component of the stratum?

- Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.





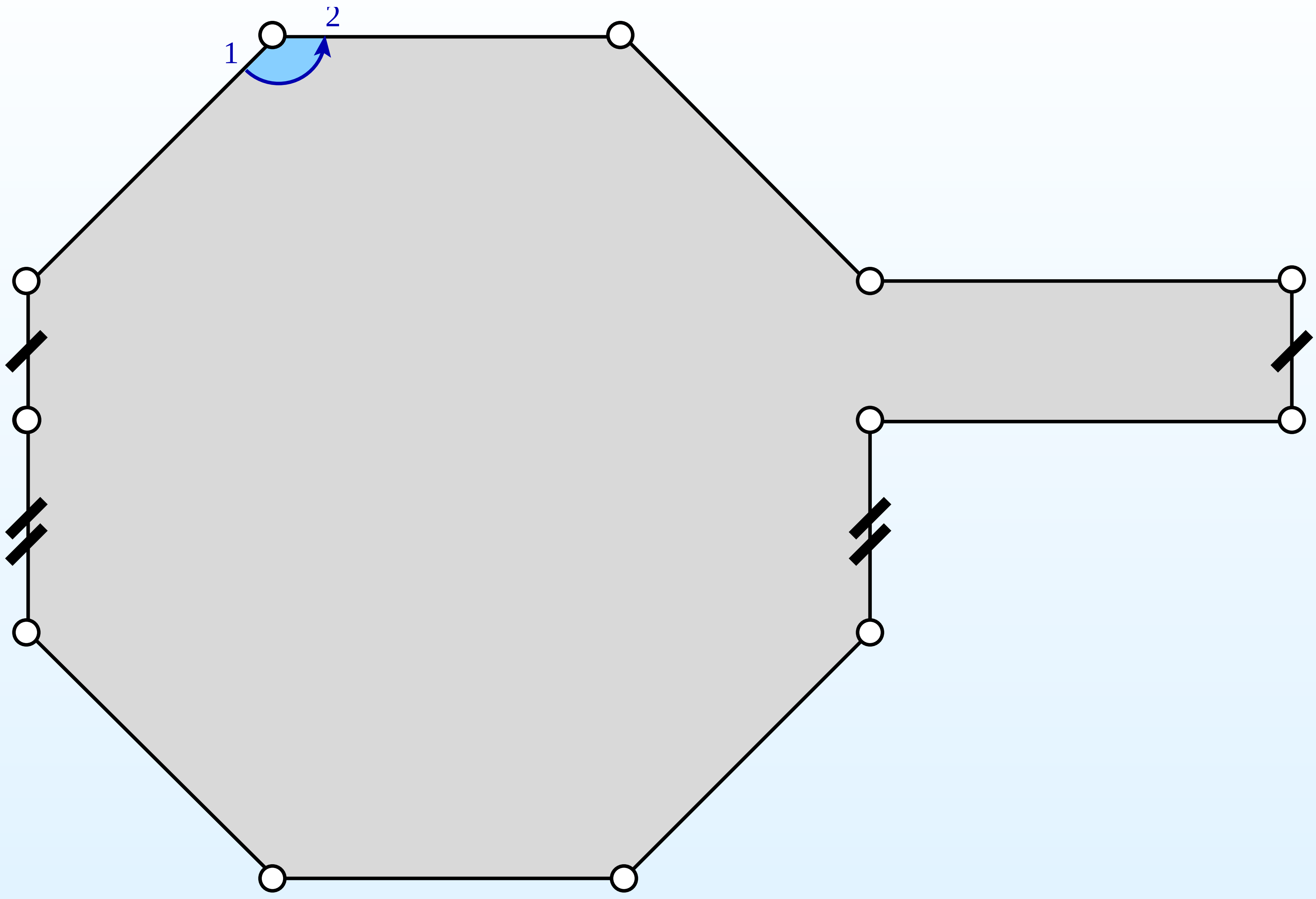
- Present a collection of closed curves representing a basis of cycles in the first homology of the surfaces. Compute their intersection numbers.

A *hyperelliptic involution* is a holomprphic involution of a Riemann surface such that the quotient over the involution is a Riemann sphere.

- Compute the parity of the spin structure for these surfaces (and notice that it is not the same).

- Determine which of the two surfaces is hyperelliptic and find the hyperelliptic involution in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.

- Proof that the following flat surface belongs to the stratum $\mathcal{H}(4)$.



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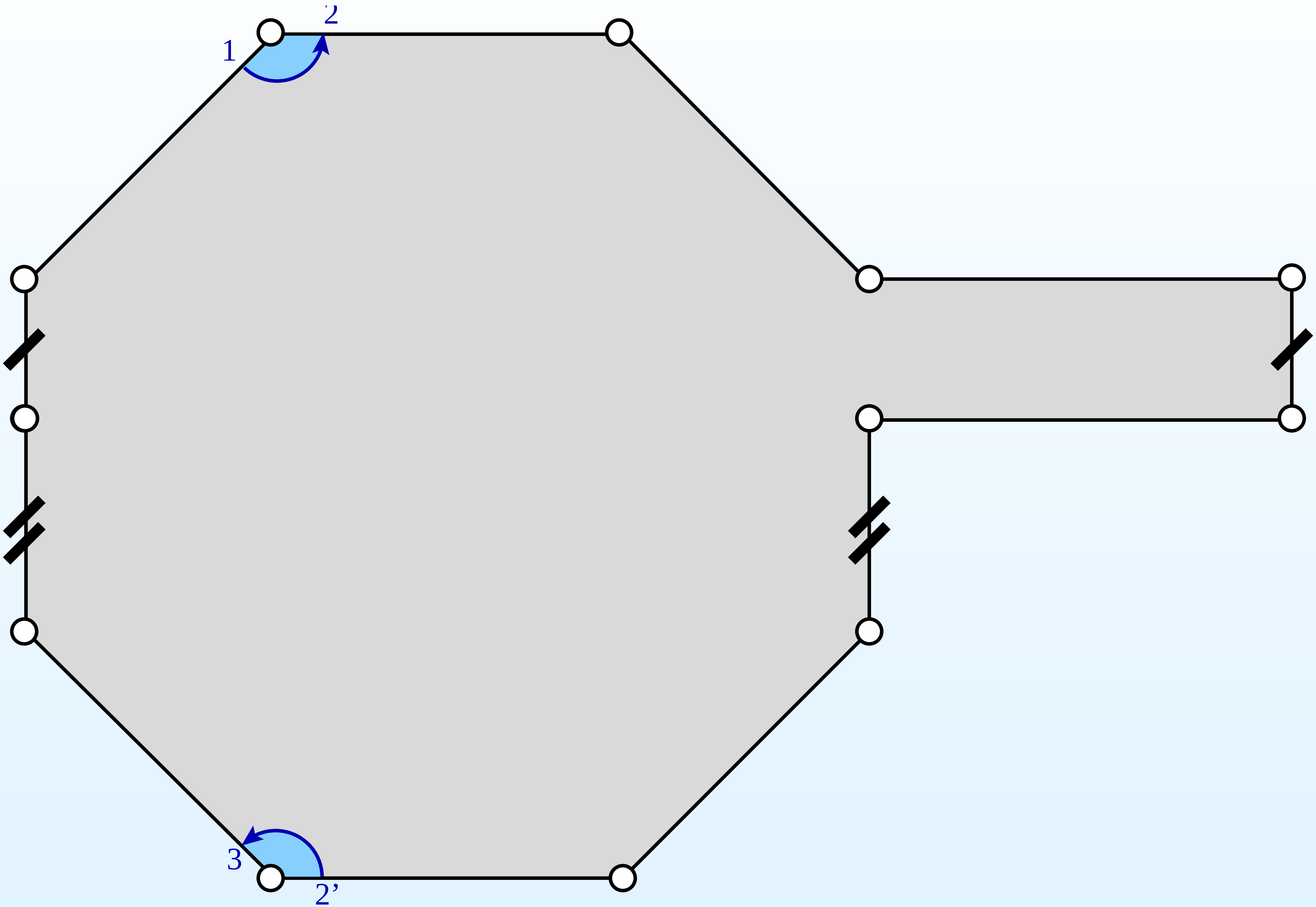

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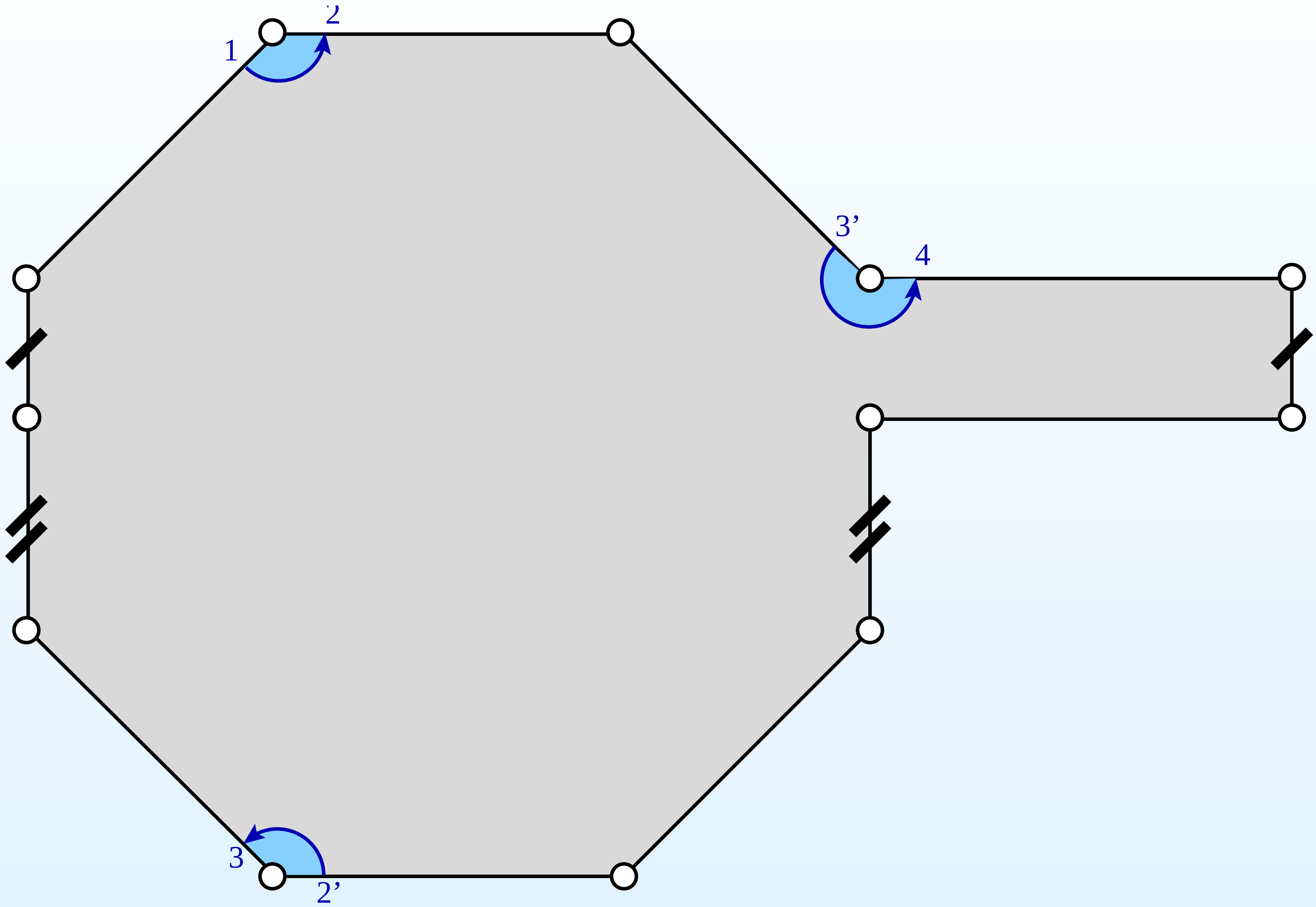

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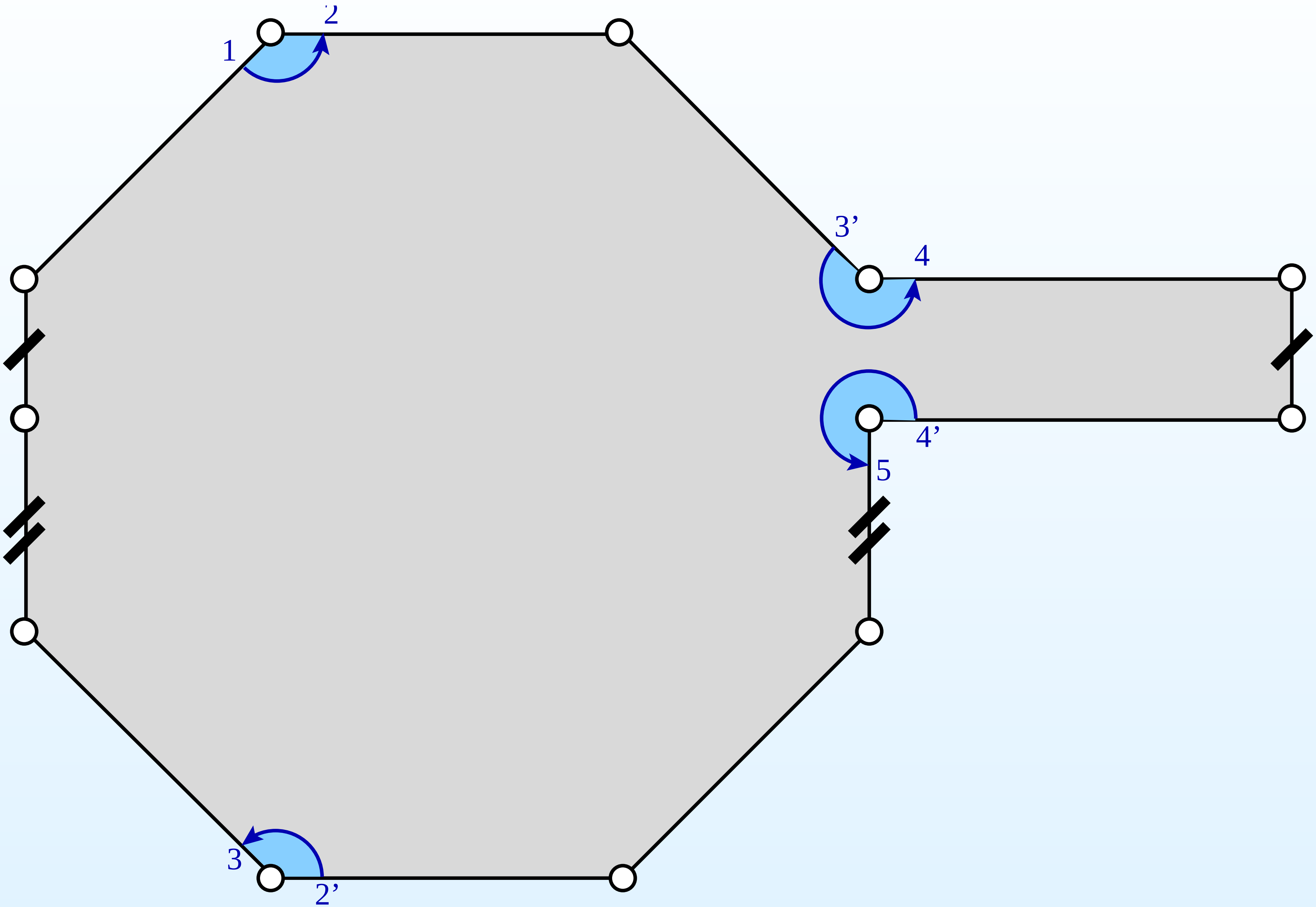

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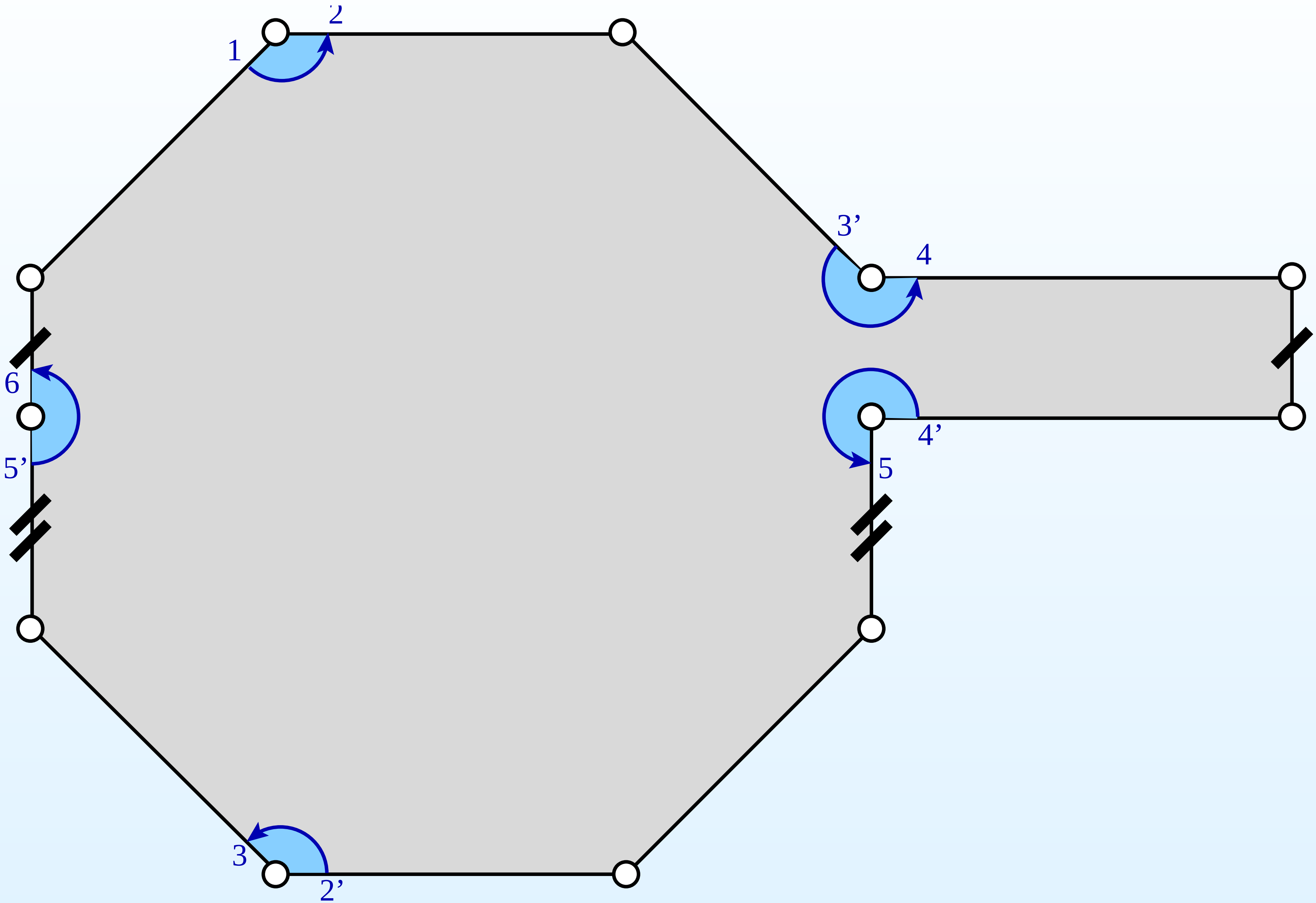

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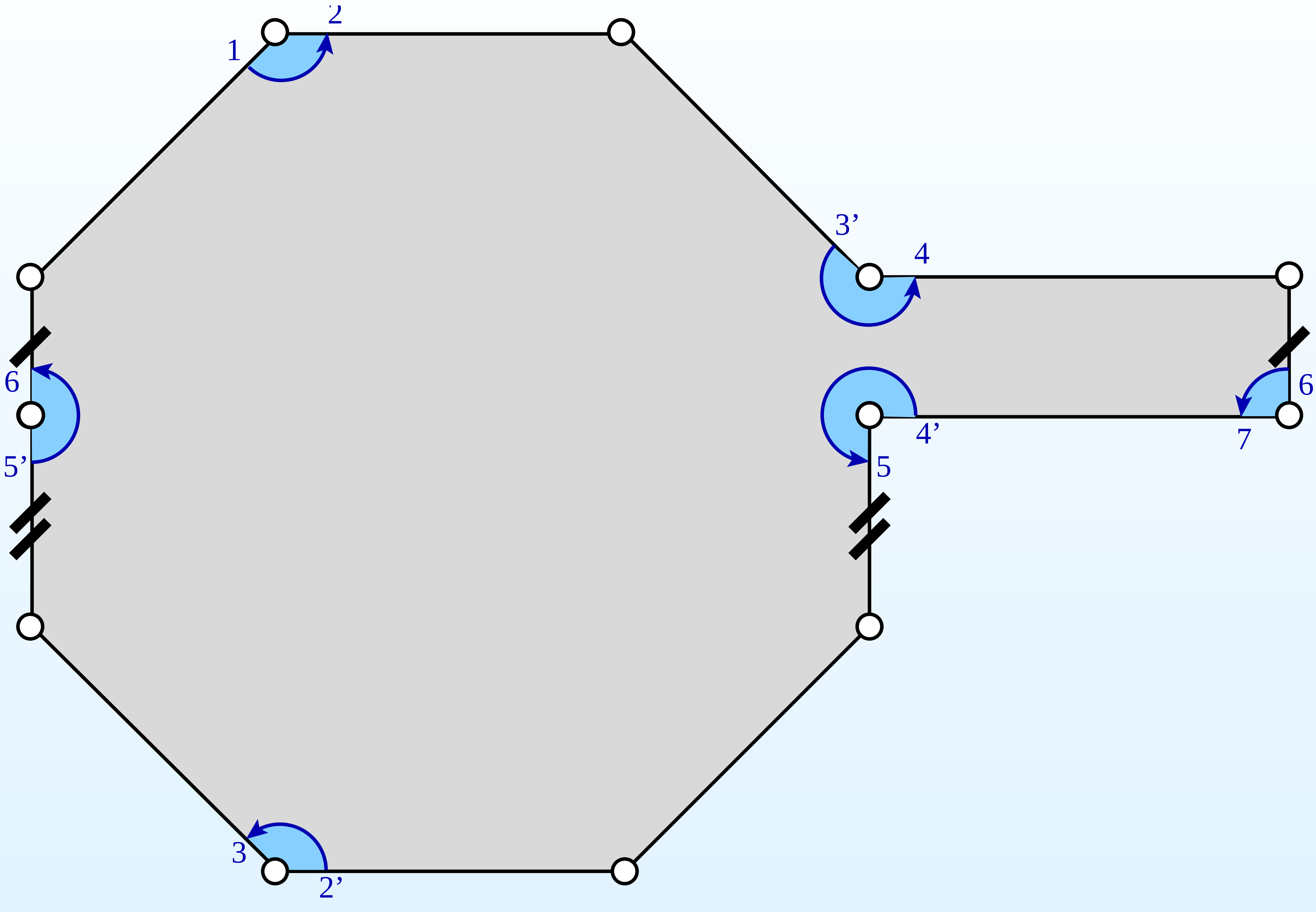

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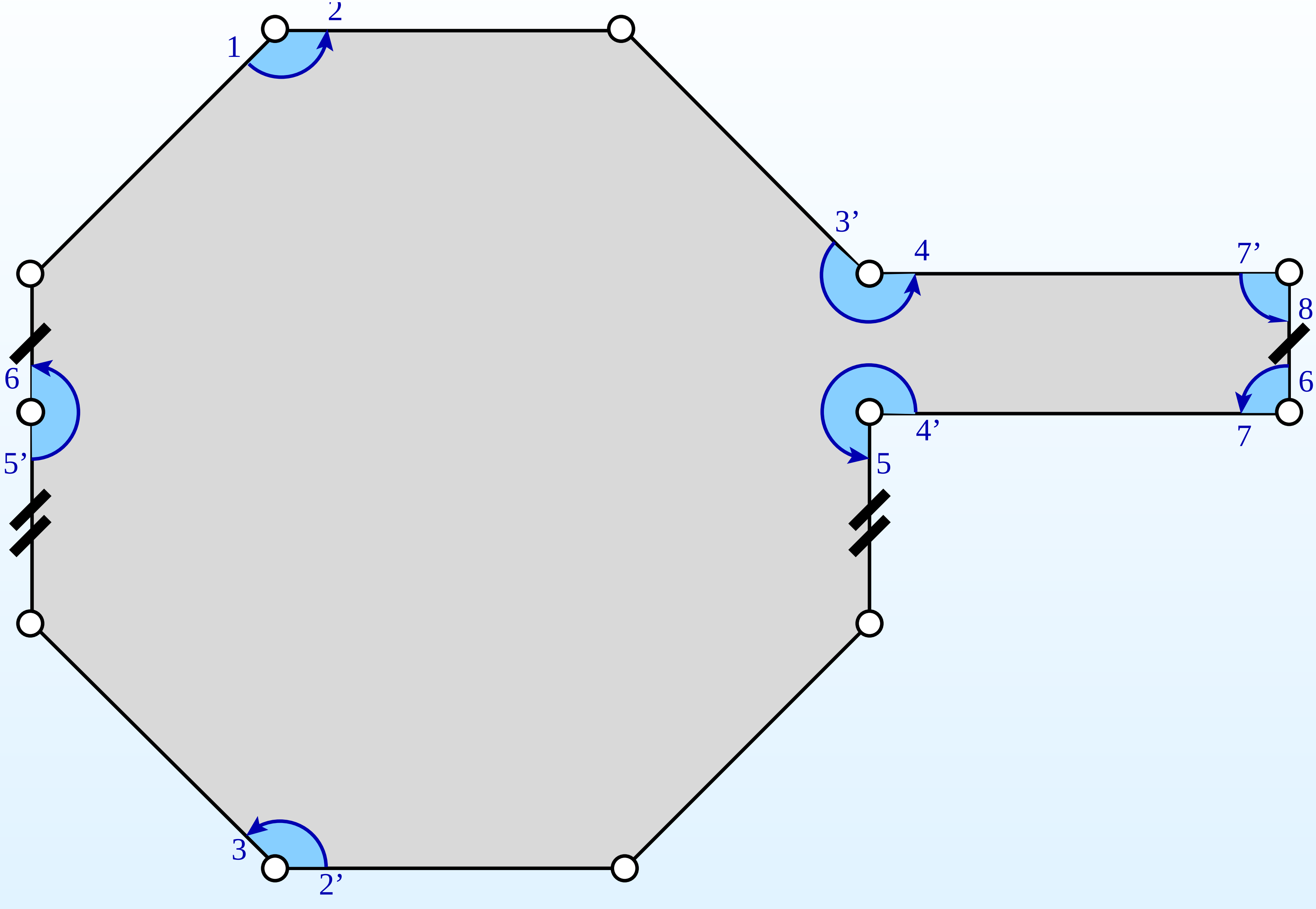

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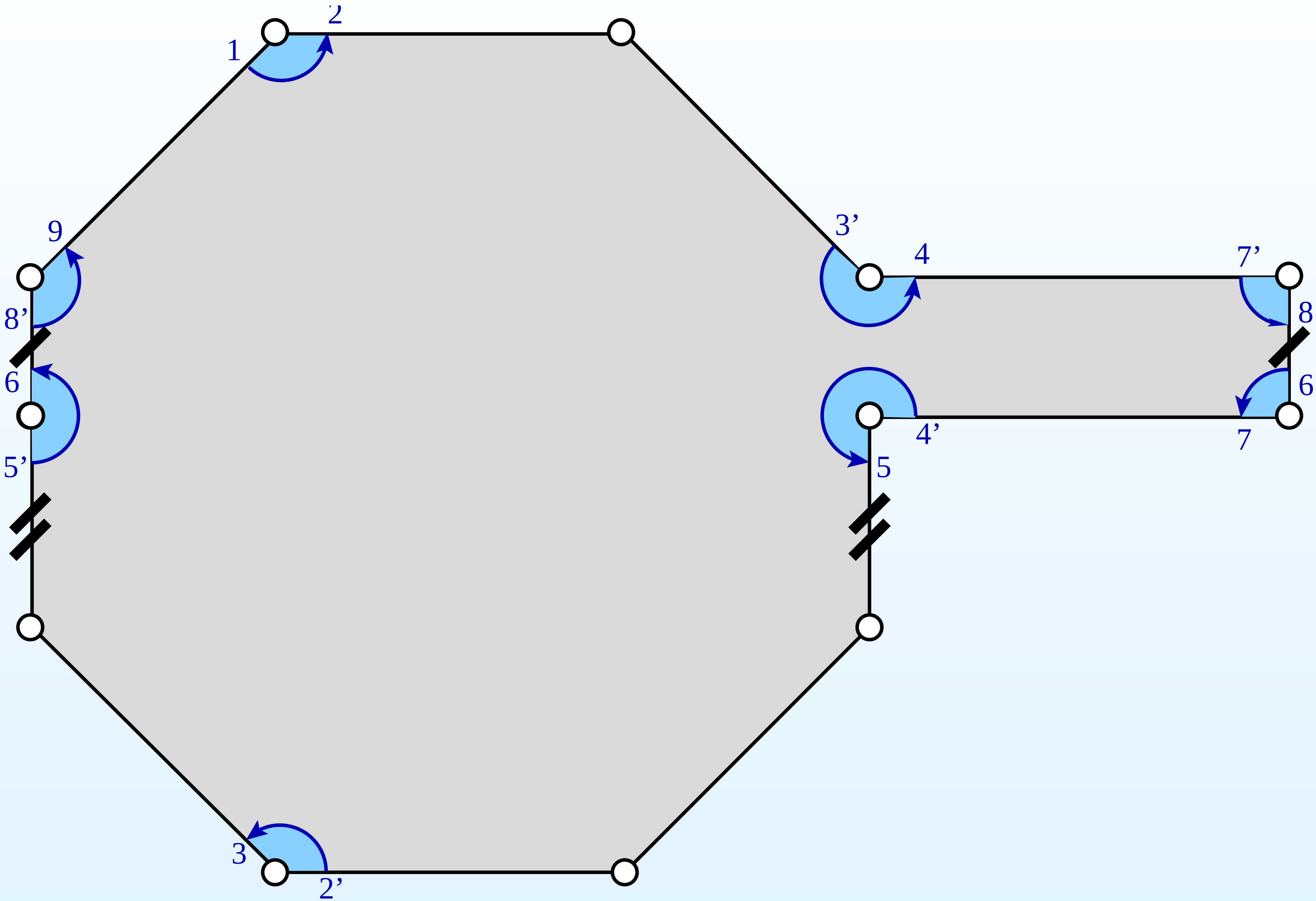

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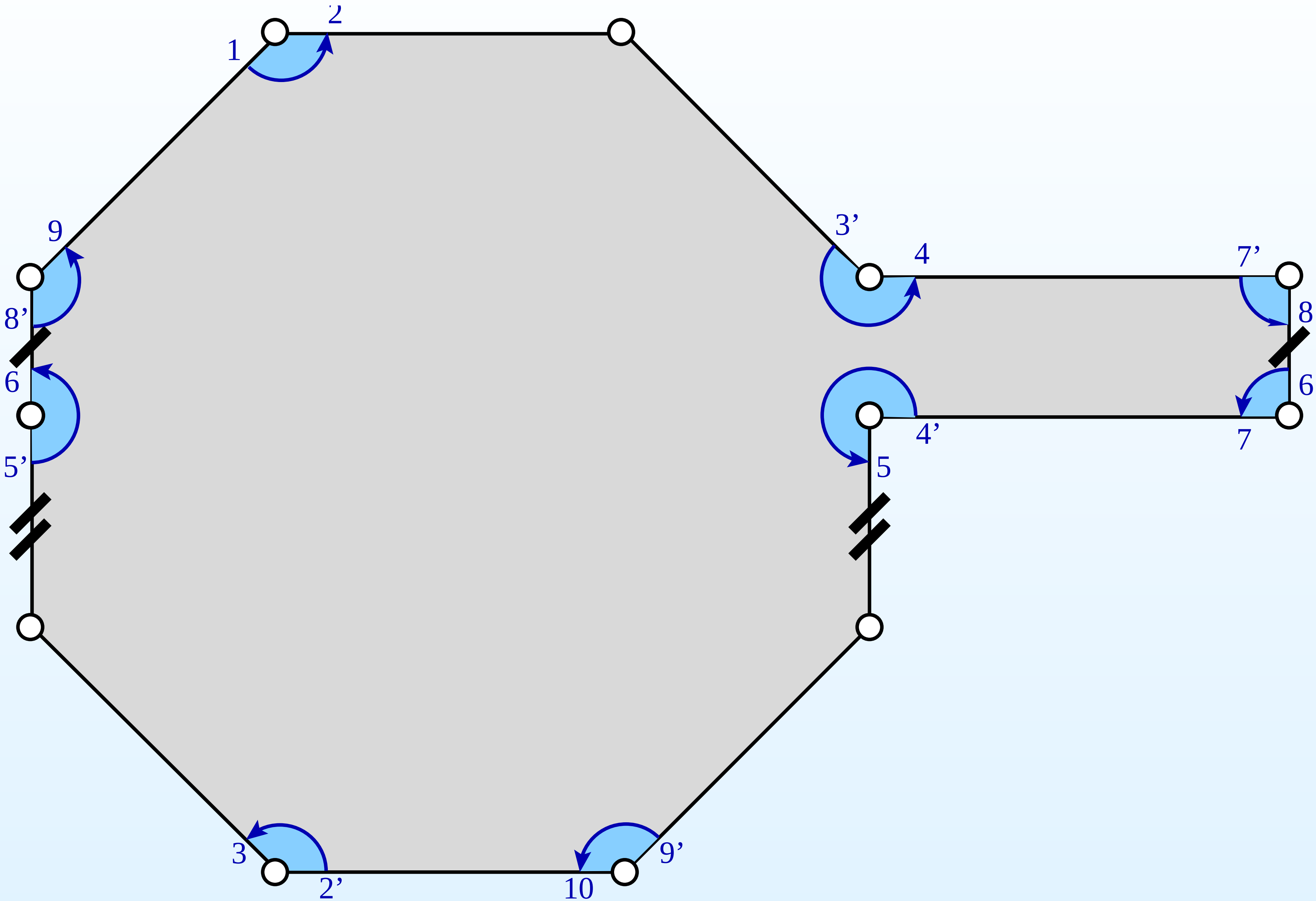

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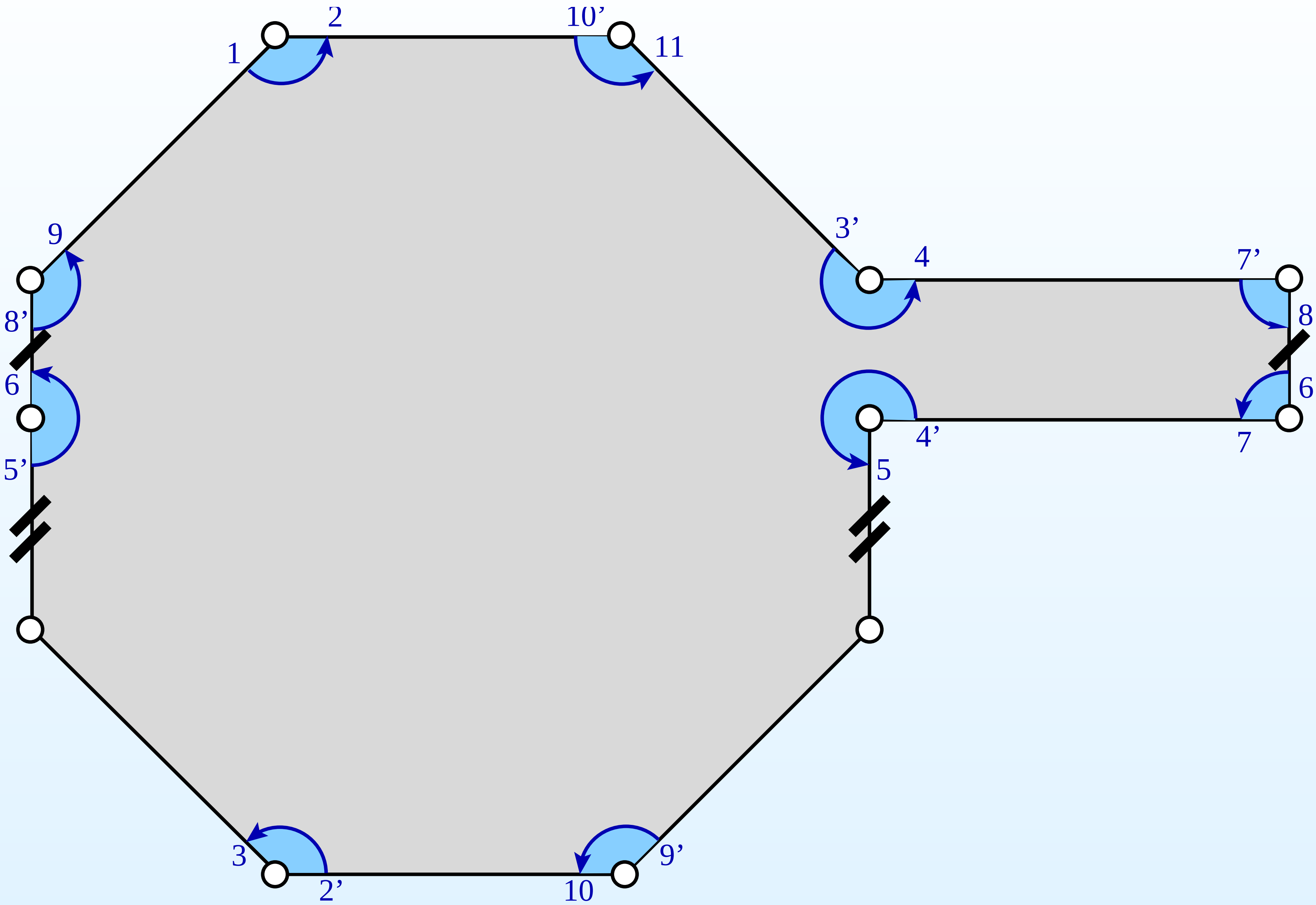

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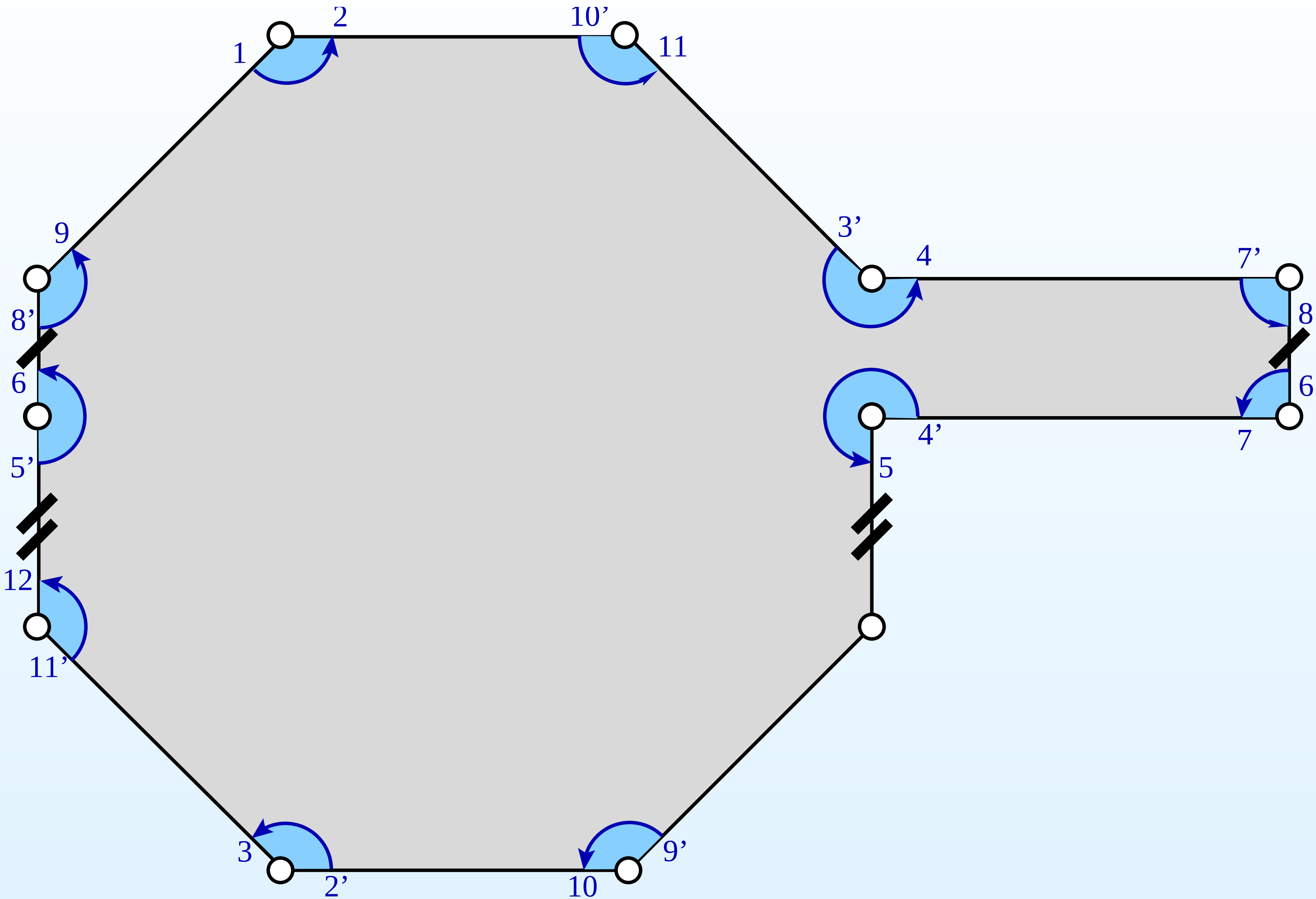

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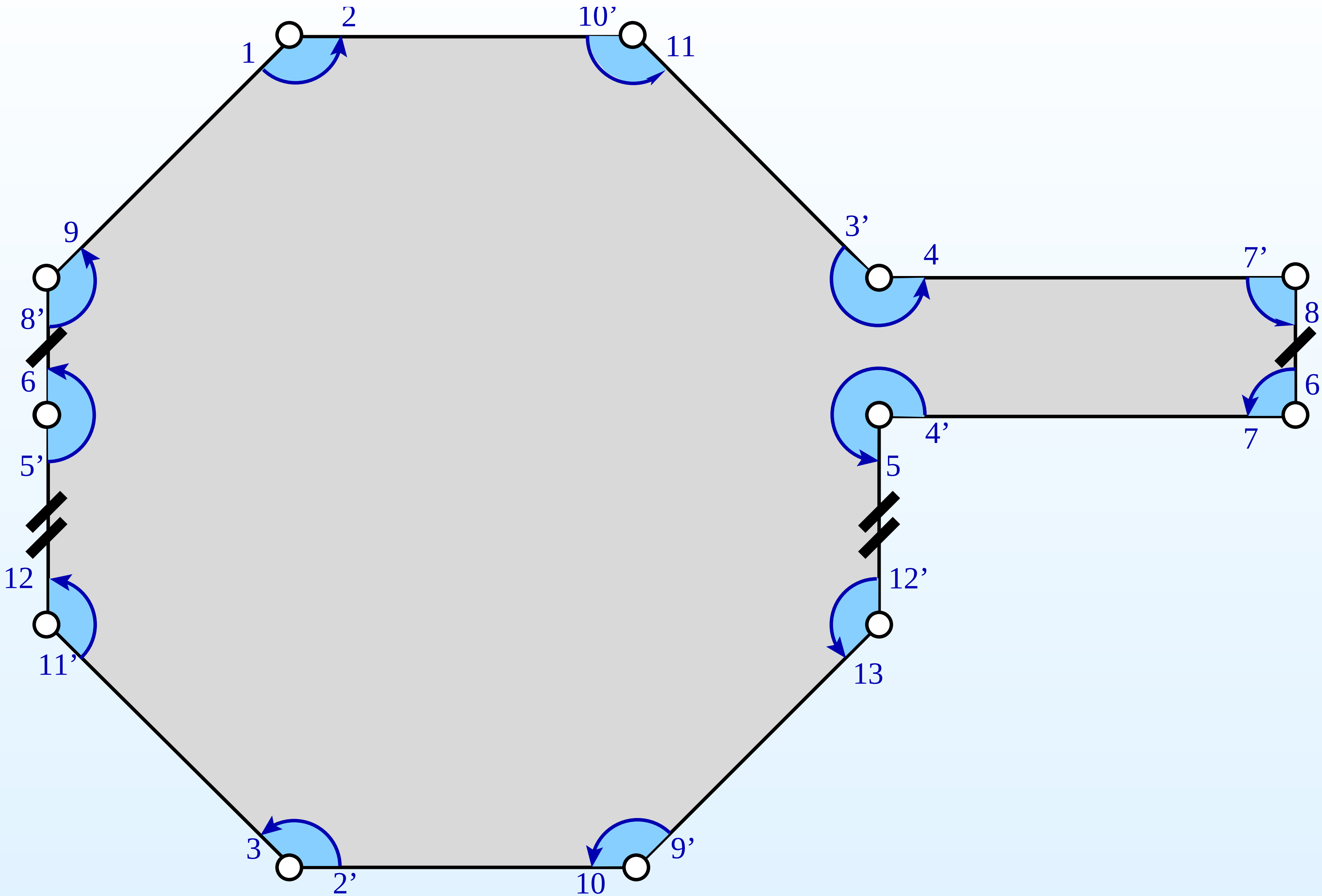

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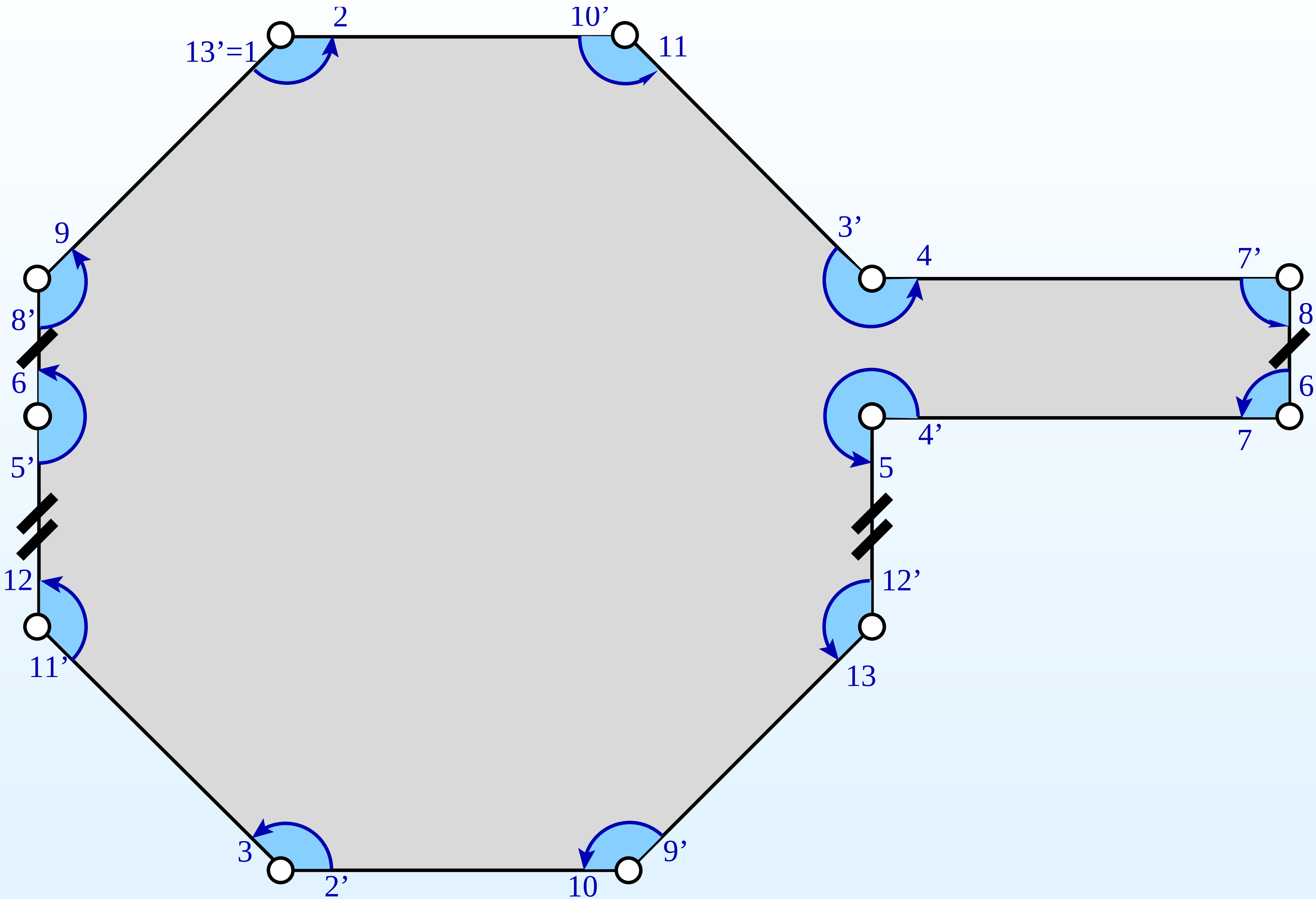
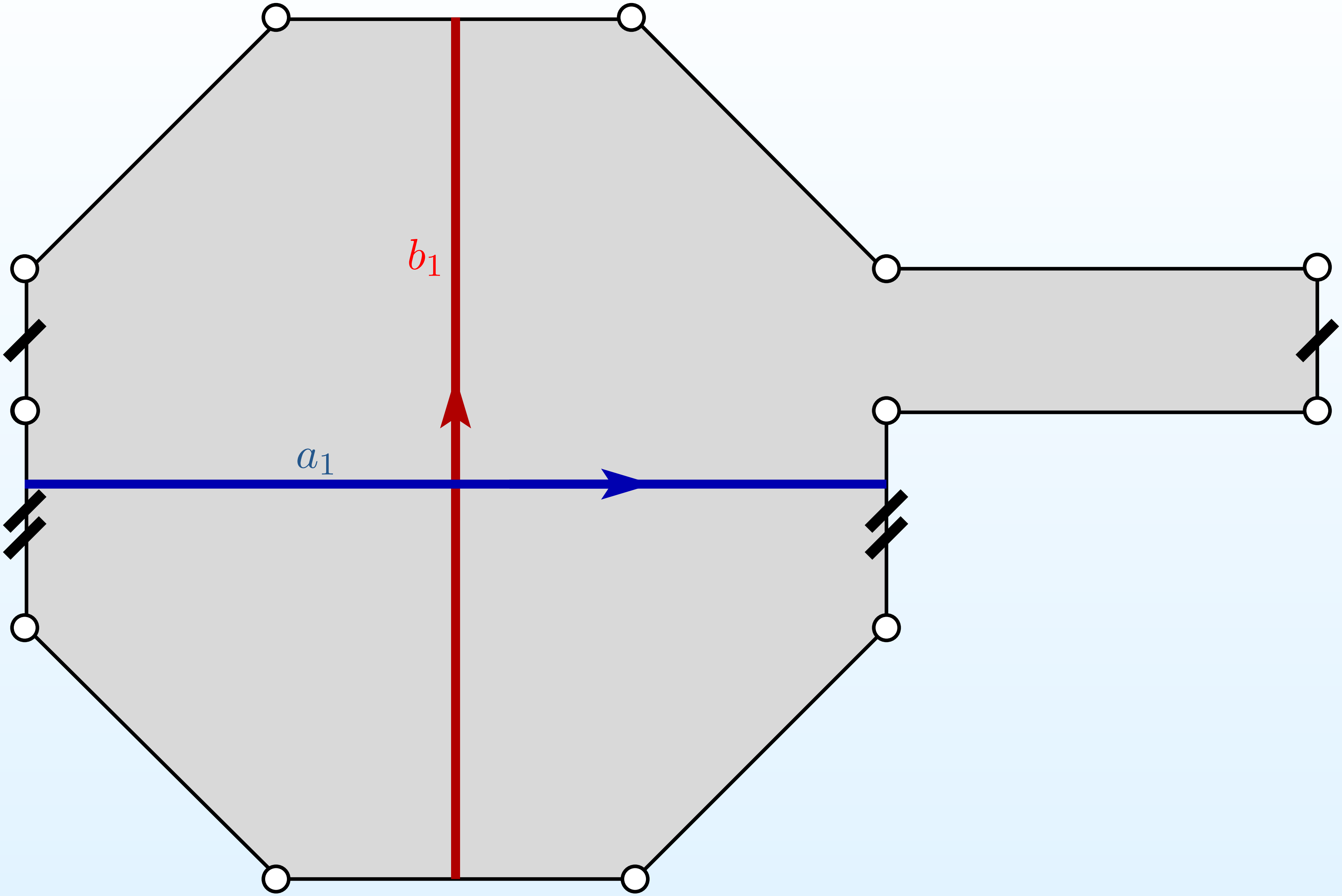


- Construction of a canonical basis of cycles.



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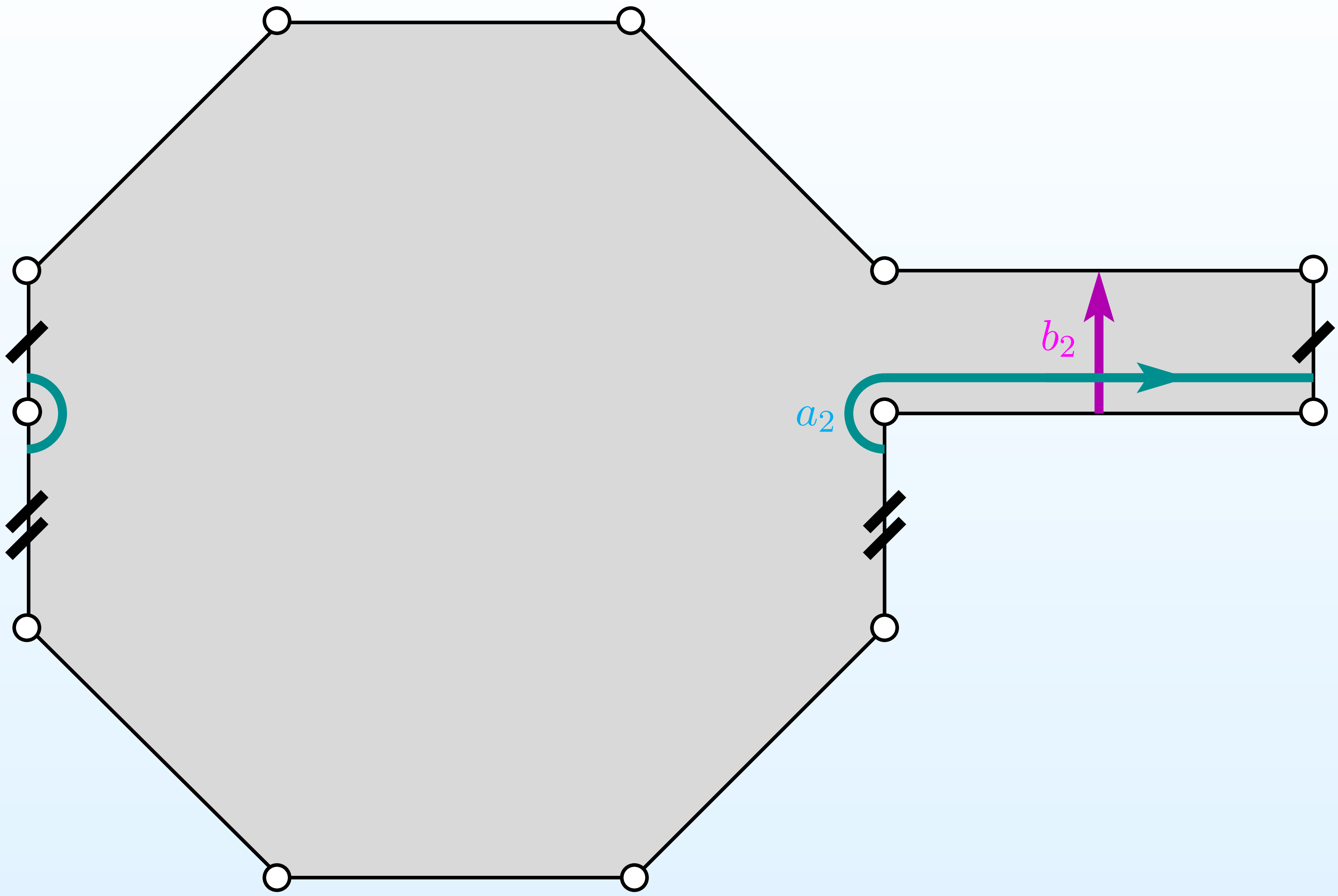

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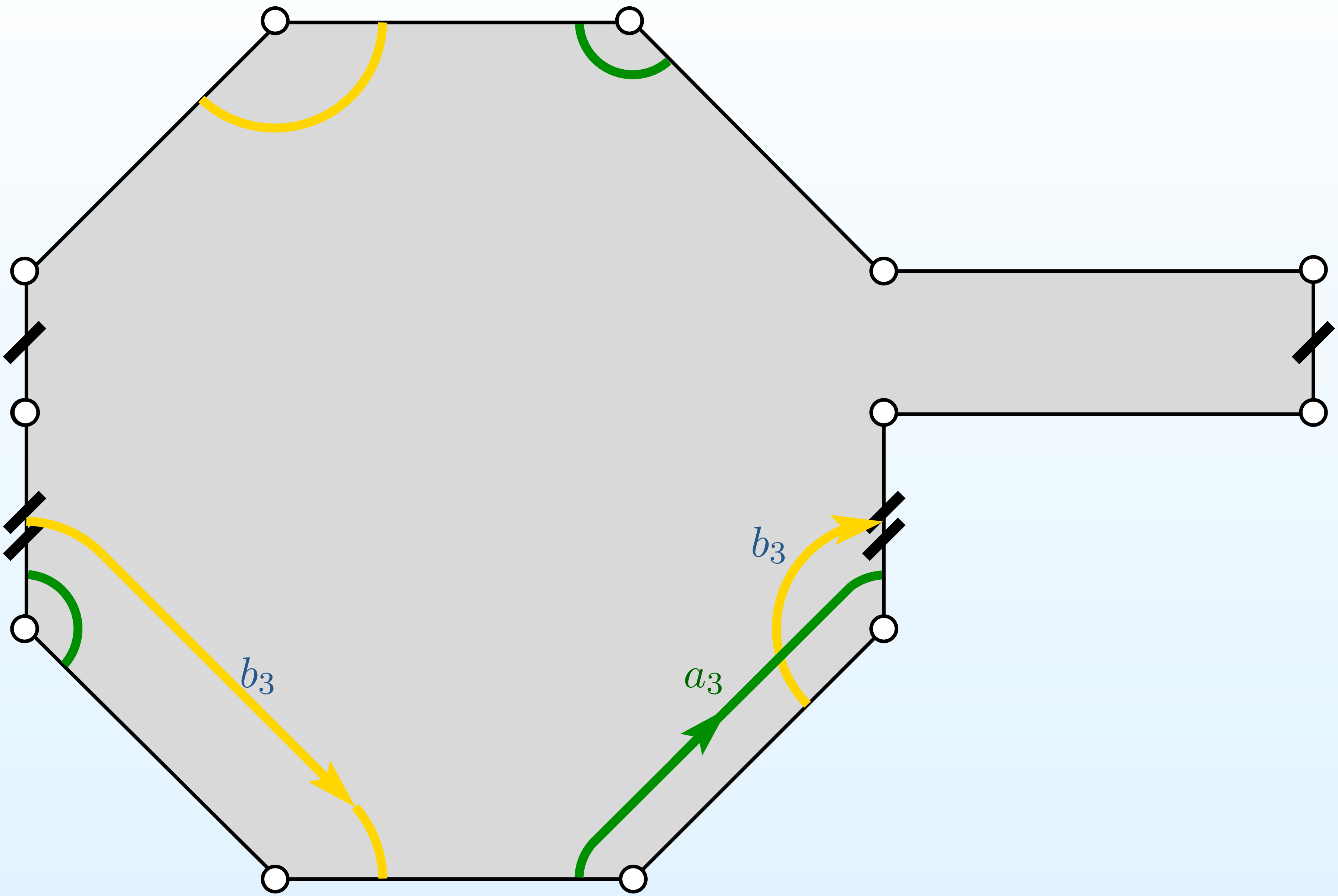

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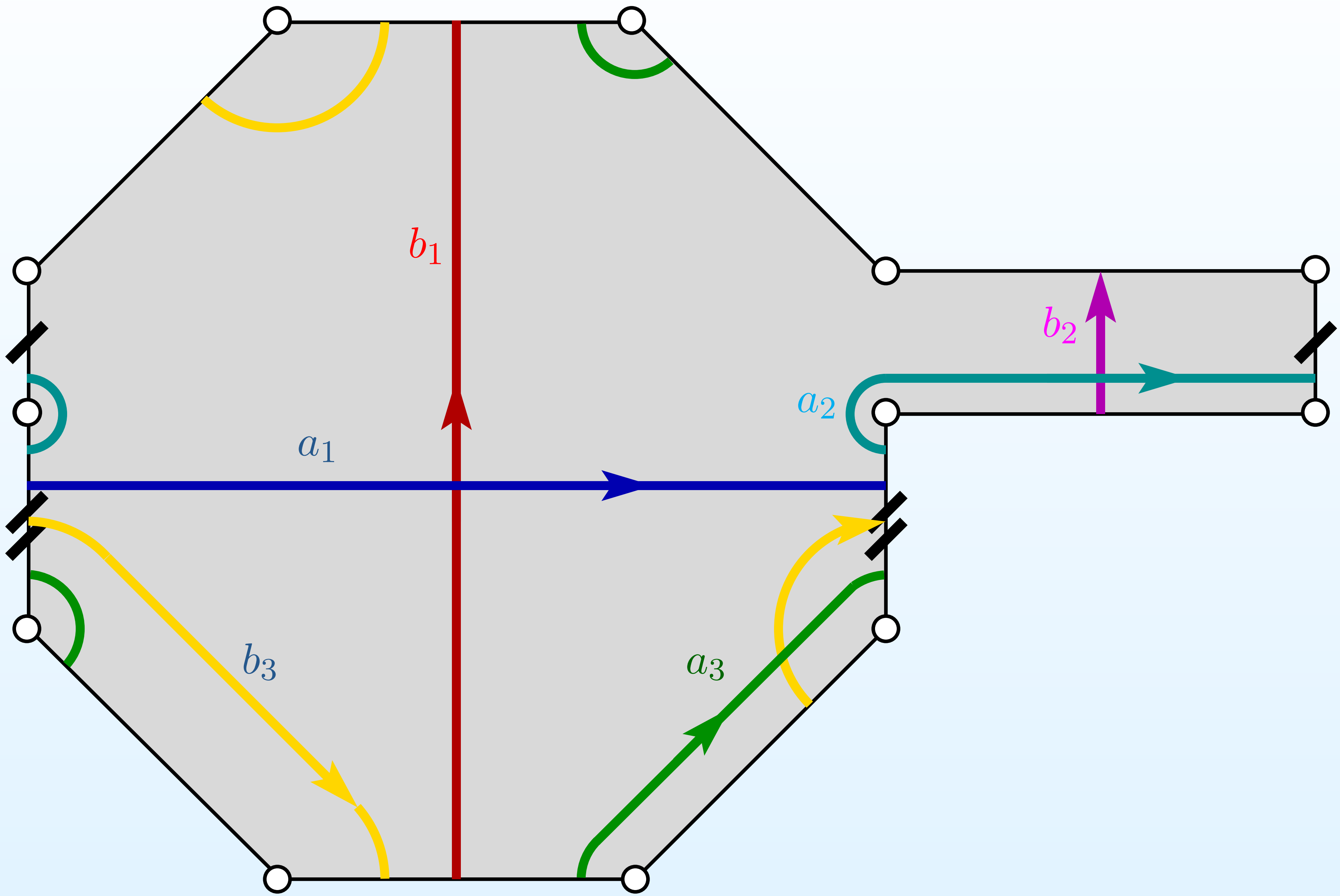
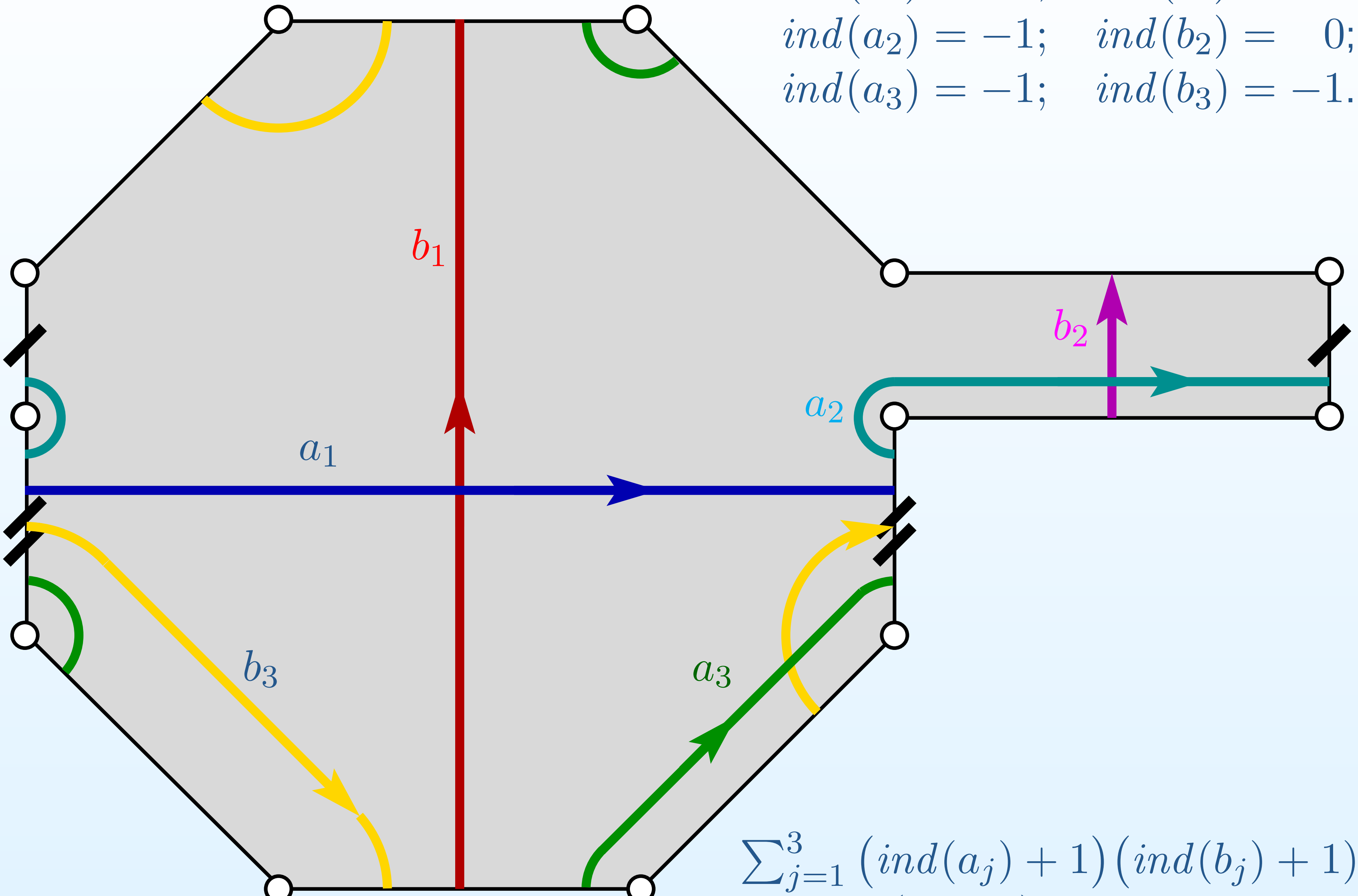


- Evaluation of the parity of a spin-structure.

$$ind(a_1) = 0; \quad ind(b_1) = 0;$$
$$ind(a_2) = -1; \quad ind(b_2) = 0;$$
$$ind(a_3) = -1; \quad ind(b_3) = -1.$$



$$\sum_{j=1}^{3} \big(ind(a_j)+1\big)\big(ind(b_j)+1\big) = 1 (\mathrm{mod}\, 2)$$

Spin-structure of minimal hyperelliptic components

Theorem (M. Kontsevich, A. Zorich, 2003) *Parity of the spin structure determined by an Abelian differential from the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ equals*

$$\varphi(\mathcal{H}^{hyp}(2g-2)) \equiv \left[\frac{g+1}{2}\right] \pmod 2 \,.$$

In particular,

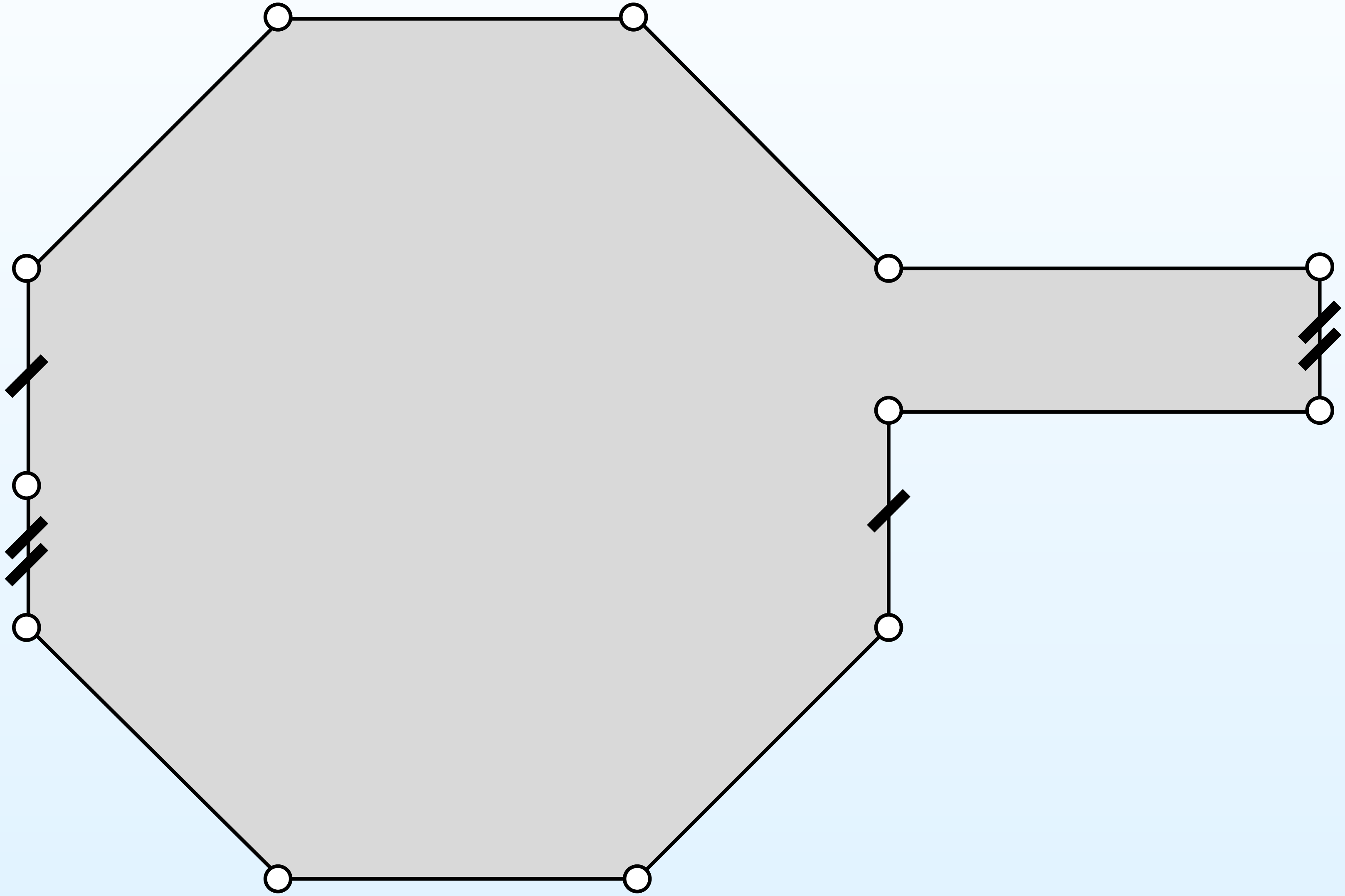
$$\varphi(\mathcal{H}^{hyp}(0)) = 1 \qquad \varphi(\mathcal{H}^{hyp}(2)) = 1 \qquad \varphi(\mathcal{H}^{hyp}(4)) = 0$$

and we conclude that the surface constructed above lives in $\mathcal{H}^{odd}(4)$.

- Find the hyperelliptic involution of the remaining surface in geometric terms.

Find the Weierstrass points (the fixed points of the hyperelliptic involution).

Check that there are $2g + 2 = 2 \cdot 3 + 2 = 8$ such points.



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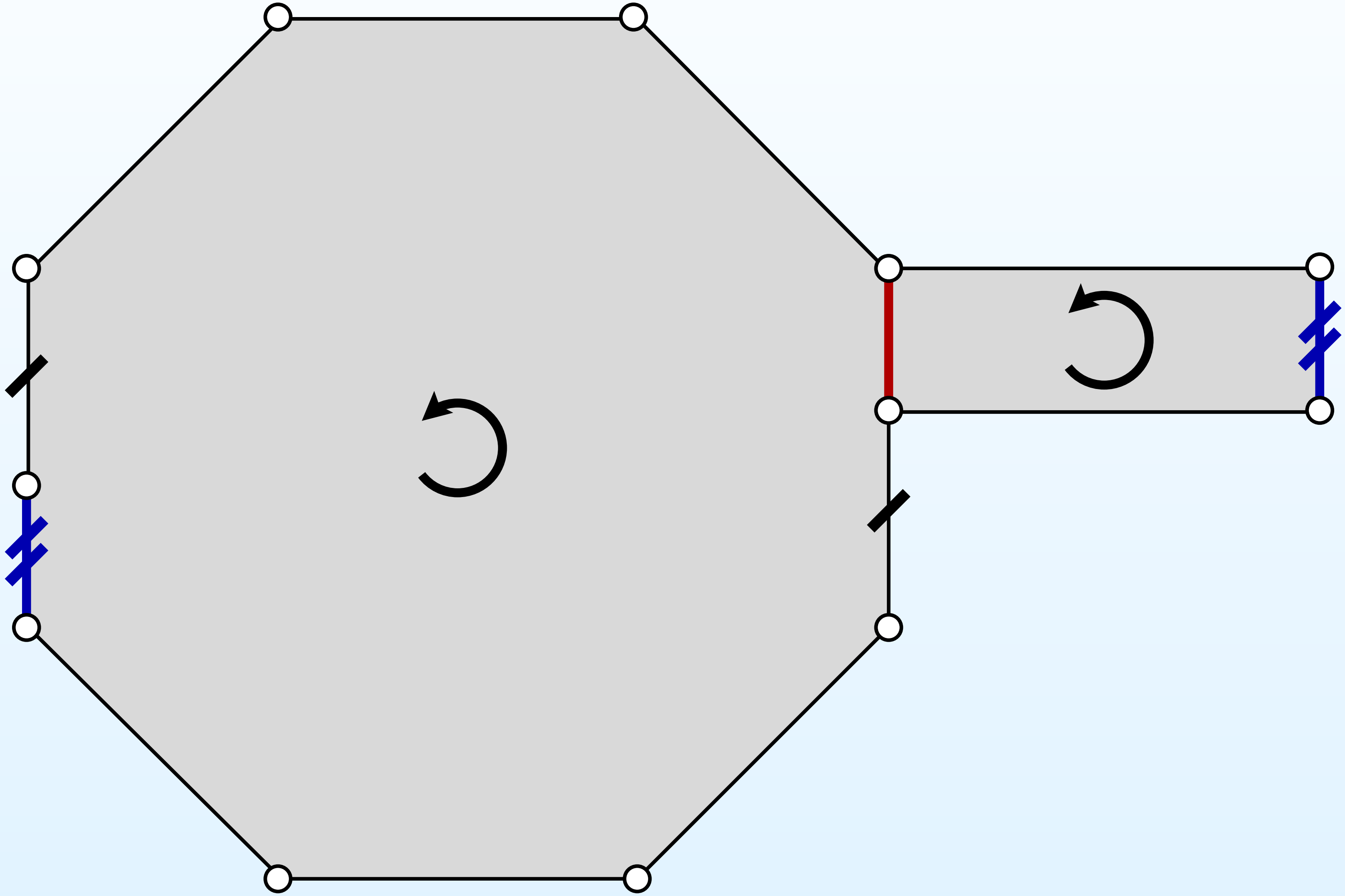

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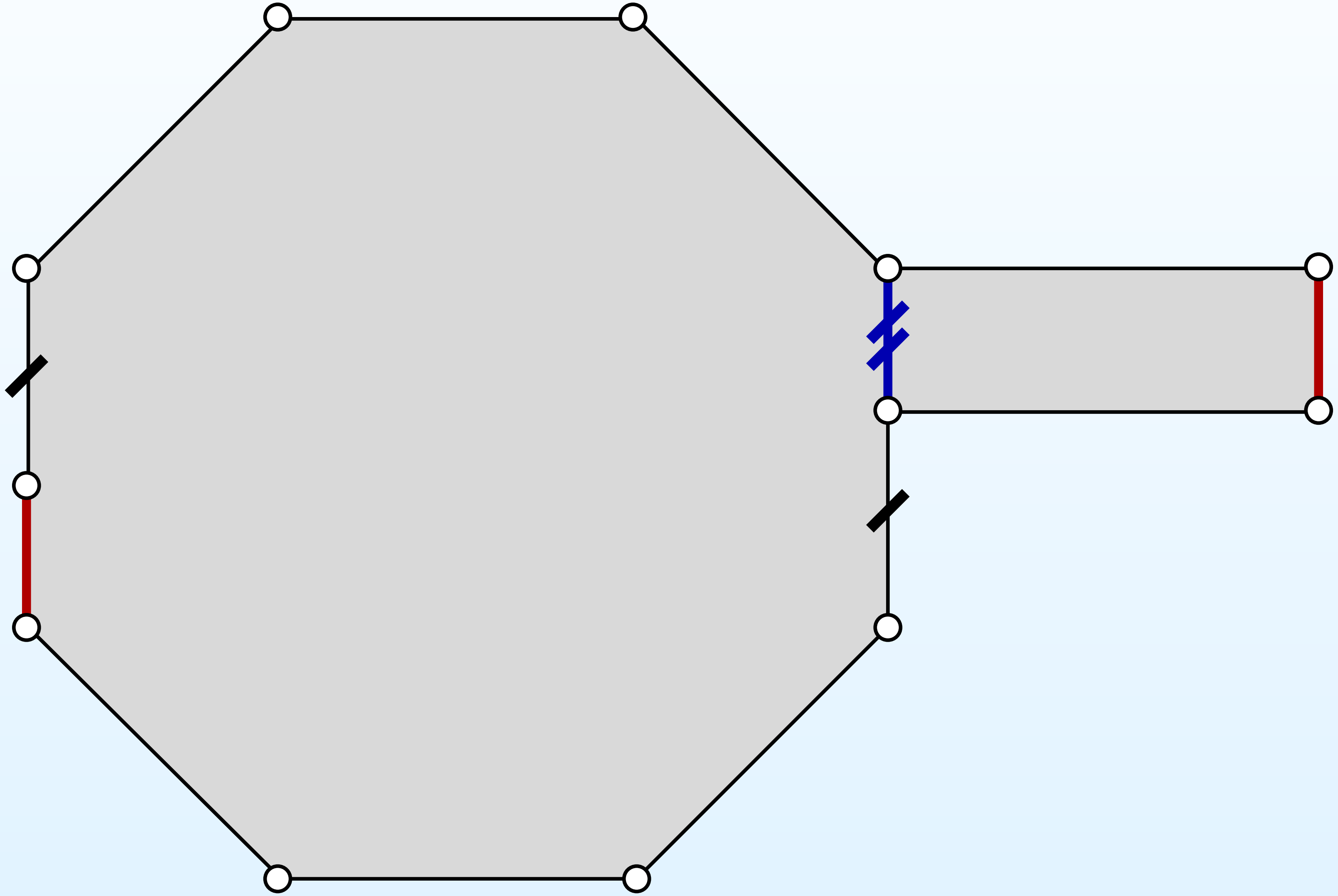

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