Dynamics and Geometry of Moduli Spaces

Lecture 9. Masur–Veech volumes
Square-tiled surfaces

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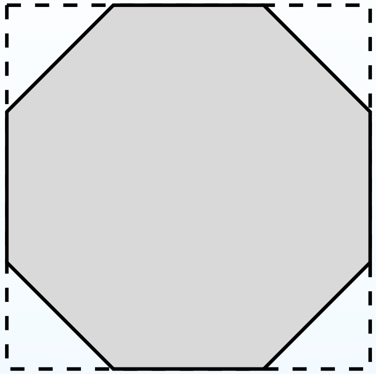
Masur–Veech volumes of the moduli spaces of Abelian differentials. Square-tiled surfaces

Reminder: translation surface of genus two



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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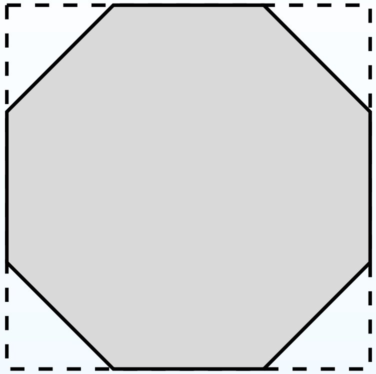

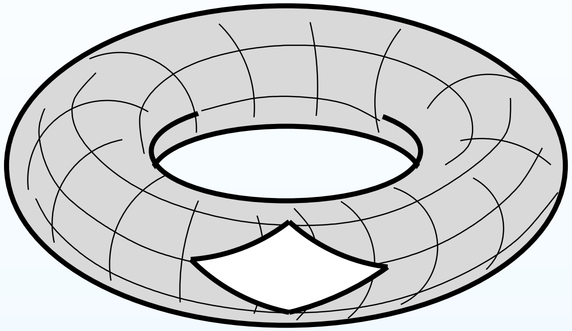

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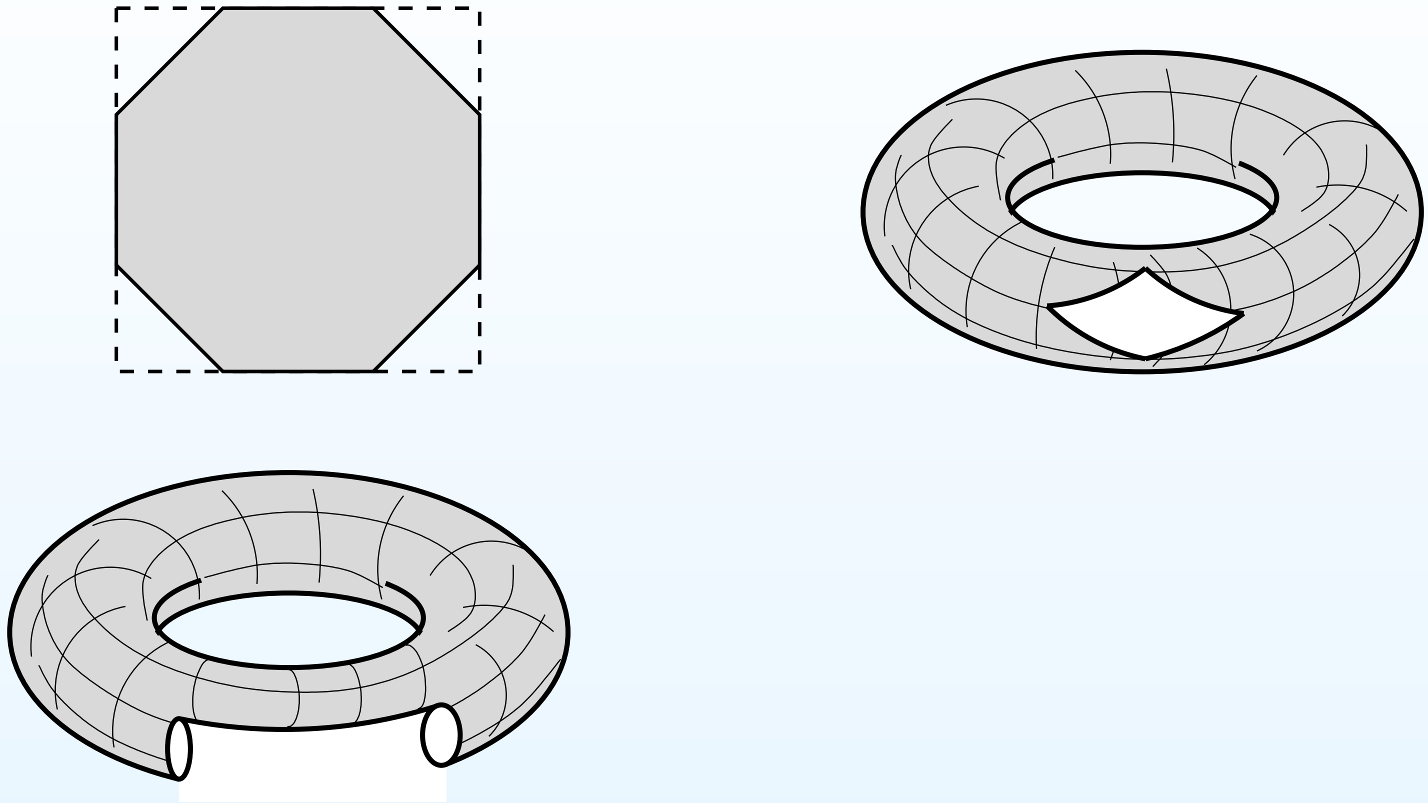

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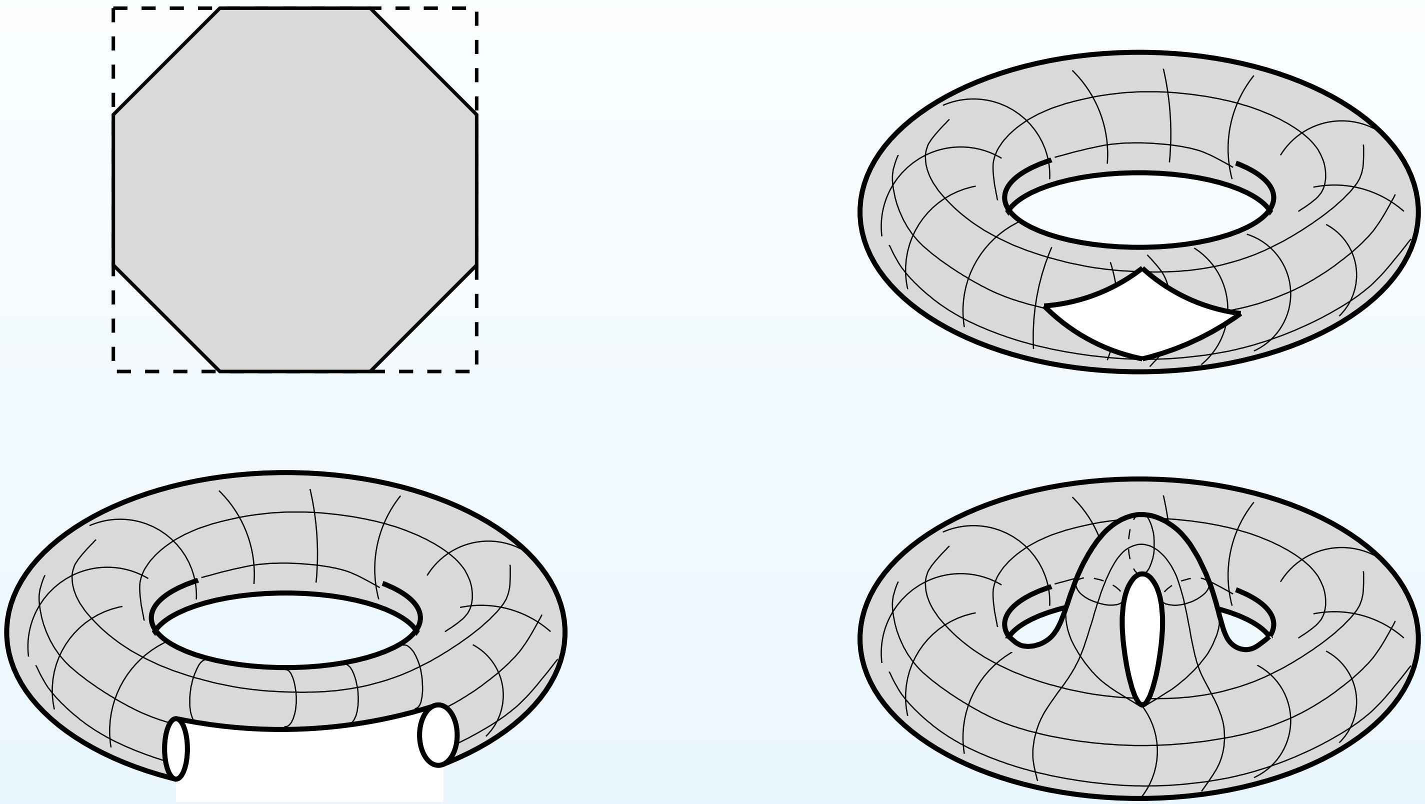

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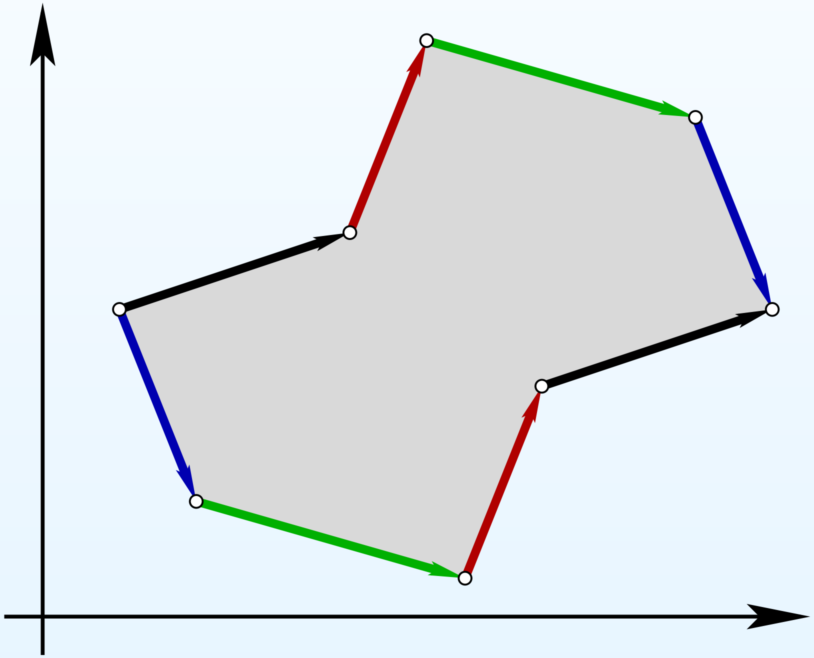
Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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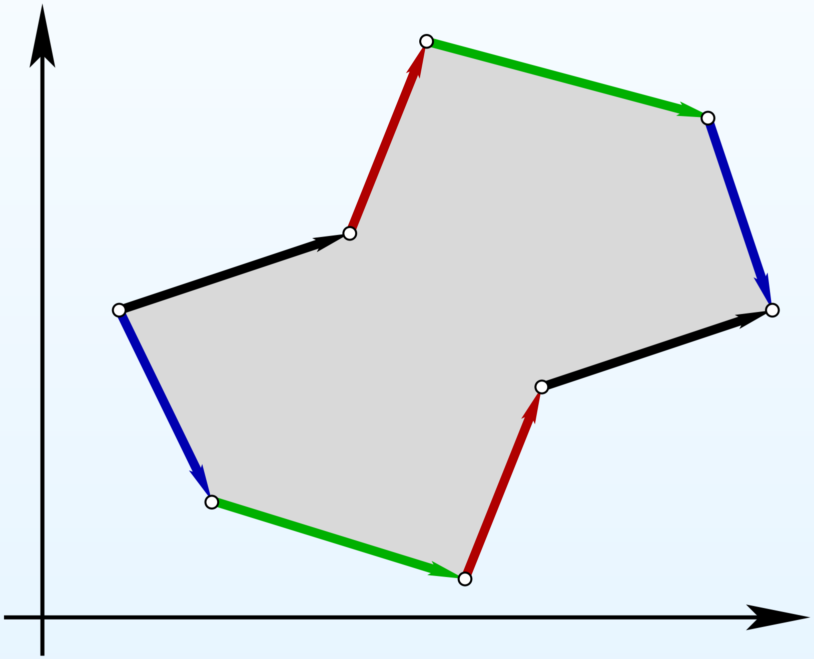

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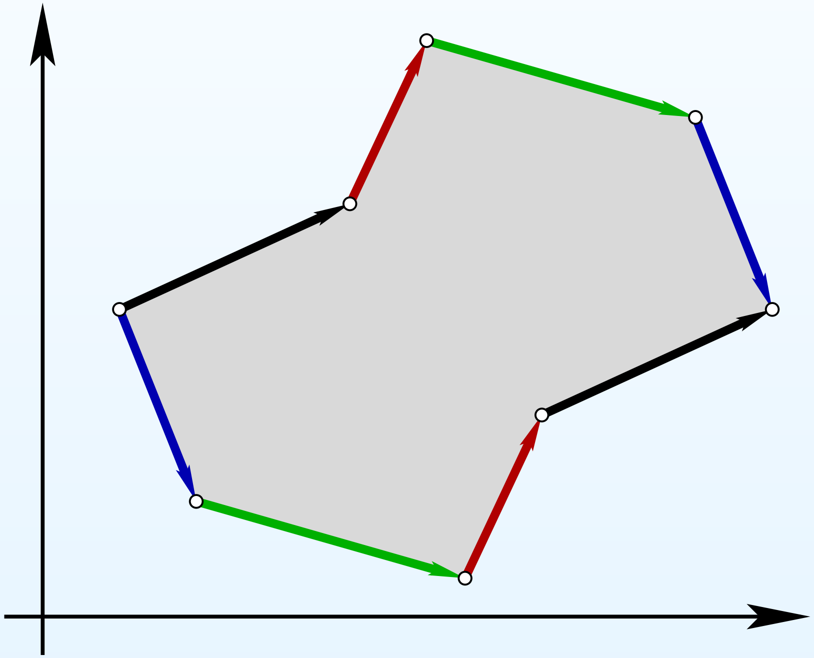

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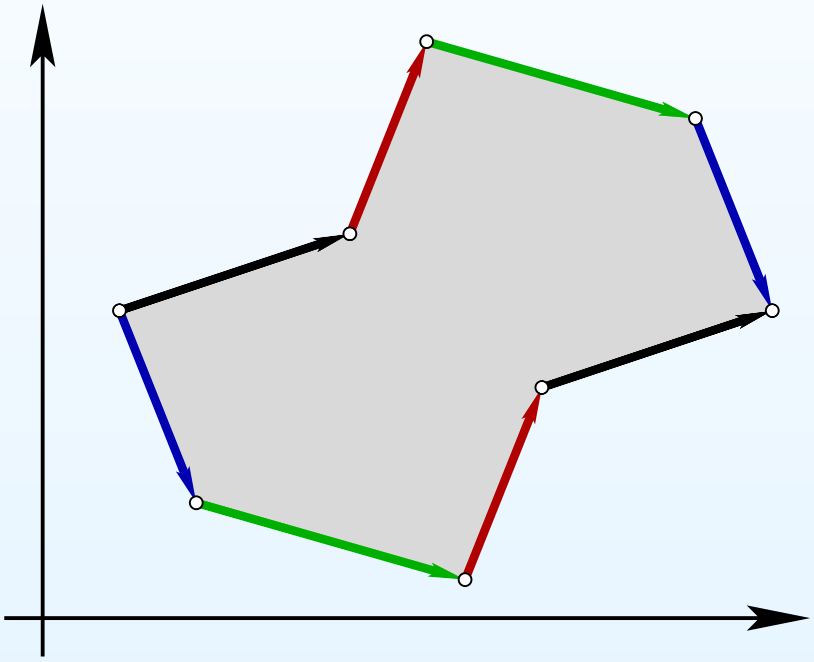
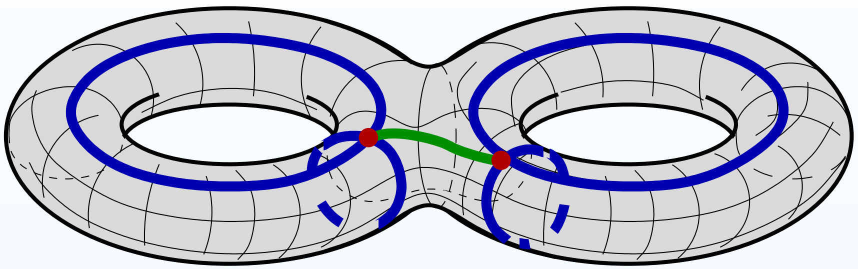
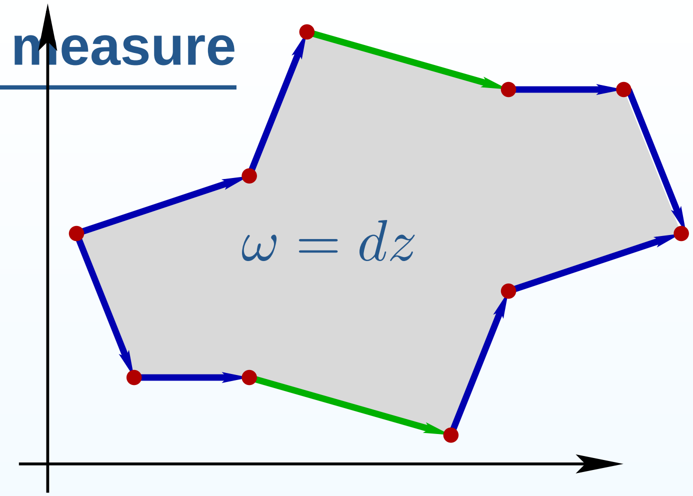


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

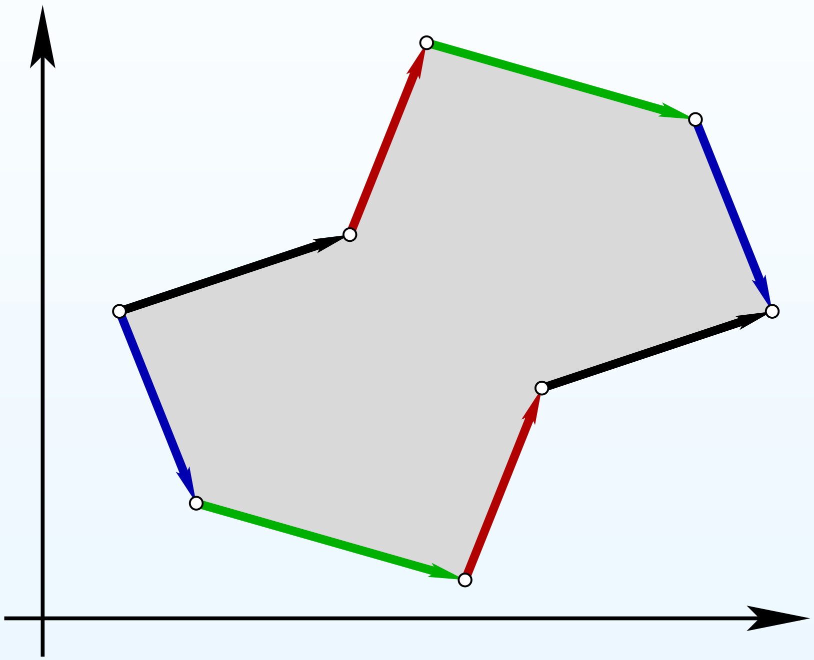
Period coordinates and Masur–Veech measure





Identifying corresponding sides V_j of a polygon by parallel translations we get a Riemann surface X and a holomorphic 1-form ω on it, where $\omega = dz$ in coordinate z on the polygon. The sides V_j become lines on S with endpoints in the collection of points $Y = \{P_1, \dots, P_k\} \subset X$ coming from vertices of the polygon. Since $d\omega = 0$ it defines a relative homology class $[\omega] \in H^1(X, Y; \mathbb{C})$: the value of $[\omega]$ on a cycle c is given by $\int_\gamma \omega$, where $[\gamma] = c$ is any collection of paths representing c. It is easy to check that vectors V_j generate $H_1(X, Y; \mathbb{C})$. Considered as complex numbers, they represent integrals $\mathbb{C} \ni V_j = \int_{V_j} dz$ of ω over the corresponding relative cycles. Thus, the collection of vectors uniquely determines $[\omega] \in H^1(X, Y; \mathbb{C})$. Reciprocally, any cohomology class in $H^1(X, Y; \mathbb{C})$ sufficiently close to $[\omega]$ defines a collection of deformed integrals over paths V_j, and, hence a deformed polygon.

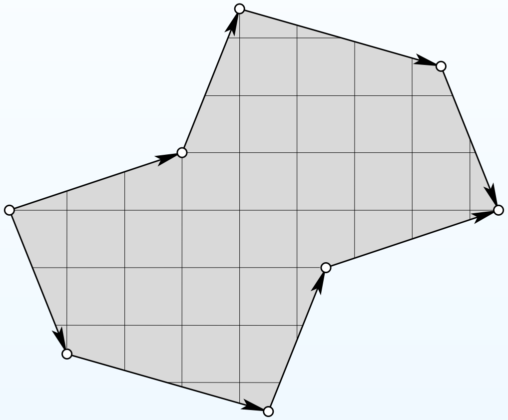
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g-2$, is modeled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any stratum $\mathcal{H}(m_1,\dots,m_n)$: we can rescale a flat surface by any positive factor r. The flat area gets rescaled by r^2.



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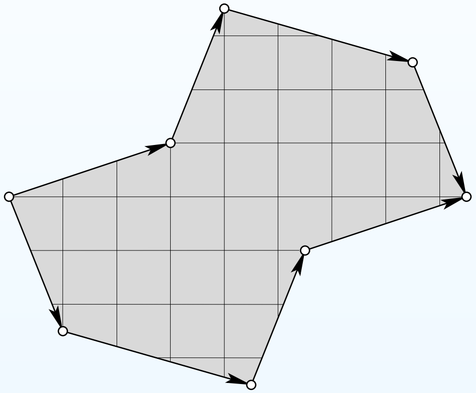

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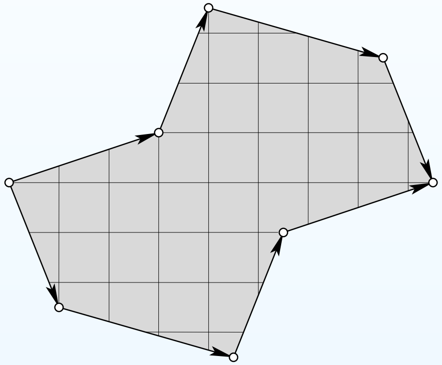

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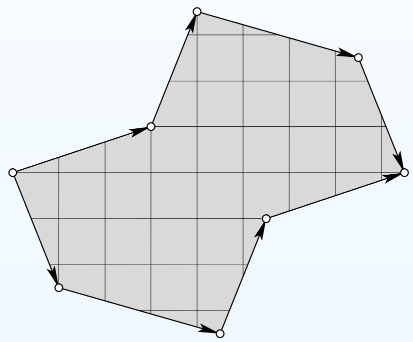

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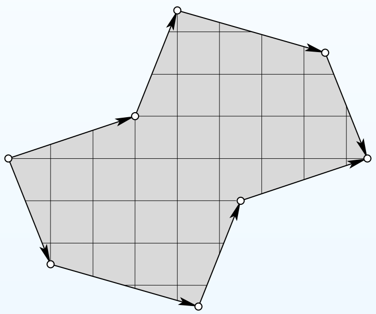

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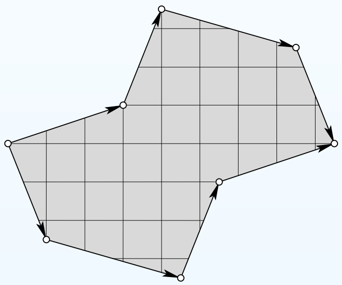

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1, \dots, m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega \wedge \bar{\omega} = \sum_{i=1}^{g}(A_i\bar{B}_i - \bar{A}_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S = (C, r \cdot \omega)$, where $r > 0$ and $(C, \omega) \in \mathcal{H}_1(m_1, \dots, m_n)$. In these "polar coordinates" the volume element disintegrates as $d\nu = r^{2d-1} dr\, d\nu_1$ where $d\nu_1$ is the induced volume element on $\mathcal{H}_1$ and $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n) = 2g + n - 1$.

Excercise. Prove that $\nu_1(\mathcal{H}_{area=1}) = 2d \cdot \nu(\mathcal{H}_{area\leq 1})$.

Period coordinates and Masur–Veech volume element

The moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g - 2$, is modelled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

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The area function defined on every stratum $\mathcal{H}(m_1, \ldots, m_n)$

$$\text{area}(C, \omega) = \frac{i}{2}\int_C \omega \wedge \bar{\omega} = \frac{i}{2}\sum_{i=1}^{g}(A_i\bar{B}_i - \bar{A}_i B_i)\,.$$

allows to define an analog of a “unit ball” $\mathcal{H}_{\le 1}$ in any stratum as a subset of those (C, ω) in $\mathcal{H}(m_1, \ldots, m_n)$, where $\operatorname{area}(C, \omega) \le 1$. (Note that in period coordinates the “unit ball” is rather the interior of a “unit hyperboloid”.)

Definition.

$$\operatorname{Vol}\mathcal{H}(m_1, \ldots, m_n) := 2d \cdot \int_{\mathcal{H}_{\le 1}} d\nu\,,$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \ldots, m_n)$.

Masur–Veech volume

Summary. Every stratum of Abelian differentials admits

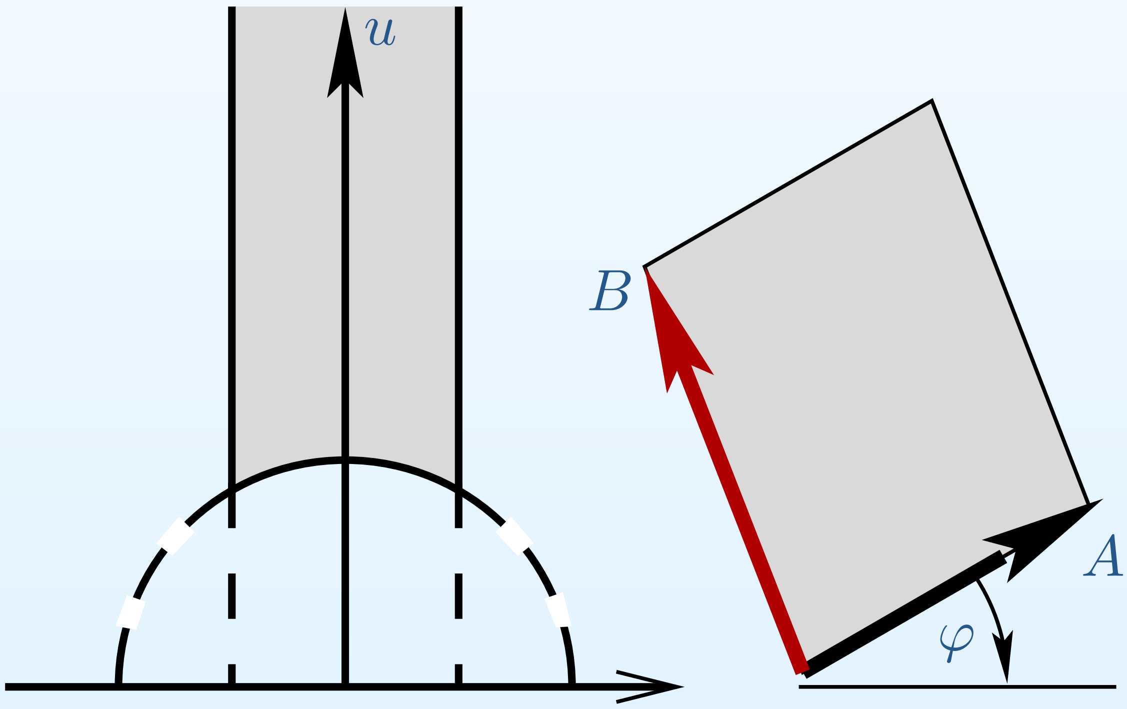
- A local structure of a vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$;
- An integer lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1, \ldots, m_n)$ or $\mathcal{Q}_1(m_1, \ldots, m_n)$ of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

Disintegration of the Masur–Veech volume element in $\mathcal{H}(0)$

Masur–Veech volume element in $\mathcal{H}(0)$

Let $A, B \in \mathbb{C}$ be periods of a holomorphic 1-form on an elliptic curve (equivalently, a pair of vectors defining a parallelogram in $\mathbb{R}^2$). Projection from the stratum $\mathcal{H}(0)$ to the modular surface $\mathrm{P}\mathcal{H}(0) = \mathcal{M}_1 = \mathbb{H}^2/\,\mathrm{PSL}(2,\mathbb{Z})$ corresponds to normalization of the A-period to 1 (equivalently, rescaling the parallelogram proportionally to make the length of the short side equal to 1 followed by a rotation making this side horizontal). We assume that B-period is chosen in such way, that we get directly to the fundamental domain.



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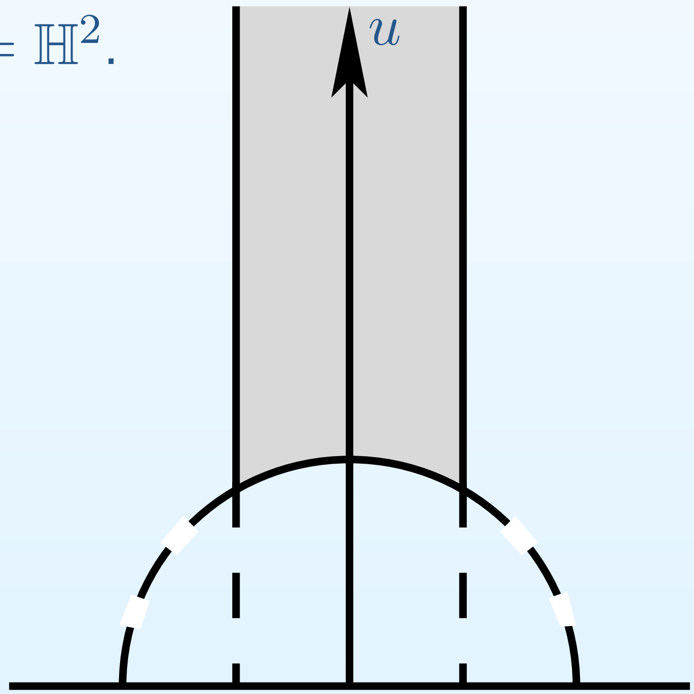
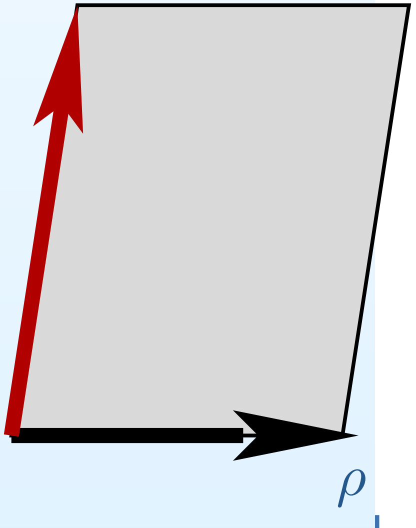
Denote by u coordinate on $\mathbb{C}^+ = \mathbb{H}^2$.
Let $A = \zeta = \rho e^{i\varphi}$, $B = \zeta \cdot u$.





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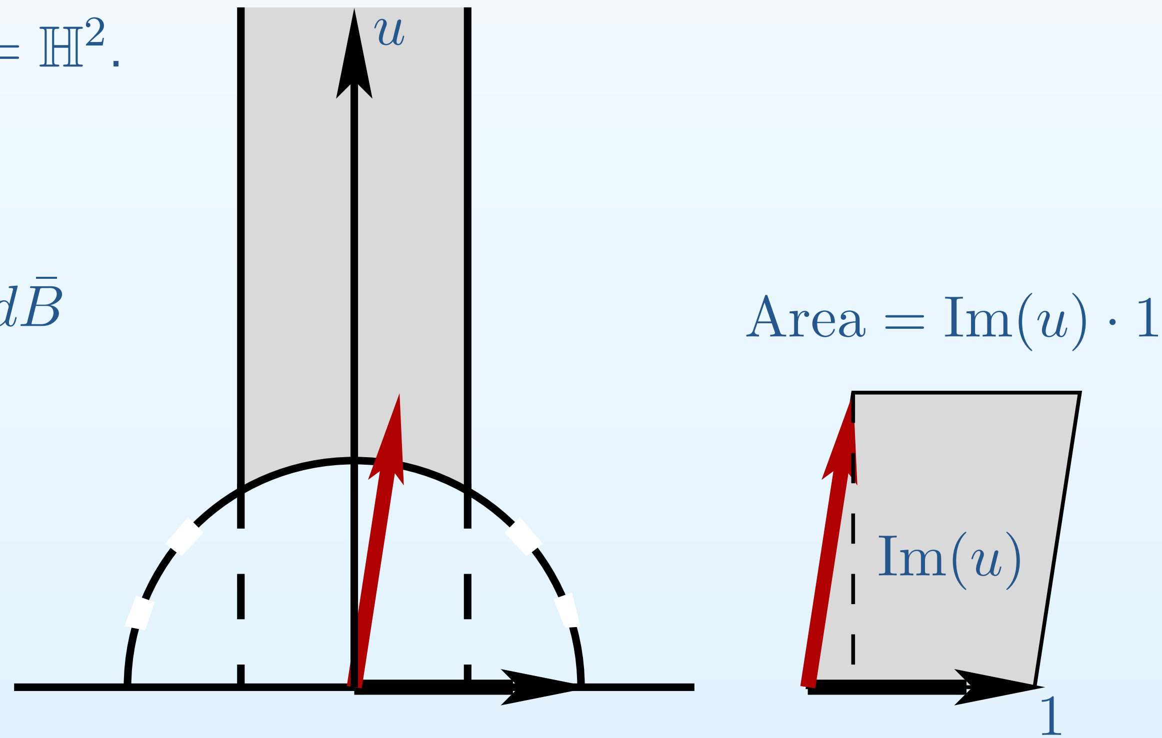

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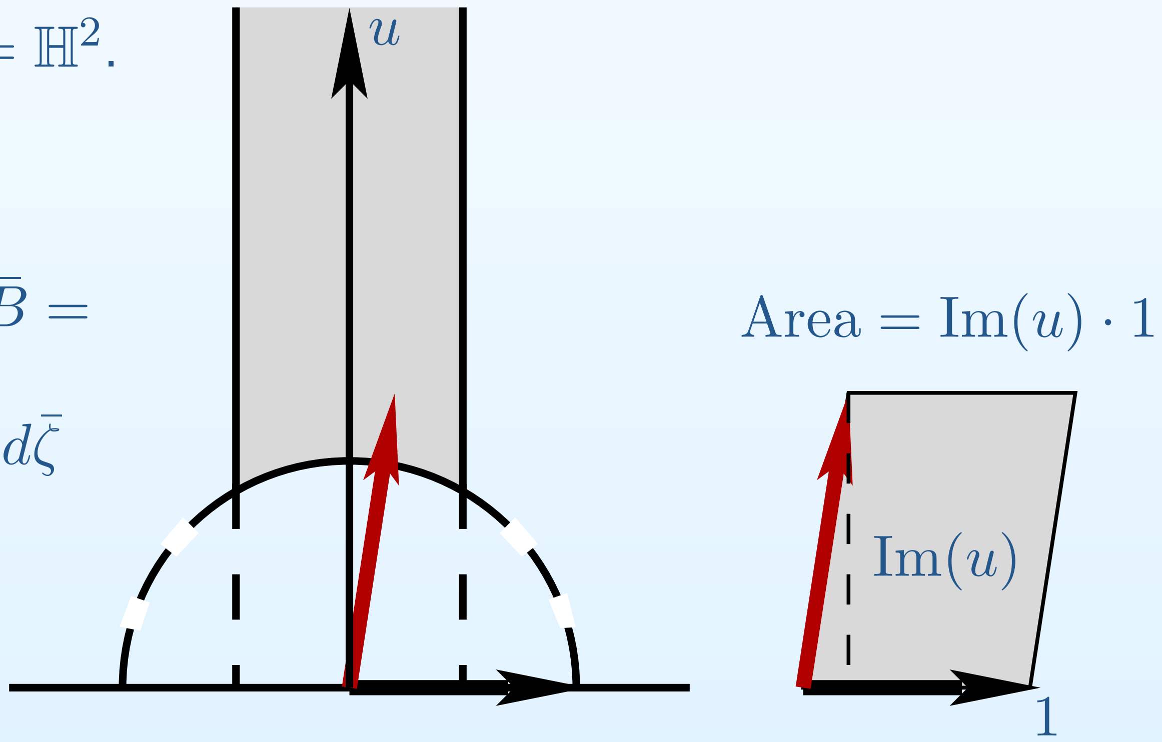

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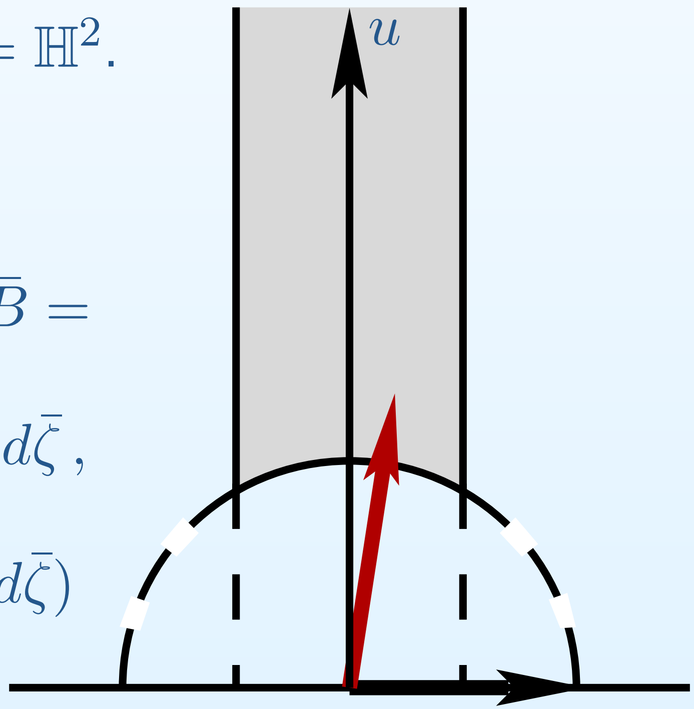

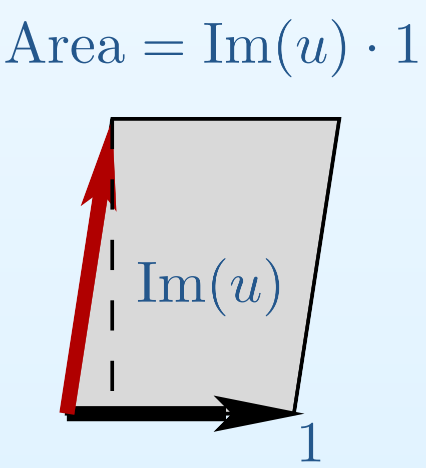

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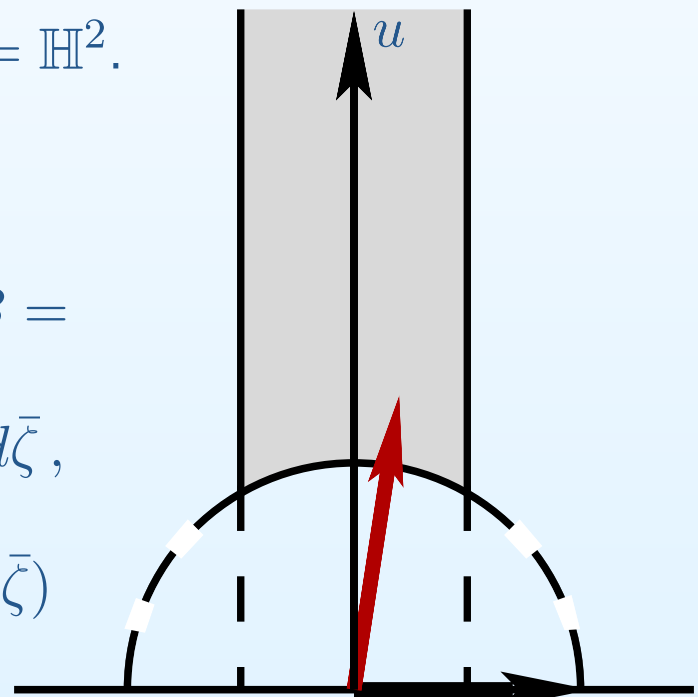

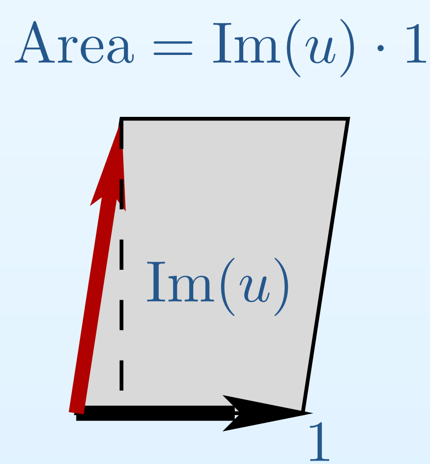

Disintegration of volume element along $\mathcal{H}(0) \to \mathcal{M}_1$

By definition of the induced volume element $d\nu_1$ on the "unit sphere" $\mathcal{H}_1(0)$ we have

$$d\nu = r^3\, dr\, d\nu_1\,.$$

where r^2 is the area of the flat torus. Recall that having rescaled the torus proportionally by a factor $\rho = |\zeta|$ we transformed its area to $\mathrm{Im}(u)$. Thus, the area of the original torus with periods A, B is $r^2 = |\zeta|^2\, \mathrm{Im}(u) = \rho^2\, \mathrm{Im}(u)$.

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We are looking for a function $f(u, \bar{u})$ such that the volume element $d\nu_1$ disintegrates as $d\nu_1 = d\varphi \wedge f(u, \bar{u})du \wedge d\bar{u}$. Replacing r with $\rho\sqrt{\mathrm{Im}(u)}$ in $d\nu$ we get

$$d\nu = r^3\, dr\, d\varphi\wedge f(u,\bar{u})du\wedge d\bar{u} = \left(\rho\sqrt{\mathrm{Im}\,u}\right)^3 \left(\sqrt{\mathrm{Im}\,u}\, d\rho\right)\wedge d\varphi\wedge f(u,\bar{u})du\wedge d\bar{u}\,.$$

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We are looking for a function $f(u, \bar{u})$ such that the volume element $d\nu_1$ disintegrates as $d\nu_1 = d\varphi \wedge f(u, \bar{u})du \wedge d\bar{u}$. Replacing r with $\rho\sqrt{\mathrm{Im}(u)}$ in $d\nu$ we get

$$d\nu = r^3\, dr\, d\varphi \wedge f(u,\bar{u})du \wedge d\bar{u} = \left(\rho\sqrt{\mathrm{Im}\, u}\right)^3 \left(\sqrt{\mathrm{Im}\, u}\, d\rho\right) \wedge d\varphi \wedge f(u,\bar{u})du \wedge d\bar{u}\,.$$

Comparing the above expression with our original formula for $d\nu$ we get

$$\frac{i}{2}\rho^3\, d\rho \wedge d\varphi \wedge du \wedge d\bar{u} = \rho^3 (\mathrm{Im}\, u)^2 f(u,\bar{u})\, d\rho \wedge (d\varphi \wedge du \wedge d\bar{u})\ ,$$

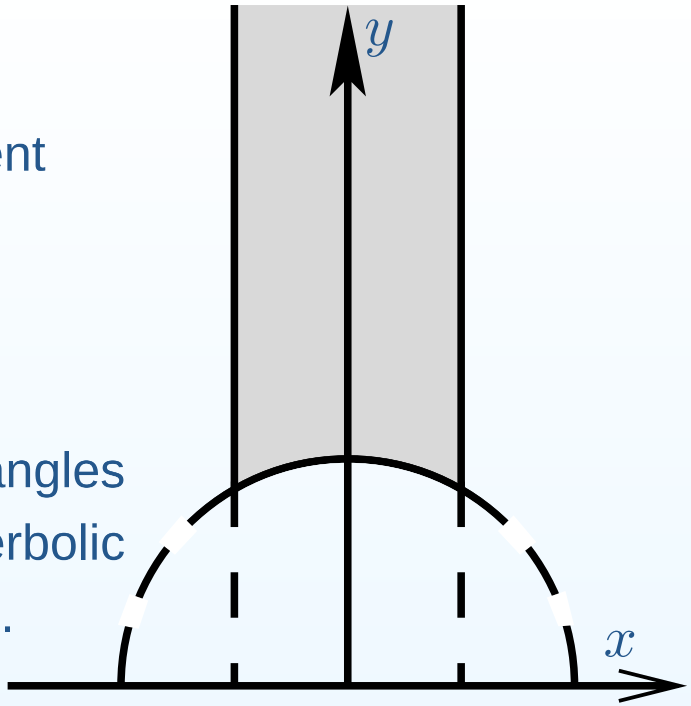
We recognize the hyperbolic volume element $f(u, \bar{u})du \wedge d\bar{u} = \dfrac{\frac{i}{2}\, du \wedge d\bar{u}}{\mathrm{Im}^2(u)}$.

Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$

Letting $u = x + iy$ we get the standard volume element

$$\frac{\frac{i}{2}\, du \wedge d\bar{u}}{\mathrm{Im}^2(u)} = \frac{dx \wedge dy}{y^2}$$

in the hyperbolic half-plane. A hyperbolic triangle with angles α, β, γ, has area $\pi - (\alpha + \beta + \gamma)$. Thus, the hyperbolic area of the modular surface is $\pi - \left(\frac{\pi}{3} + \frac{\pi}{3} + 0\right) = \frac{\pi}{3}$.



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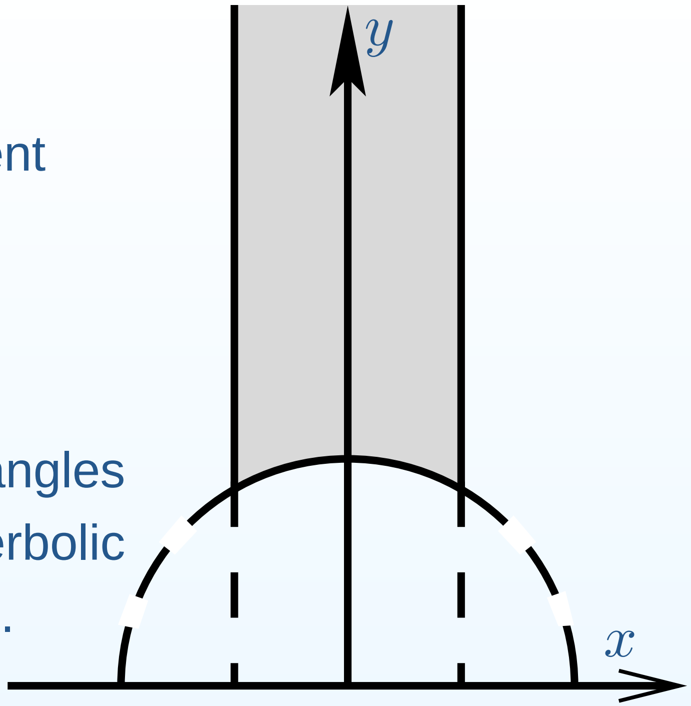
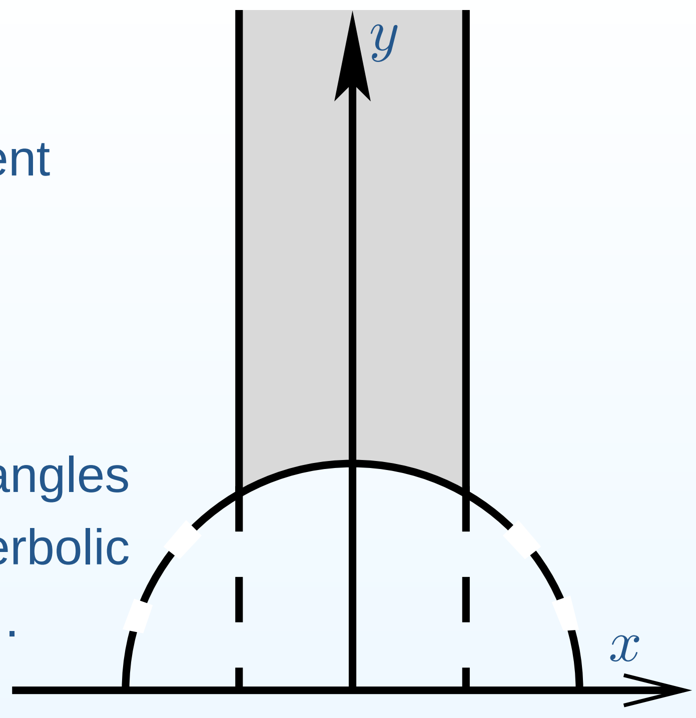


Our disintegration formula shows, that the Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$ equals the hyperbolic area of the modular surface times the measure of the circle S^1 responsible for the choice of the vertical direction on the torus.

Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$



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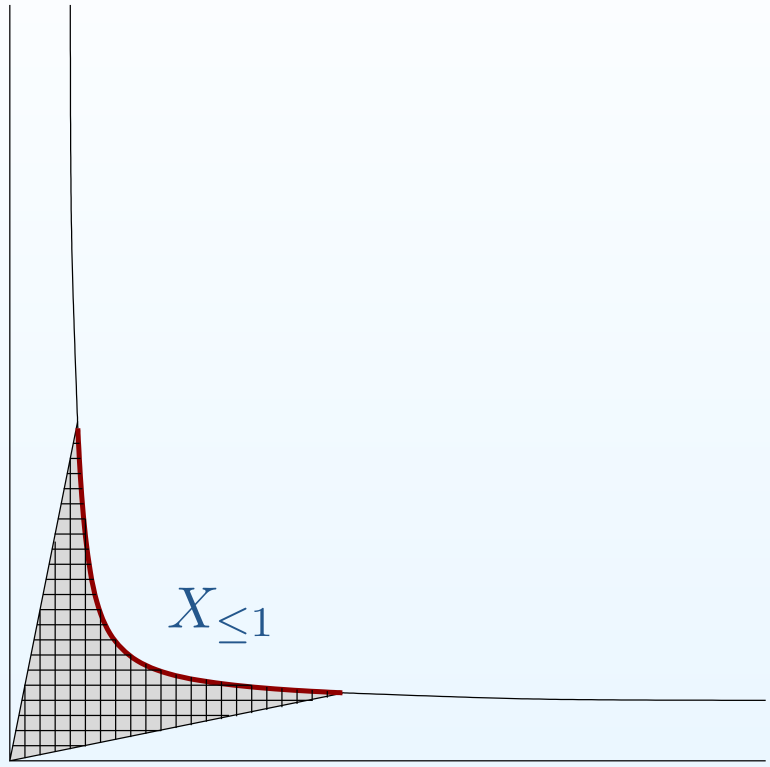
Our disintegration formula shows, that the Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$ equals the hyperbolic area of the modular surface times the measure of the circle S^1 responsible for the choice of the vertical direction on the torus. Observe that every flat torus admits an involution (central symmetry of its parallelogram pattern). Hence, directions ϕ and $-\phi$ give rise to isomorphic "polarized" flat tori and thus the measure of S^1 equals π and not 2π. We get

$$\mathrm{Vol}(\mathcal{H}(0)) = \frac{\pi}{3} \cdot \pi = \frac{\pi^2}{3}\,.$$

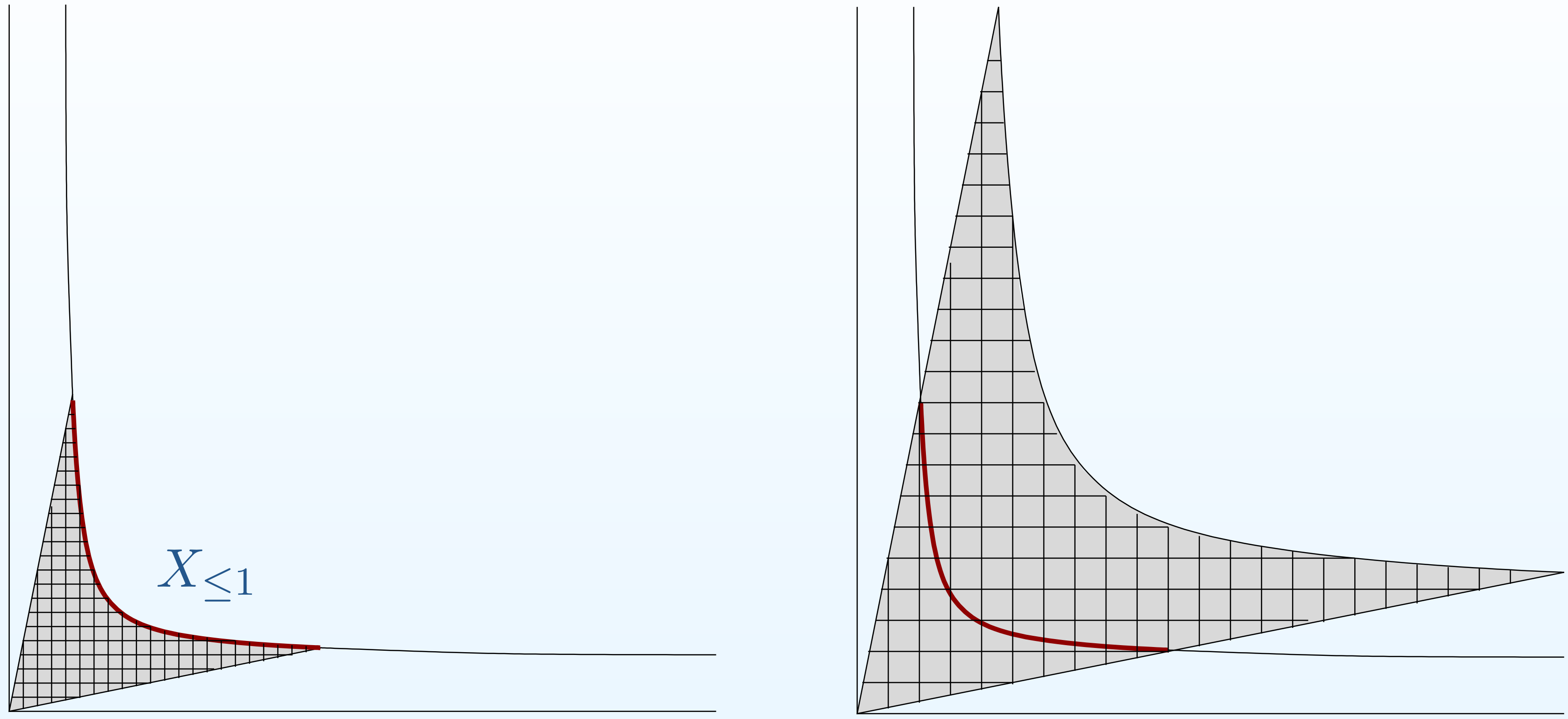
Square-tiled surfaces as integer points of the modular space

Counting volume by counting integer points in a large cone



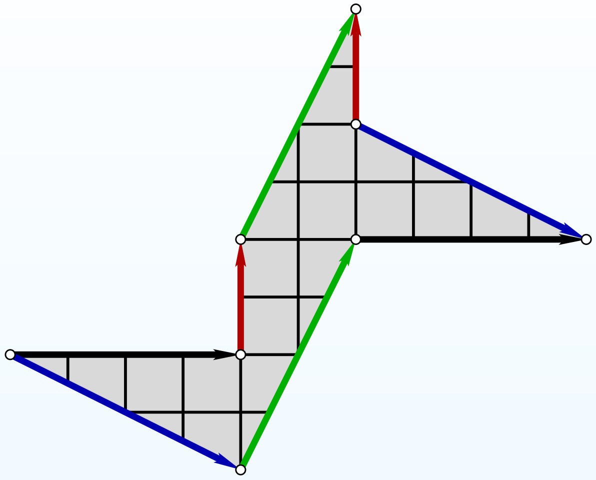
To count volume of the cone $X_{\leq 1}$ one can take an ε-grid and count the number of lattice points inside it.

Counting volume by counting integer points in a large cone



To count volume of the cone $X_{\leq 1}$ one can take an ε-grid and count the number of lattice points inside it.
Counting points of the ε-grid in the cone $X_{\leq 1}$ is the same as counting integer points in the proportionally rescaled cone $X_{\leq 1/\varepsilon}$.

Integer points as square-tiled surfaces



Integer points in period coordinates are represented by *square-tiled surfaces*.

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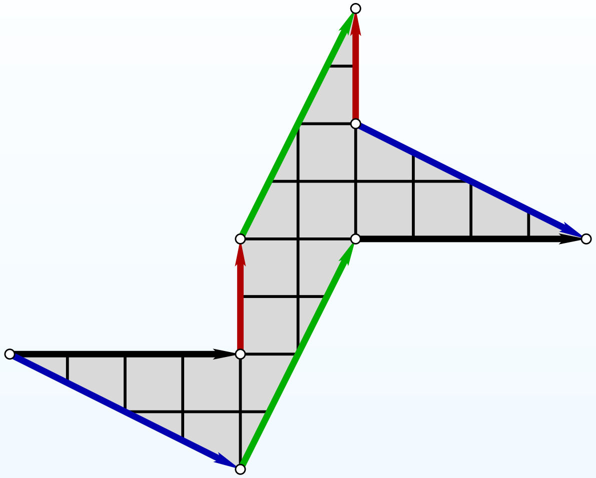

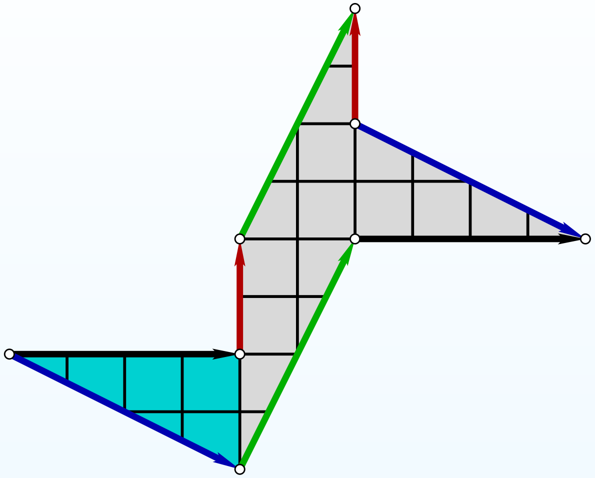

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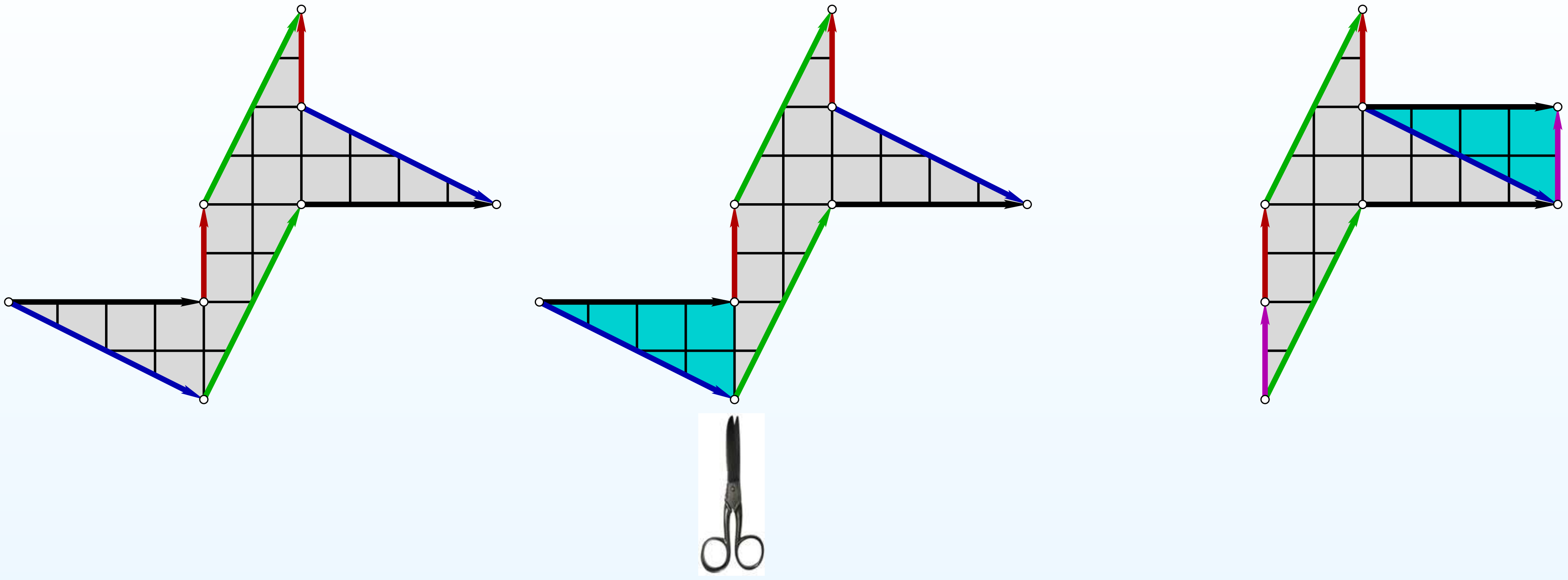

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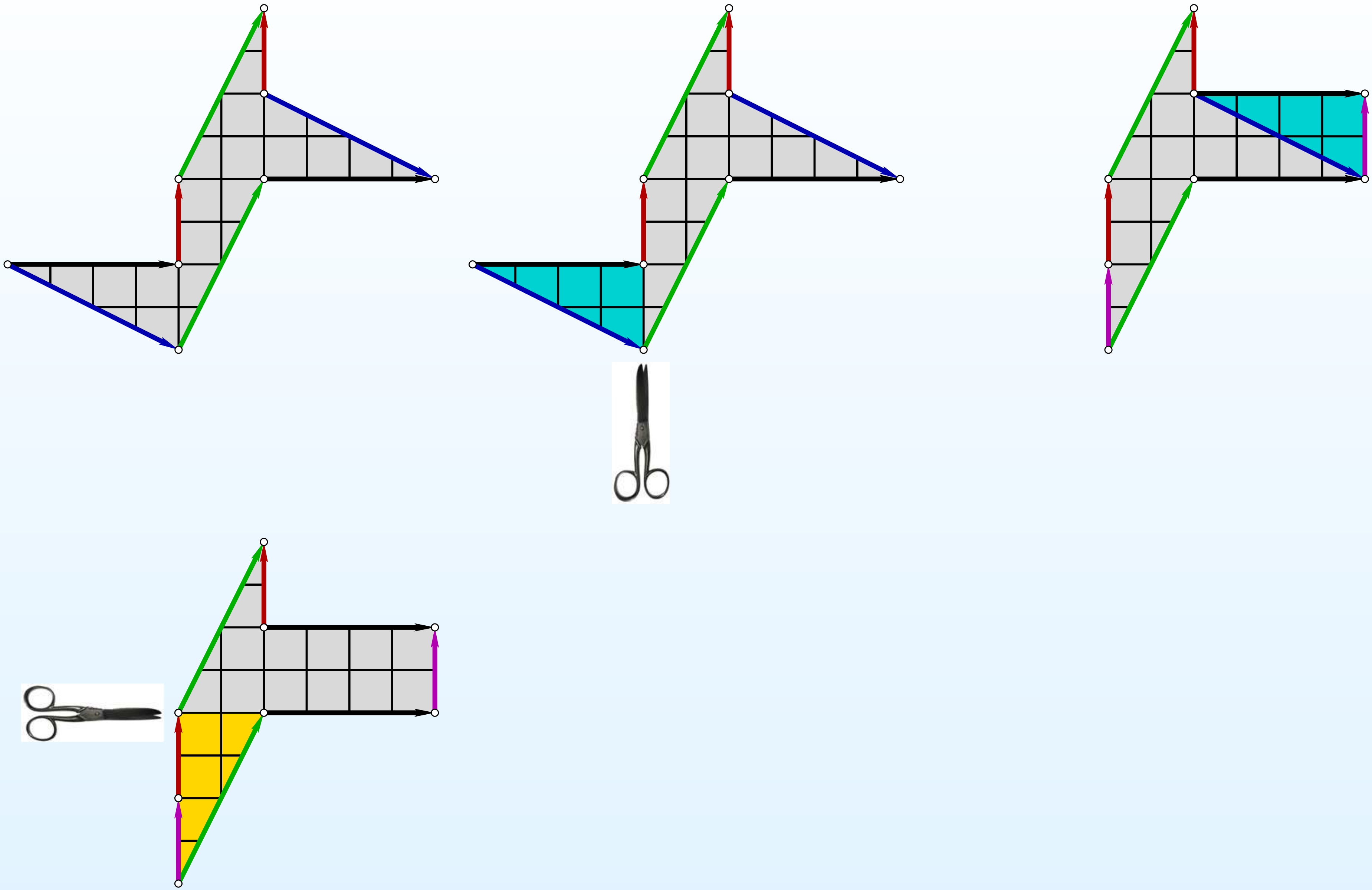

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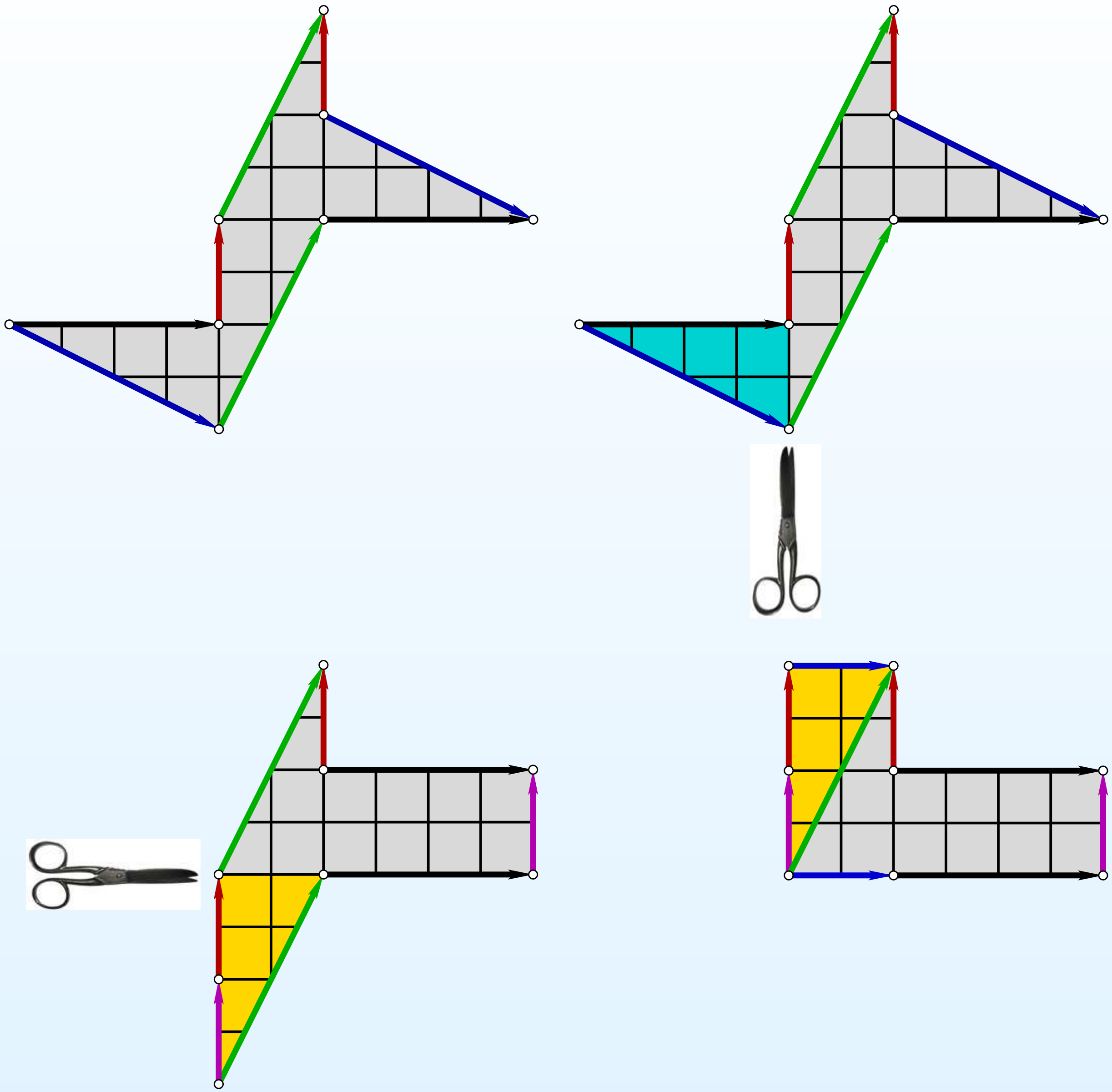

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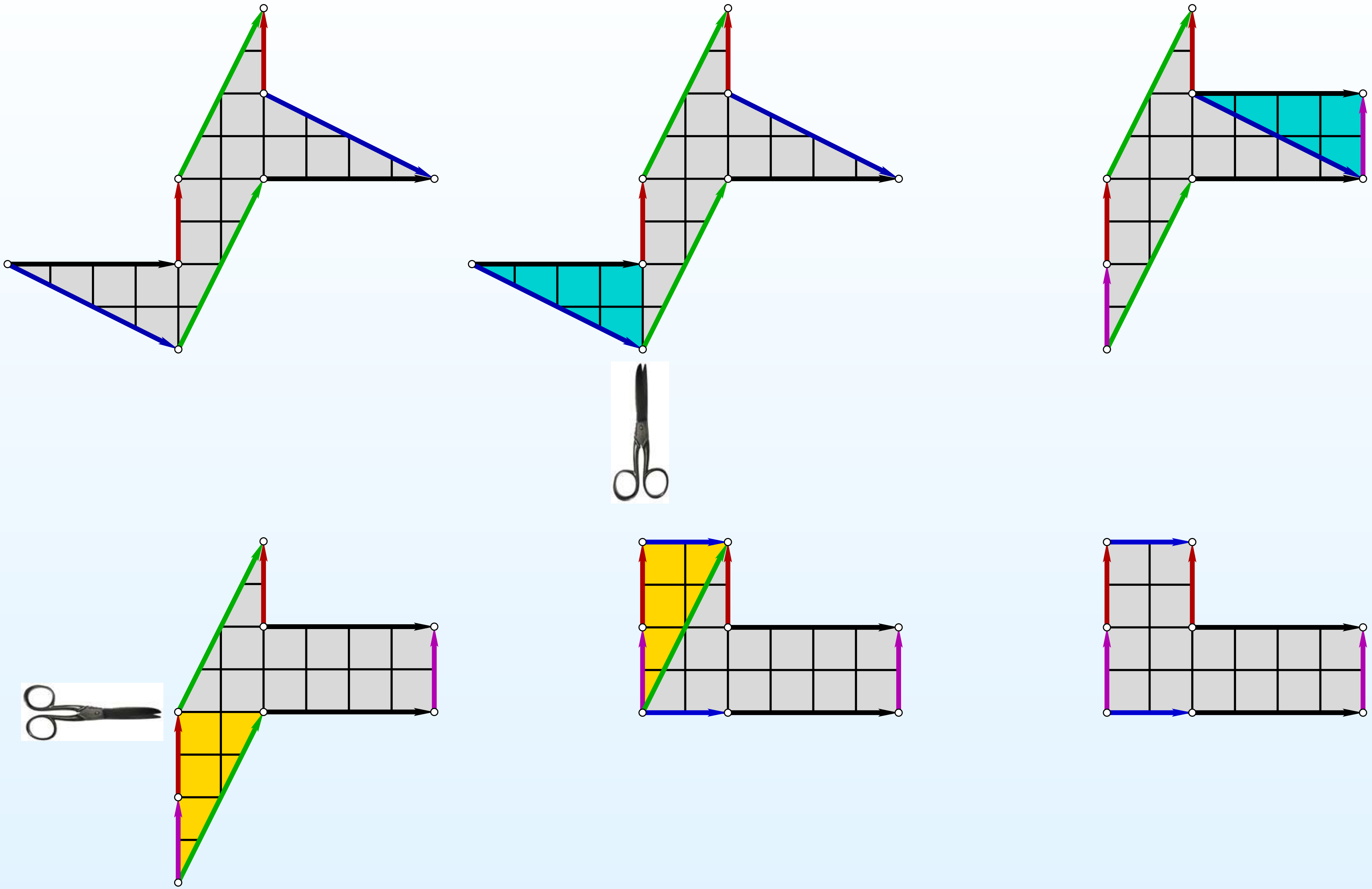

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Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point. Let P_1 be a zero of ω and $P \in C$ any point of the Riemann surface C. Define

$$\begin{array}{cccc} p: & P & \mapsto & \int_{P_1}^{P} \omega \pmod{\mathbb{Z} \oplus i\mathbb{Z}} \\ p: & C & \to & \mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \end{array}$$

The ramification points of the cover p are exactly the zeroes of ω.

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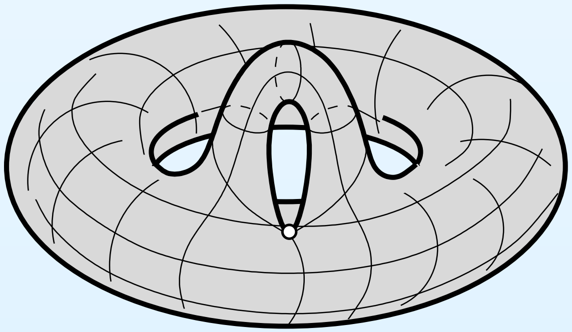
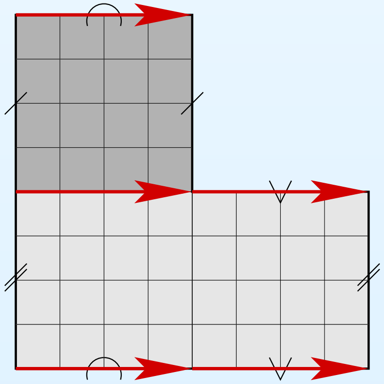
Choosing the standard unit square pattern for $\mathbb{T}$ we get induced tiling of (C, ω) by unit squares which form horizontal and vertical cylinders. The square-tiled surface of genus two in the picture has 2 maximal horizontal cylinders filled with periodic geodesics.





Masur–Veech volumes through count of square-tiled surfaces

Square-tiled surfaces which represent integer points in a “ball $\mathcal{H}_{area\le N}$ of radius N” in a given stratum $\mathcal{H}$ of Abelian differentials are the ones, tiled with at most N unit squares. Denote the corresponding set by $\mathcal{ST}_N(\mathcal{H})$. We have,

$$\nu(\mathcal{H}_{area\le N}) \sim \operatorname{card}(\mathcal{ST}_N(\mathcal{H}))\,.$$

By homogeneity of the Masur–Veech volume element ν we get

$$\nu(\mathcal{H}_{area\le R}) = R^d\cdot\nu(\mathcal{H}_{area\le 1})\,,$$

where

$$d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)=2g+n-1\,.$$

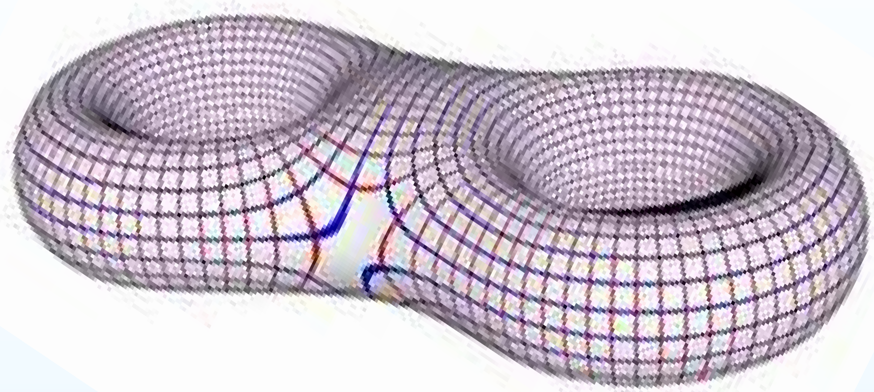
Thus,

$$\nu(\mathcal{H}_{\le 1})=\lim_{N\to+\infty}\frac{\operatorname{card}(\mathcal{ST}_N)}{N^d}\,.$$

By definition of the Masur–Veech volume $\operatorname{Vol}\mathcal{H}_1$ of the “ unit sphere $\mathcal{H}_1$ $=\mathcal{H}_{area=1}$”, we have $\operatorname{Vol}\mathcal{H}_1=2d\cdot\nu(\mathcal{H}_{area\le 1})$. Combining, we get

$$\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n):=2(2g+n-1)\cdot\lim_{N\to+\infty}\frac{\operatorname{card}(\mathcal{ST}_N(\mathcal{H}(m_1,\dots,m_n)))}{N^d}\,.$$

Count of square-tiled surfaces



Picture created by Jian Jiang

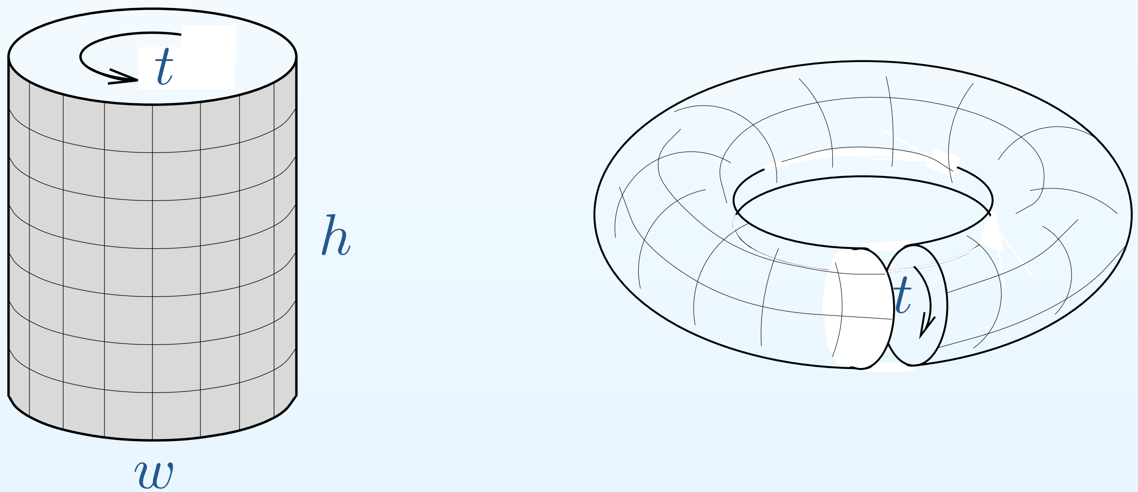
We reduced evaluation of the Masur–Veech volumes $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ to a combination of the following two related problems:

• Describe all combinatorial types of square-tiled surfaces in any given stratum $\mathcal{H}(m_1,\dots,m_n)$.
• Count the leading term in the asymptotics of the number of square-tiled surfaces of any given combinatorial type tiled with at most N squares when $N \to +\infty$.

Count of square-tiled surfaces through separatrix diagrams

Baby case: decomposition of a square-tiled torus

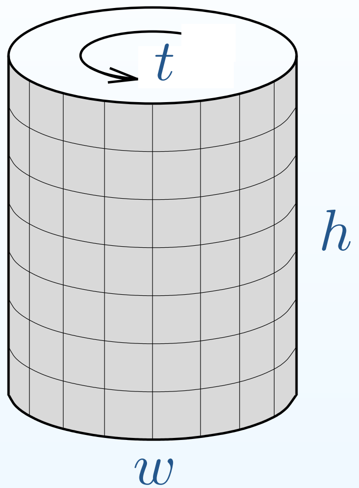
Let us count the number $\mathrm{card}(\mathcal{ST}_N(\mathcal{H}(0))$ of square-tiled tori tiled by at most $N \gg 1$ squares. Cutting a square-tiled torus by a horizontal waist curve we get a cylinder of integer height h. A waist curve of the cylinder has integer length w. The number of squares in the tiling equals $w \cdot h$.

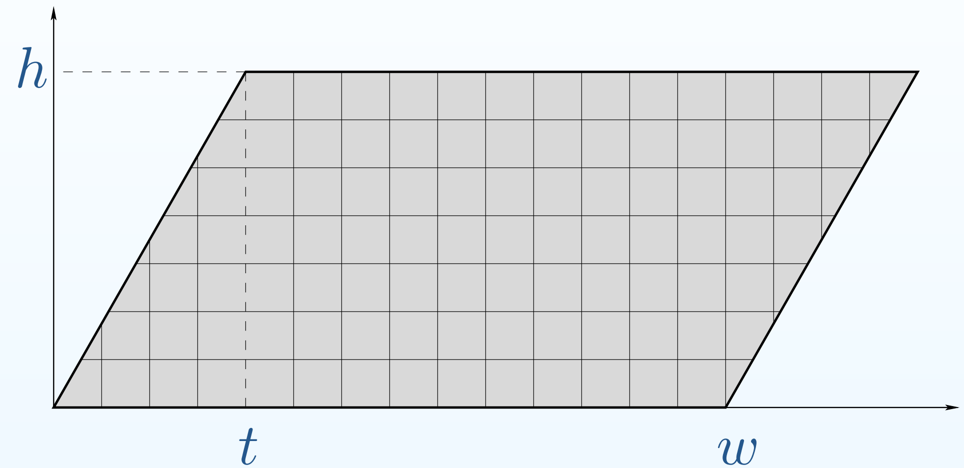


The way, in which two boundary components of the cylinder are identified, is described by an integer *twist* t which can take any value in $\{0, 1, \ldots, w-1\}$. Thus, for any fixed $w, h \in \mathbb{N}$ we get exactly w distinct square-tiled tori.

Baby case: counting square-tiled tori

We get the following leading term for $\operatorname{card}(\mathcal{ST}_N(\mathcal{H}(0))$ as $N \to +\infty$.





$$\operatorname{card}(\mathcal{ST}_N(\mathcal{H}(0)) = \sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w = \sum_{\substack{w,h\in\mathbb{N}\\ w\le \frac{N}{h}}} w \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2}$$

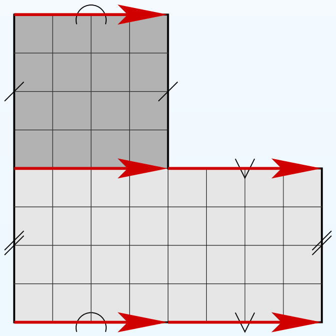
$$= \frac{N^2}{2}\cdot\zeta(2) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

Our formula gives

$$\operatorname{Vol}\mathcal{H}_1(0) := 2d\cdot\lim_{N\to+\infty}\frac{\operatorname{card}(\mathcal{ST}_N(\mathcal{H}(0)))}{N^d} = \frac{\pi^2}{3}.$$

Critical graph (separatrix diagram)

Note that all leaves of the horizontal (vertical) foliation on a square-tiled surface are closed. The *critical graph* Γ (*separatrix diagram*) is the union of all horizontal critical leaves. Vertices of Γ are represented by the conical points; the edges of Γ are formed by horizontal saddle connections (red in the picture).



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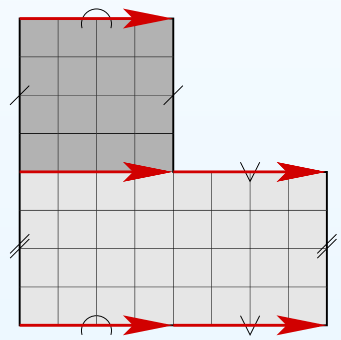

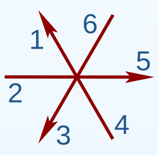

Let us construct the critical graph step-by-step. In our example the cone angle is 6π, so there are three outgoing separatrix rays and three incoming rays; they are alternated with respect to the natural cyclic order.

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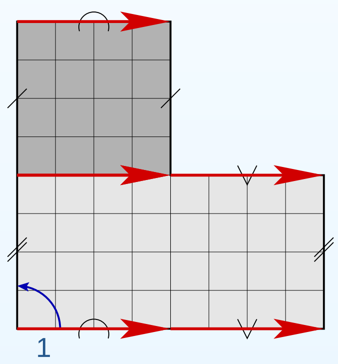

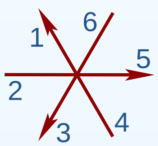

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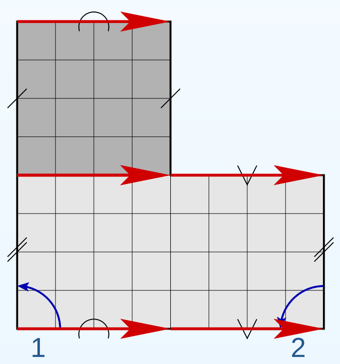

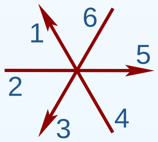

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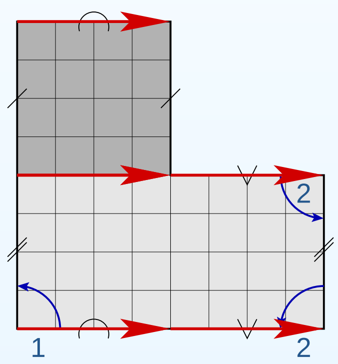

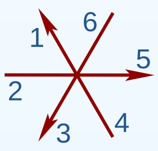

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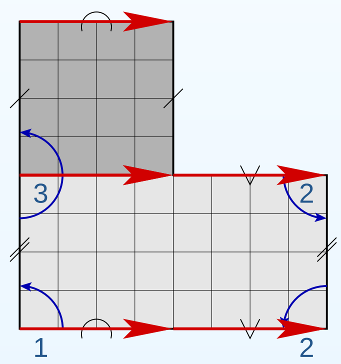

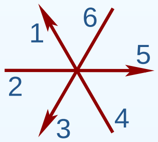

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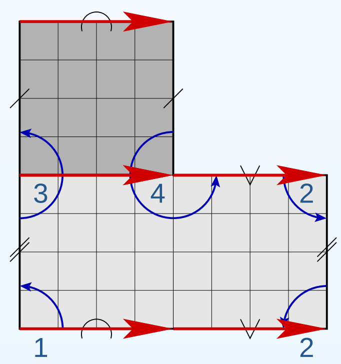

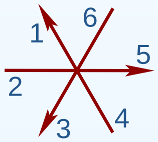

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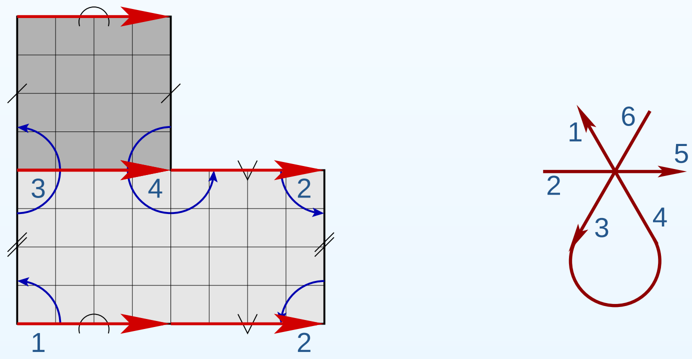

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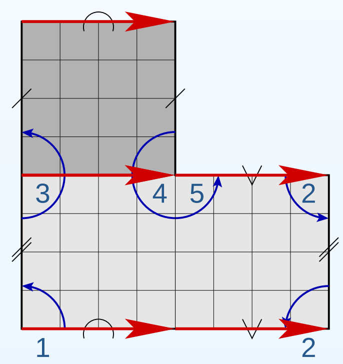

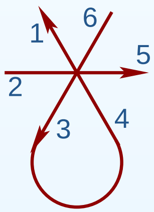

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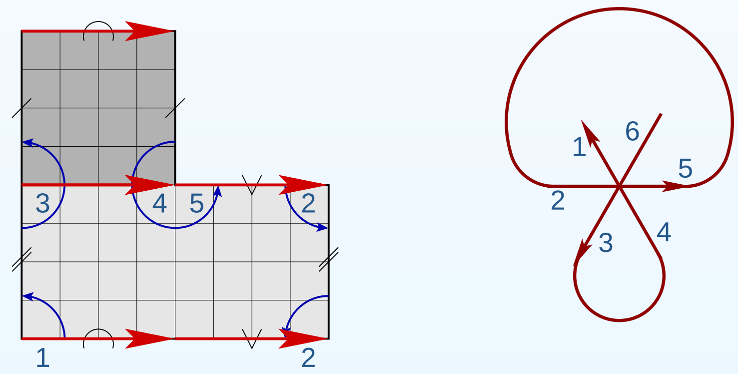

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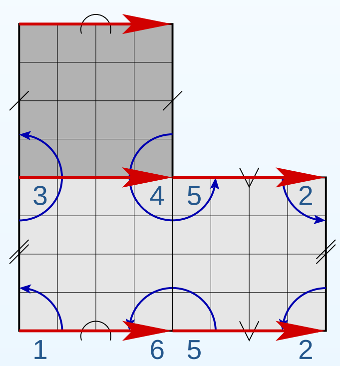

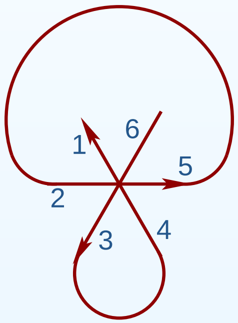

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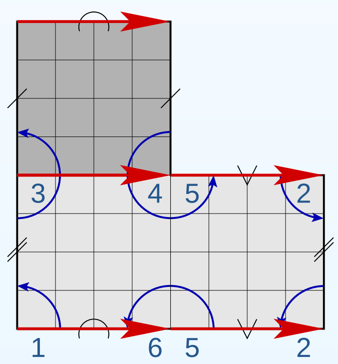

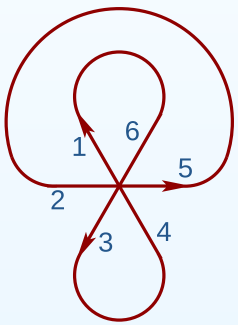

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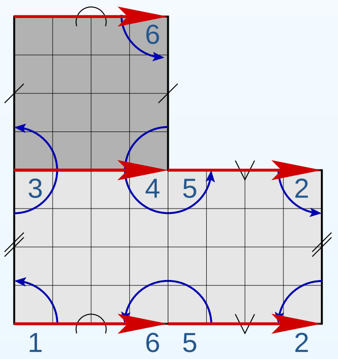

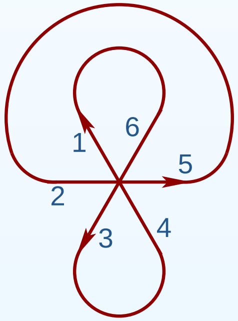

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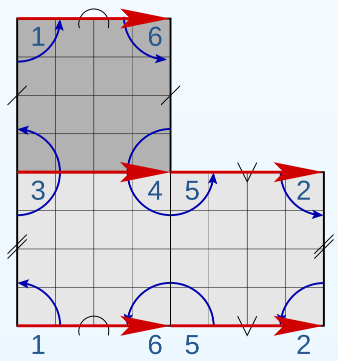

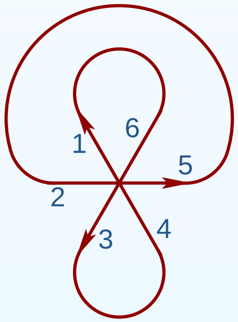

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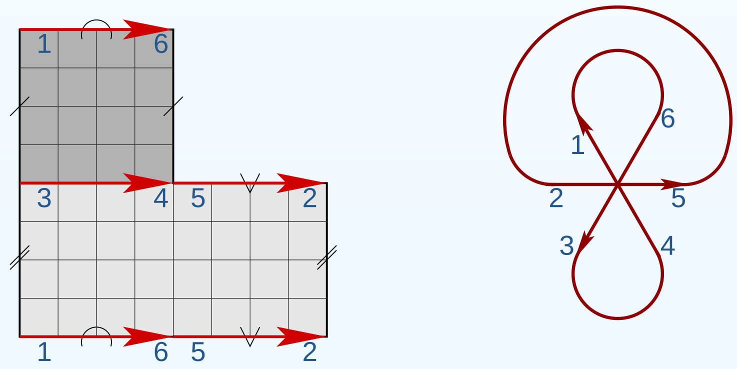

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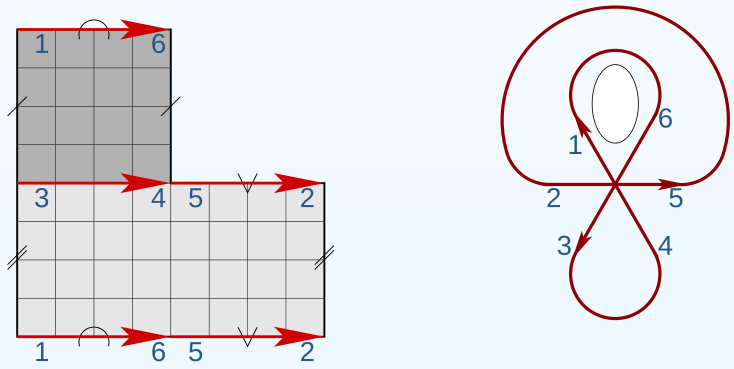

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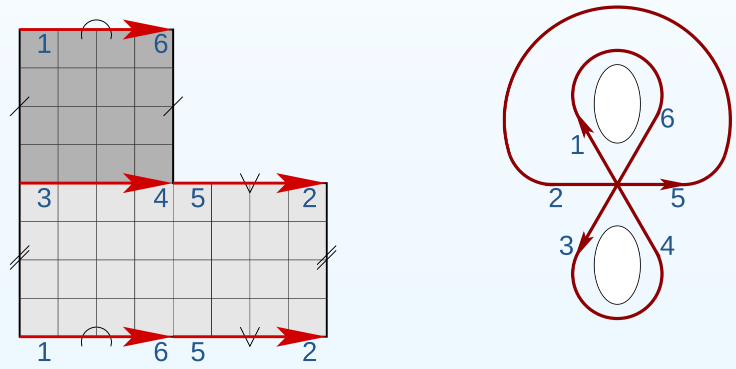

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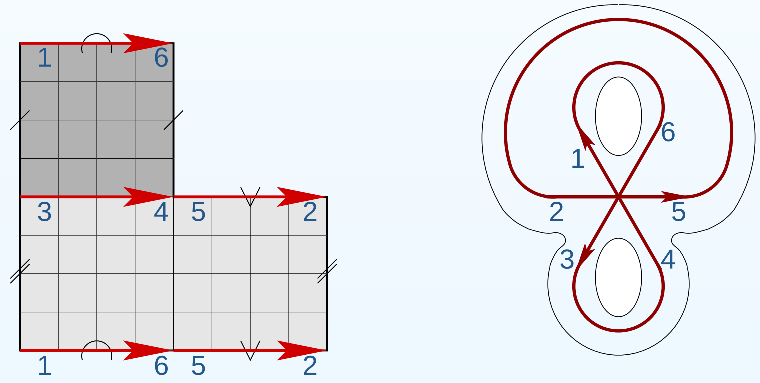

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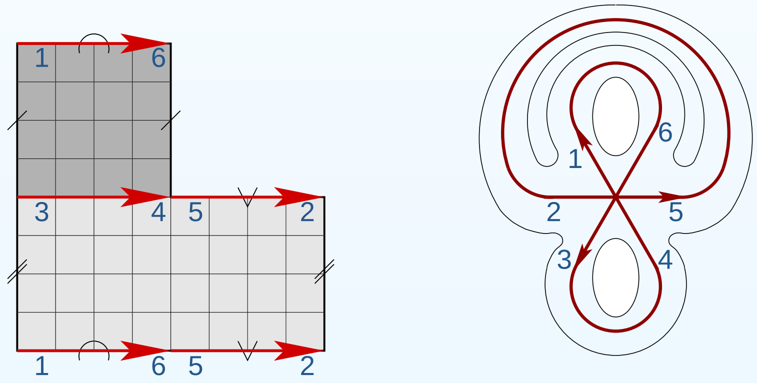

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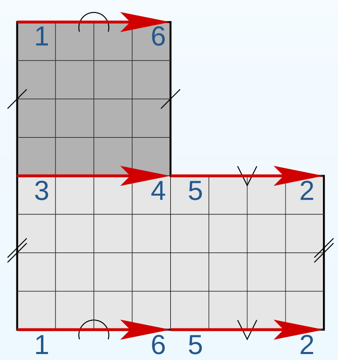

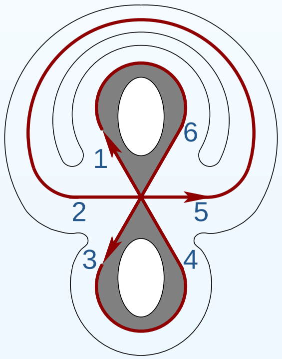

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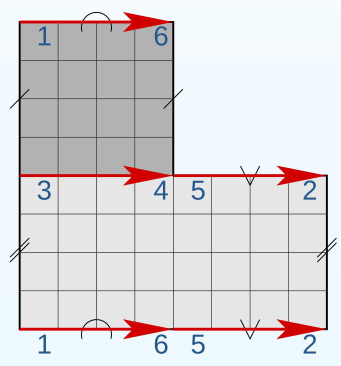

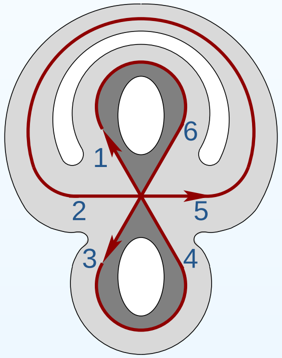

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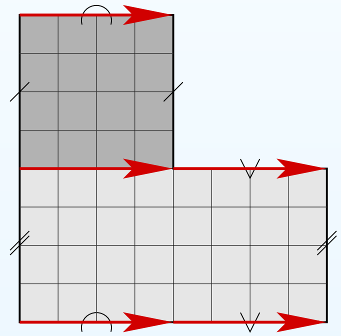
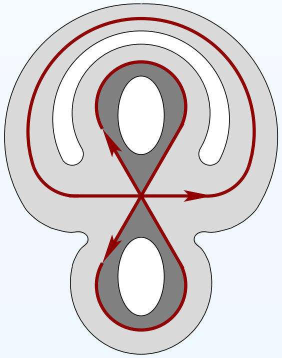




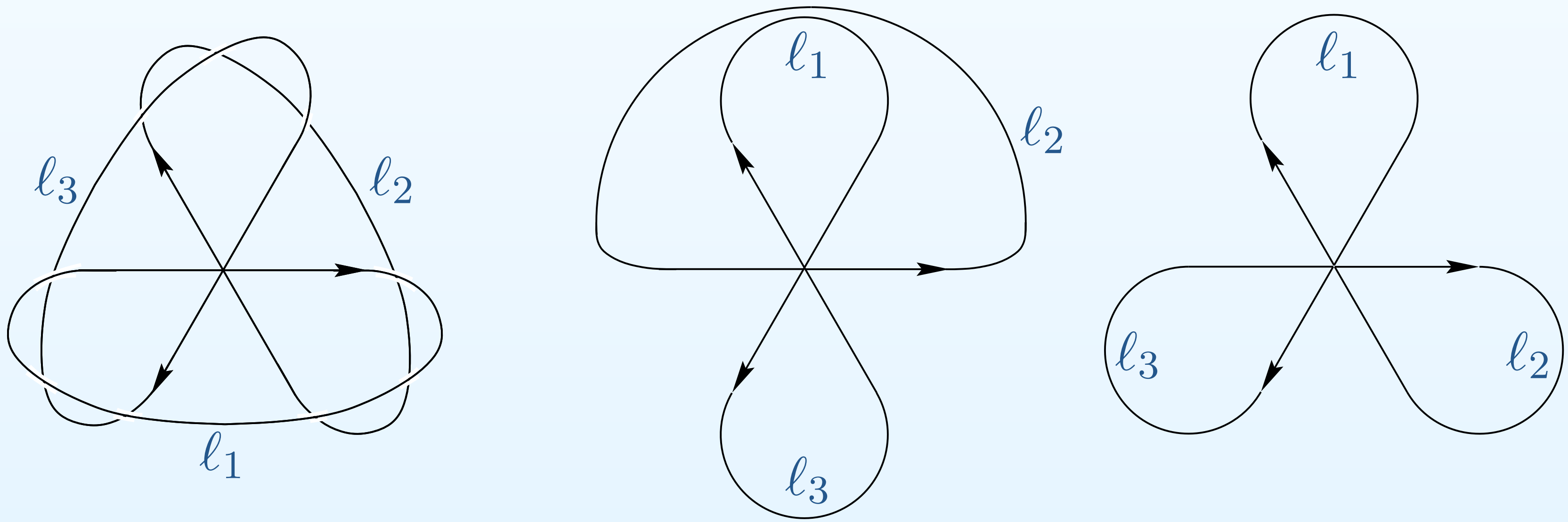
A *critical graph* Γ is an *oriented ribbon graph* endowed with the following structure:

1. The orientation of edges at any vertex is alternated with respect to the cyclic order of edges at this vertex.

2. The complement $S - \Gamma$ is a finite disjoint union of flat cylinders foliated by oriented circles. Thus, the set of boundary components of the ribbon graph is decomposed into pairs: to each pair of boundary components we glue a cylinder, and there is one positively oriented and one negatively oriented boundary component in each pair.

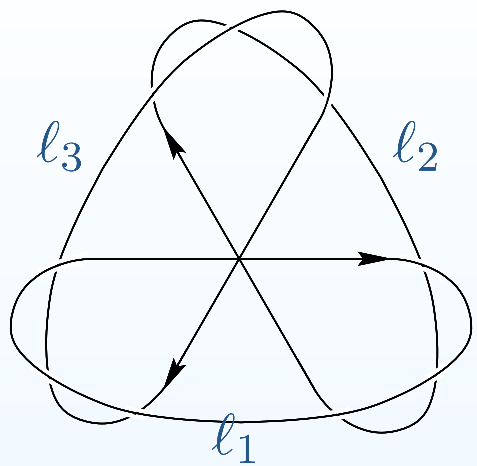
Realizable separatrix diagrams

Note, however, that not all ribbon graphs as above correspond to actual flat surfaces. A flat metric endows saddle connections with positive lengths ℓ_i. The left graph is realizable for any lengths ℓ_1, ℓ_2, ℓ_3. The middle one — only when $\ell_1 = \ell_3$. The rightmost one is never realizable: pairs of boundary components bounding the same cylinder have to have equal length, and we cannot find a pair for the component of length $\ell_1 + \ell_2 + \ell_3$.



Lemma. *The set of all square-tiled surfaces sharing any realizable separatrix diagram provides a nontrivial contribution to the volume of the corresponding stratum.*

Volume computation for $\mathcal{H}(2)$: the 1-cylinder diagram



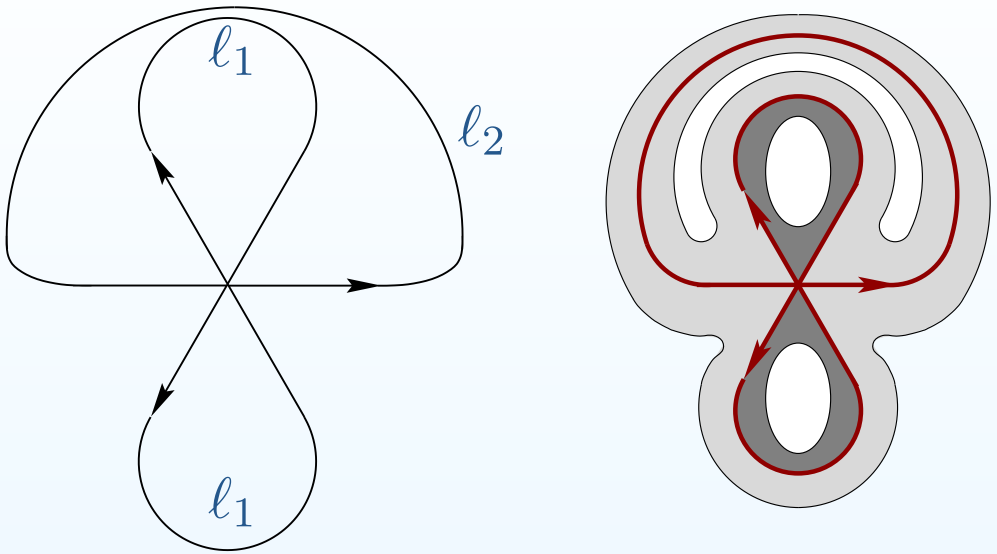
Single cylinder

$$\frac{1}{3}\sum_{\substack{\ell_1,\ell_2,\ell_3,h\in\mathbb{N}\\(\ell_1+\ell_2+\ell_3)h\le N}}(\ell_1+\ell_2+\ell_3) \approx \frac{1}{3}\sum_{\substack{w,h\in\mathbb{N}\\w\cdot h\le N}} w\cdot\frac{w^2}{2} = \frac{1}{6}\sum_{\substack{w,h\in\mathbb{N}\\w\le\frac{N}{h}}} w^3$$

$$\approx \frac{1}{6}\sum_{h\in\mathbb{N}}\frac{1}{4}\cdot\left(\frac{N}{h}\right)^4 = \frac{N^4}{24}\cdot\sum_{h\in\mathbb{N}}\frac{1}{h^4}$$

$$= \frac{N^4}{24}\cdot\zeta(4) = \frac{N^4}{24}\cdot\frac{\pi^4}{90}\,.$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram



$$\sum_{\substack{\ell_1,\ell_2,h_1,h_2\in\mathbb{N}\\ \ell_1 h_1+(\ell_1+\ell_2)h_2\le N}} \ell_1(\ell_1+\ell_2) = \sum_{\substack{\ell_1,\ell_2,h_1,h_2\in\mathbb{N}\\ \ell_1(h_1+h_2)+\ell_2 h_2\le N}} (\ell_1^2+\ell_1\ell_2) =$$

$$= \sum_{h_1,h_2\in\mathbb{N}} \sum_{\substack{\ell_1,\ell_2\in\mathbb{N}\\ \frac{\ell_1(h_1+h_2)}{N}+\frac{\ell_2 h_2}{N}\le 1}} (\ell_1^2+\ell_1\ell_2)\,.$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram

For any fixed h_1, h_2 we can replace the sum with respect to ℓ_1, ℓ_2 by the integral. Let $x_1 := \ell_1 \cdot \frac{h_1+h_2}{N}$ and $x_2 := \ell_2 \cdot \frac{h_2}{N}$ be the new variables, where h_1, h_2 are considered as parameters. After this change of variables our sums with respect to ℓ_1, ℓ_2 become the integral with respect to x_1, x_2, where we integrate over the simplex $\Delta = \{x_1 + x_2 \le 1 : x_1 \ge 0; x_2 \ge 0\}$:

$$\sum_{\substack{\ell_1,\ell_2\in\mathbb{N}\\ \frac{\ell_1(h_1+h_2)}{N}+\frac{\ell_2 h_2}{N}\le 1}} (\ell_1^2+\ell_1\ell_2) \approx$$

$$\approx \int_\Delta \left[\left(\frac{x_1 N}{h_1+h_2}\right)^2 + \left(\frac{x_1 N}{h_1+h_2}\right)\left(\frac{x_2 N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2}dx_1\right)\left(\frac{N}{h_2}dx_2\right) .$$

Multiple zeta-values

We will need the values of the sums

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1,\ldots,n_k \geq 1} \frac{1}{n_1^{s_1}(n_1+n_2)^{s_2}\ldots(n_1+\cdots+n_k)^{s_k}}$$

at positive integers s_j, where $s_k \geq 2$. They are called *multiple zeta-values* and have beautiful properties, which recently attracted a lot of attention by Brown, Cartier, Deligne, Drinfeld, Écalle, Goncharov, Kontsevich, Zagier, to give only some names. We already used *zeta values* as

$$\zeta(2) = \frac{\pi^2}{6}; \qquad \zeta(4) = \frac{\pi^4}{90}; \qquad \zeta(2n) = \frac{p}{q} \cdot \pi^{2n}, \quad \text{where } p, q \in \mathbb{N}\,.$$

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Conjecturally $\pi, \zeta(3), \zeta(5), \dots$ are algebraically independent over $\mathbb{Q}$. However, multiple zeta values satisfy numerous relations, some of them were discovered already by Euler, for example

$$\zeta(1,3) = \frac{1}{4}\zeta(4); \qquad \zeta(2,2) = \frac{3}{4}\zeta(4)\,.$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram

$$\sum_{h_1,h_2} \int_\Delta \left[\left(\frac{x_1 N}{h_1+h_2}\right)^2 + \left(\frac{x_1 N}{h_1+h_2}\right)\left(\frac{x_2 N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2}dx_1\right)\left(\frac{N}{h_2}dx_2\right)$$

$$= N^4 \left[\int_\Delta x_1^2\, dx_1 dx_2 \cdot \sum_{h_1,h_2\in\mathbb{N}} \frac{1}{h_2(h_1+h_2)^3}\right.$$

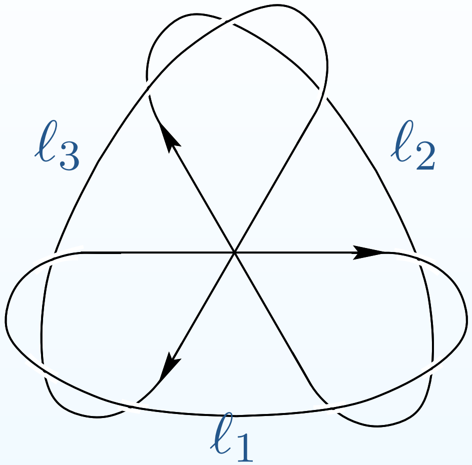
$$\left. + \int_\Delta x_1 x_2\, dx_1 dx_2 \cdot \sum_{h_1,h_2\in\mathbb{N}} \frac{1}{h_2^2(h_1+h_2)^2}\right]$$

$$= \frac{N^4}{24}[2\cdot\zeta(1,3)+\zeta(2,2)] = \frac{N^4}{24}\left[2\cdot\frac{\zeta(4)}{4}+\frac{3\zeta(4)}{4}\right]$$

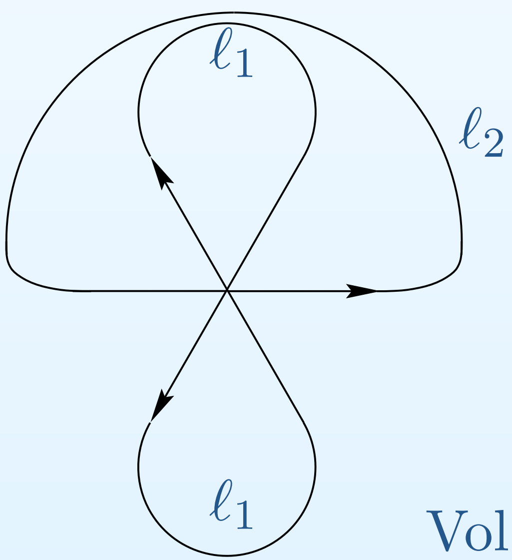
$$= \frac{N^4}{24}\cdot\frac{5}{4}\cdot\frac{\pi^4}{90}.$$

where we used the identities $\zeta(1,3)=\frac{1}{4}\zeta(4), \quad \zeta(2,2)=\frac{3}{4}\zeta(4)$ and the values $\int_\Delta x_1^2\, dx_1 dx_2 = 2\int_\Delta x_1 x_2\, dx_1 dx_2 = 2\cdot\frac{1}{4!}$.

Volume computation for $\mathcal{H}(2)$: summary



$$\frac{1}{3}\sum_{\substack{\ell_1,\ell_2,\ell_3,h\in\mathbb{N}\\(\ell_1+\ell_2+\ell_3)h\le N}}(\ell_1+\ell_2+\ell_3)\approx\frac{N^4}{24}\cdot\zeta(4)$$



$$\sum_{\substack{\ell_1,\ell_2,h_1,h_2\\ \ell_1h_1+(\ell_1+\ell_2)h_2\le N}}\ell_1(\ell_1+\ell_2)$$

$$=\frac{N^4}{24}\big[2\cdot\zeta(1,3)+\zeta(2,2)\big]=\frac{N^4}{24}\cdot\frac{5}{4}\cdot\zeta(4)$$

$$\operatorname{Vol}(\mathcal{H}_1(2))=\lim_{N\to\infty}\frac{2\cdot 4}{N^4}\cdot\text{(Number of surfaces)}=\frac{\pi^4}{120}$$

Contributions $\operatorname{Vol}_k \mathcal{H}(3,1)$ of k-cylinder surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$\operatorname{Vol}_1 \mathcal{H}(3,1) = \frac{\zeta(7)}{15}$$

$$\operatorname{Vol}_2 \mathcal{H}(3,1) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{45}$$

$$\begin{aligned}\operatorname{Vol}_3 \mathcal{H}(3,1) = \frac{1}{90}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\
&+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\
&- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\
&+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\
&- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\
&- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$\operatorname{Vol}_4 \mathcal{H}(3,1) = \frac{2\zeta(2)}{45}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big).$$

After simplification

Multiple zeta values satisfy numerous relations. After simplification (which is now accessible through a SAGE package) we get

$$
\begin{aligned}
\mathrm{Vol}_1\,\mathcal{H}(3,1) &= 1/15\cdot\zeta(7)\\
\mathrm{Vol}_2\,\mathcal{H}(3,1) &= -7/135\cdot\zeta(1,6)+1/135\cdot\zeta(2,5)+23/135\cdot\zeta(7)\\
\mathrm{Vol}_3\,\mathcal{H}(3,1) &= -2/15\cdot\zeta(1,6)-2/45\cdot\zeta(2,5)+1/5\cdot\zeta(6)-4/45\cdot\zeta(7)\\
\mathrm{Vol}_4(\mathcal{H}(3,1) &= 5/27\cdot\zeta(1,6)+1/27\cdot\zeta(2,5)+7/45\cdot\zeta(6)-4/27\cdot\zeta(7)
\end{aligned}
$$

Conjecturally, multiple zeta values involved in these simplified expressions are linearly independent over rational numbers. However, the total contribution is a rational multiple of π^{2g} in accordance with the general result by A. Eskin and A. Okounkov, 2001:

$$\mathrm{Vol}\,\mathcal{H}(3,1)=\mathrm{Vol}_1\,\mathcal{H}(3,1)+\cdots+\mathrm{Vol}_4\,\mathcal{H}(3,1)=\frac{16}{42525}\pi^6$$

Volumes of some low-dimensional strata

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{H}_1(\emptyset)) &= 2\cdot\zeta(2) &&= \frac{1}{3}\cdot\pi^2\\
\mathrm{Vol}(\mathcal{H}_1(2)) &= \frac{2}{3!}\cdot\frac{9}{4}\cdot\zeta(4) &&= \frac{1}{120}\cdot\pi^4\\
\mathrm{Vol}(\mathcal{H}_1(1,1)) &= \frac{1}{4!}\cdot 4\cdot\zeta(4) &&= \frac{1}{135}\cdot\pi^4\\
\mathrm{Vol}(\mathcal{H}_1^{hyp}(4)) &= \frac{2}{5!}\cdot\frac{135}{16}\cdot\zeta(6) &&= \frac{1}{6720}\cdot\pi^6\\
\mathrm{Vol}(\mathcal{H}_1^{odd}(4)) &= \frac{2}{5!}\cdot\frac{70}{3}\cdot\zeta(6) &&= \frac{1}{2430}\cdot\pi^6\\
\mathrm{Vol}(\mathcal{H}_1(1,3)) &= \frac{2}{6!}\cdot 128\cdot\zeta(6) &&= \frac{16}{42525}\cdot\pi^6\\
\mathrm{Vol}(\mathcal{H}_1^{hyp}(6)) &= \frac{2}{7!}\cdot\frac{2625}{64}\cdot\zeta(8) &&= \frac{1}{580608}\cdot\pi^8
\end{aligned}
$$

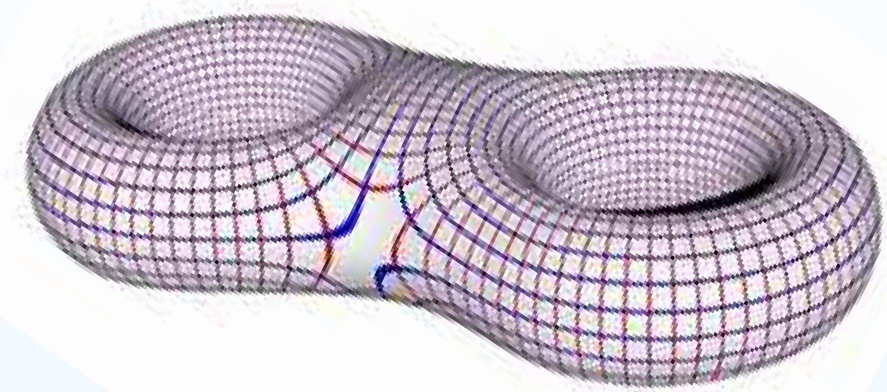
Volumes through multiple zeta values

Conjecture. *Prove that for any connected component of any stratum the contribution to the Masur–Veech volume coming from square-tiled having exatly* k *horizontal cylinders is a linear combination with rational coefficients of multiple zeta values.*

Stronger Conjecture. *Prove the that contribution to the Masur–Veech volume coming from square-tiled corresponding to any fixed separatrix diagram is a linear combination with rational coefficients of multiple zeta values.*

The latter statement is elementary for 1-cylinder separatrix diagrams, simple for 2-cylinder diagrams. It is already a nontrivial theorem (proved by B. Allombert and V. Delecroix) for 3-cylinder diagrams.

Homework assignment



Picture created by Jian Jiang

Questions.

- *To what stratum belongs this square-tiled surface?*
- *Find all realizabe separatrix diagrams for this stratum.*
- *To which of the found diagrams corresponds the square-tiled surface from the picture?*

Outline of approaches to Masur–Veech volumes

Masur–Veech volumes of strata of Abeliand differentials: a historical retrospective

● Around 1998. Masur–Veech volumes of several low-dimensional strata of Abelian differentials were evaluated by M. Kontsevich and A. Zorich through straightforward count of square-tiled surfaces.

● Around 2001. A. Eskin and A. Okounkov found a much more efficient approach based on quasimodularity of an associated generating function. A. Eskin wrote a computer code giving volumes of all strata in genera at most 10 and of some strata in genera up to 200.

● 2020. D. Chen, M. Möller, A. Sauvaget and D. Zagier obtained very important advances based on recent BCGGM smooth compactification of the moduli space of Abelian differentials. They developed intersection theory of relevant moduli spaces and found a recursive formula for volumes.

● 2018–2020. D. Chen–M. Möller–A. Sauvaget–D. Zagier and independently A. Aggarwal obtained spectacular results on large genus asymptotics of Masur–Veech volumes uniform for all strata stratum of Abelian differentials proving a conjecture by A. Eskin and of A. Zorich based on their numerical experiments from 2003.

Masur–Veech volumes of strata of quadratic differentials: a brief historical retrospective

The knowledge of Masur–Veech volumes $\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)$ of strata of *quadratic* differentials is still limited.

• Around 1998-2000. Masur–Veech volumes of several low-dimensional strata of quadratic differentials were evaluated by A. Zorich through straightforward count of square-tiled surfaces.

• 2001. A. Eskin and A. Okounkov found a much more efficient approach based on quasimodularity of the generating function counting *pillowcase covers*. However, the resulting expressions contain huge tables of characters of the symmetric group, which makes the computation inefficient. The algorithm is more involved than for Abelian differentials.

• 2016. The algorithm of A. Eskin and A. Okounkov was implemented by E. Goujard. She wrote a code and computed volumes of all strata up to dimension 12.

Masur–Veech volumes of strata of quadratic differentials: a brief historical retrospective

- 2016. J. Athreya–A. Eskin–A. Zorich obtained a close expression (conjectured by M. Kontsevich) for the Masur–Veech volume of any stratum in genus zero through the formula of A. Eskin–M. Kontsevich–A. Zorich for the sum of Lyapunov exponents combined with some combinatorial considerations.
- 2019. V. Delecroix–E. Goujard–P. Zograf–A. Zorich computed volumes of the principal strata (the ones containing only simple zeroes and poles) in terms of Witten–Kontsevich correlators.
- 2019. D. Chen–M. Möller–A. Sauvaget expressed volumes of the principal strata in terms of certain Hodge integrals.
- 2019. J. Andersen–G. Borot–S. Charbonnier–V. Delecroix–A. Giacchetto–D. Lewanski–C. Wheeler used the DGZZ-formula to compute volumes through topological recursion.
- 2020. M. Kazarian and independently Di Yang–D. Zagier–Y. Zhang developed efficient recursion for the Hodge integrals involved in the CMS-formula.
- 2021. A. Aggarwal derived the large genus asymptotics for the volumes of principal strata conjectured by V. Delecroix–E. Goujard–P. Zograf–A. Zorich.

Open problem: volumes of strata of quadratic differentials

Let $\boldsymbol{d} = (d_1, \dots, d_n)$ be an unordered partition of a positive integer number $4g-4$ divisible by 4 into a sum $|\boldsymbol{d}| = d_1 + \dots + d_n = 4g-4$, where $d_i \in \{-1, 0, 1, 2, \dots\}$ for $i = 1, \dots, n$. Denote by $\hat{\Pi}_{4g-4}$ the set of those partitions as above, which satisfy the additional requirement that the number of entries $d_i = -1$ in $\boldsymbol{d}$ is at most $\log(g)$.

Open problem. *Find the Masur–Veech volume of strata $\mathcal{Q}(d_1, \dots, d_n)$ of meromorphic quadratic differentials with at most simple poles when at least one od d_i is even. Prove the following conjectural asymptotic formula (currently proved by A. Aggarwal only for the principal stratum): for any $\boldsymbol{d} \in \hat{\Pi}_{4g-4}$ one has*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_n) = \frac{4}{\pi} \cdot \prod_{i=1}^{n} \frac{2^{d_i+2}}{d_i+2} \cdot \big(1 + \varepsilon_1(\boldsymbol{d})\big)\,,$$

where

$$\lim_{g \to \infty} \max_{\boldsymbol{d} \in \hat{\Pi}_{4g-4}} |\varepsilon_1(\boldsymbol{d})| = 0\,.$$

For strata of dimension up to 12 the volumes are found by E. Goujard using Eskin–Okounkov algorithm.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