Geometry and dynamics in moduli spaces

Lecture 7. Train tracks. Integral measured laminations. Idea of the proof of Mirzakhani's count of simple closed geodesics

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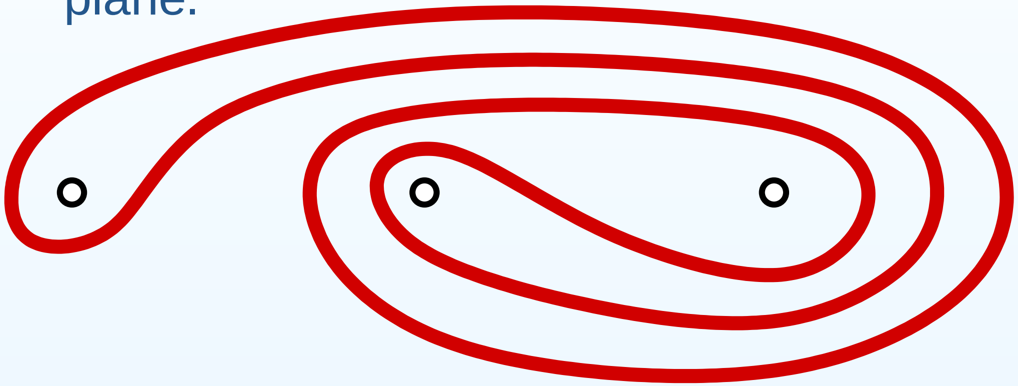
Université Paris Cité

March 24, 2026

Space of multicurves

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



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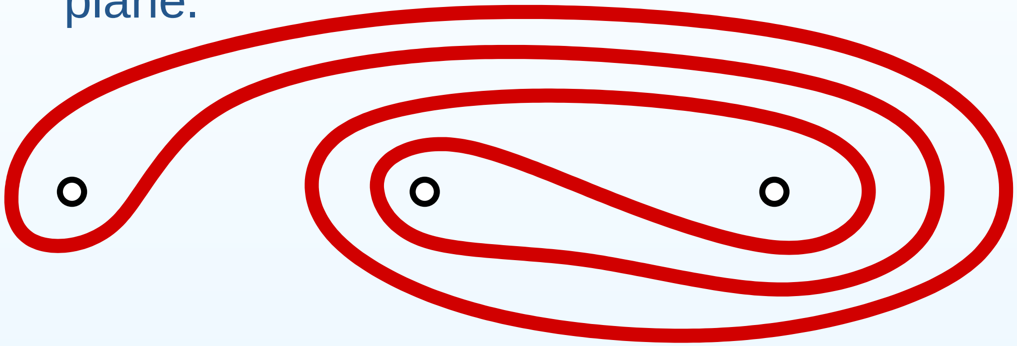

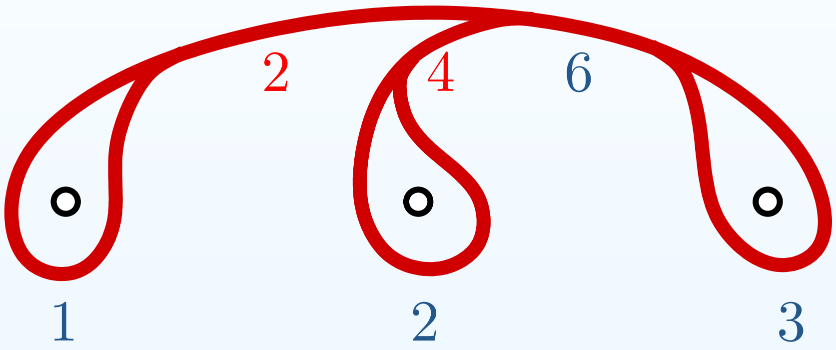

We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

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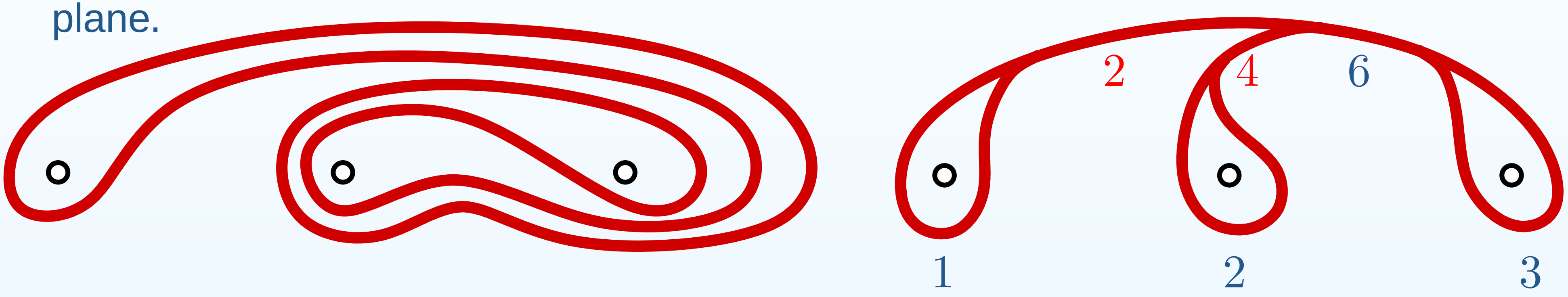

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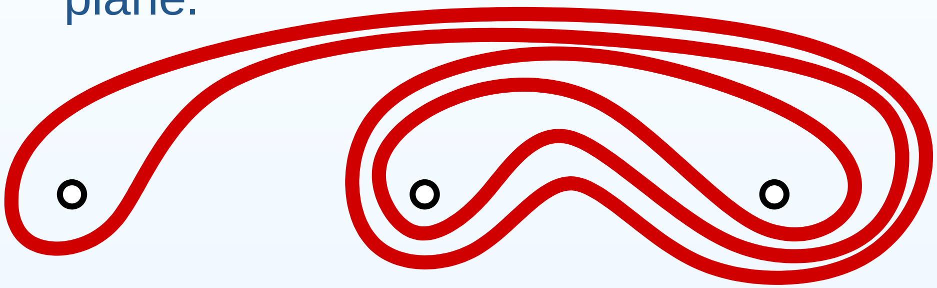

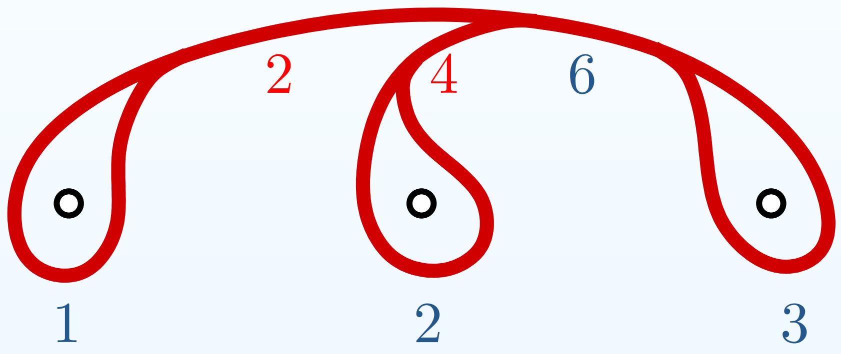

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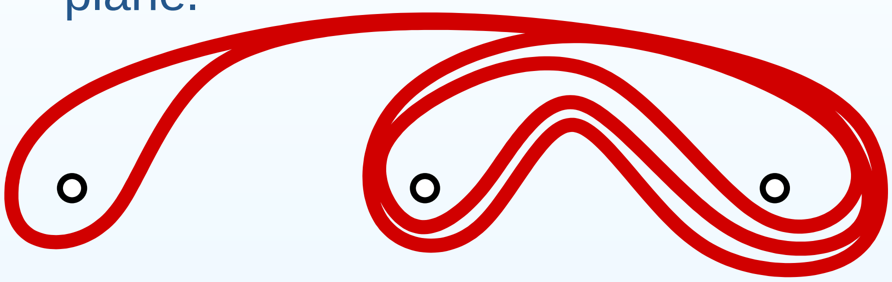

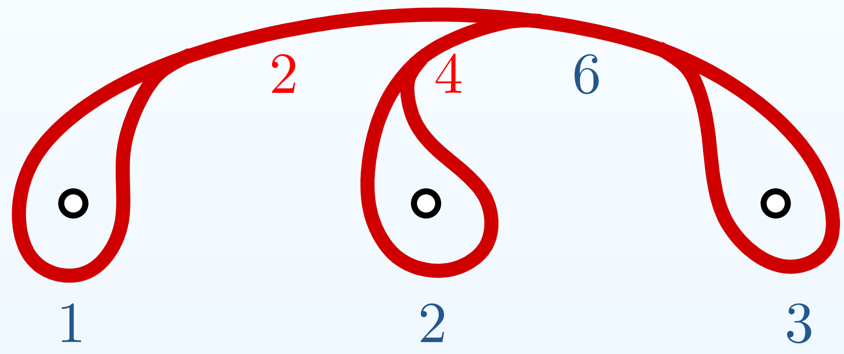

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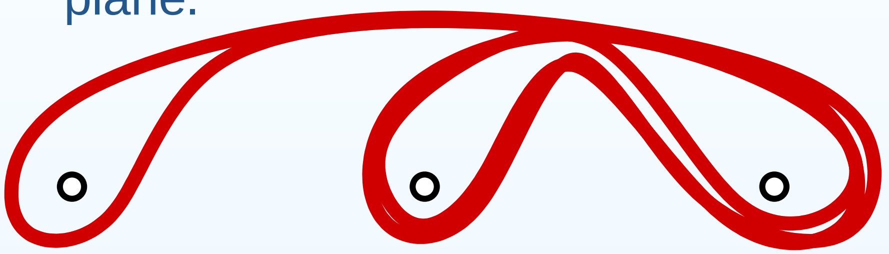

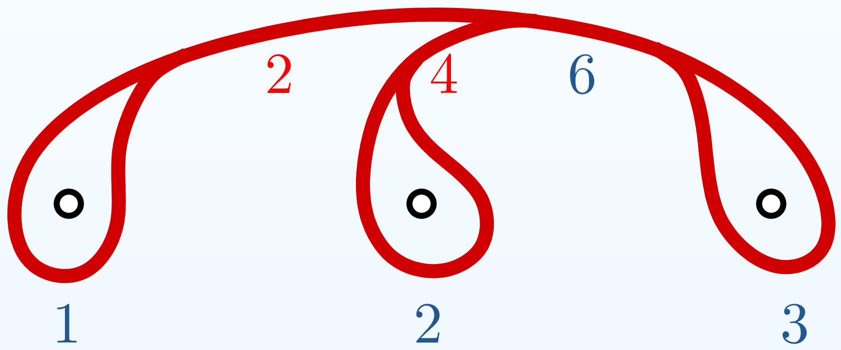

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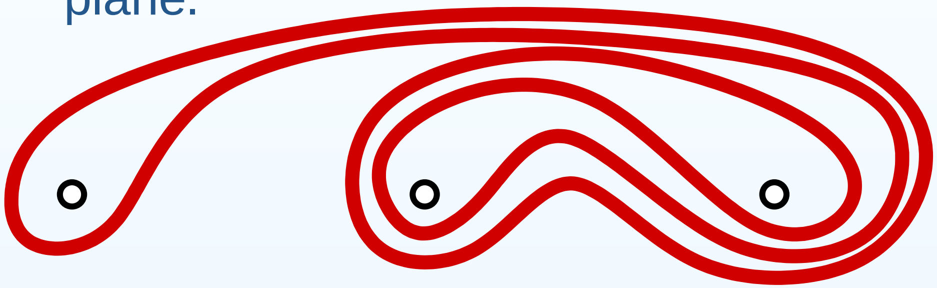

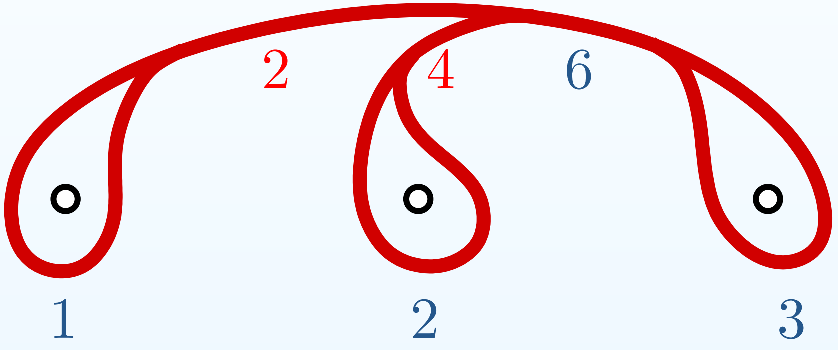

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Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

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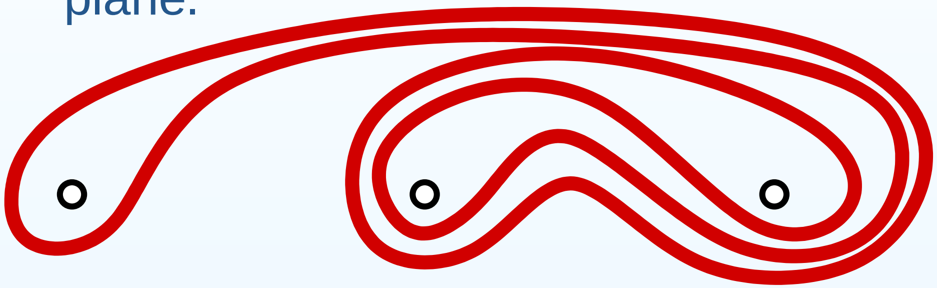

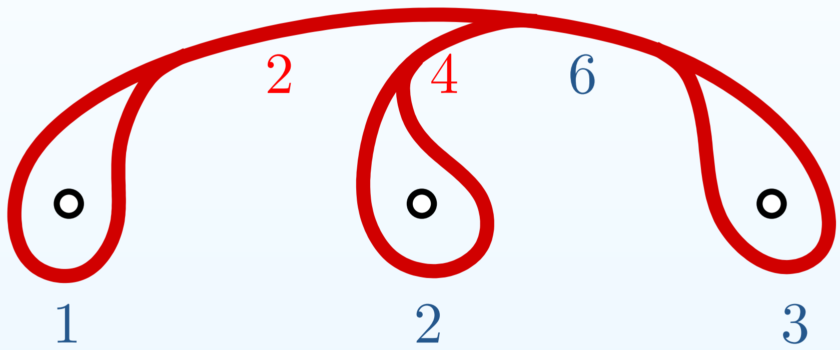

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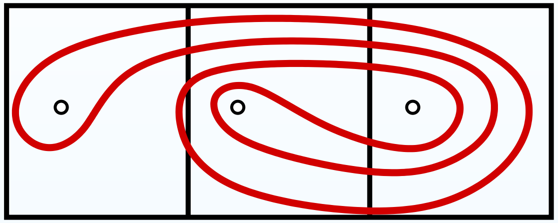
Note that the two weights in red uniquely determine all other weights.

Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



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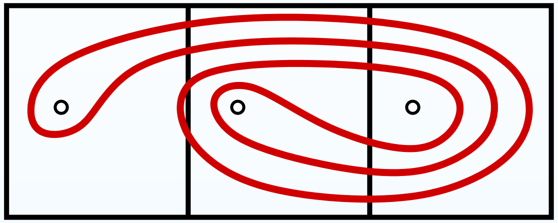

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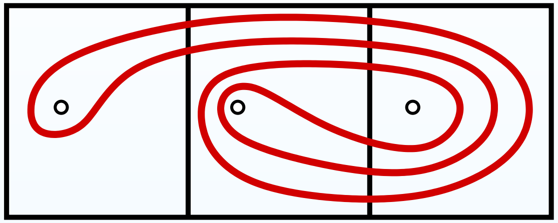

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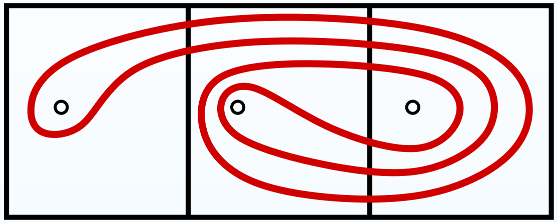

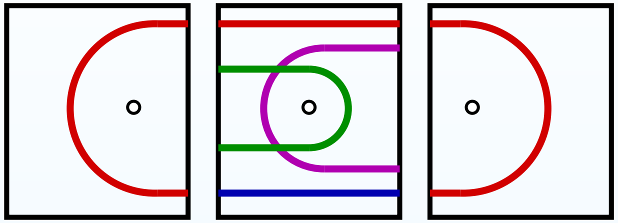

By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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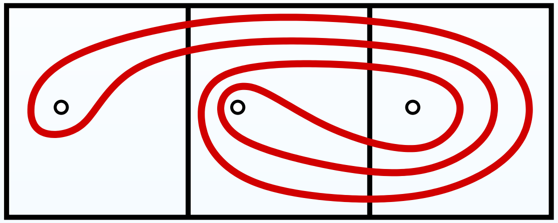
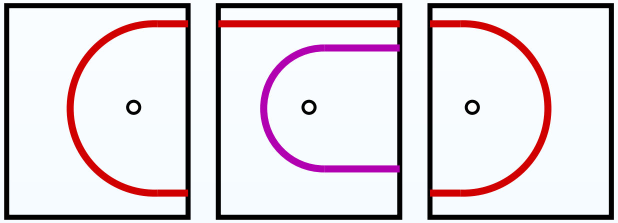
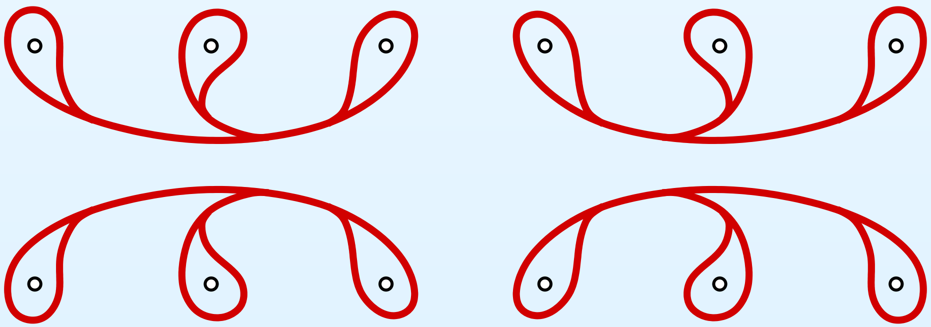
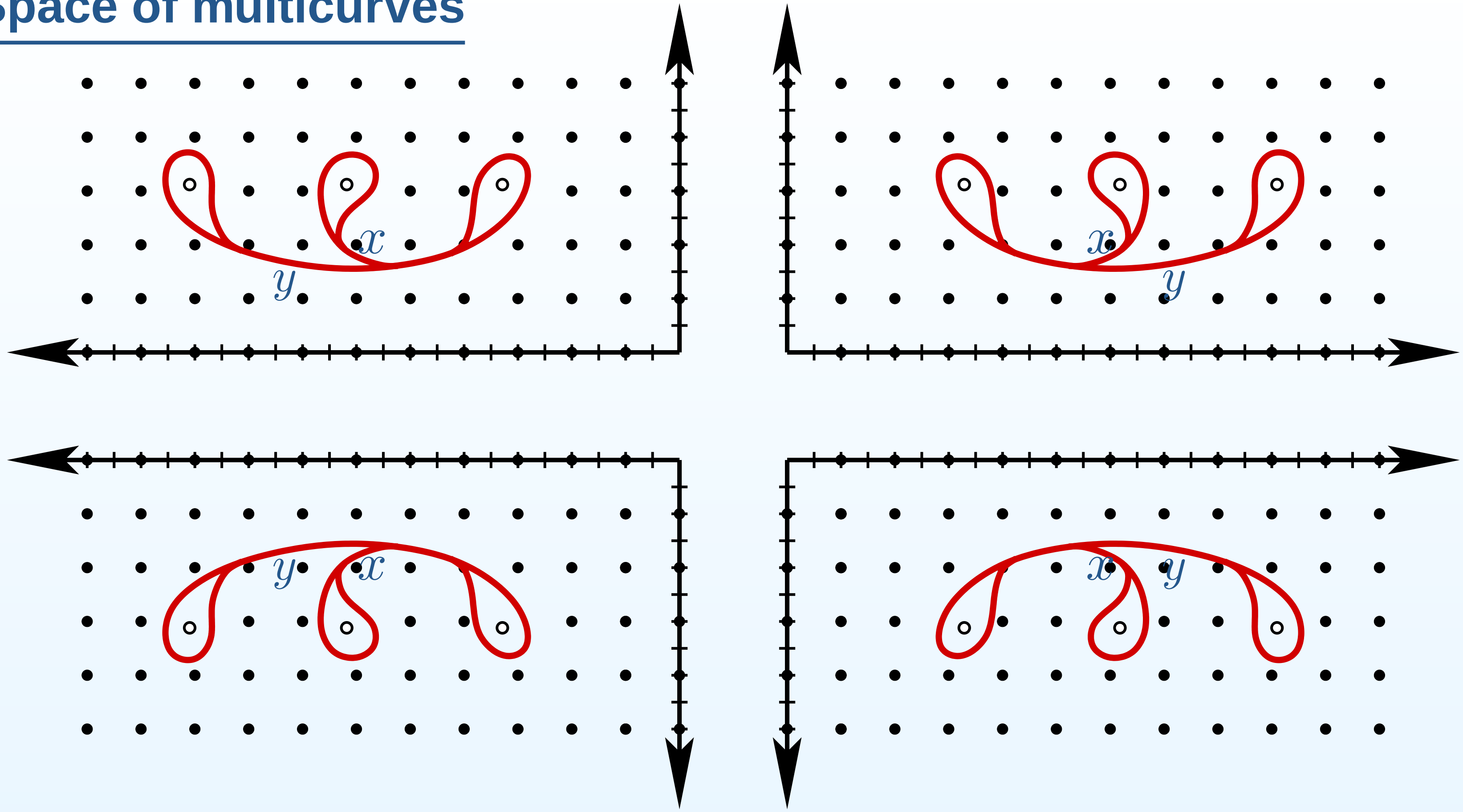




By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those. Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve is carried by one of the following four train tracks:



Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

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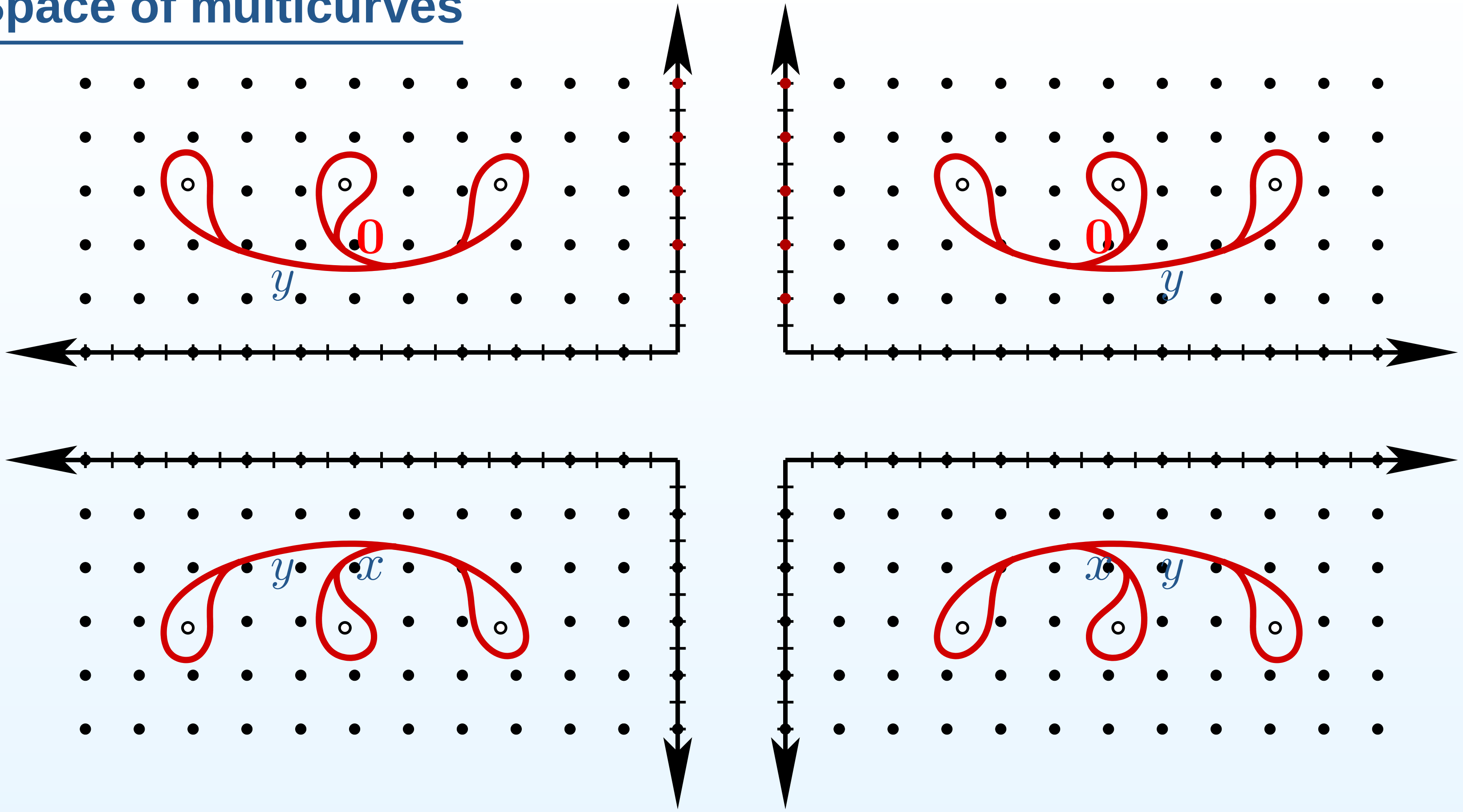

Weight zero on an edge of a train track tells that such edge can be deleted.
This implies that pairs of quadrants should be identified along their edges.

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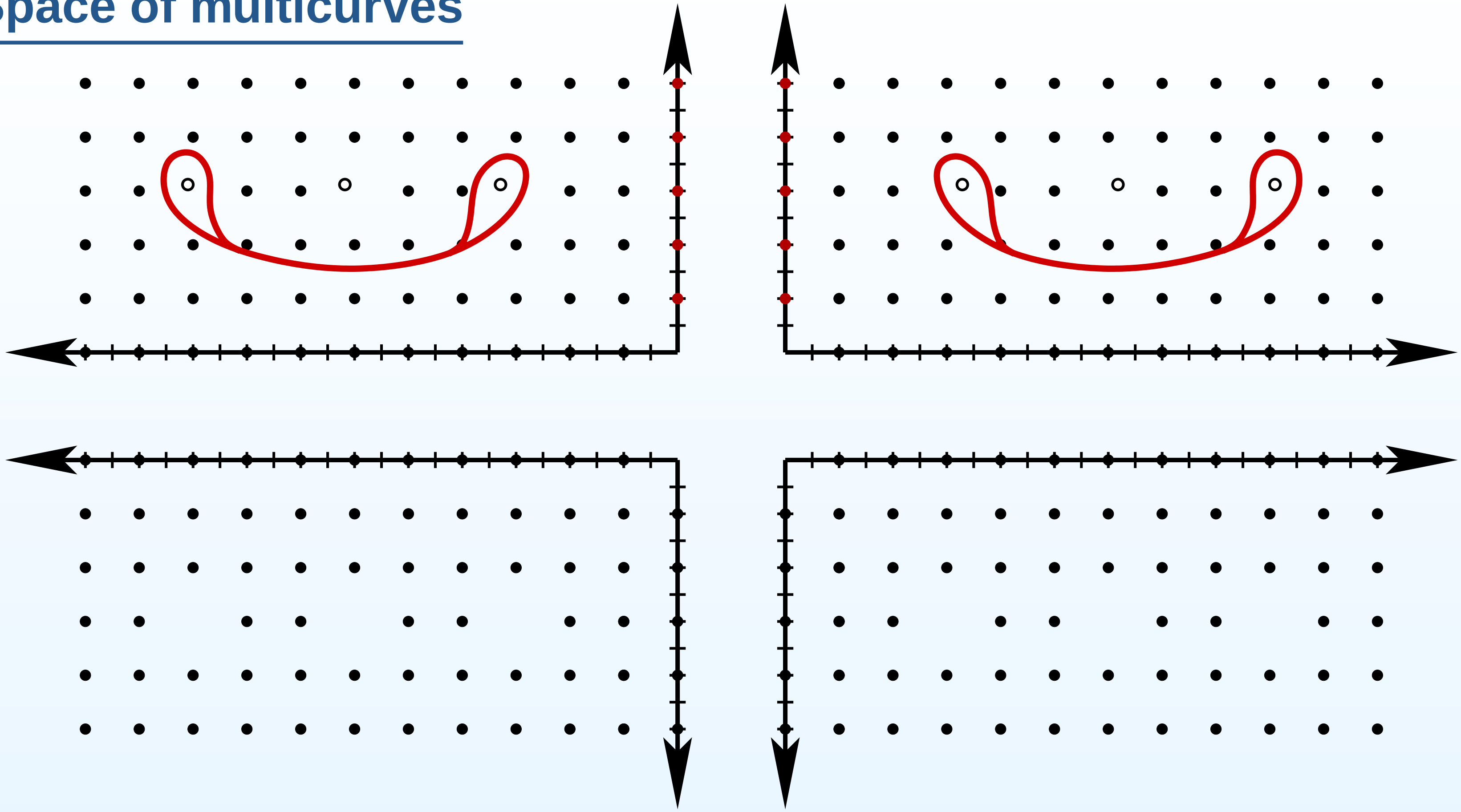

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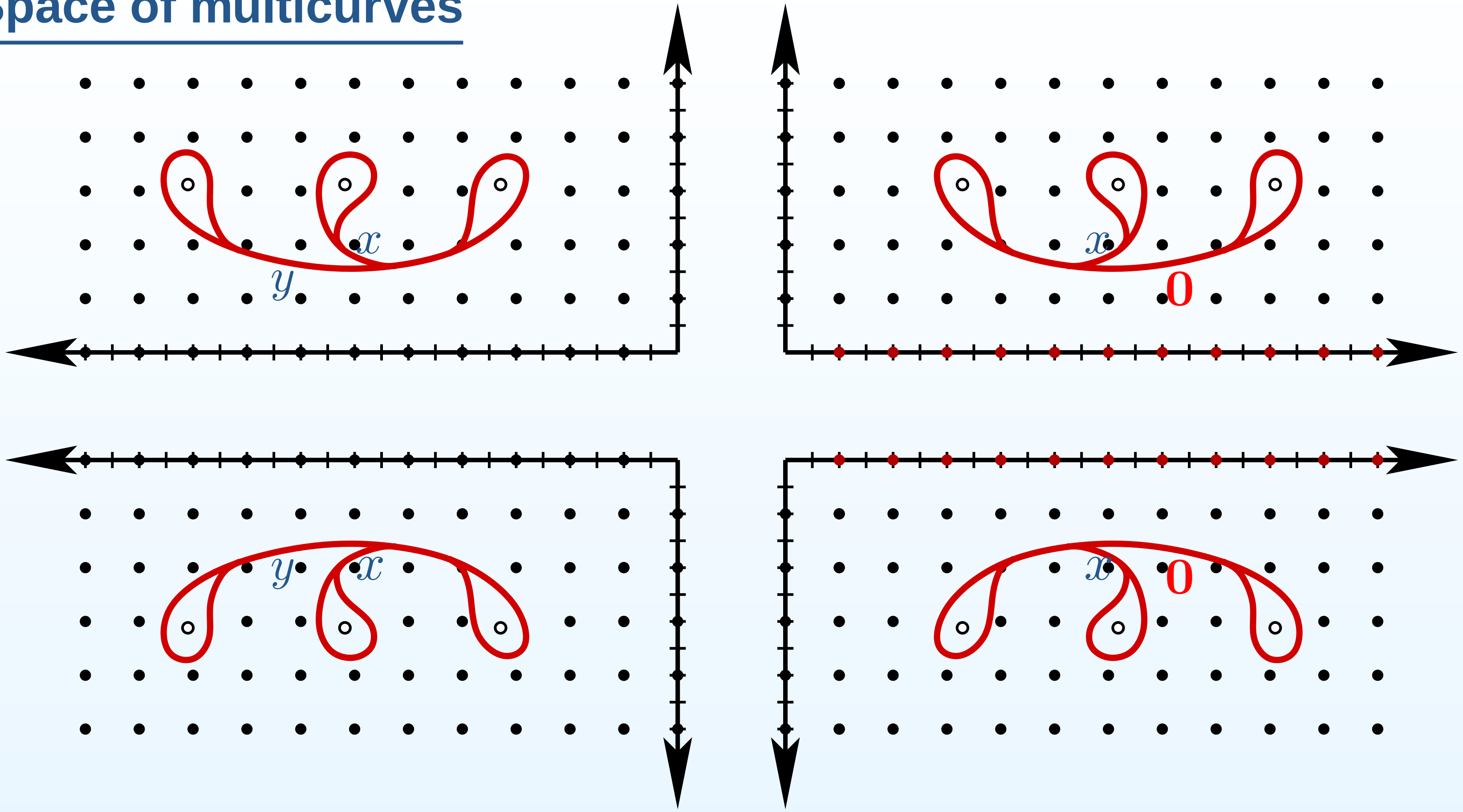

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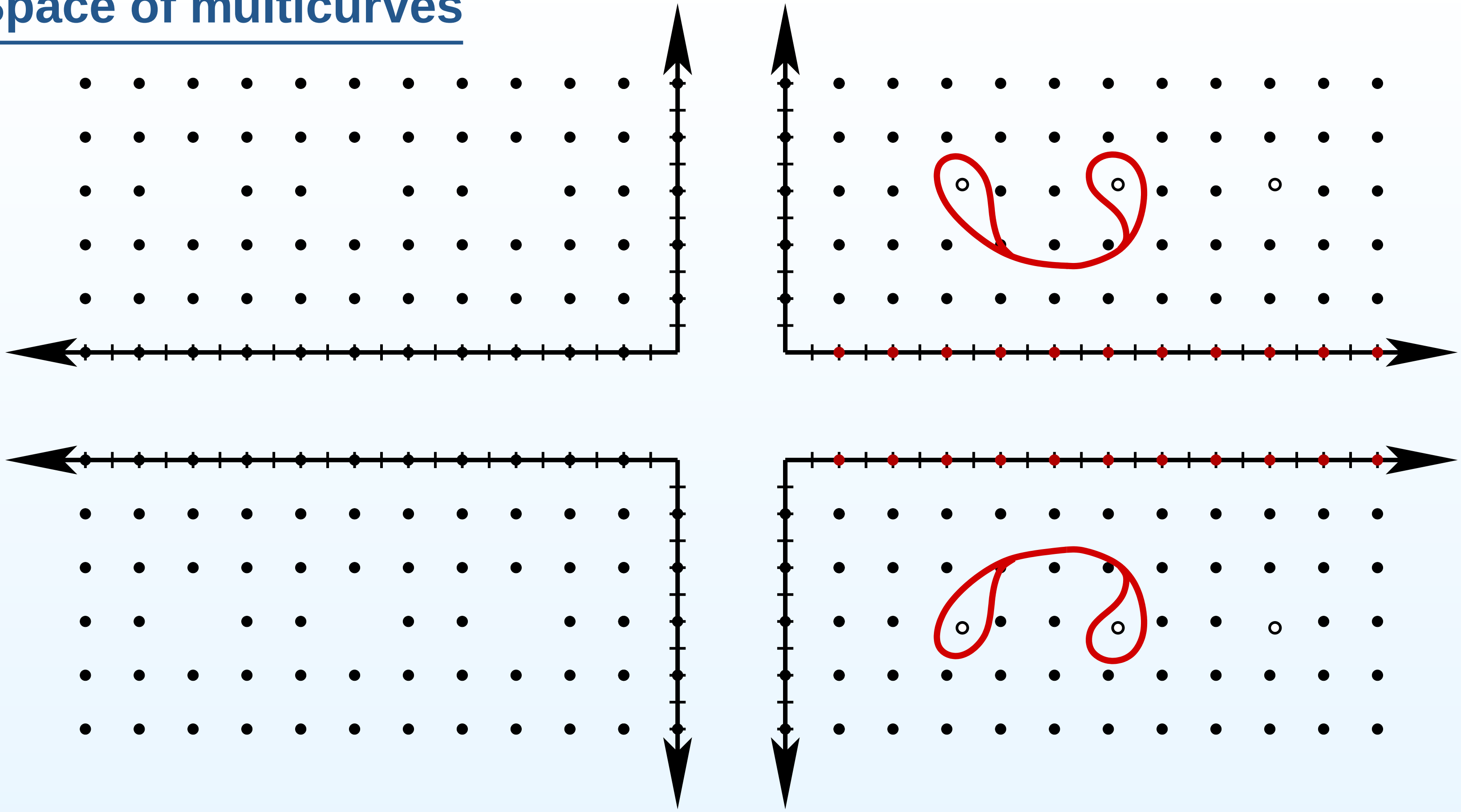

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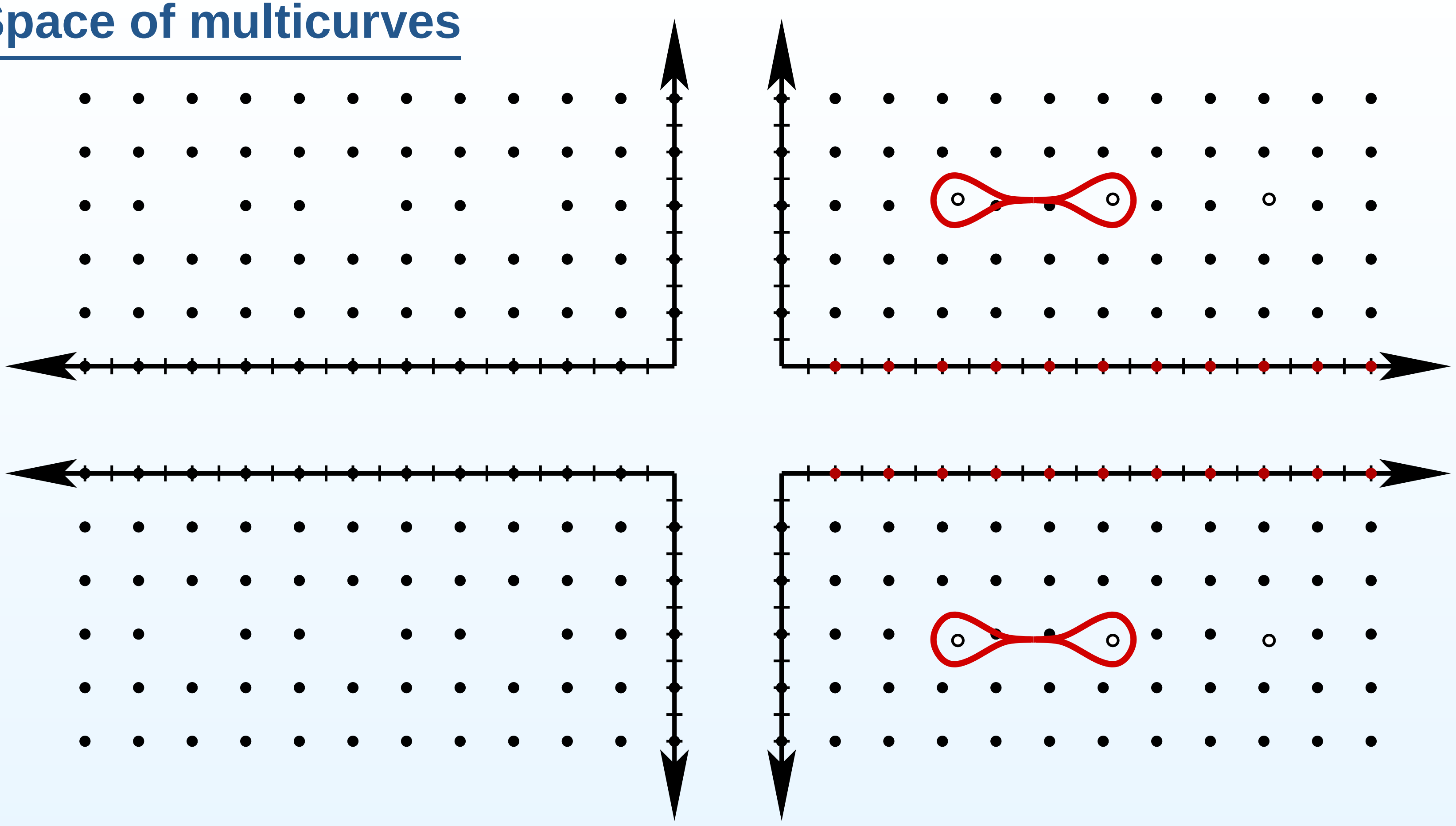

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The resulting space is homeomorphic to $\mathbb{R}^2$. We do not have the structure of a vector space on this $\mathbb{R}^2$, but we have a structure of a *polyhedral cone*. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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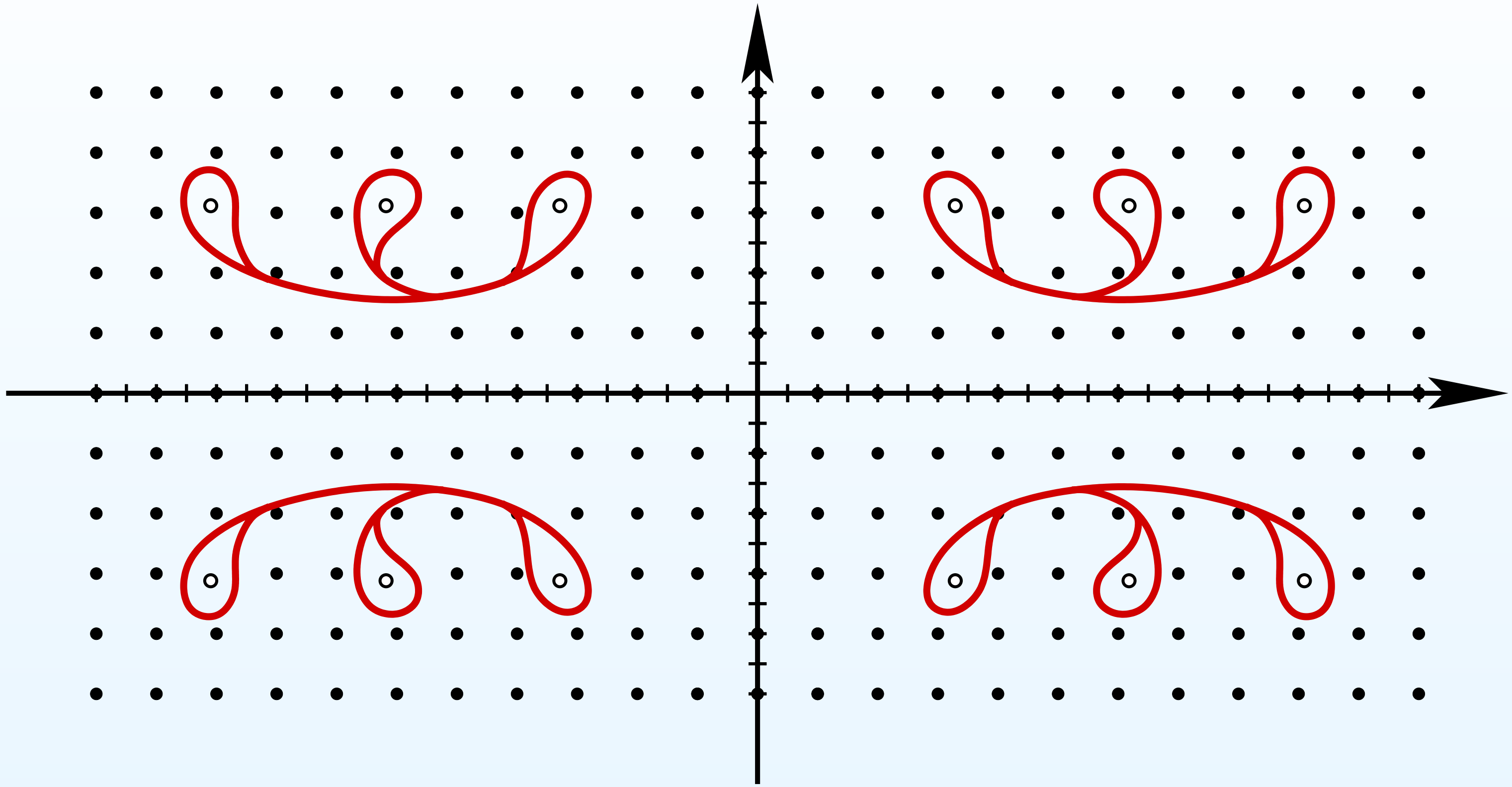

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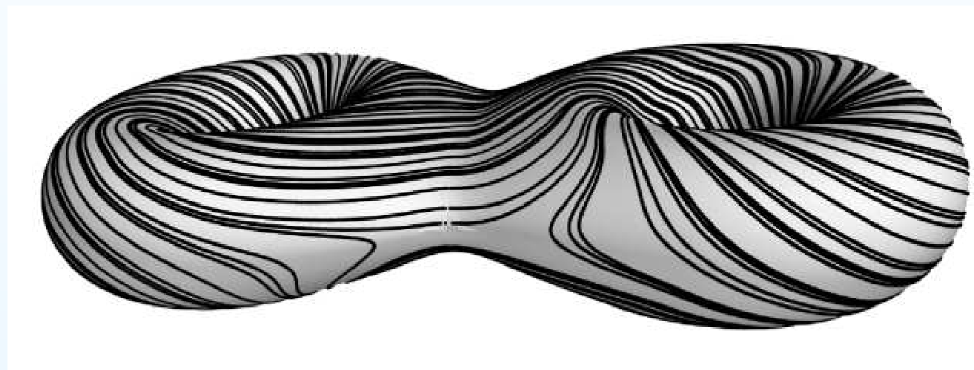
Thurston's and Mirzakhani's measures on $\mathcal{ML}_{g,n}$

Orbits of multicurves

Thurston suggested to consider simple closed multicurves as integral points in the piecewise-linear space of measured laminations. All integral multicurves are partitioned in orbits under action of the mapping class group.

A general multicurve ρ:



the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

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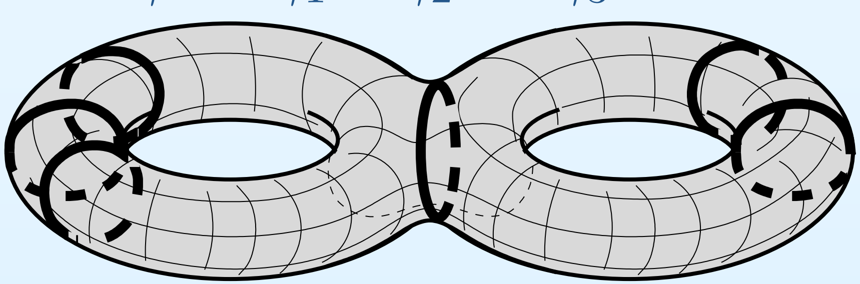

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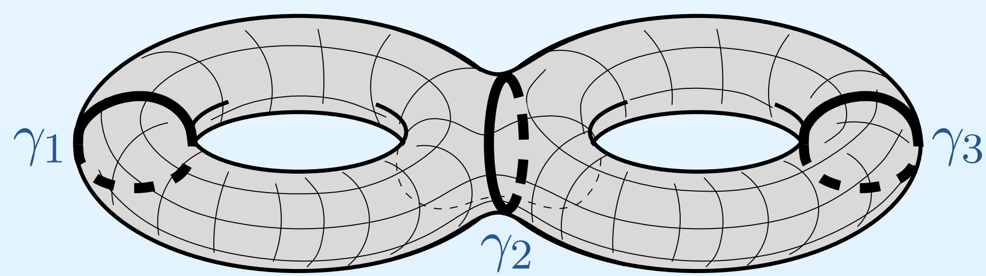
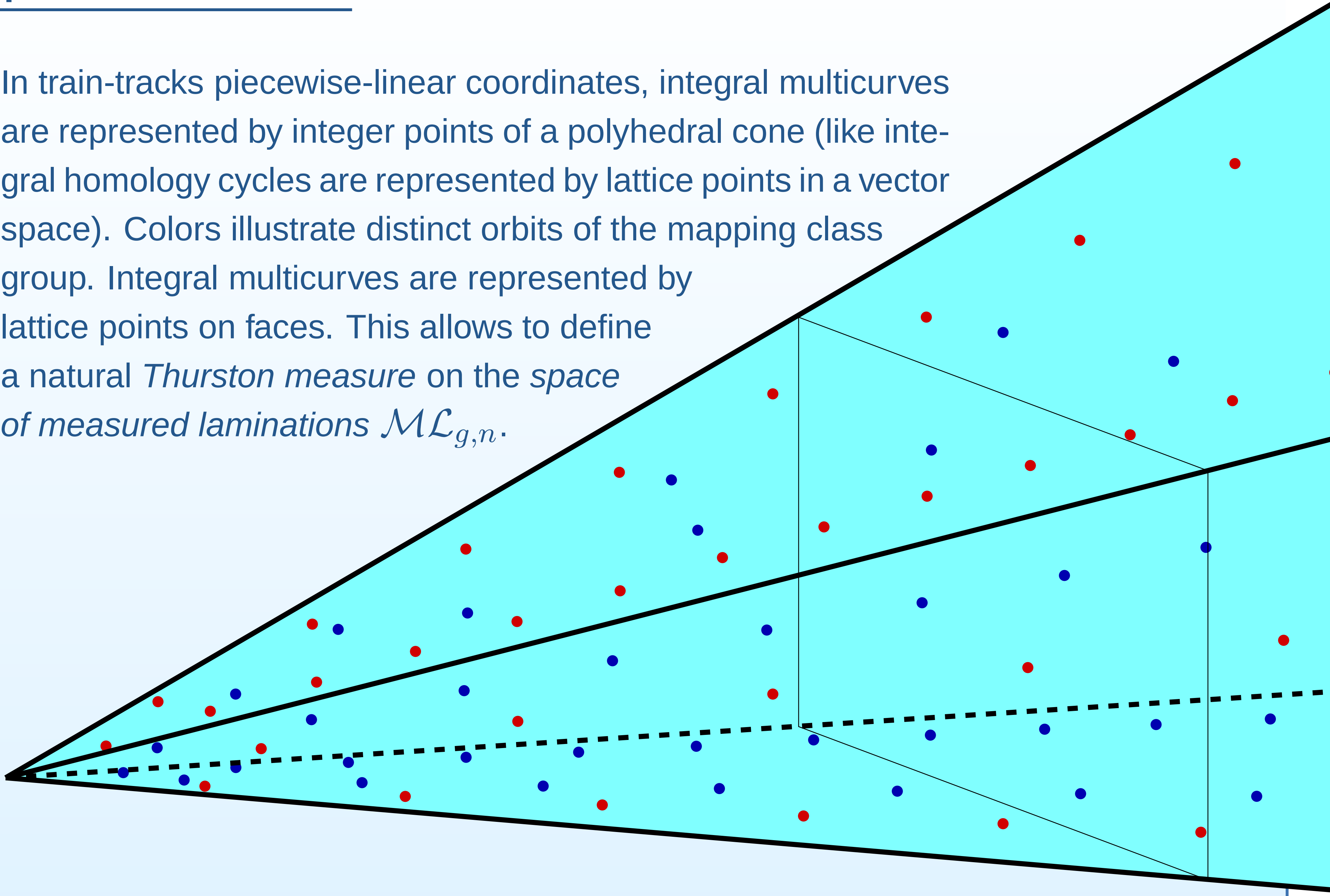


Space of multicurves

In train-tracks piecewise-linear coordinates, integral multicurves are represented by integer points of a polyhedral cone (like integral homology cycles are represented by lattice points in a vector space). Colors illustrate distinct orbits of the mapping class group. Integral multicurves are represented by lattice points on faces. This allows to define a natural *Thurston measure* on the *space of measured laminations* $\mathcal{ML}_{g,n}$.



Thurston measure on $\mathcal{ML}_{g,n}$

One can give sense not only to integer or rational, but to *all* points of the corresponding polyhedral cone and, following Bill Thurston, define a *space of measured laminations* $\mathcal{ML}_{g,n}$. Train track charts define *piecewise linear structure* on $\mathcal{ML}_{g,n}$. Integral multicurves define an integral lattice $\mathcal{ML}_{g,n}(\mathbb{Z})$ in $\mathcal{ML}_{g,n}$. This lattice is defined independently of coordinates. The lattice $\mathcal{ML}_{g,n}(\mathbb{Z})$ provides canonical normalization of the linear volume form μ_{Th} on $\mathcal{ML}_{g,n}$ in which the fundamental domain of the lattice has unit volume.

One can check that the action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is piecewise-linear. It clearly sends multi-curves to multi-curves. Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves. Hence, the action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the “integral lattice” $\mathcal{ML}_{g,n}(\mathbb{Z})$. Hence, it preserves the Thurston’s measure μ_{Th}.

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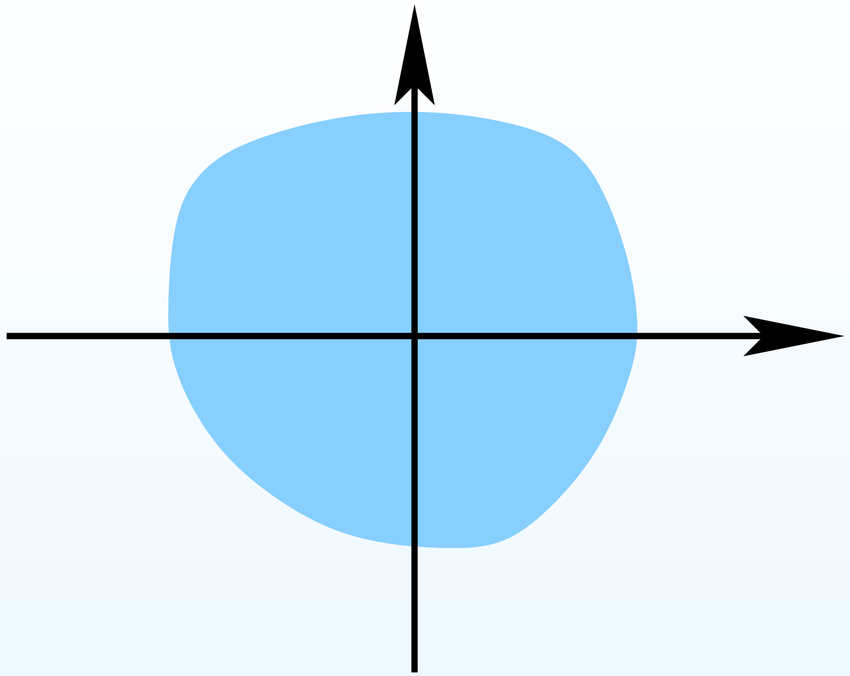
Theorem (H. Masur’85). *The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is ergodic with respect to the Lebesgue measure class (i.e. any measurable subset of $\mathcal{ML}_{g,n}$ invariant under $\mathrm{Mod}_{g,n}$ has measure zero or its complement has measure zero). Any $\mathrm{Mod}_{g,n}$-invariant measure in the Lebesgue measure class is just Thurston measure rescaled by some constant factor.*

Lebesgue measure of a set

By definition, the Lebesgue measure $\mu(U)$ of a set $U \subset \mathbb{R}^n$ is defined as the limit of the normalized number of points of the ε-grid which get to U:

$$\mu(U) := \lim_{\varepsilon \to 0} \varepsilon^n \cdot \mathrm{card}(U \cap \varepsilon \mathbb{Z}^n) .$$



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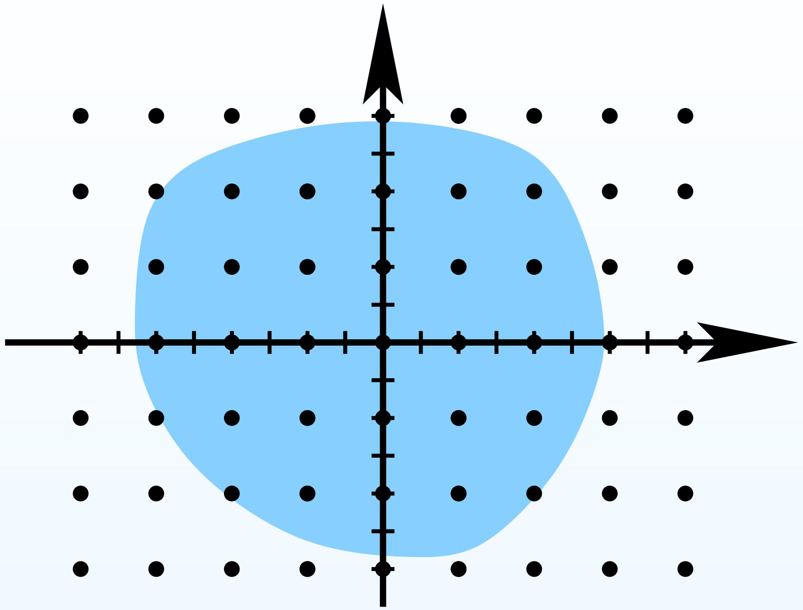

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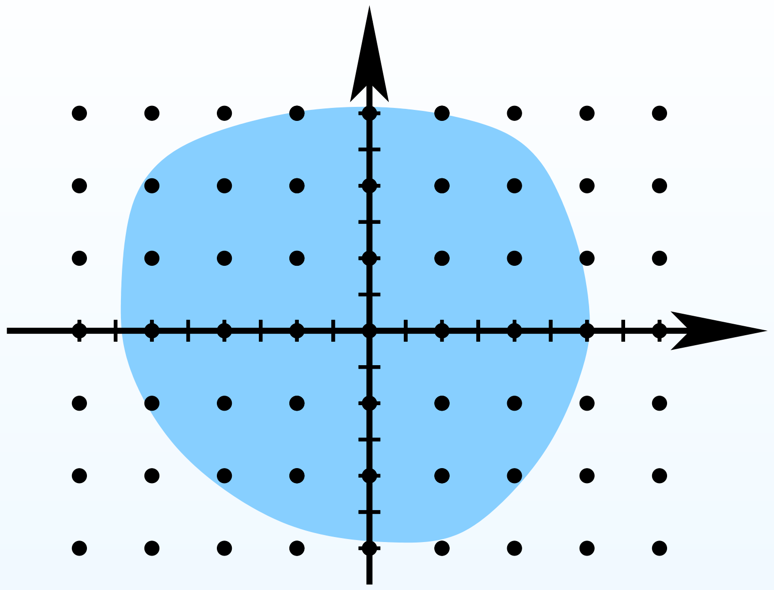

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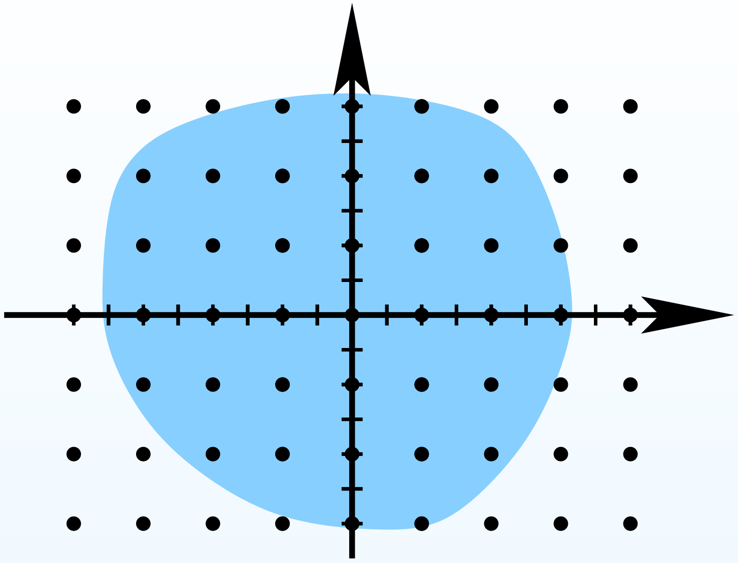

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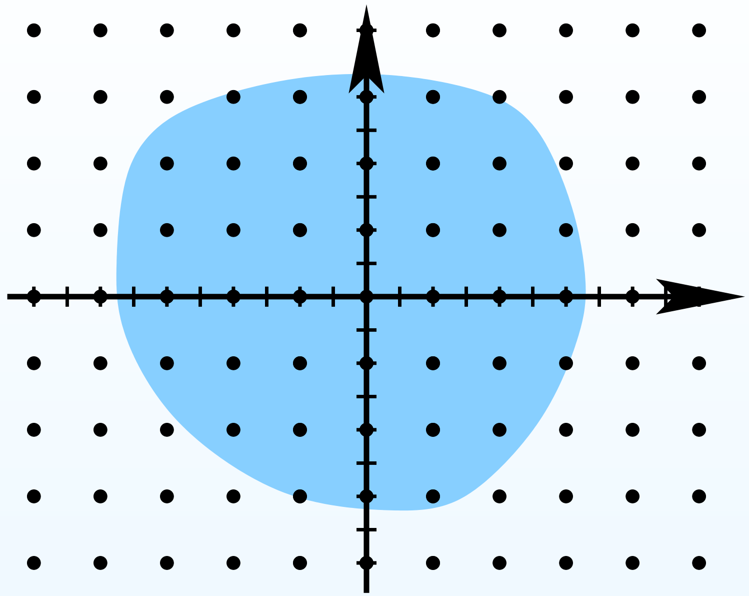

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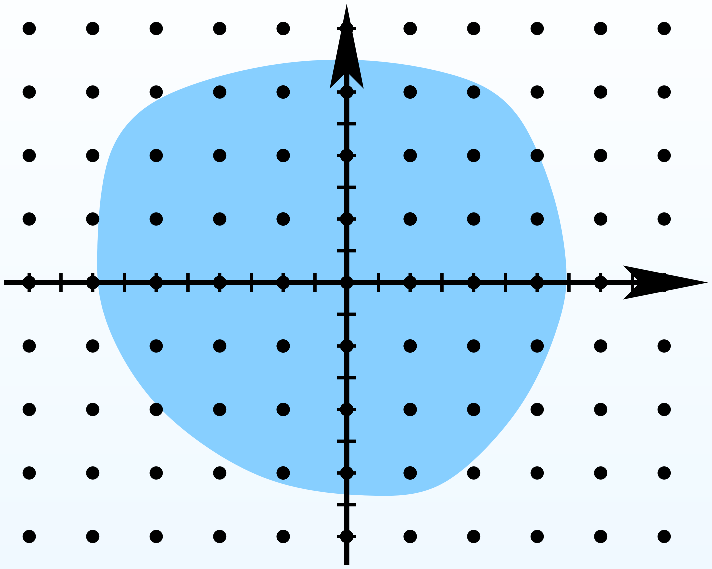

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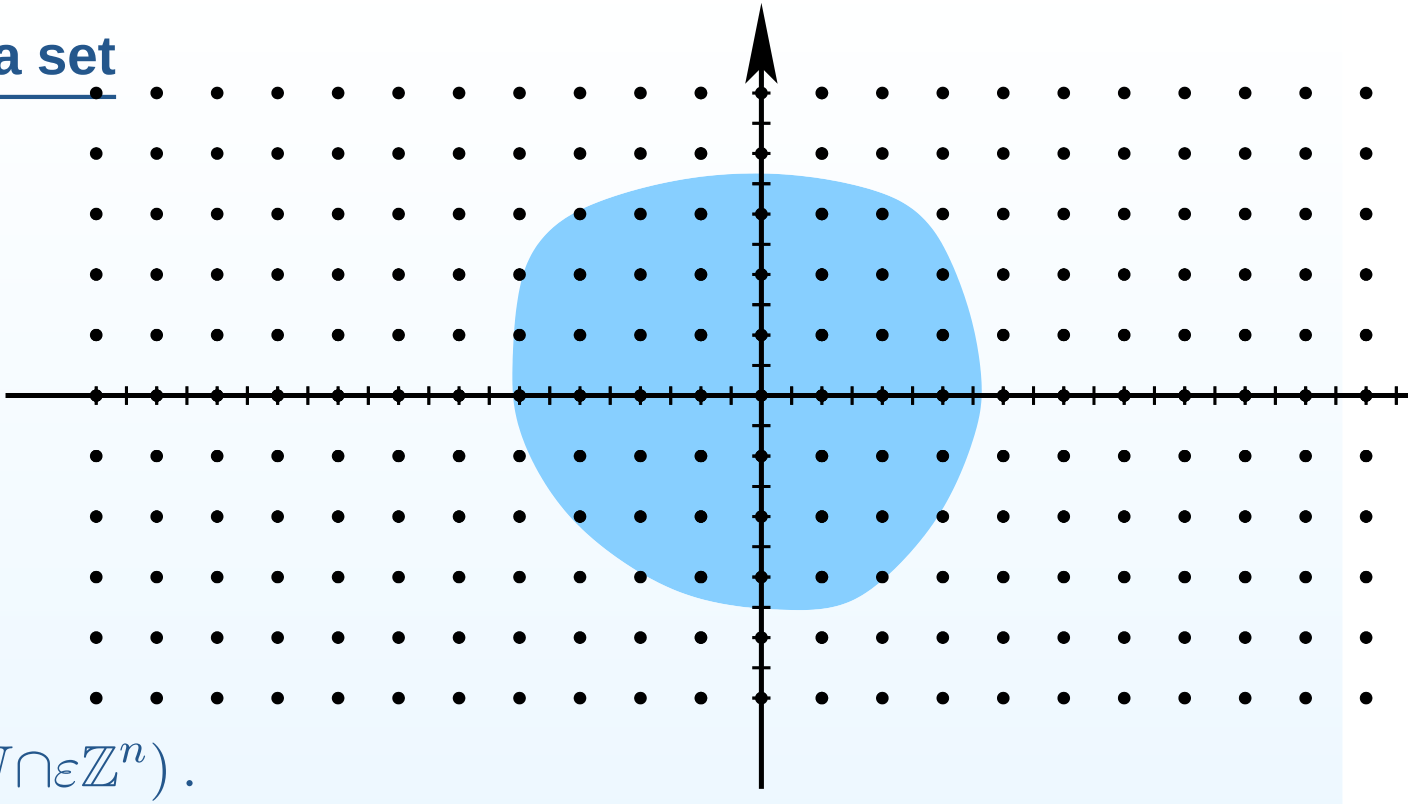

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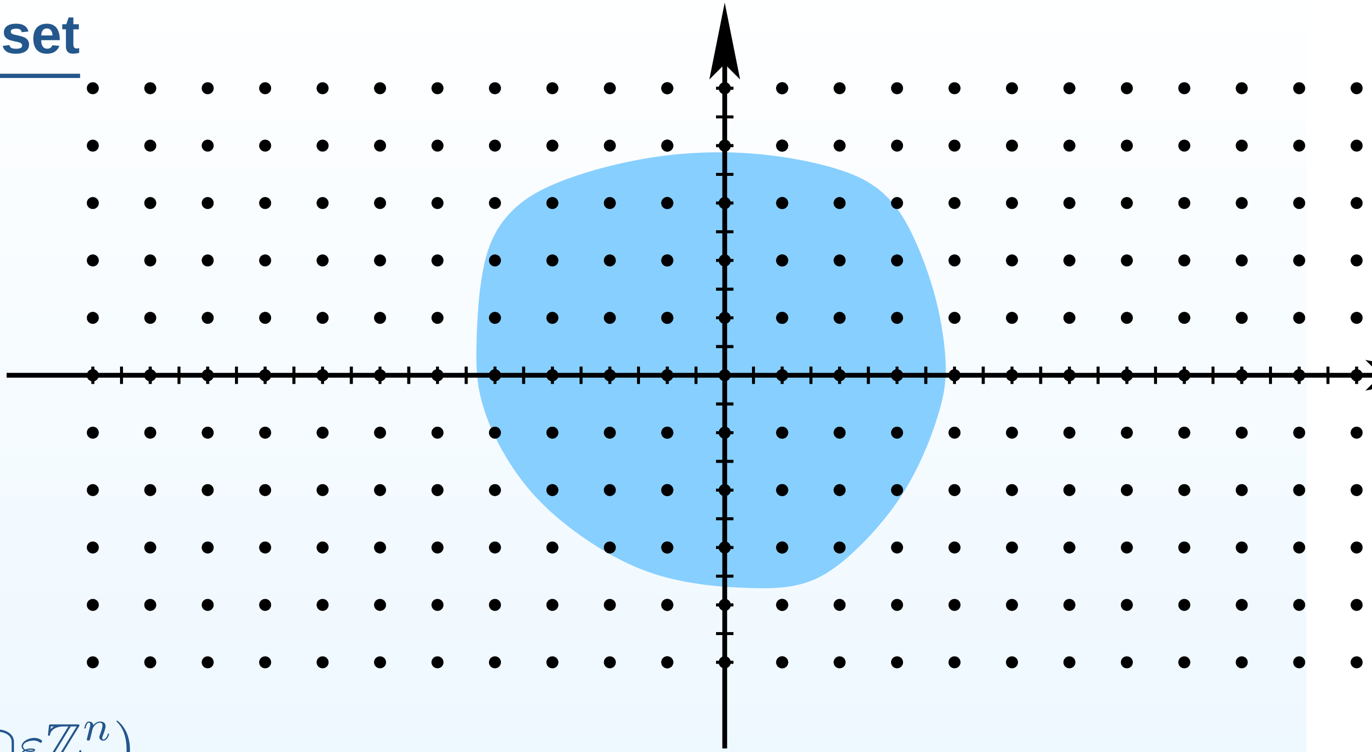

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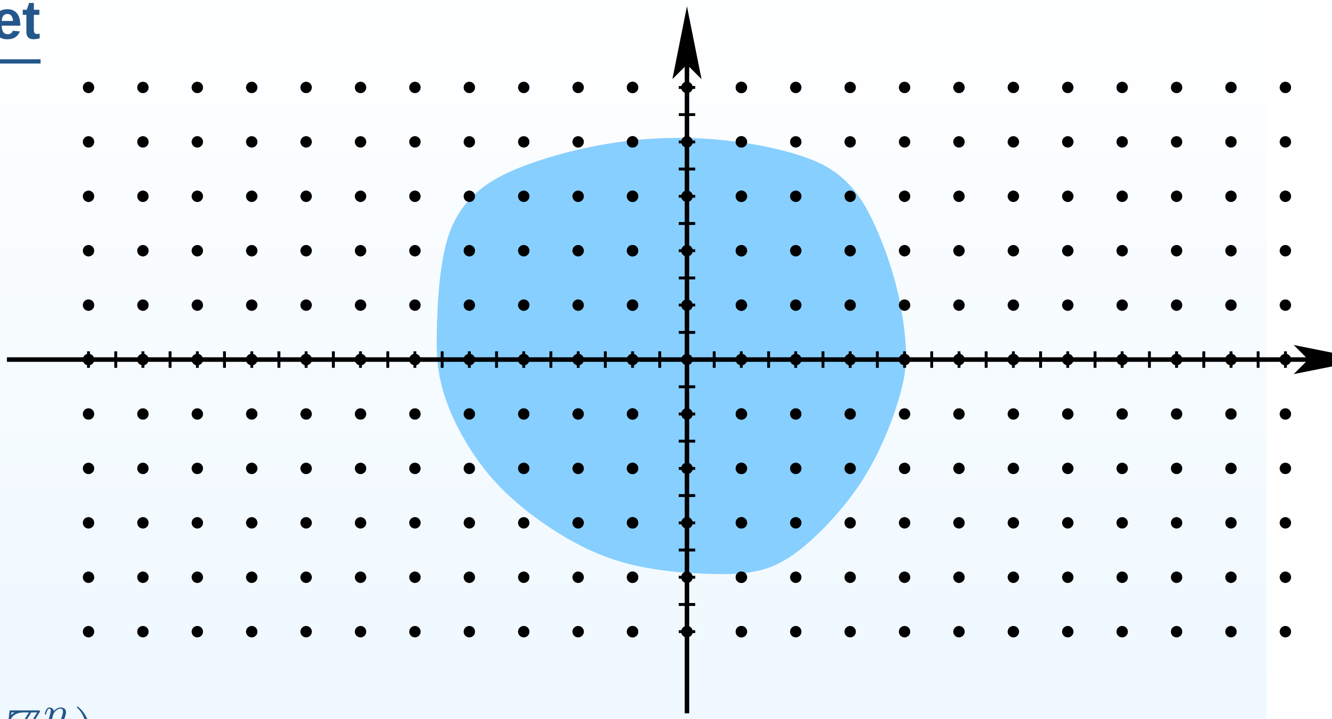

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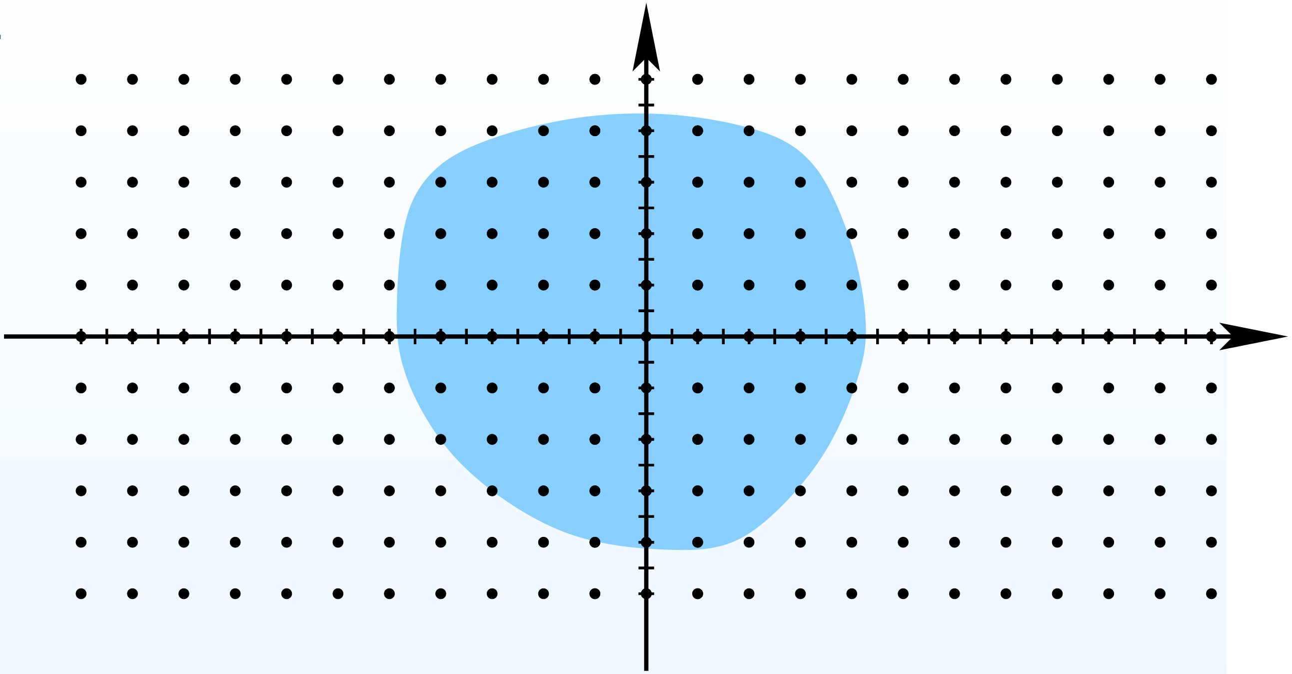

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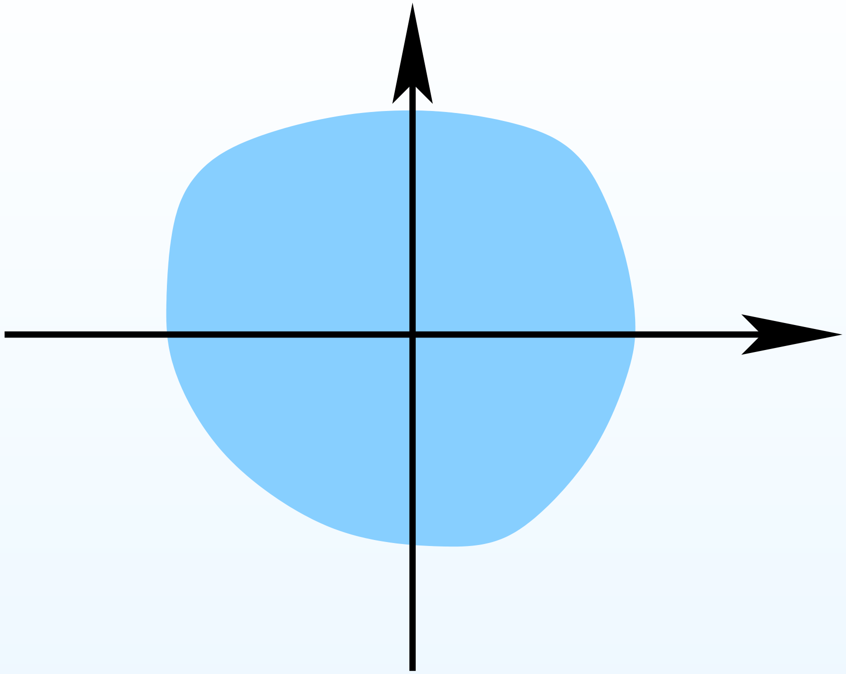

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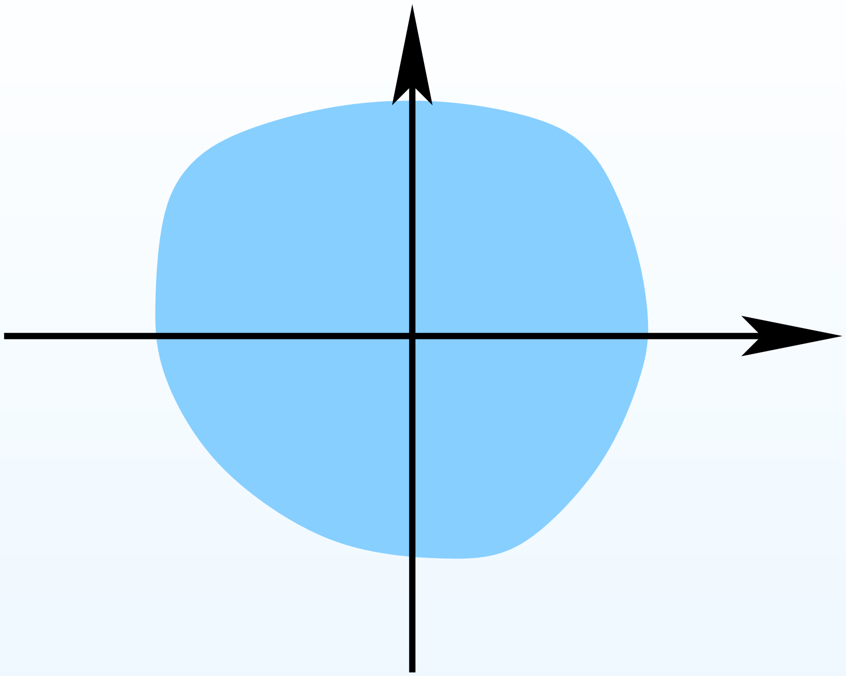

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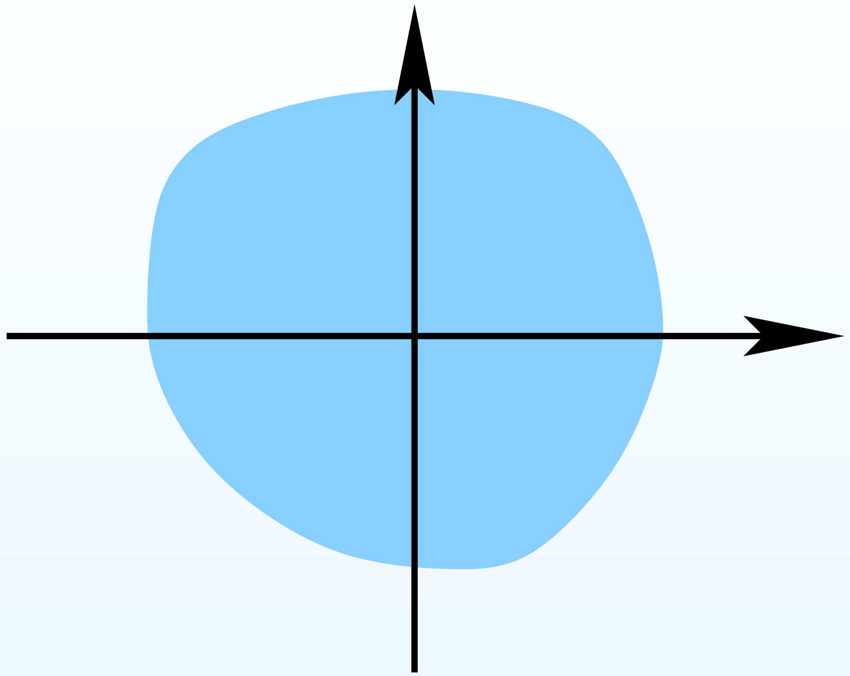

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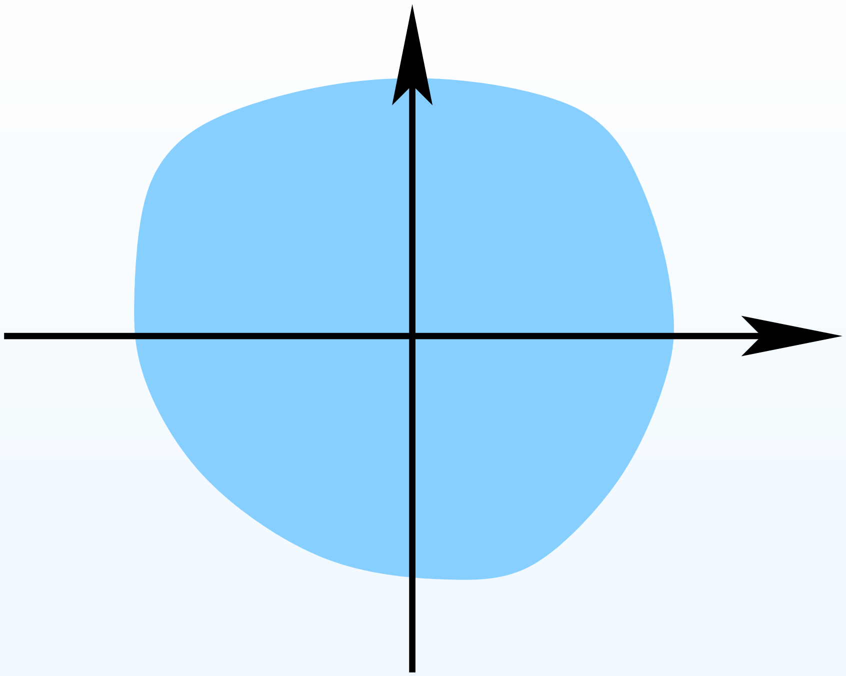

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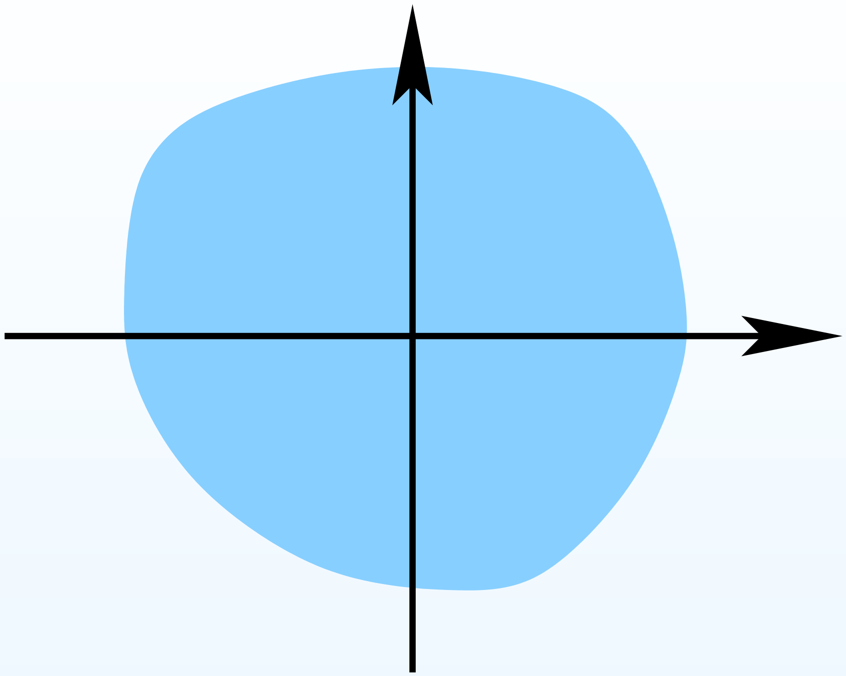

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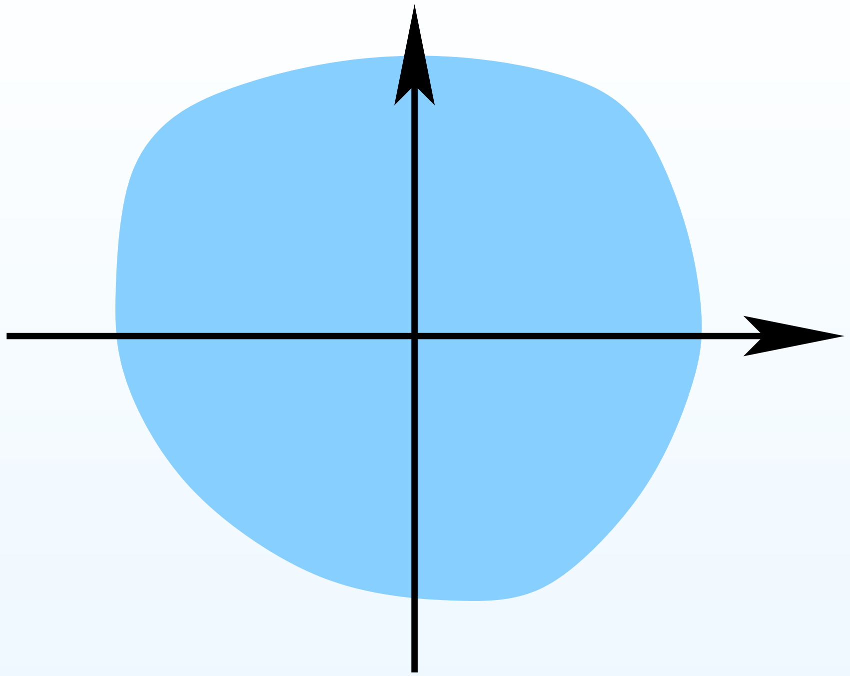

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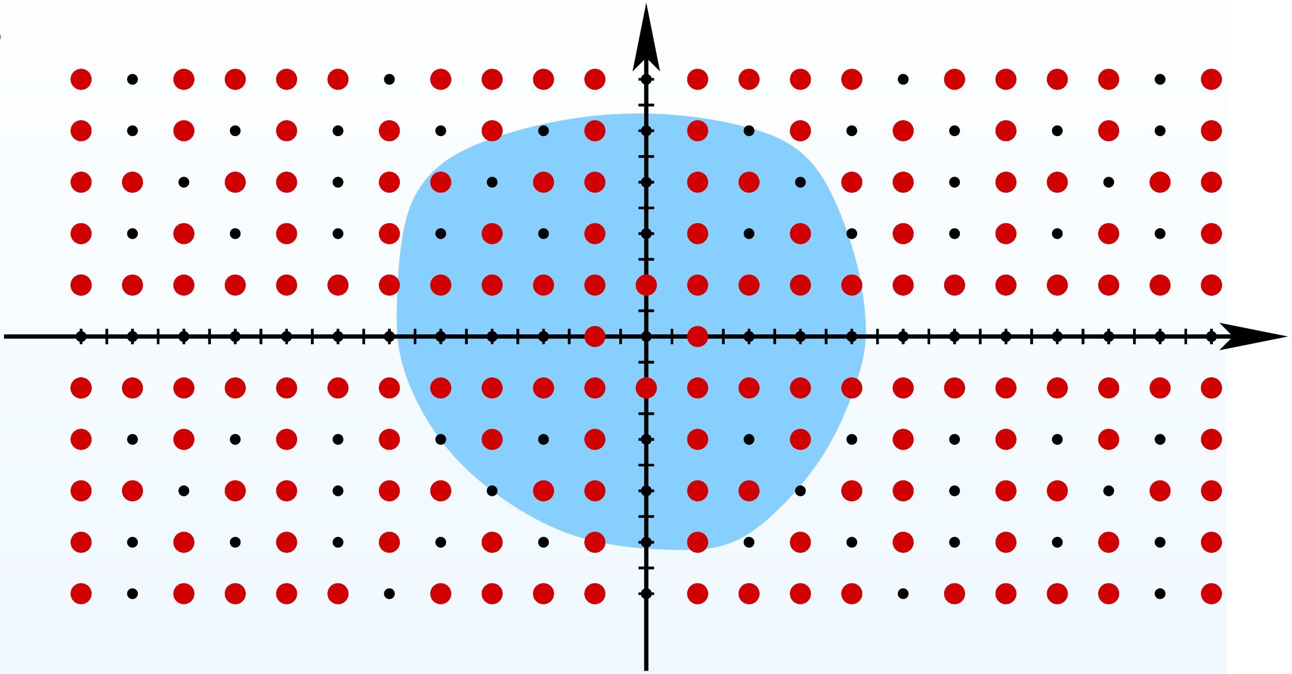
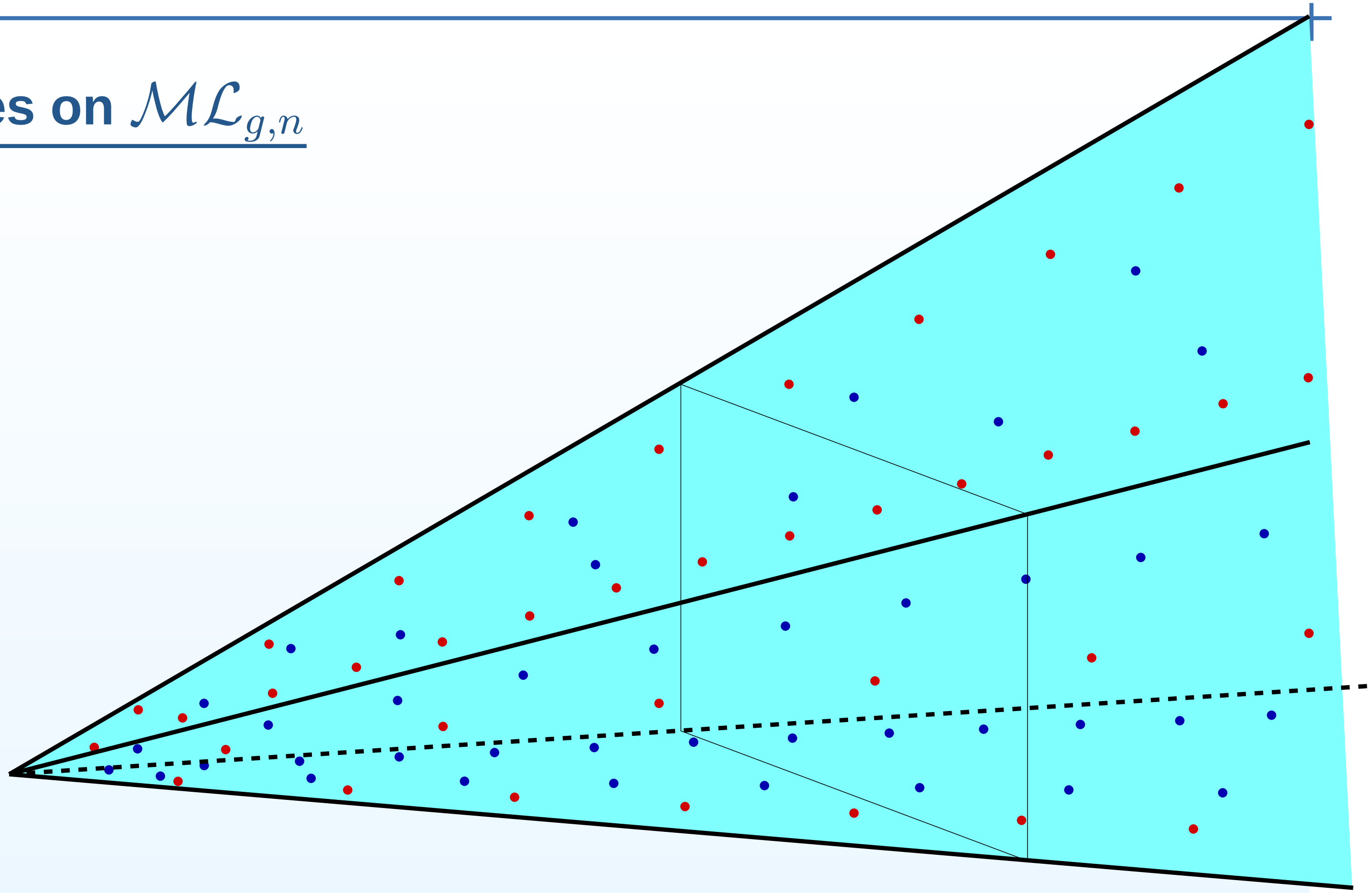


Finally, instead of using the entire lattice $\mathbb{Z}^n$ we can use any sublattice $\mathbb{L}^n \subset \mathbb{Z}^n$ having some positive *uniform* density in $\mathbb{Z}^n$.

For example, the set of coprime integral points in $\mathbb{Z}^2$ has density $\dfrac{6}{\pi^2}$ and can be also used to define the Lebesgue measure (scaled by the factor $\dfrac{6}{\pi^2}$) in any of the two ways discussed above.

Mirzakhani's measures on $\mathcal{ML}_{g,n}$



Choose some integral multicurve γ, say, a simple closed curve on $S_{g,n}$. The subset $\mathcal{O}_\gamma := \mathrm{Mod}_{g,n} \cdot \gamma$ can be seen as an analog of coprime integral points in $\mathcal{ML}_{g,n}$. The insight of Mirzakhani was to realize that replacing the discrete set $\mathcal{ML}_{g,n}(\mathbb{Z})$ with the subset $\mathcal{O}_\gamma$ we get a new measure on $\mathcal{ML}_{g,n}$ which is proportional to the Thurston measure μ_{Th} with coefficient depending only on the homotopy type of γ.

Mirzakhani's measures on $\mathcal{ML}_{g,n}$

More formally: the Thurston measure of a subset $U \subset \mathcal{ML}_{g,n}$ is defined as

$$\mu_{\mathrm{Th}}(U) := \lim_{L\to+\infty} \frac{\operatorname{card}\{L \cdot U \cap \mathcal{ML}_{g,n}(\mathbb{Z})\}}{L^{6g-6+2n}}\,.$$

Mirzakhani defines a new measure μ_γ as

$$\mu_\gamma(U) := \lim_{L\to+\infty} \frac{\operatorname{card}\{L \cdot U \cap \mathcal{O}_\gamma\}}{L^{6g-6+2n}}\,.$$

(More accurately: using compactness of the space of measures, we get a weak convergence for sequences $\{L_i\}$.) For any U we have $\mu_\gamma(U) \le \mu_{\mathrm{Th}}(U)$ since $\mathcal{O}_\gamma \subset \mathcal{ML}_{g,n}(\mathbb{Z})$, so μ_γ belongs to the Lebesgue measure class of Thurston's measure. By construction μ_γ is $\mathrm{Mod}_{g,n}$-invariant. Ergodicity of μ_{Th} implies that $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ where $k_\gamma = const$; it does not depend on U.

It remains to prove, however, that k_γ, which formally depends on a subsequence of scales $\{L_i\}_i$, is one and the same for all subsequences; that $k_\gamma > 0$ for any topological type of a multicurve; and to compute k_γ.

Proof of the main result

Length function and unit ball

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space. One can extend the length function to simple closed multicurves $\ell_{\sum a_i\gamma_i} = \sum a_i \ell_{\gamma_i}(X)$ by linearity. By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t\cdot\lambda}(X) = t \cdot \ell_\lambda(X)$.

Each hyperbolic metric X defines its own "unit ball" B_X in $\mathcal{ML}_{g,n}$:

$$B_X := \{\lambda \in \mathcal{ML}_{g,n} \,|\, \ell_\lambda(X) \le 1\}\,.$$

By definition of μ_{Th}, the Thurston volume of the unit ball is equal to the normalized number of integral points in a "ball of radius L" associated to X:

$$\mu_{\mathrm{Th}}(B_X) = \lim_{L\to+\infty} \frac{\mathrm{card}\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \,|\, \ell_\lambda(X) \le L\}}{L^{6g-6+2n}}\,.$$

Denote by $b_{g,n} = \int_{\mathcal{M}_{g,n}} \mu_{\mathrm{Th}}(B_X)\, dX$ the average of Thurston volume of unit balls.

Summary of notations

- X — a hyperbolic surface in $\mathcal{M}_{g,n}$.
- $s_X(L,\gamma)$ — the number of geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.
- $P(L\,,\gamma) := \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\,dX$ — the polynomial in L providing the *average* number of geodesic multicurves of topological type $[\gamma]$ and of hyperbolic length at most L over all hyperbolic surfaces $X \in \mathcal{M}_{g,n}$.
- $c(\gamma)$ — the coefficient of the leading term $L^{6g-6+2n}$ of the polynomial $P(L\,,\gamma)$.

- $B(X)$ — "Unit ball" in $\mathcal{ML}_{g,n}$ defined by means of the length function $\ell_X(\alpha)$, where $\alpha \in \mathcal{ML}_{g,n}$.
- $\mu_{\mathrm{Th}}(B(X)) := \lim_{L\to+\infty} \dfrac{\mathrm{card}\{L\cdot B_X \cap \mathcal{ML}(\mathbb{Z})\}}{L^{6g-6+2n}}$ is the Thurston measure of the unit ball $B(X)$
- $\mu_\gamma(B(X)) := \lim_{L\to+\infty} \dfrac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\!\cdot\!\gamma\}}{L^{6g-6+2n}}$ is the Mirzakhani measure of the unit ball $B(X)$ defined by the sublattice $\mathrm{Mod}_{g,n}\!\cdot\!\gamma \subset \mathcal{ML}(\mathbb{Z})$.

Theorem which we aim to prove

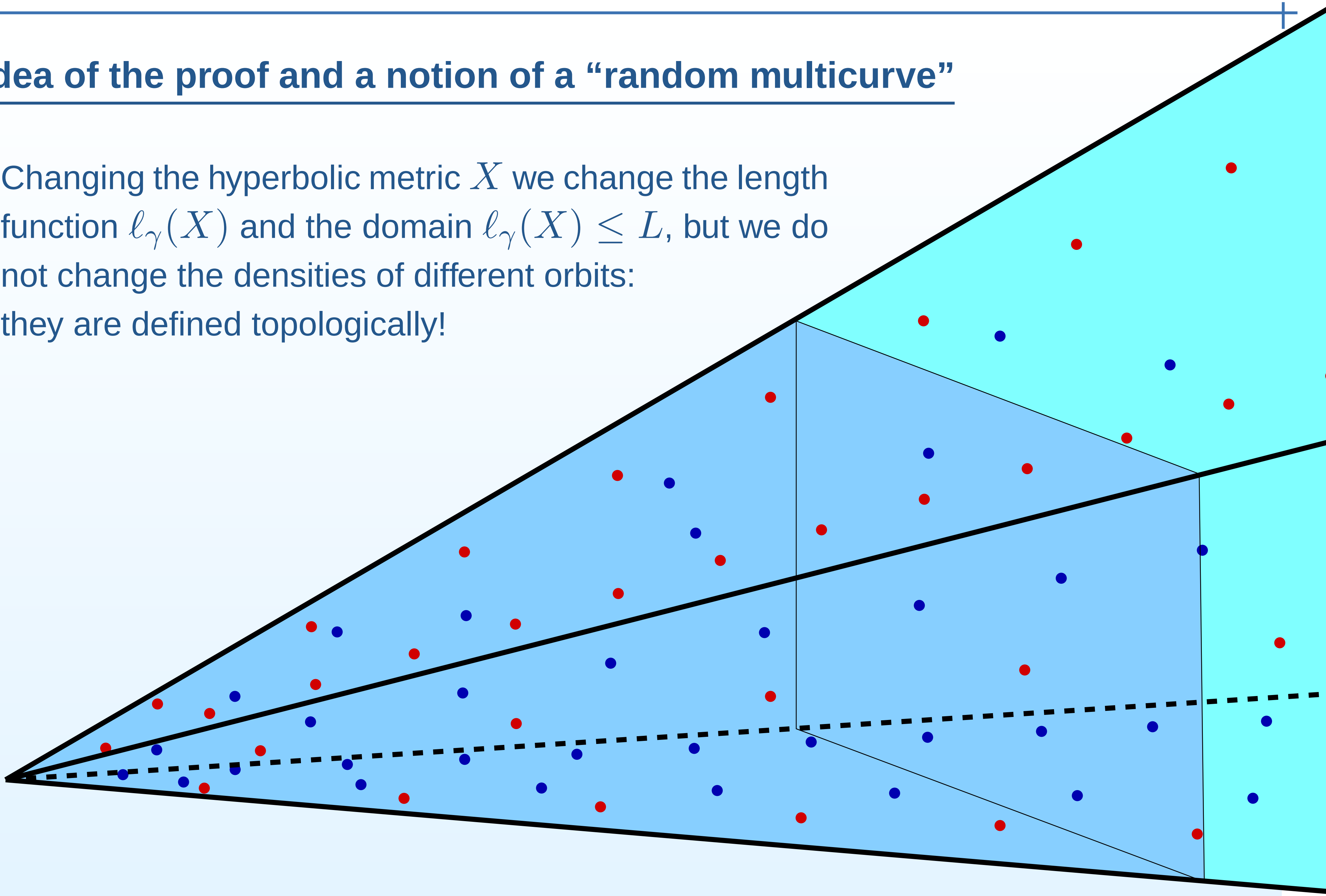
Theorem (M. Mirzakhani'08). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (the average value of B_X over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ. M. Mirzakhani expressed $c(\gamma)$ in terms of the Witten–Kontsevich correlators.

Idea of the proof and a notion of a "random multicurve"

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \leq L$, but we do not change the densities of different orbits: they are defined topologically!



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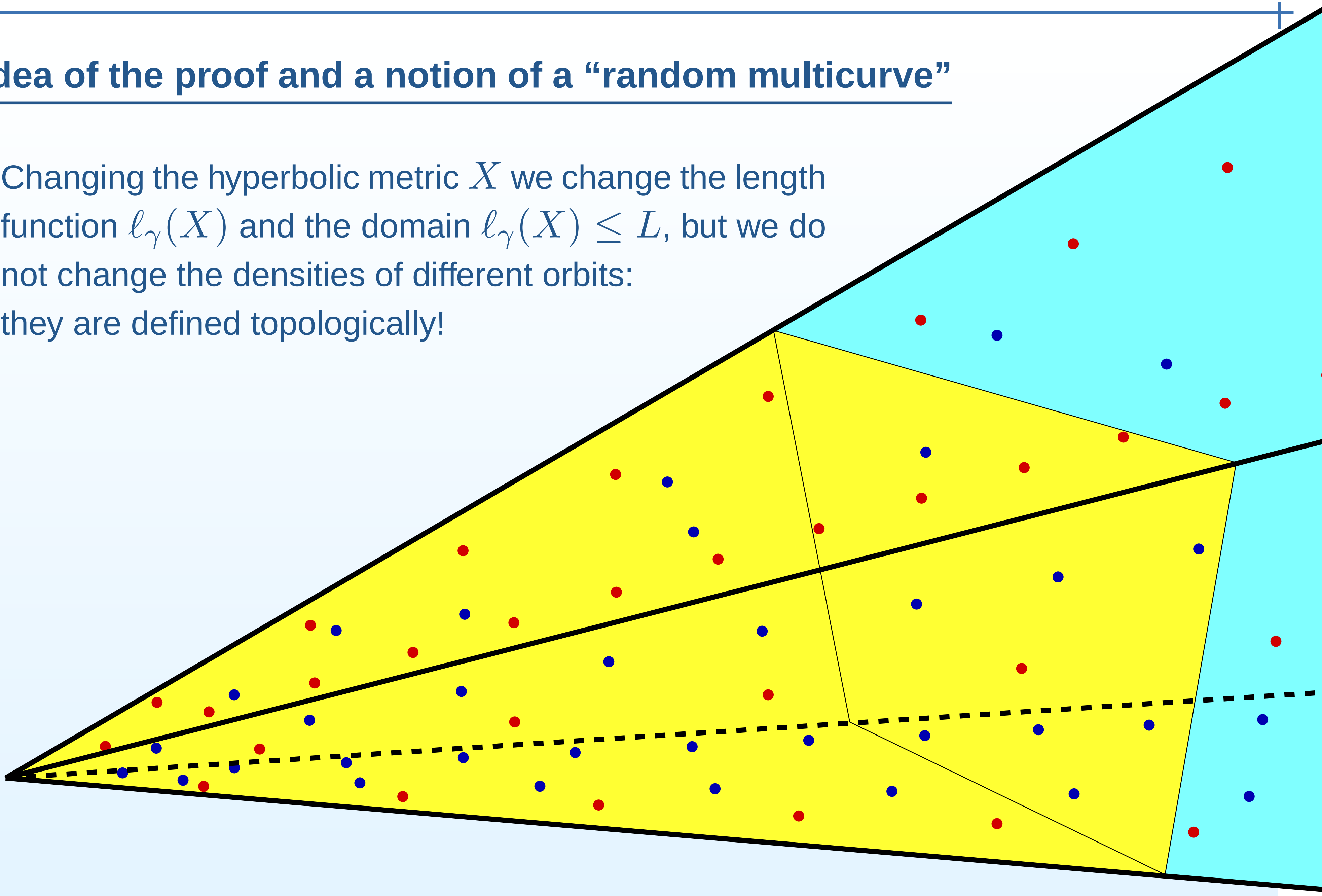

Proof and computation of k_γ

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the "unit ball" B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,.$$

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Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\, dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\, dX =$$

$$= \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}}\, dX = \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}}\, dX =$$

$$= \lim_{L\to+\infty} \frac{1}{L^{6g-6+2n}} \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX = \lim_{L\to+\infty} \frac{P(L,\gamma)}{L^{6g-6+2n}} = c(\gamma)\,,$$

so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzakhani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

Uniform density of coprime integer points

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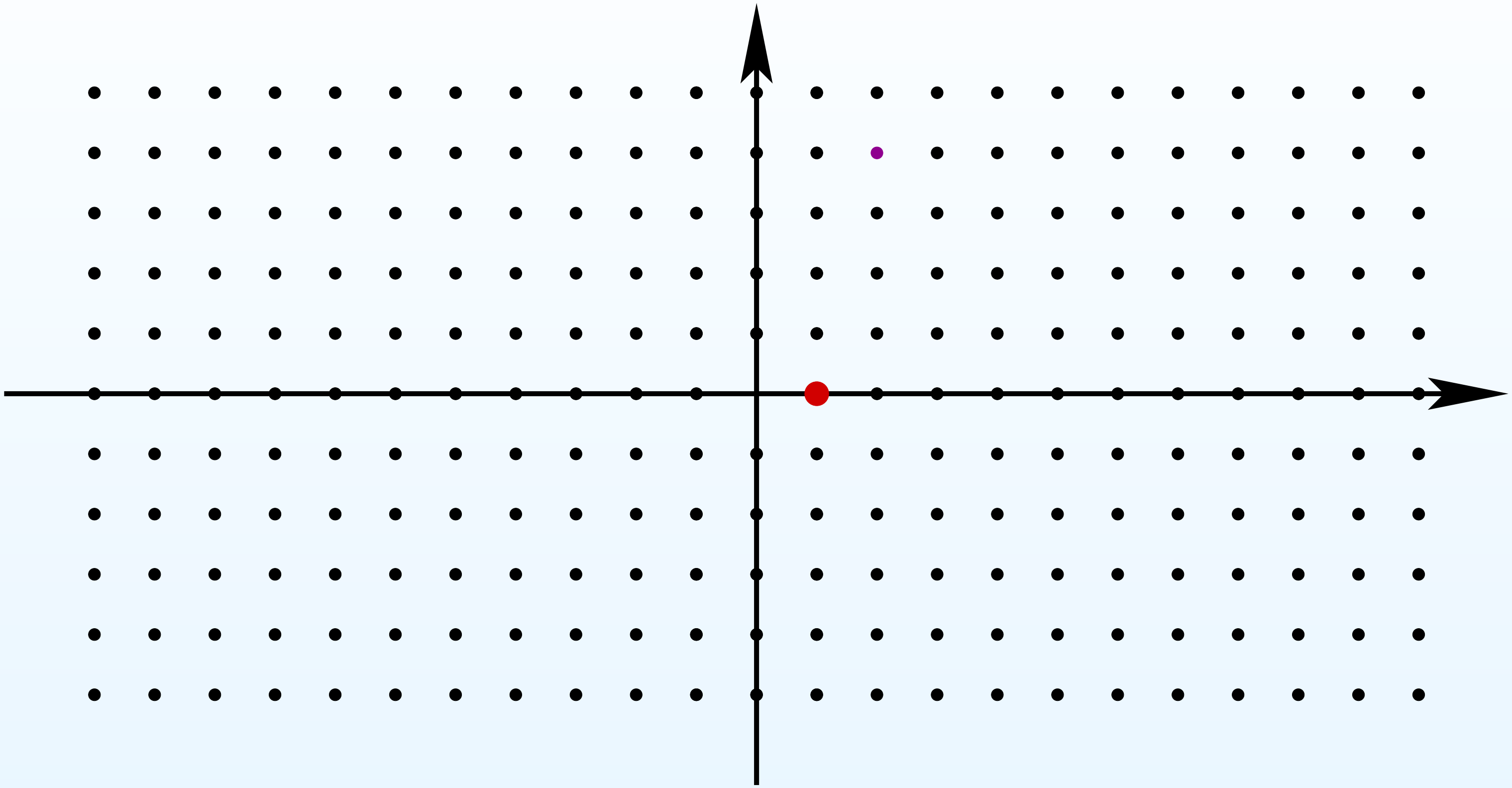

The set of *coprime* points $(p,q) \in \mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ such that $\mathrm{pgcd}(p,q) = 1$ is an $\mathrm{SL}(2,\mathbb{Z})$ orbit of $(1,0)$. In train-track coordiantes it can be identified with the mapping class group orbit $\mathrm{Mod}_{1,1} \cdot [\gamma]$ of a simple closed curve γ in $\mathcal{ML}_{1,1}$. We have proved that this subset has nonzero *uniform density* $k_1 = \frac{c(\gamma)}{b_{1,1}}$ in the ambient lattice of all integral measured laminations.

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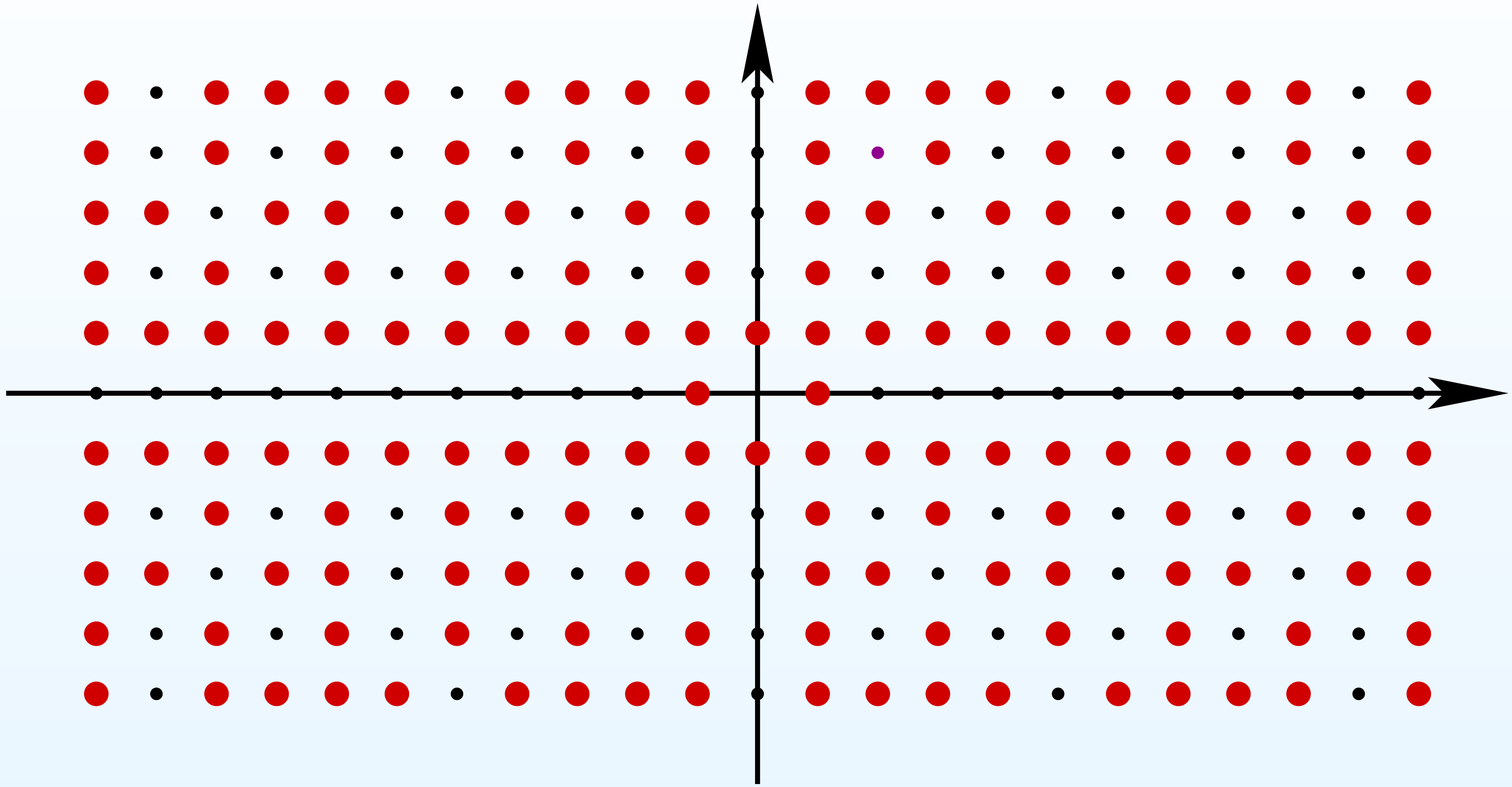

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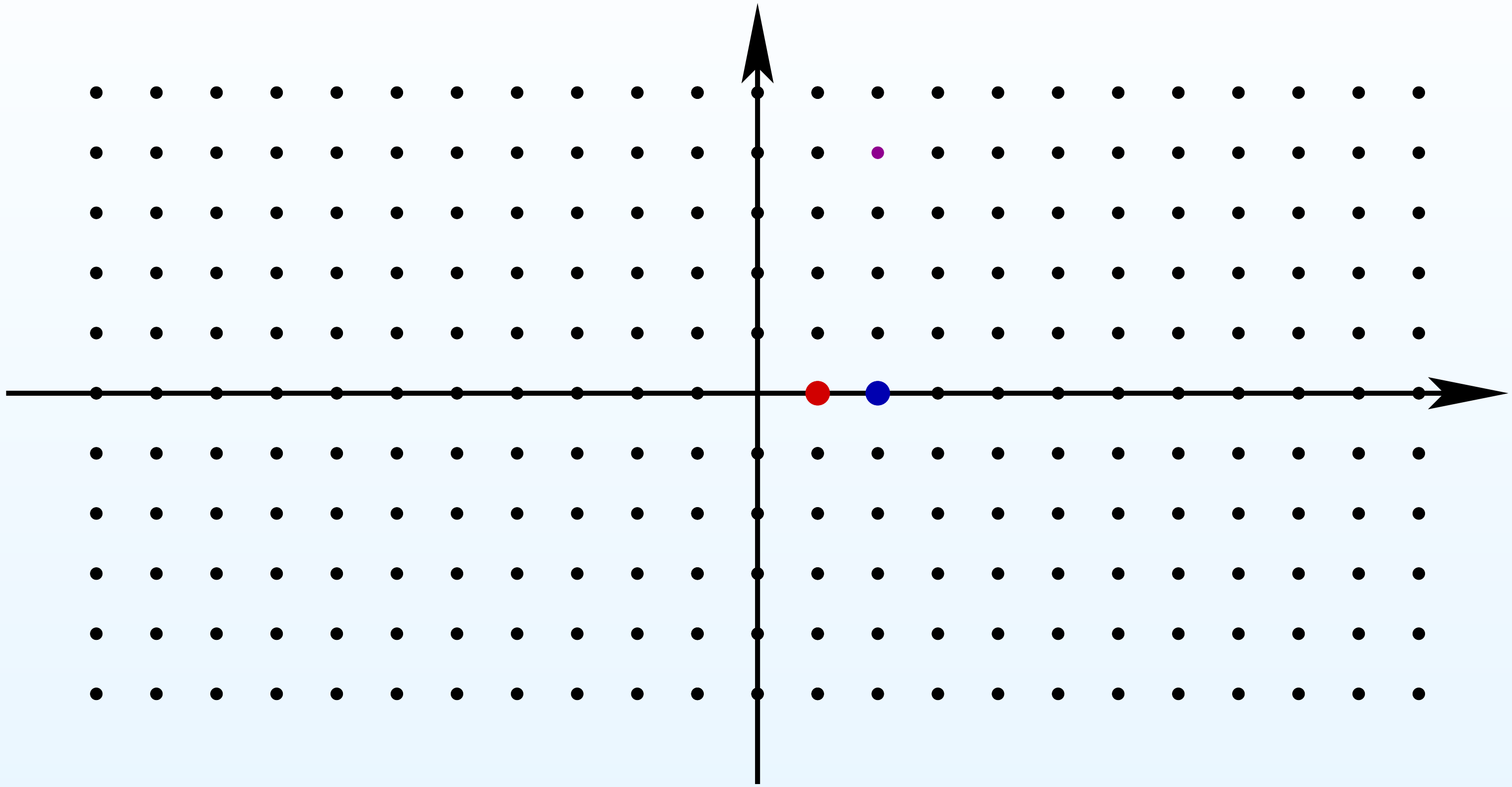

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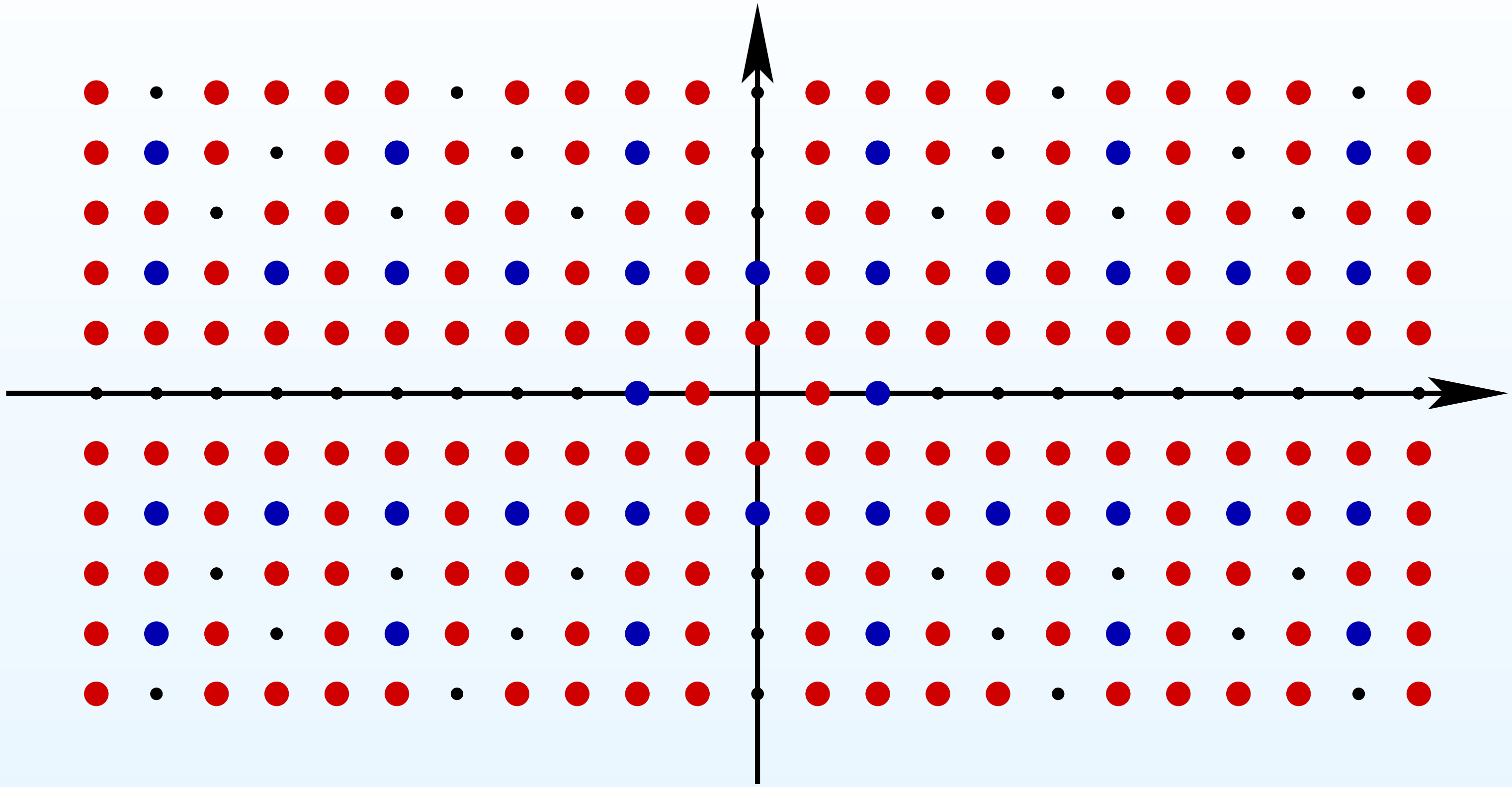

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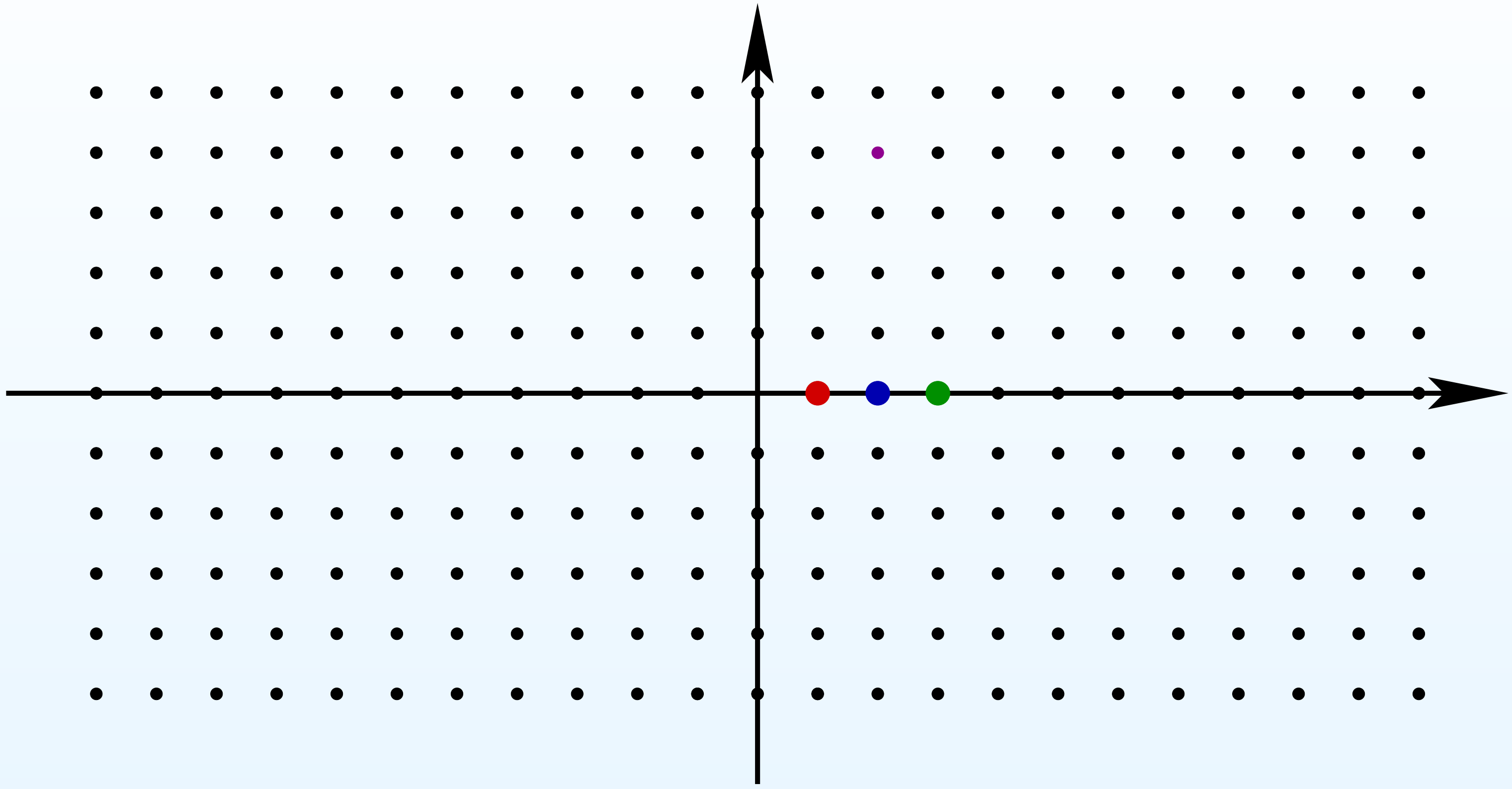

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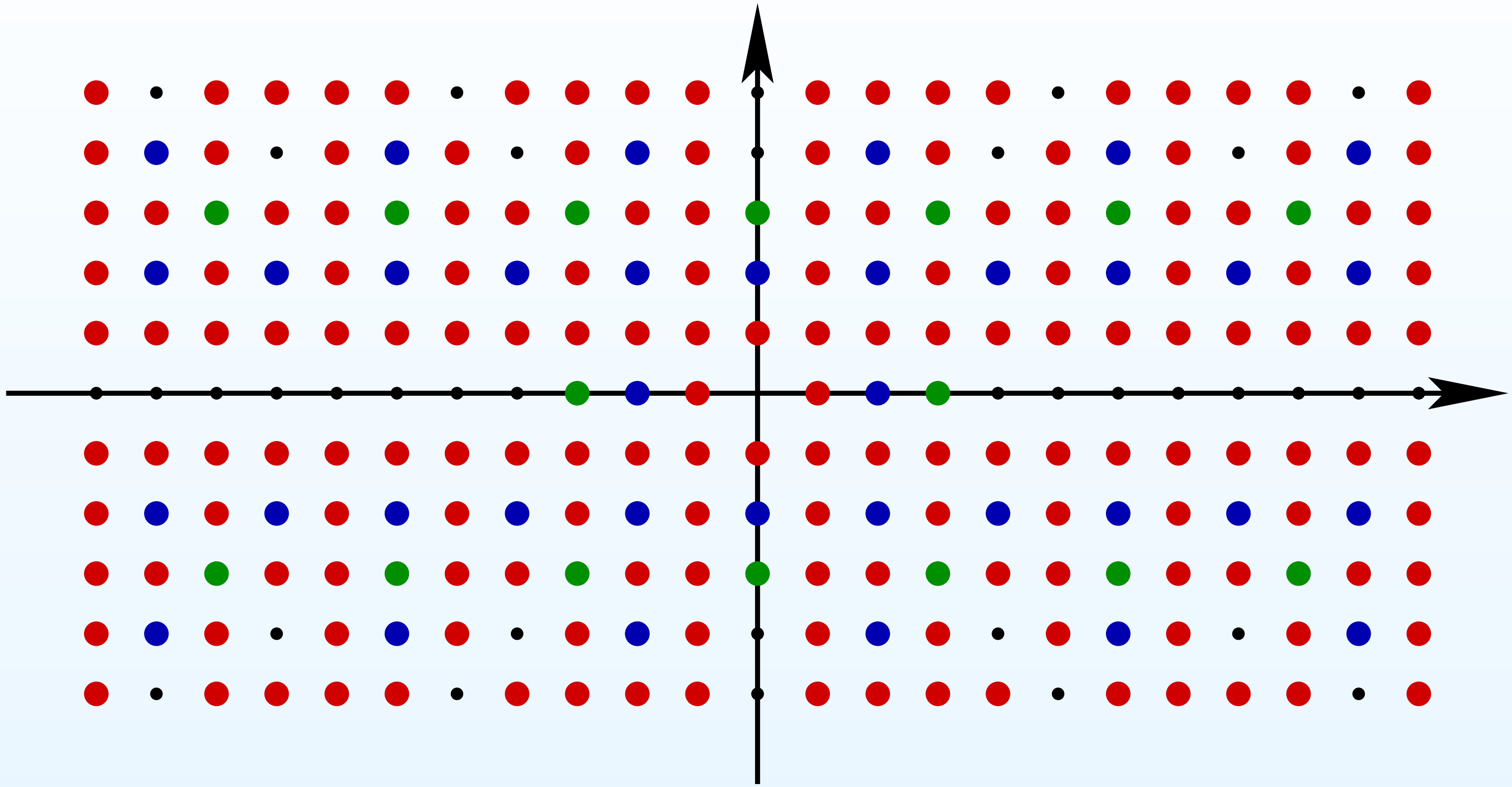

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Uniform density of coprime integer points

The disjoint union of all these orbits gives all the lattice

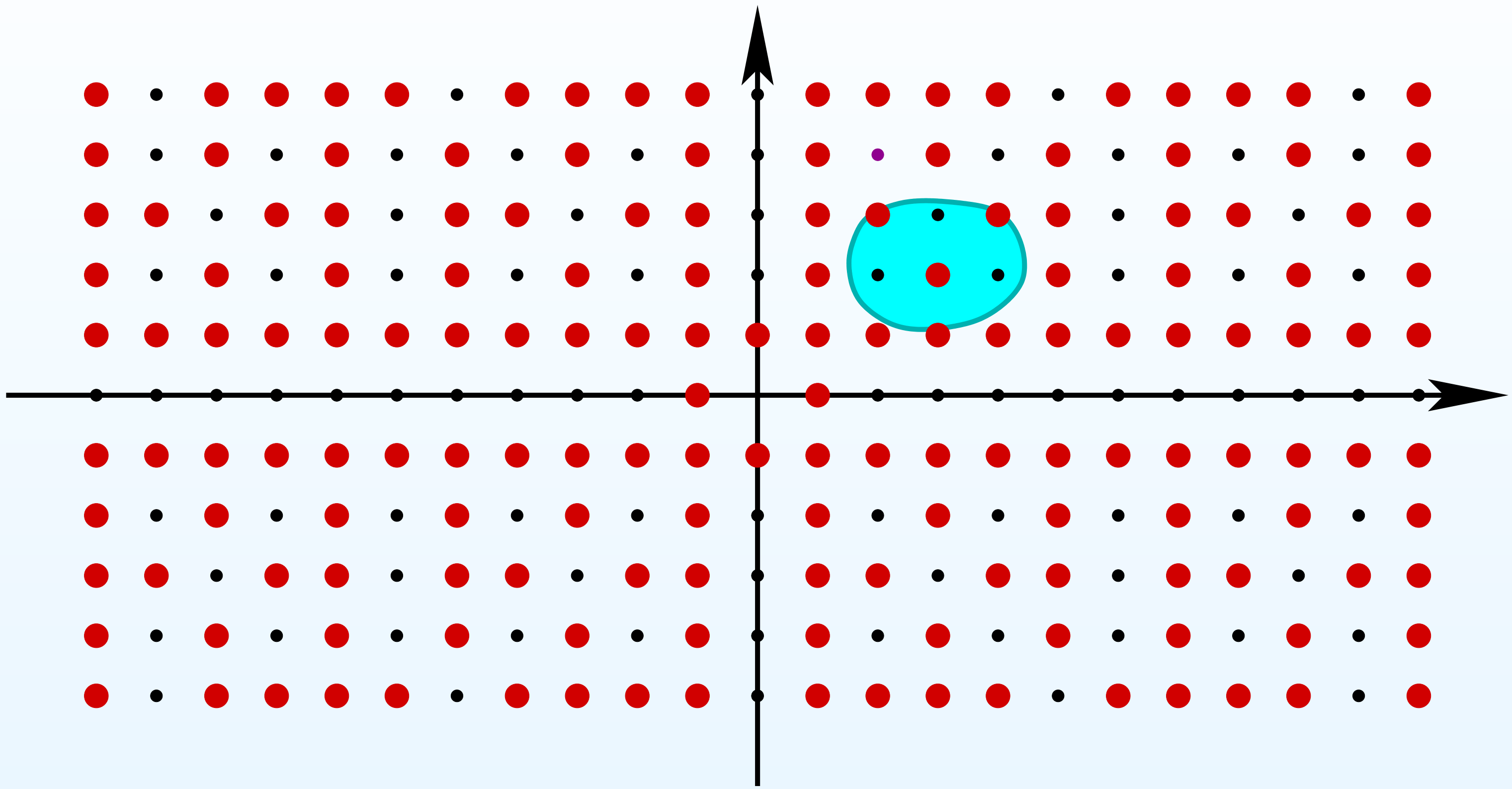
$$\sqcup_{n=1}^{\infty} \mathrm{SL}(2,\mathbb{Z}) \cdot (n,0) = \mathbb{Z} \oplus \mathbb{Z}\,.$$

Thus the sum of the densities gives the full density, $k_1 + k_2 + \cdots = 1$. Since $k_n = \frac{1}{n^2} \cdot k_1$, we get

$$k_1 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) = 1$$

and, hence, $k_1 = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.

Uniform density of coprime integer points



Note that though we can measure the Lebesgue measure of a set by counting the number of coprime points inside its dilatations (which is the definition of a *uniform density*), it is known that going exponentially far from the origin, i.e. at the distance of the order e^R or more one can find islands of radius R without a single coprime point, where R is any positive number.

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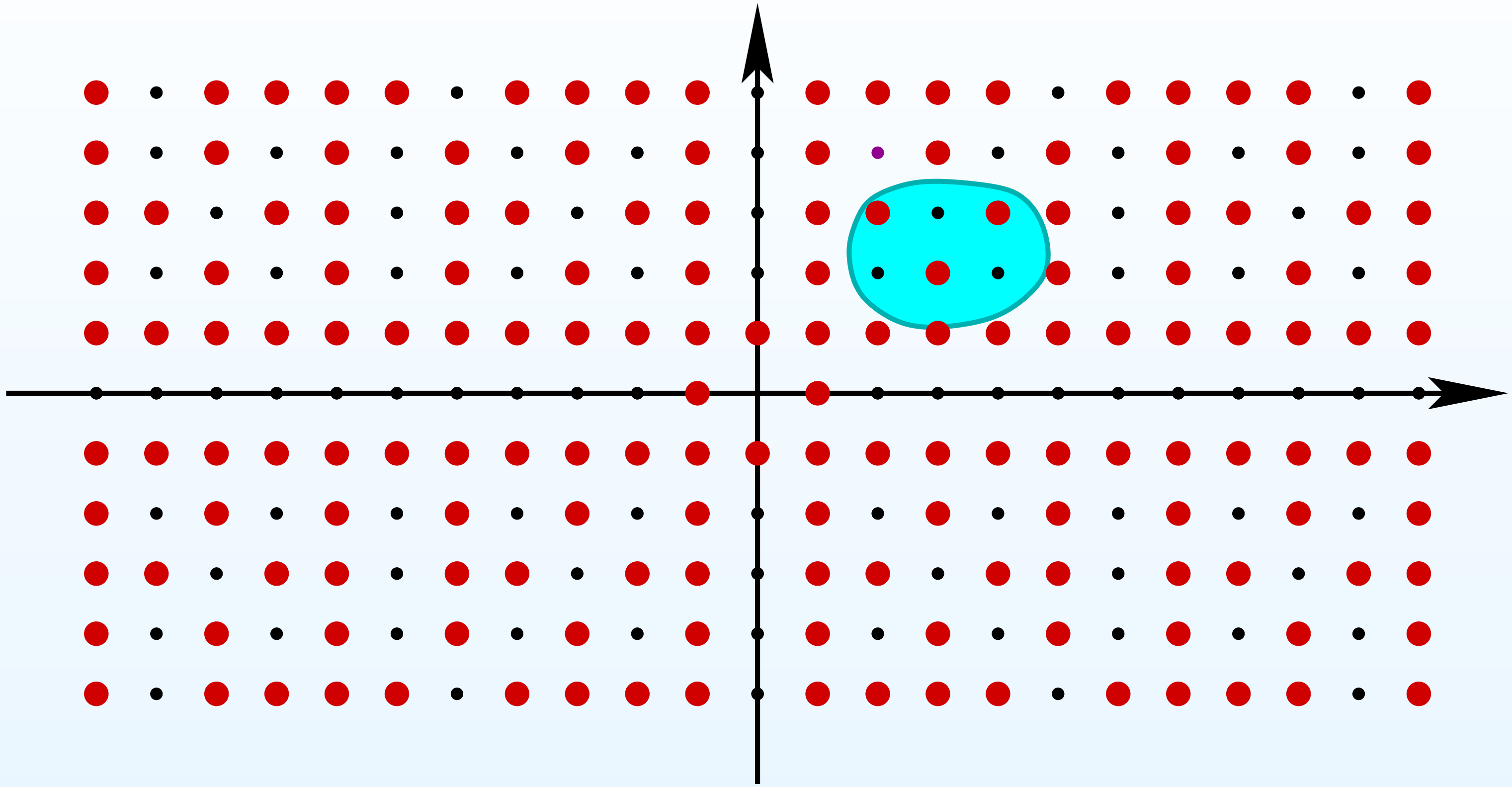

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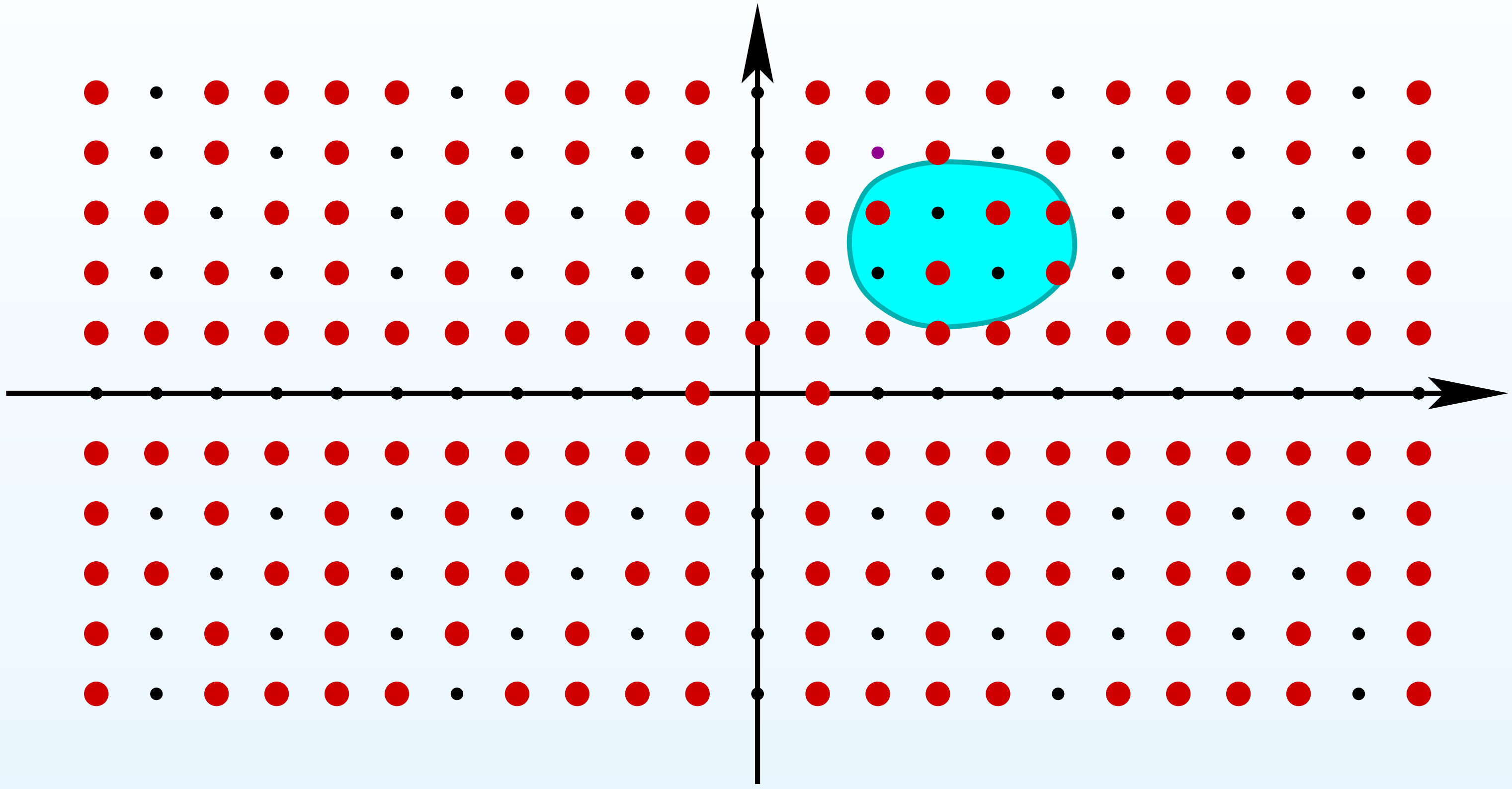

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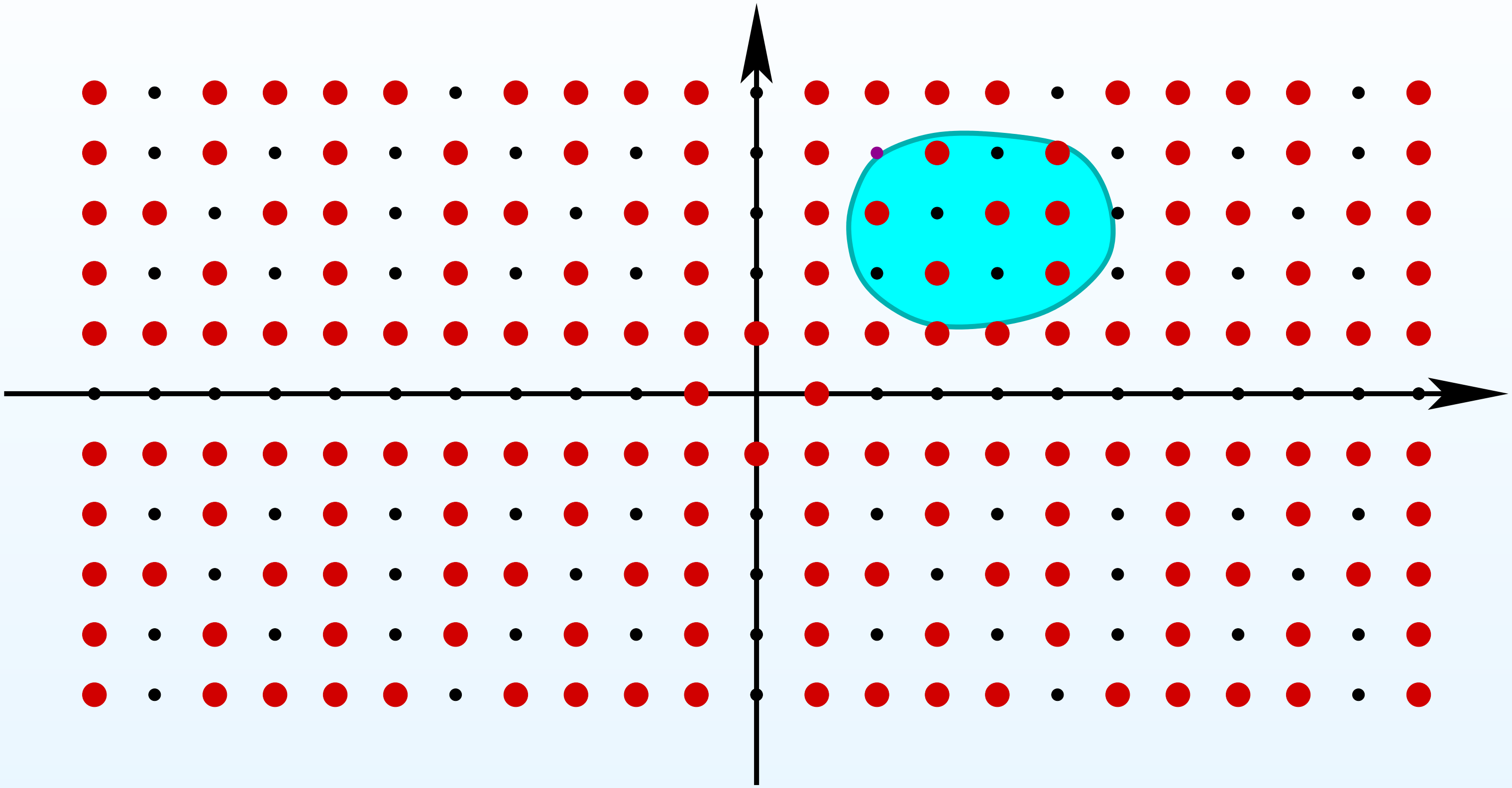

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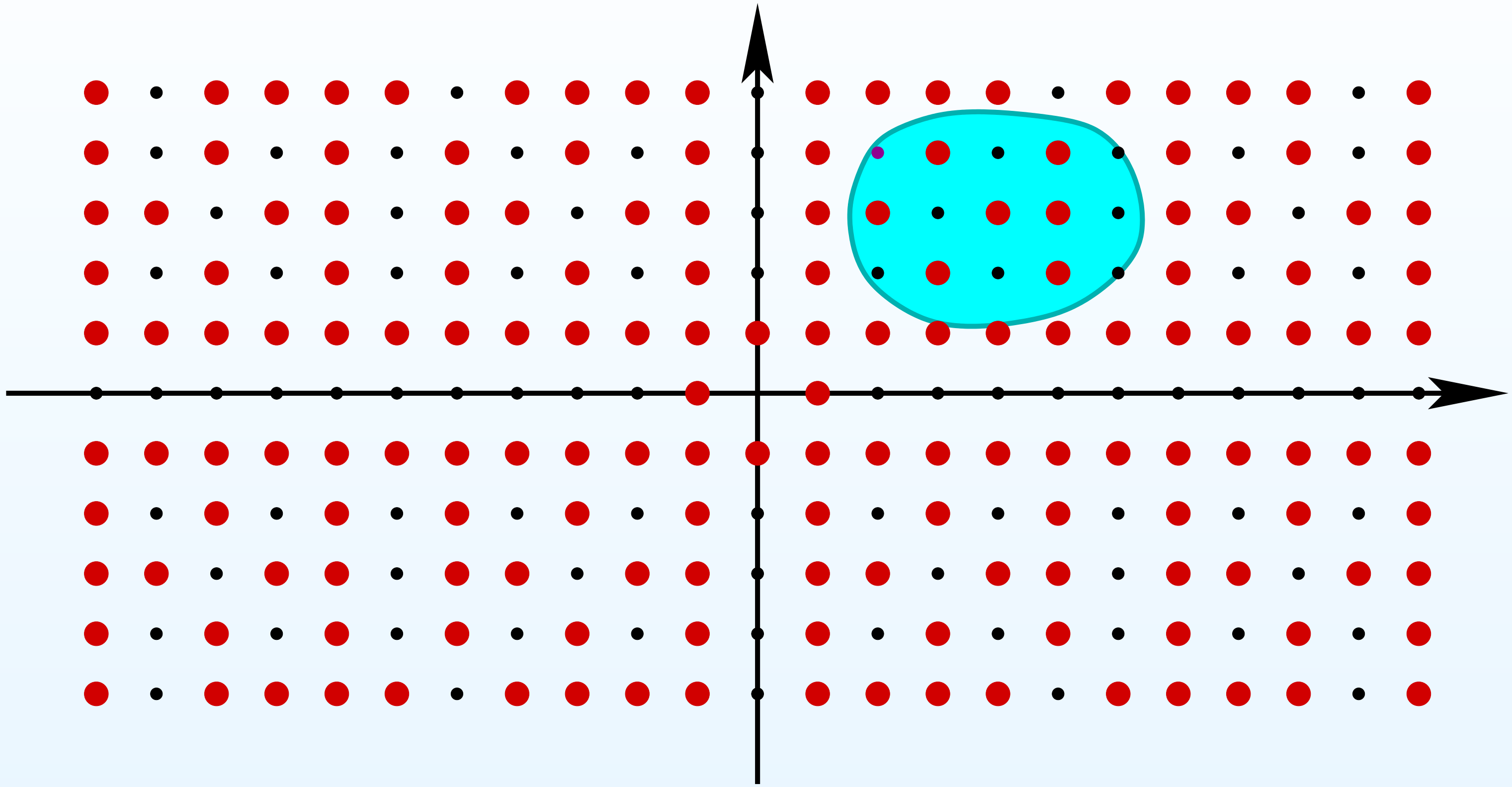

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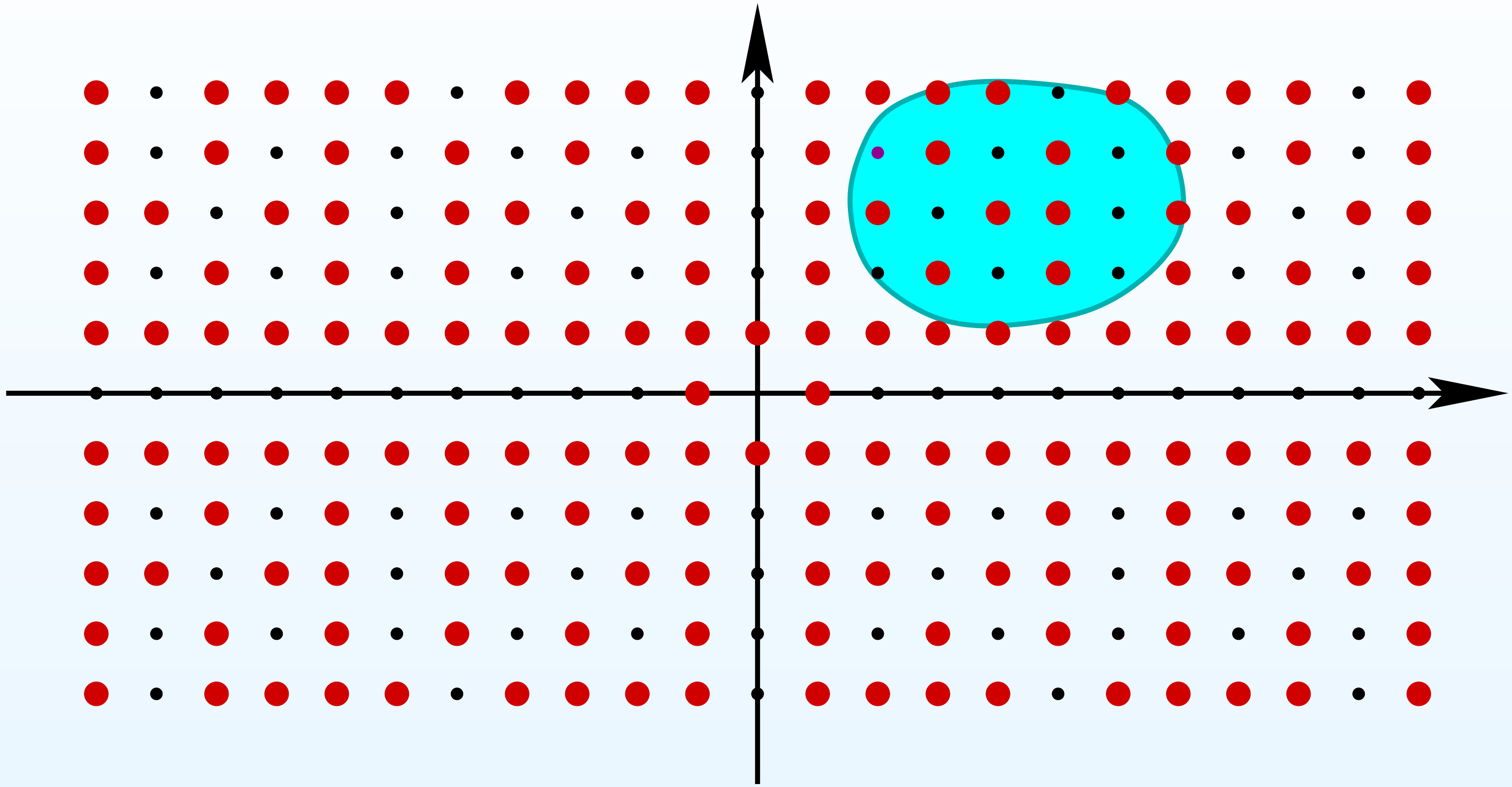

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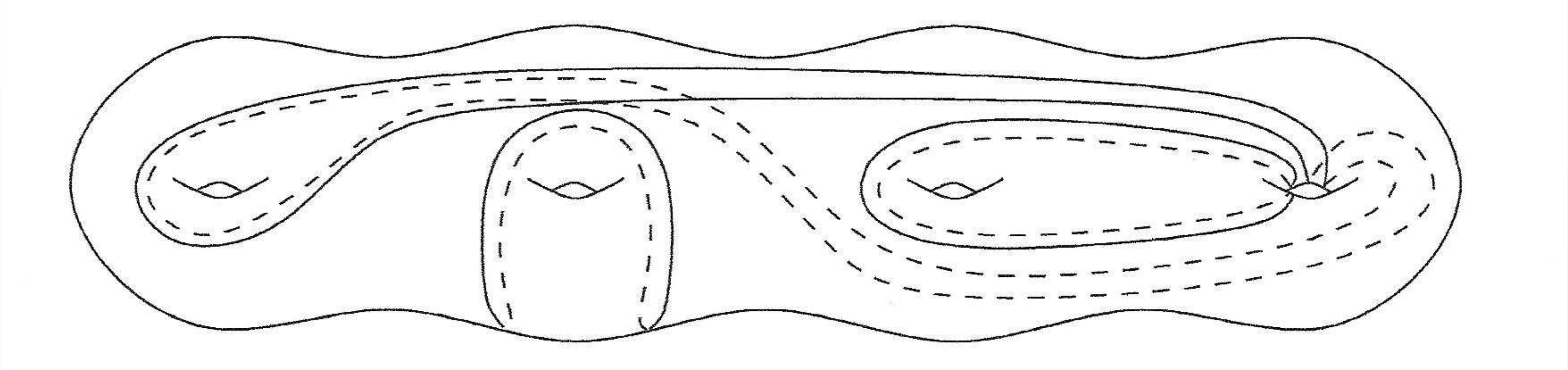
Exercises

Separating curves

Exercise. *Prove that all curves presented at the picture are separating.*

Hint: choose an appropriate basis of cycles and verify that intersection numbers of each curve with all basic cycles are zero.



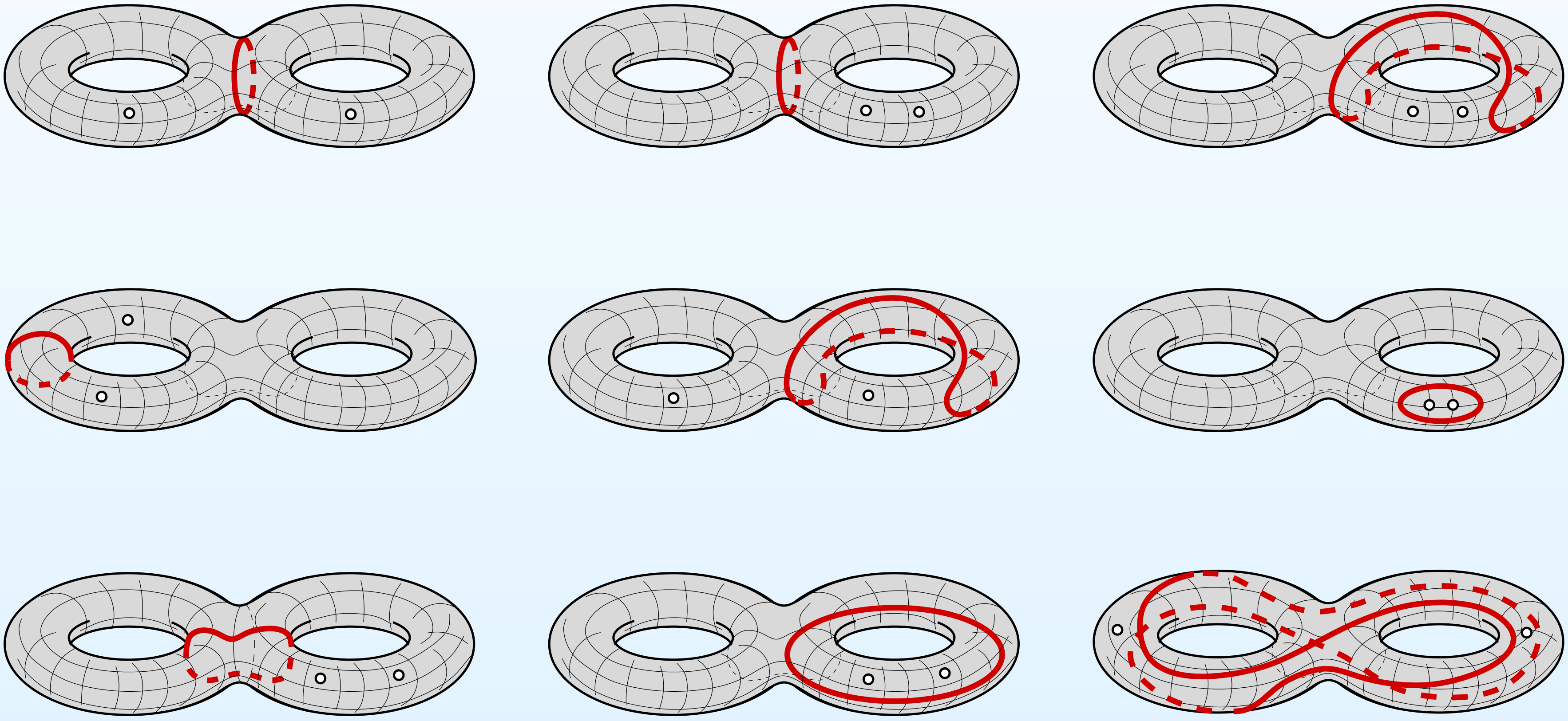
The picture is taken from the book of B. Farb and D. Margalit “A Primer on Mapping Class Groups”.

Exercise. *Detect which curves are essential and which essential curves belong to the same orbit of the mapping class group.*

Orbits of the mapping class group

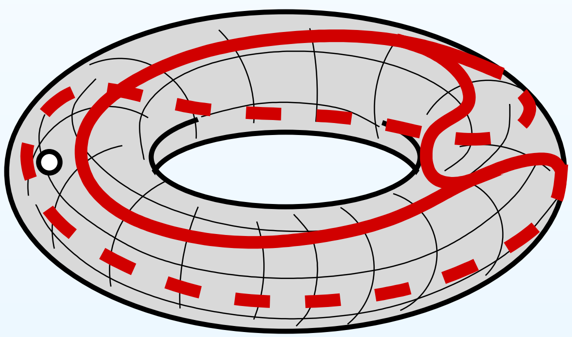
Select all simple closed curves in the picture below which might be isotopic to simple closed hyperbolic geodesics on a twice-punctured surface of genus two.

How many distinct orbits of $\mathrm{Mod}_{2,2}$ they represent? Indicate which curves correspond to which orbit.

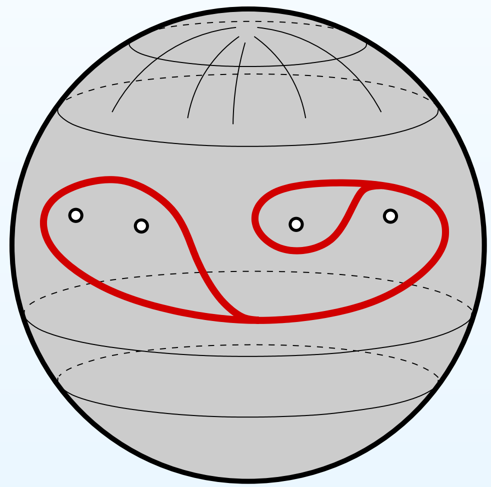


Train-tracks

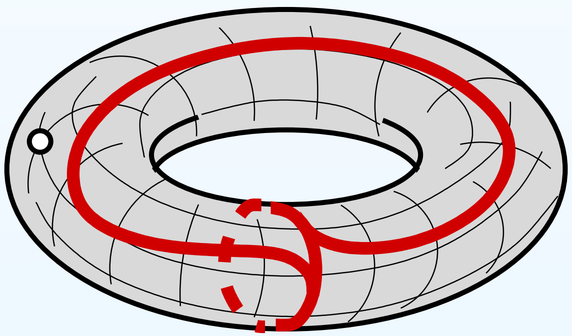
Which of the given train-tracks τ_1, τ_2, τ_3 might carry a simple closed hyperbolic geodesic? Indicate some legitimate weights if you claim that the train track carries a simple closed hyperbolic geodesic.



τ_1



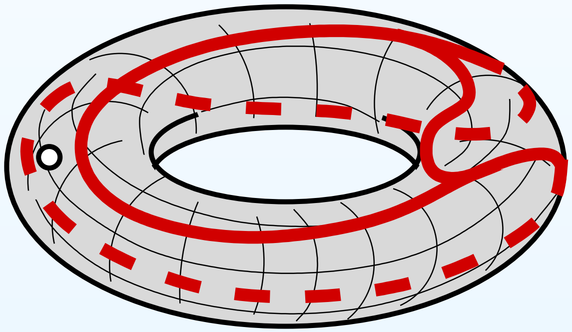
τ_2



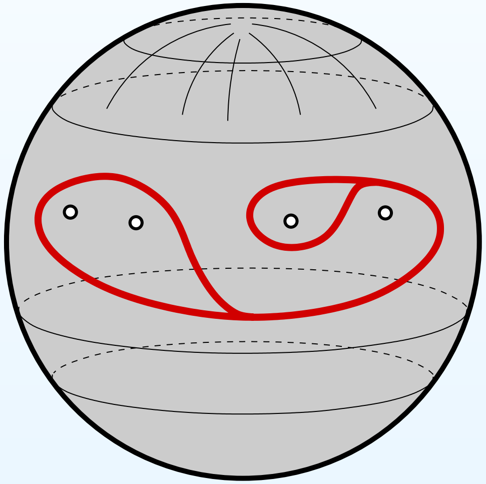
τ_3

Train-tracks

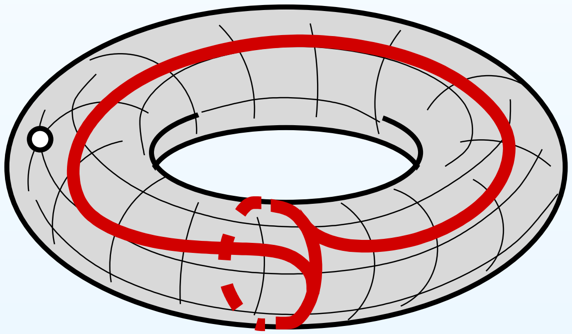
Which of the given train-tracks τ_1, τ_2, τ_3 might carry a simple closed hyperbolic geodesic? Indicate some legitimate weights if you claim that the train track carries a simple closed hyperbolic geodesic.



τ_1



τ_2



τ_3

Can any of the given train-tracks τ_1, τ_2, τ_3 carry *different* simple closed hyperbolic geodesic? Indicate the corresponding different legitimate collections of weights if you claim that.

Playing with train tracks

The picture on the left presents the four "basic" train tracks $\tau_1, \ldots, \tau_4$ on a four-punctured sphere.

a) Find which of these four basic train-tracks (if any) carries the simple closed curve γ on top of the right picture. Indicate the resulting weights.

b) Find a sequence of "zips" and/or "unzips" which deform the given train track τ on the bottom of the right picture to one of the four "basic" train tracks $\tau_1, \ldots, \tau_4$. Indicate the weights of all branches of the initial and of the final train tracks.