Integration on moduli spaces

Notes for the course: "Panorama of geometry and dynamics of moduli spaces"

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The goal of this lecture is to explain Mirzakhani's integration formula for the Weil–Petersson volume form on moduli spaces of hyperbolic surfaces [Mir07].

1. Spaces of lattices

Before we get to this, we will discuss integration over spaces of lattices and Siegel's integration formula. We do this because this formula shares many similarities with Mirzakhani's integration formula, except it's about spaces of lattices. Siegel's integration formula has a further analogue in the (two-dimensional) flat world [Vee98, Theorem 0.5.I] that we shall not discuss today.

First of all, recall that a **lattice** $\Lambda < (\mathbb{R}^n, +)$ is a discrete subgroup of finite covolume (the volume of $\mathbb{R}^n/\Lambda$). Another way of saying the same thing is to say it's a subgroup of the form

$$\operatorname{span}_{\mathbb{Z}}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} m_i \, v_i \, : \, m_i \in \mathbb{Z}, \; i = 1, \ldots, n \right\},$$

where $(v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$. If $\Lambda < \mathbb{R}^n$ is a lattice, then $\mathbb{R}^n/\Lambda$ is an n-dimensional torus, that comes with a natural metric, which descends from the standard Euclidean metric on $\mathbb{R}^n$. The volume of this metric equals

$$\operatorname{vol}(\mathbb{R}^n/\Lambda) = \left|\det(v_1, \ldots, v_n)\right|.$$

As such the space of n-dimensional lattices of covolume 1 can be identified with

$$X_n = \operatorname{SL}(n, \mathbb{R}) / \operatorname{SL}(n, \mathbb{Z}).$$

Where the point $[A] \in X_n$ represents the latice $A \cdot \mathbb{Z}^n$.

Since $\operatorname{SL}(n, \mathbb{R})$ is a Lie group (an example of a locally compact group), it comes with a **Haar measure** – a Borel measure that is invariant under right-multiplication – that we shall denote μ. This measure is unique up to scaling. We will normalize it such that $\operatorname{vol}(X_n) = 1$. Note that the latter quantity if well defined by multiplication invariance. Moreover, it turns out that, in the special case of $\operatorname{SL}(n, \mathbb{R})$, μ is also left-invariant.

Given a Lebesgue integrable function $f : \mathbb{R}^n \to \mathbb{R}$, its **Siegel transform** is the function $S(f) : X_n \to \mathbb{R}$ given by

$$S(f)([A]) = \sum_{v \in \mathbb{Z}^n - \{0\}} f(A \cdot v)$$

We can think of $\mathbb{Z}^n$ both as the fundamental group of the n-torus and as the set of free homotopy classes of non-trivial closed curves in the n-torus. This means that the sum appearing on the right can be seen as the sum over all homotopy classes of closed geodesics in the torus $A \cdot \mathbb{Z}^n$. So, if we for example set $f(v) = \chi_{[0,L]}(||v||)$, where $\chi_{[0,L]}$ denotes the indicator function of the interval $[0, L]$, then f counts the number of homotopy classes of closed geodesics (or lattice vectors) of length at most L on the the torus (or lattice) associated to $[A] \in X_n$.

Siegel's integral formula [Sie45] tells us how to integrate such functions over the moduli space of lattices:

Theorem 1.1 (Siegel)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue integrable. Then*

$$\int_{X_n} S(f)([A])\, d\mu([A]) = \int_{\mathbb{R}^n} f(x)dx.$$

We can also specialize to a specific type of geodesics on a torus. For example, a vector $v \in \mathbb{Z}^n$ is called **primitive** if the greatest common divisor of its coordinates equals 1. Primitive vectors form a single $\mathrm{SL}(n, \mathbb{Z})$-orbit on $\mathbb{Z}^n$. In topological terms, $\mathrm{SL}(n, \mathbb{Z})$ can be identified with the **mapping class group** of the n-torus: the group of homotopy classes of orientation preserving self-diffeomorphisms of $\mathbb{R}^n/\mathbb{Z}^n$. The **primitive Siegel transform** of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is the function $S_p(f) : X_n \to \mathbb{R}$ given by

$$S_p(f)([A]) = \sum_{v \in \mathbb{Z}^n \text{ primitive}} f(A \cdot v)$$

The primitive integration formula reads:

Theorem 1.2 (Siegel)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue integrable. Then*

$$\int_{X_n} S_p(f)([A])\, d\mu([A]) = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f(x)dx,$$

where $\zeta : \mathbb{C} - \{1\} \to \mathbb{C}$ denotes the Riemann ζ-function.

As a consequence of this formula, we can for instance count the number of short vectors in a random lattice of covolume 1. For instance, write $N_L^p : X_n \to \mathbb{N}$ for the random variable that counts the number of primitive vectors of length at most L in a lattice. Then, because $N_L^p = S_p(\chi_{B(0,L)})$, where $\chi_{B(0,L)}$ denotes the indicator function of the ball of radius L around the origin in $\mathbb{R}^n$, we obtain

$$\mathbb{E}(N_L^p) = \frac{1}{\zeta(n)} \operatorname{vol}(B(0, L)).$$

2. Hyperbolic surfaces

Now we get to our main goal today: proving Mirzakhani's integration formula for the Weil–Petersson volume form on moduli spaces of hyperbolic surfaces with boundary. A priori, these spaces are nothing like the spaces X_n, they're not at all locally homogeneous, nonetheless the formula one obtains is remarkably similar to Siegel's formula. It also has applications to the study of random hyperbolic surfaces (see for instance [Mir13]) that we won't get to today and is also used as a tool in Mirzakhani's counting of the number of short simple closed geodesics [Mir08], which will be discussed later on in the course.

2.1. **Symplectic forms on moduli spaces of surfaces with boundary.** We start by recalling some definitions and generalizing them to surfaces with boundary. From now on we will take the hyperbolic perspective. In what follows Σ_g and $\Sigma_{g,n}$ will denote a closed oriented surface of genus g and a compact oriented surface of genus g with n boundary components respectively. The boundary components of $\Sigma_{g,n}$ will be denoted $\beta_1, \ldots \beta_n$

Given $L = (L_1, \ldots, L_n)$, the **Teichmüller space** $\mathcal{T}_{g,n}(L)$ is defined as

$$\mathcal{T}_{g,n}(L) = \left\{ (X,f) \; : \; \begin{array}{c} X \text{ a hyperbolic surface with totally geodesic boundary,} \\ f : \Sigma_{g,n} \to X \text{ an orientation preserving diffeomorphism} \\ \text{such that } f(\beta_i) \text{ has length } L_i \text{ for } i = 1, \ldots, n \end{array} \right\} \Big/ \sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if and only if there exists an isometry $\varphi : X_1 \to X_2$ such that $f_2^{-1} \circ \varphi \circ f_1 : \Sigma_{g,n} \to \Sigma_{g,n}$ is homotopic to the identity. We allow $L_i = 0$, in which case we require the surface to have a cusp. The **mapping class group**

$$\mathrm{MCG}(\Sigma_{g,n}) = \left\{ \varphi : \Sigma_{g,n} \to \Sigma_{g,n} \; : \; \begin{array}{c} \varphi \text{ an orientation preserving} \\ \text{diffeomorphism such that} \\ \varphi(\beta_i) = \beta_i, \; i = 1, \ldots, n \end{array} \right\} \Big/ \text{homotopy}$$

acts on $\mathcal{T}_{g,n}(L)$ by $[\varphi] \cdot [X, f] = [X, f \circ \varphi^{-1}]$. The quotient is the **moduli space**

$$\mathcal{M}_{g,n}(L) = \mathcal{T}_{g,n}(L) / \, \mathrm{MCG}(\Sigma_{g,n}).$$

Given a pants decomposition $\mathcal{P} = \{\alpha_1, \ldots, \alpha_{3g+n-3}\}$ of $\Sigma_{g,n}$, we obtain global **Fenchel–Nielsen coordinates** $(\ell_i, \tau_i)_{i=1}^{3g+n-3}$ on $\mathcal{T}_{g,n}(L)$. Here the ℓ_i's and τ_i's denote the lengths and twists of the α_i's respectively. Goldman [Gol84] proved that these Teichmüller spaces (and many more spaces like them) come with a $\mathrm{MCG}(\Sigma_{g,n})$-invariant symplectic form ω_{WP} and Wolpert [Wol82] proved that Fenchel–Nielsen coordinates are Darboux coordinates for it. That is

$$\omega_{\mathrm{WP}} = \frac{1}{2} \sum_{\alpha \in \mathcal{P}} d\ell_\alpha \wedge d\tau_\alpha.$$

This is often called **Wolpert's magical formula**. The letters WP here stand for Weil–Petersson, because the form coincides (again by work of Wolpert) with the symplectic form coming from the Weil–Petersson Kähler form whenever the latter is defined.

Because the form ω_{WP} is mapping class group invariant, it descends to $\mathcal{M}_{g,n}(L)$.

2.2. **The volume form.** Our goal is to integrate functions on $\mathcal{M}_{g,n}(L)$ with respect to the volume form [1]:

$$d\,\mathrm{vol}_{\mathrm{WP}} = \frac{2^{3g-3}}{(3g-3)!} \wedge^{3g-3} \omega_{\mathrm{WP}}$$

The most natural thing to try would of course be to find a nice fundamental domain for the $\mathrm{MCG}(\Sigma_g)$-action on $\mathcal{T}_g$ and try to integrate $d\,\mathrm{vol}_{\mathrm{WP}}$ over it. However, it turns out that describing such a fundamental domain is a really hard problem. So, we need something else, which is where Mirzakhani's integration formula comes in.

[1] We are actually skipping over a little issue when we say this: like X_n above, $\mathcal{M}_{g,n}(L)$ is *not* a manifold, it is only an orbifold. In particular, it does not have a well-defined tangent space at each point. There are multiple ways out of this, that we will discuss when the issue comes up.

2.3. **Finiteness of total volume.** Before we get to this, we first sketch a proof that the toal volume of moduli space is finite. This is not immediately clear, because $\mathcal{M}_g$ is not compact. However, it follows from Wolpert's magical formula together with a theorem by Bers:

Theorem 2.1. *Let $g \geq 2$. Then*

$$\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g) < \infty.$$

Proof sketch. Bers [Ber74] proved that there exists a constant[2] $B_g > 0$ so that every closed orientable surface of genus g admits a pants decomposition in which all curves have length at most B_g.

There are finitely many $\mathrm{MCG}(\Sigma_g)$-orbits of (topological) pants decompositions of Σ_g. Let $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ be a set of representatives of these orbits (one representative per orbit). Then

$$\bigcup_{i=1}^{k} \{\, X \in \mathcal{T}_g \,:\, \ell_\alpha(X) \leq B_g, 0 \leq \tau_\alpha(X) \leq \ell_\alpha(X) \;\forall \alpha \in \mathcal{P}_i \,\}$$

contains a fundamental domain for the mapping class group. On the other hand, it is finite a collection of bounded sets in $\mathcal{T}_g$ and hence has finite volume. □

Since there are estimates on B_g, the proof above can even be improved in order to bound V_g from above. For us, it just indicates that the quest for the value of V_g is not pointless.

2.4. **The Mirzakhani transform.** In this section we will define the hyperbolic analogue of the Siegel transform. This will yield a specific type of functions on $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ that Mirzakhani called geometric functions.

Definition 2.2 (The Mirzakhani transform)**.** Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L_1, \ldots, L_n \in \mathbb{R}_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Finally, let $f : \mathbb{R}_+^k \to \mathbb{R}$ be a function. Then we define the **Mirzakhani transform** of f to be the function

$$M_\Gamma(f) : \mathcal{M}_{g,n}(L_1, \ldots, L_n) \to \mathbb{R}$$

given by

$$M_\Gamma(f)(X) = \sum_{(\alpha_1, \ldots, \alpha_k) \in \mathrm{MCG}(\Sigma_{g,n}) \cdot \Gamma} f(\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X)).$$

Like in the case of the Siegel transform, even though the individual length functions are not well defined on $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$, the sum above is.

[2] What the value of B_g is, is an open problem. It is not even clear at which rate it grows as a function of g. The best known lower bound on this rate is $\approx \sqrt{g}$ [Bus10, Chapter 5] and the best known upper bound is $\approx g$ [Par14, Par24]

Example 2.3. Consider an non-separating simple closed curve α on $\Sigma_{g,n}$ (like the curve in Figure 1). Moreover, let $\chi_{[a,b]} : \mathbb{R}_+ \to \mathbb{R}$ denote the characteristic function of the interval $[a,b]$. That is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$.

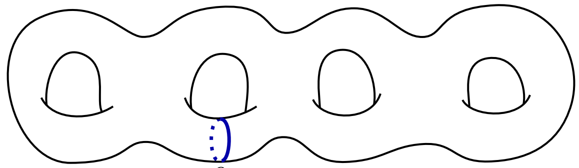

FIGURE 1. A non-separating simple closed curve α

First of all note that the orbit $\mathrm{MCG}(\Sigma_{g,n}) \cdot \alpha$ is the set of all non-separating simple closed curves on $\Sigma_{g,n}$. So

$$M_\alpha(\chi_{[a,b]})(X) = \sum_{\substack{\gamma \text{ a non-separating simple} \\ \text{closed geodesic on } X}} \chi_{[a,b]}(\ell_\gamma(X)) = \# \left\{ \begin{array}{c} \text{non-separating simple} \\ \text{closed geodesics on } X \text{ with} \\ \text{length in } [a,b] \end{array} \right\}$$

for all $X \in \mathcal{M}_{g,n}(L_1, \ldots, L_n)$.

2.5. **Mirzakhani's integration formula.** We are now ready to state Mirzakhani's integration formula. We first introduce some terminology.

Definition 2.4. Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L_1, \ldots, L_n \in \mathbb{R}_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Define

$$\mathrm{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma) = \bigcap_{i=1}^{k} \mathrm{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i).$$

and let $\mathrm{Stab}^+_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i)$ denote the subgroup of the mapping class group that also preserves the left and right hand side of γ_i, or equivalently, an orientation on it. The **Symmetry group** of Γ is the group

$$\mathrm{Sym}(\Gamma) = \mathrm{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma) \Big/ \bigcap_{i=1}^{k} \mathrm{Stab}^+_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i).$$

Moreover, we set

$$t(\Gamma) = \#\{\text{connected components of } \Sigma_{g,n} \setminus \Gamma \text{ that are homeomorphic to } \Sigma_{1,1}\}.$$

and for $x \in [0,\infty)^k$, we will denote the moduli space of hyperbolic metrics on $\Sigma_{g,n} \setminus \Gamma$ whose boundary lengths are given by L and (x,x) for the boundary components coming from Γ, by $\mathcal{M}(\Sigma_{g,n} \setminus \Gamma, L, x, x)$.

We now have:

Theorem 2.5 (Mirzakhani's integration formula)**.** *Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L \in \mathbb{R}^k_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Finally, let $f : \mathbb{R}^k_+ \to \mathbb{R}$ be integrable. Then*

$$\int\limits_{\mathcal{M}_{g,n}(L)} M_\Gamma(f)(X)\, d\,\mathrm{vol}_{\mathrm{WP}}(X)$$

$$= C_\Gamma \int\limits_{\mathbb{R}^k_+} f(x)\,\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma_{g,n} \setminus \Gamma, L, x, x) \cdot x_1 \cdot x_2 \cdots x_k \, dx_1 \cdots dx_k,$$

where C_Γ is a constant. If $(g, n) \notin \{(1, 1), (2, 0)\}$ then

$$C_\Gamma = \frac{1}{2^{t(\Gamma)} \# \mathrm{Sym}(\Gamma)}.$$

2.6. **Stabilizers, moduli spaces and level sets.** Before we derive Mirzakhani's integration formula, we first introduce some more moduli spaces that will be used in the proof.

Definition 2.6. Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L_1, \ldots, L_n \in \mathbb{R}_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Define

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma = \mathcal{T}_{g,n}(L_1, \ldots, L_n) / \operatorname{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma).$$

Furthermore, we will write

$$\pi^\Gamma : \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma \to \mathcal{M}_{g,n}(L_1, \ldots, L_n)$$

for the projection map.

In words: in $\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma$, the boundary components *and* the curves in Γ are still marked. Informally, we can also write

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma = \left\{ (X, \eta) \; : \; \begin{array}{c} X \in \mathcal{M}_{g,n}(L_1, \ldots, L_n), \\ \eta \in \mathcal{M}(\Sigma_{g,n}) \cdot \Gamma \text{ realized by closed geodesics} \end{array} \right\}.$$

In particular, the length functions of the curves in Γ are well defined on $\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma$. As such we can make sense of level sets for the length functions on Γ:

Definition 2.7. Given $a \in \mathbb{R}^k_+$, we define

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma[a] := \left\{ (X, \eta) \in \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma \; : \; \ell_{\eta_i}(X) = a_i, \; i = 1, \ldots, k \right\}.$$

Because ω_{WP} is $\mathrm{MCG}(\Sigma_{g,n})$ invariant, $\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma$ also comes with a symplectic form. The level sets above come with k natural Hamiltonian flows

$$\phi_i^t : \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma[a] \to \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma[a],$$

obtained by twisting around the i^{th} curve in Γ. The fact that this flow is Hamiltonian is direct from Wolpert's magical formula. Note that this flow induces an action of a k-dimensional torus $\mathbb{T}^k$ on $\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]$. Let us write

$$\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^* = \mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]/\mathbb{T}^k.$$

Since the flows are Hamiltonian, they preserve the symplectic form and hence $\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^*$ comes with a symplectic form too.

Now note that we have a natural map

$$\mathcal{M}(\Sigma_{g,n}\setminus\Gamma, L_1,\ldots,L_n,a,a) \to \mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^*$$

obtained by "gluing Γ together". The reason that a appears twice is that every curve in Γ gives rise to two boundary components. This map is

$$\left[\mathrm{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma) : \bigcap_{i=1}^{k}\mathrm{Stab}^+_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i)\right] \text{ - to - } 1,$$

because of the fact that boundary components are marked. Here $\mathrm{Stab}^+_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i)$ denotes the subgroup of the mapping class group that also preserves the left and right hand side of γ_i, or equivalently, an orientation on it.

So, the identification we spoke about in the beginning of this section is:

Lemma 2.8. *The map*

$$\mathcal{M}(\Sigma_{g,n}\setminus\Gamma, L_1,\ldots,L_n,a,a) \to \mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^*$$

locally preserves $d\,\mathrm{vol}_{\mathrm{WP}}$.

Proof. This is direct from Wolpert's magical formula. □

2.7. Proof of the integration formula.

Proof. First of all, let us define a function

$$\widehat{f} : \mathcal{M}_{g,n}(L)^\Gamma \to \mathbb{R}$$

by

$$\widehat{f}(X,\eta) = F(\ell_{\eta_1}(X),\ldots,\ell_{\eta_k}(X))$$

for all $(X,\eta)\in\mathcal{M}_{g,n}(L)$. We then have

$$M_\Gamma(f)(X) = \sum_{(X,\eta)\in(\pi^\Gamma)^{-1}(X)} \widehat{f}(X,\eta)$$

and hence

$$\int\limits_{\mathcal{M}_{g,n}(L)} M_\Gamma(f)(X)\, d\operatorname{vol}_{\mathrm{WP}}(X) = \int\limits_{\mathcal{M}_{g,n}(L)} \sum_{(X,\eta)\in(\pi^\Gamma)^{-1}(X)} \widehat{f}(X,\eta) d\operatorname{vol}_{\mathrm{WP}}(X)$$

$$= \int\limits_{\mathcal{M}_{g,n}(L)^\Gamma} \widehat{f}(X,\eta) d\operatorname{vol}_{\mathrm{WP}}(X,\eta)$$

$$= \int_{\mathbb{R}_+^k} \int\limits_{\mathcal{M}_{g,n}(L)^\Gamma[a]} \widehat{f}(a_1,\dots a_k) d\operatorname{vol}_{\mathrm{WP}}(X,\eta) da_1\cdots da_k$$

$$= \int_{\mathbb{R}_+^k} \widehat{f}(a_1,\dots a_k)\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]) da_1\cdots da_k.$$

Now we claim that

$$\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]) = 2^{-M(\Gamma)} a_1\cdots a_k \cdot \operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]^*).$$

The reason for this is that generically,

$$\varphi_i^t(X,\eta) \neq (X,\eta) \text{ for all } 0 < t < \ell_{\gamma_i}(X).$$

On the other hand, this does describe the whole fibre of the quotient map

$$\mathcal{M}_{g,n}(L)^\Gamma[a] \to \mathcal{M}_{g,n}(L)^\Gamma[a]^*.$$

The only exception is if γ_i cuts off a one-holed torus. Every one holed torus has an order 2 symmetry. As such we only need to go up to $a_i/2$ in order to describe the whole fibre. Note that if $\Sigma\setminus\Gamma$ consists of pairs of pants, then the only deformations of the elements in $\mathcal{M}_{g,n}(L)^\Gamma[a]$ are twists along the curves in Γ. As such, we need to set

$$\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{0,3}(L_1,L_2,L_3) = 1$$

in order to make our claim work.

All in all we get

$$\int\limits_{\mathcal{M}_{g,n}(L)} M_\Gamma(f)(X) d\operatorname{vol}_{\mathrm{WP}}(X)$$

$$= 2^{-t(\Gamma)} \int_{\mathbb{R}_+^k} \widehat{f}(a_1,\dots a_k)\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]^*) a_1\cdots a_k \cdot da_1\cdots da_k$$

$$= \frac{1}{2^{t(\Gamma)}\operatorname{Sym}(\Gamma)} \int_{\mathbb{R}_+^k} \widehat{f}(a_1,\dots a_k)\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma_{g,n}\setminus\Gamma, L, a, a)) a_1\cdots a_k \cdot da_1\cdots da_k,$$

using Lemma 2.8. □

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