Geometry and dynamics in moduli spaces

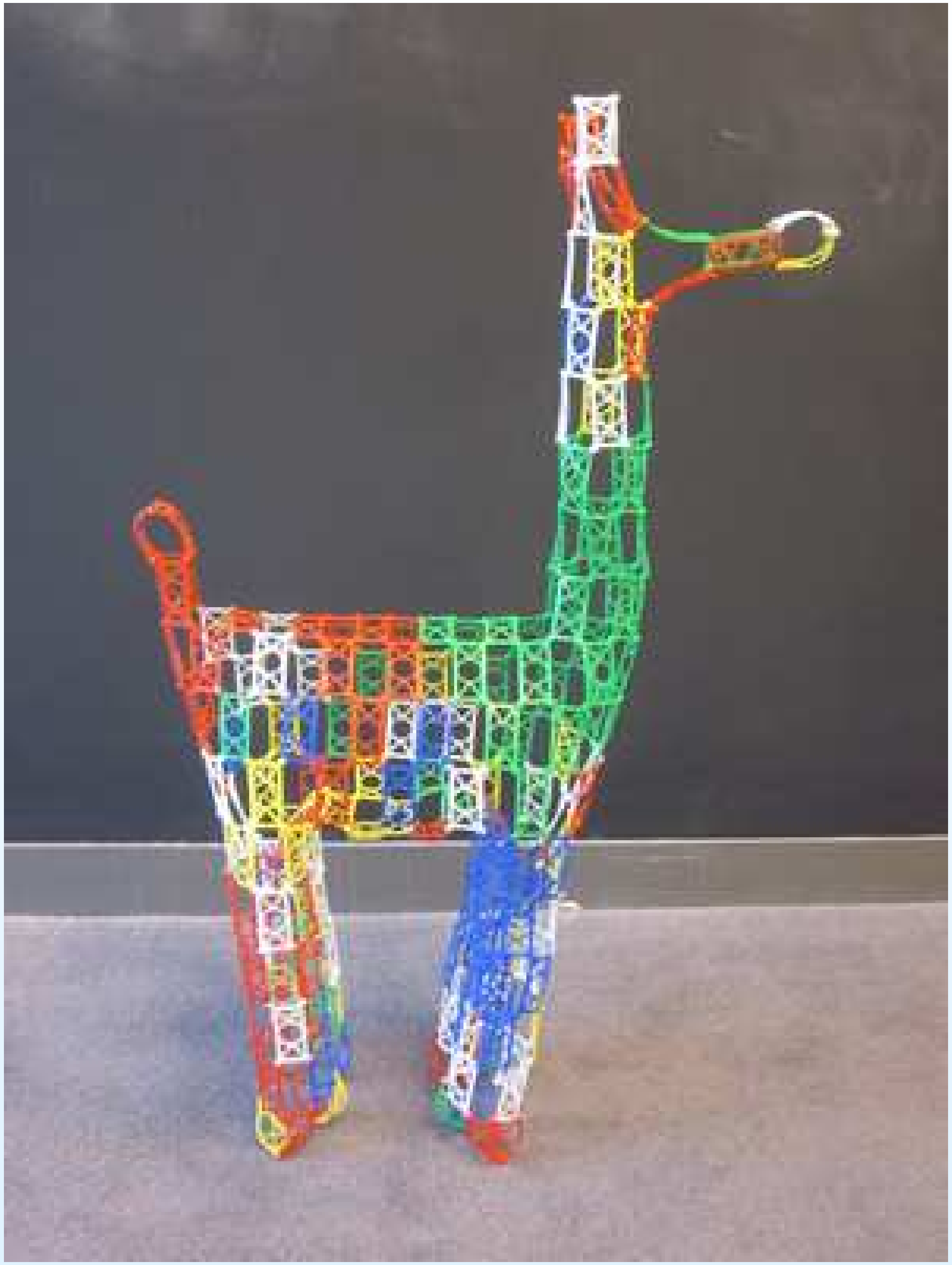
Lecture 8. Count of square-tiled surfaces and of simple closed hyperbolic geodesics after joint works with V. Delecroix, E. Goujard and P. Zograf

Anton Zorich
Université Paris Cité

March 26, 2026

Count of square-tiled surfaces. (Masur–Veech volume of the moduli space of quadratic differentials)

A square-tiled surface



Square-tiled surfaces: formal definition

Take a finite set of copies of identical oriented squares for which two opposite sides are chosen to be horizontal and the remaining two sides are declared to be vertical. Identify pairs of sides of the squares by isometries in such way that horizontal sides are glued to horizontal sides and vertical sides to vertical. We get a topological surface S without boundary. We consider only those surfaces obtained in this way which are connected and oriented. The form dz^2 on each square is compatible with the gluing and endows S with a complex structure and with a non-zero quadratic differential $q = dz^2$ with at most simple poles. We call such a surface a *square-tiled surface*.

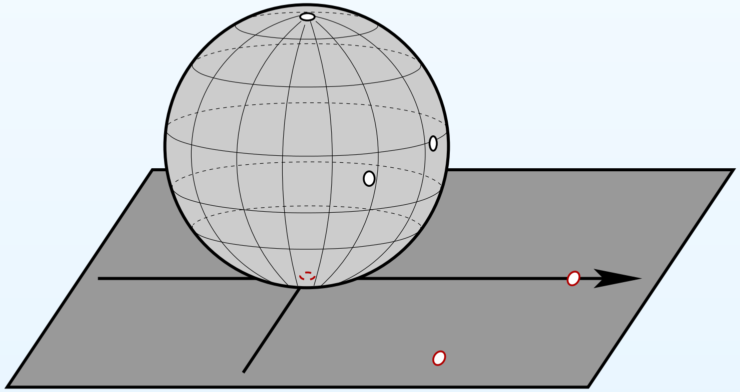
Square-tiled surfaces: formal definition

Take a finite set of copies of identical oriented squares for which two opposite sides are chosen to be horizontal and the remaining two sides are declared to be vertical. Identify pairs of sides of the squares by isometries in such way that horizontal sides are glued to horizontal sides and vertical sides to vertical. We get a topological surface S without boundary. We consider only those surfaces obtained in this way which are connected and oriented. The form dz^2 on each square is compatible with the gluing and endows S with a complex structure and with a non-zero quadratic differential $q = dz^2$ with at most simple poles. We call such a surface a *square-tiled surface*.

Fix the genus g of the surface and the number n of corners with cone angle π (the ones adjacent to exactly two squares). The question on the asymptotic number of such square-tiled surfaces tiled with at most $N \gg 1$ squares (connected covers over $\mathbb{C}\mathrm{P}^1$ ramified over 4 points and having prescribed ramification profile) is equivalent to evaluation of the Masur–Veech volume of the moduli space of quadratic differentials.

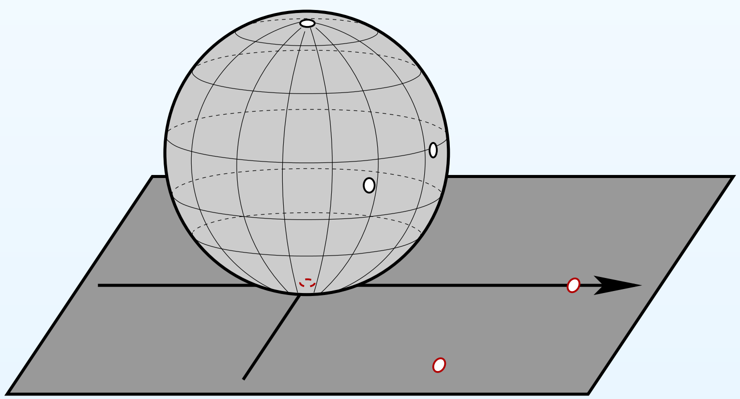
Families of hyperbolic surfaces

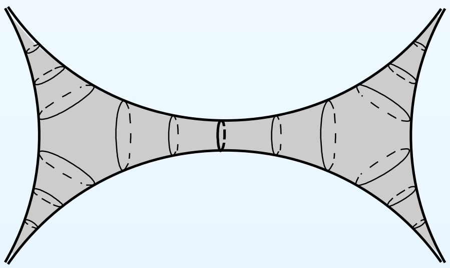
Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).



Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

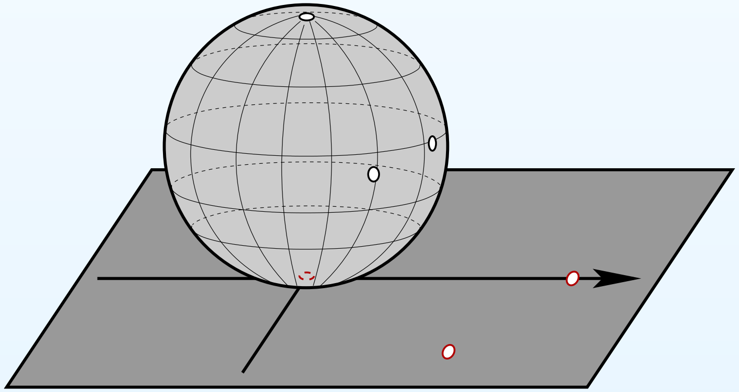


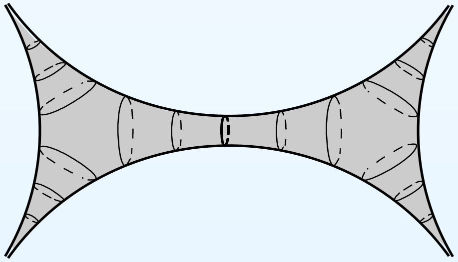


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

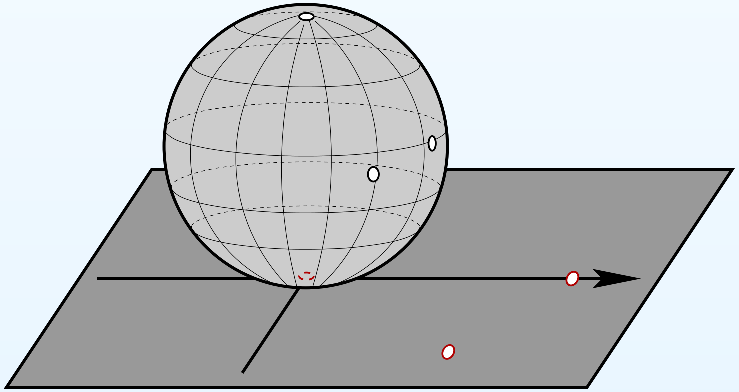


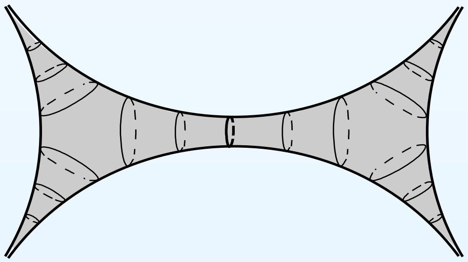


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

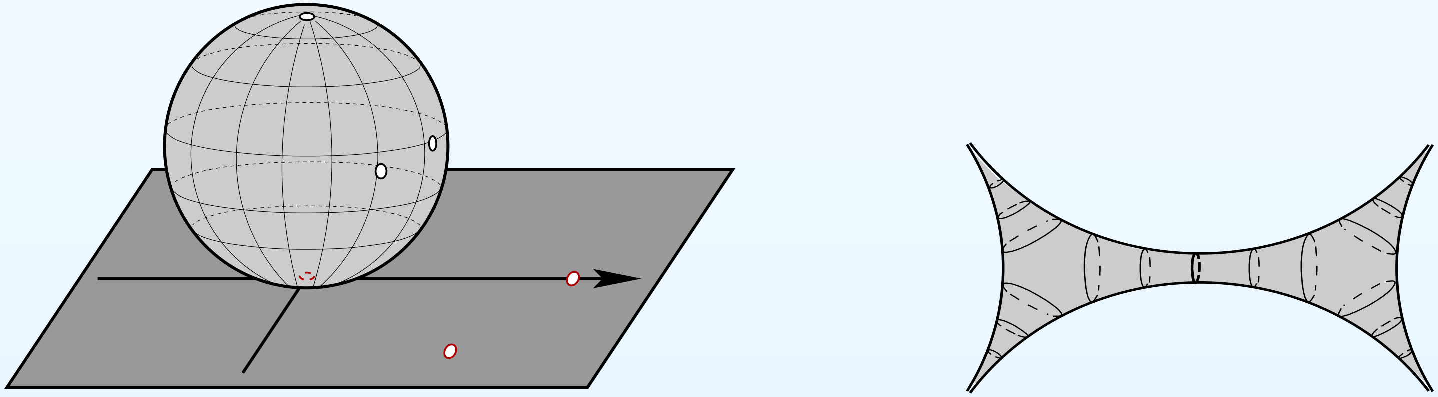




By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

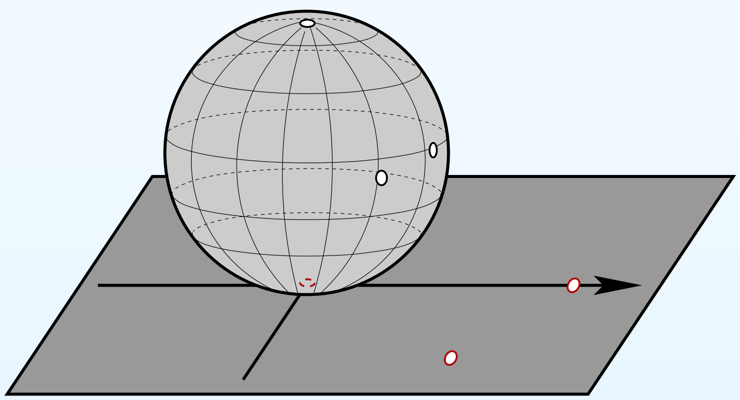
Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

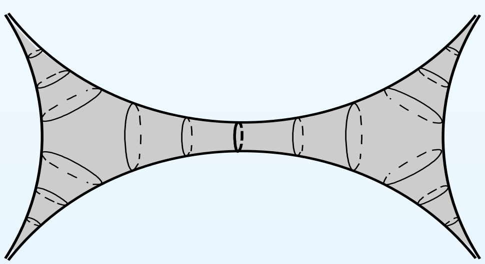


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

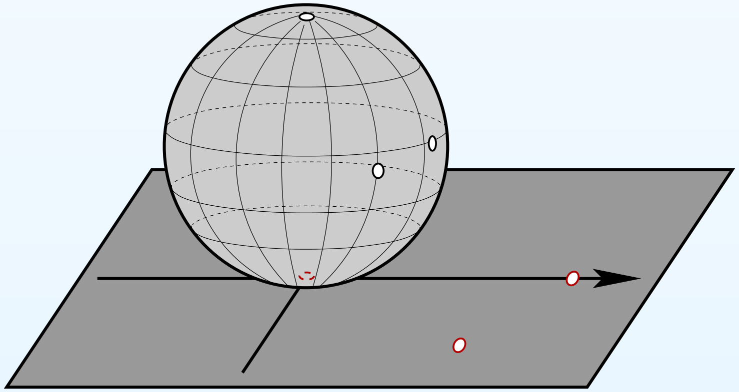


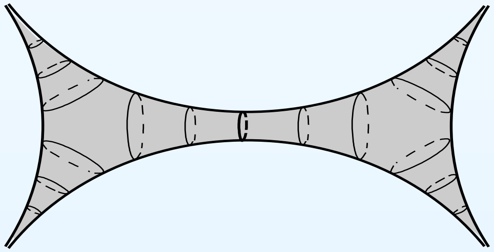


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

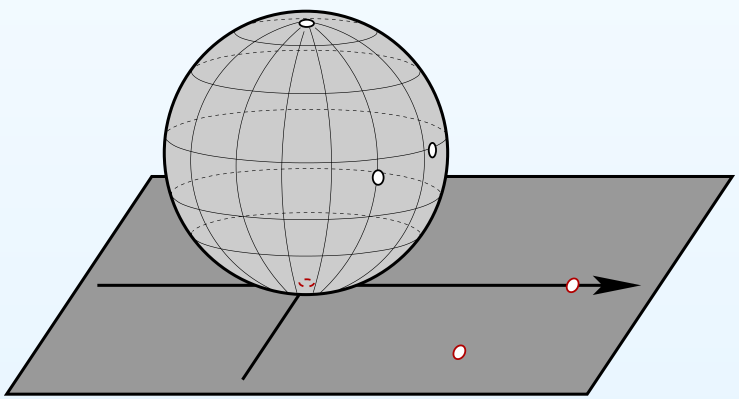


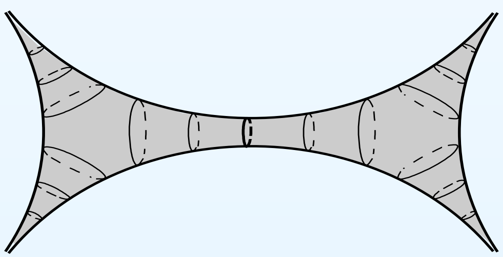


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

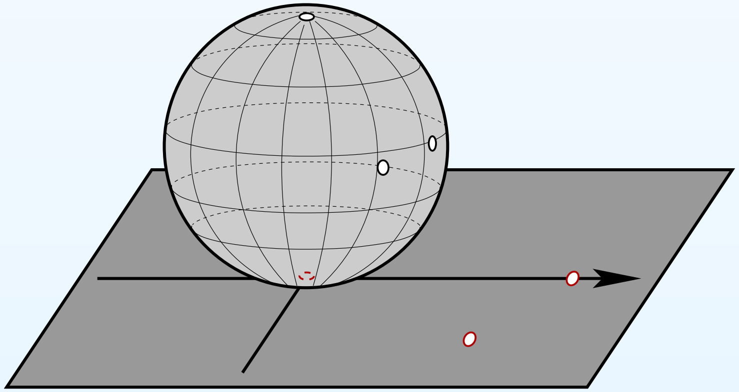


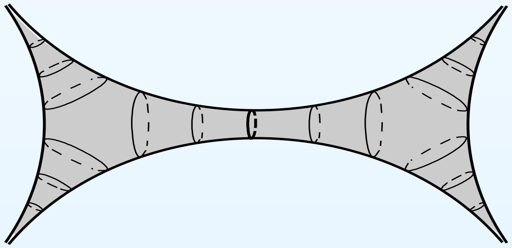


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

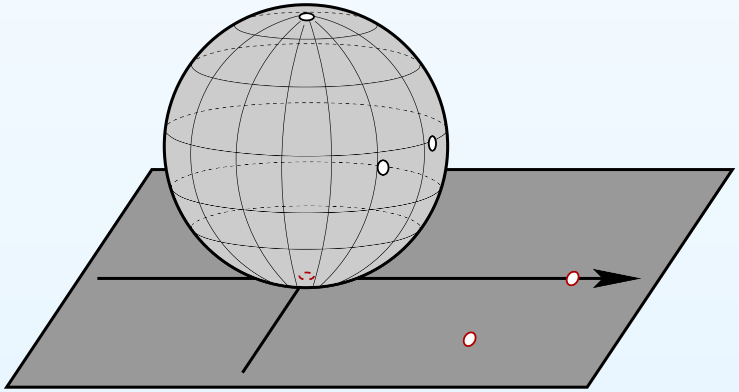


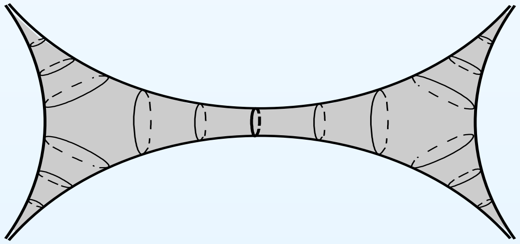


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

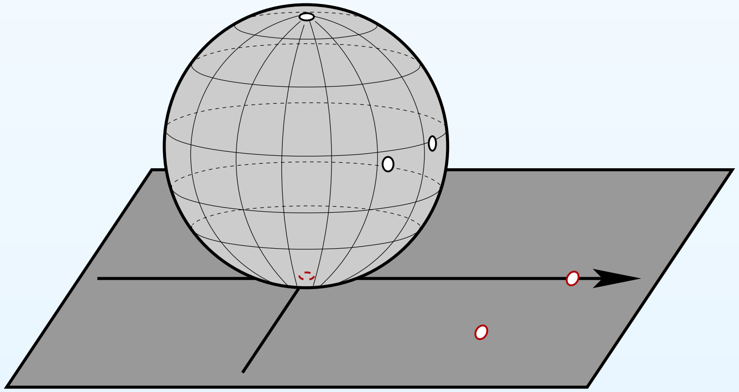


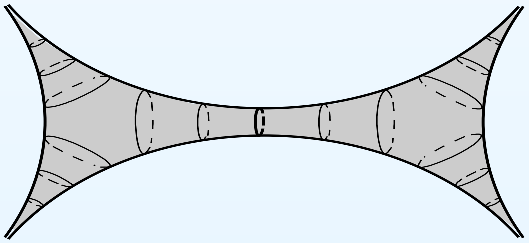


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).

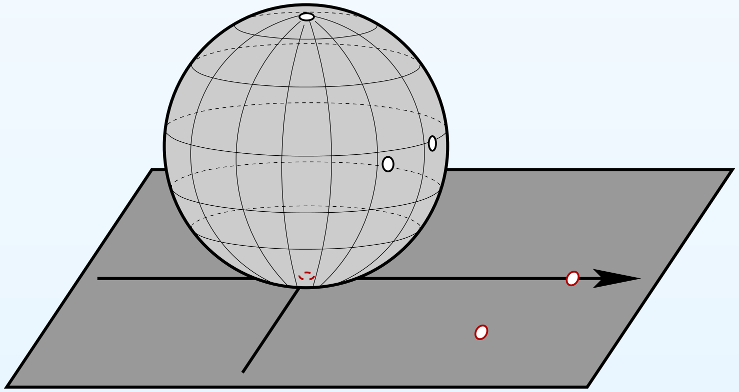


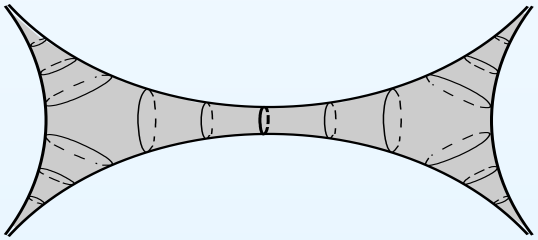


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

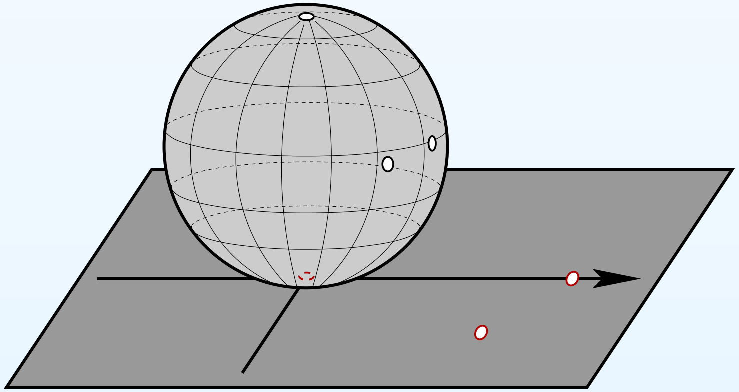


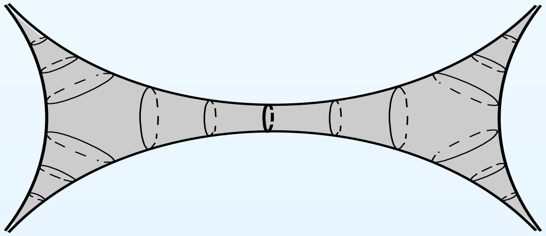


By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using appropriate holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{C}\mathrm{P}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{C}\mathrm{P}^1$ (up to a holomorphic diffeomorphism).

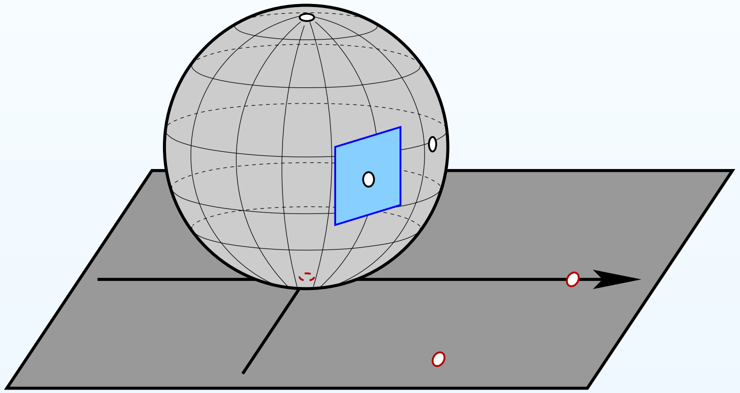




By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

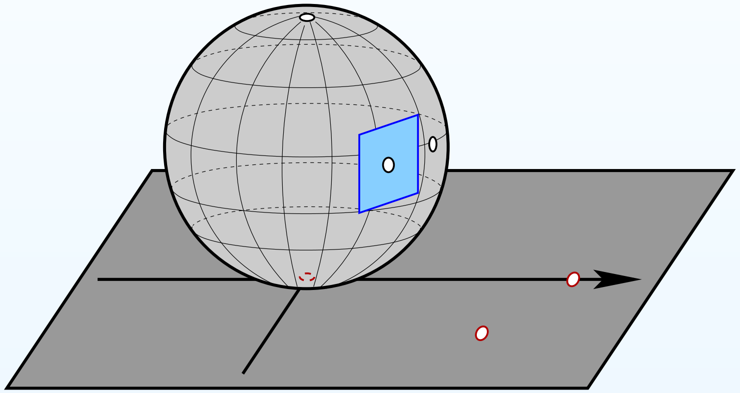
Tautological line bundles

As before, fix the points x_1, x_2, x_3 (by sending them to 0, 1 and ∞). Take the tangent plane to the point x_4. Considering $\mathbb{C}\mathrm{P}^1$ as a complex *curve* (instead of a real surface), the tangent *plane* should be seen rather as a tangent *line*.



Tautological line bundles

As before, fix the points x_1, x_2, x_3 (by sending them to 0, 1 and ∞). Take the tangent plane to the point x_4. Considering $\mathbb{C}\mathrm{P}^1$ as a complex *curve* (instead of a real surface), the tangent *plane* should be seen rather as a tangent *line*.



When we move the point x_4, the tangent line also moves. Recall that a location of the point x_4 parameterizes the moduli space $\mathcal{M}_{0,4} = \mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$. We get a nice family of complex lines parameterized by points of $\mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$. This family, actually, forms a holomorphic line bundle over $\mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$. Taking cotangent planes (i.e. complex cotangent lines) we get a holomorphic line bundle, which has its proper name: it is called the *tautological* line bundle.

ψ-classes

There is a smart way to extend the tautological line bundle to the punctures $\{0, 1, \infty\}$ and to get a holomorphic line bundle already over the compactification $\overline{\mathcal{M}_{0,4}} = \mathbb{C}\mathrm{P}^1$.

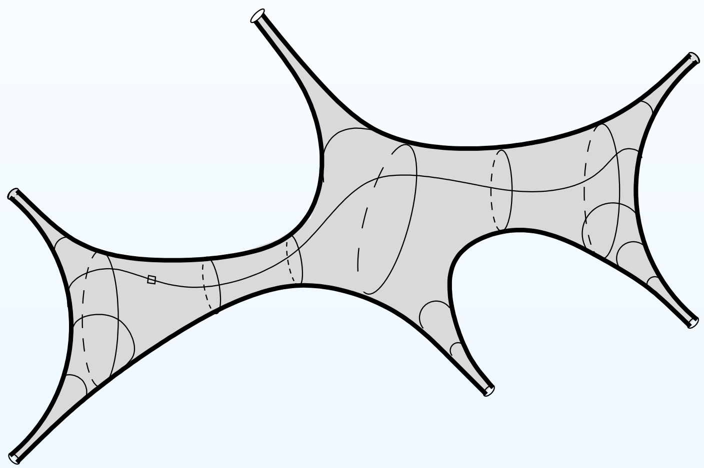
Every holomorphic line bundle defines a natural closed 2-form on the base. The closed 2-form, or rather its cohomology class, associated to the tautological bundle is so important that it also has a proper name: it is called ψ_4, where the index “4” indicates that we used the point x_4. In a complete analogy, we could also construct ψ_1, ψ_2, ψ_3. It can be checked that integrating ψ_i over $\overline{\mathcal{M}_{0,4}} = \mathbb{C}\mathrm{P}^1$ we get

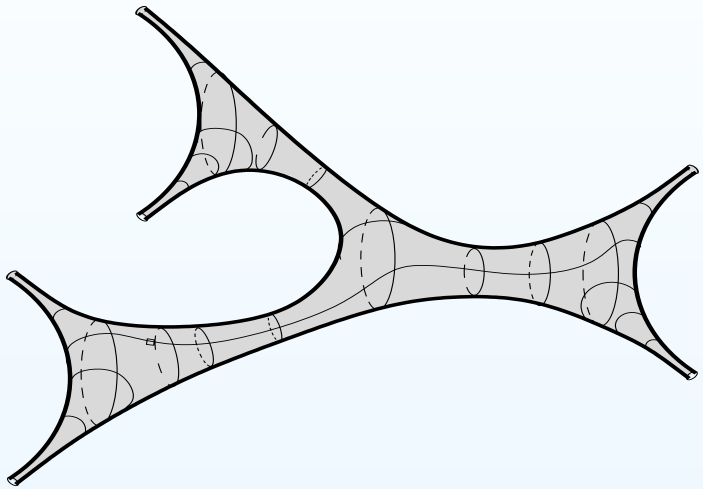
$$\int_{\overline{\mathcal{M}}_{0,4}} \psi_i = \int_{\mathbb{C}\mathrm{P}^1} \psi_i = 1 \quad for \quad i = 1, \ldots, 4\,.$$

By historical reasons such kind of integrals are called *intersection numbers*. In our case any such integral counts the number of intersection points of the zero section of the line bundle with any transverse holomorphic section.

Moduli space $\mathcal{M}_{g,n}$

Similarly, we can consider the moduli space $\mathcal{M}_{0,n}$ of spheres with n cusps.





The space $\mathcal{M}_{g,n}$ of configurations of n distinct points on a smooth closed orientable Riemann surface of genus $g > 0$ is even richer. While the sphere admits only one complex structure, a surface of genus $g \geq 2$ admits complex $(3g-3)$-dimensional family of complex structures. As in the case of the Riemann sphere, complex structures on a smooth surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points. For genus $g \geq 2$ one can let $n = 0$ and consider the space $\mathcal{M}_g = \mathcal{M}_{g,0}$ of hyperbolic surfaces without cusps.

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g-3+n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational “*intersection number*” (or the “*correlator*” in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \, .$$

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g-3+n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in* 2*-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\, \mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

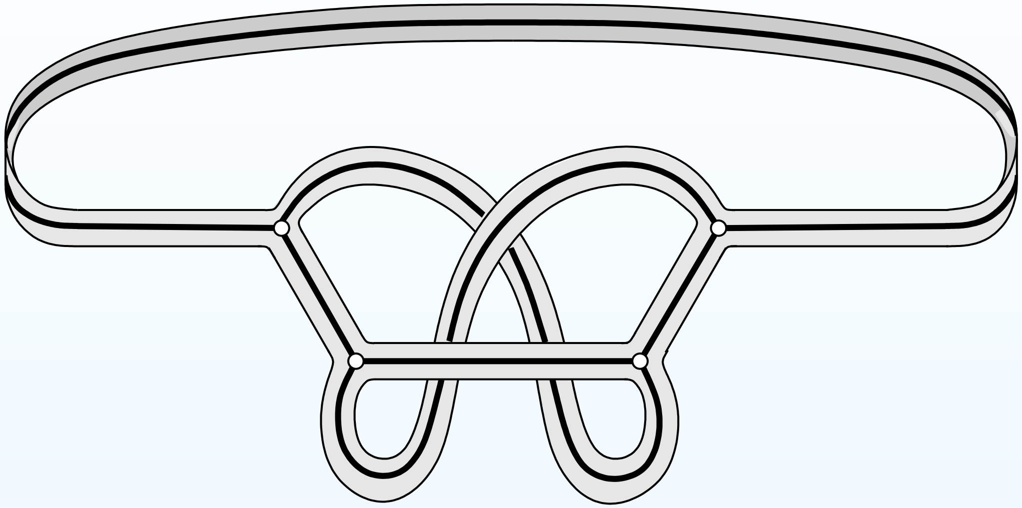
$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 1$ with $n = 2$ boundary components. Assigning lengths to all edges of the core graph, we endow each boundary component with an induced length defined as the sum of the lengths of edges which it follows.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorems of Kontsevich and Norbury count them.

Count of metric ribbon graphs

Theorem (Kontsevich'92; in this stronger form — Norbury'10). *Consider a collection of positive integers $b_1, \ldots, b_n$ such that $\sum_{i=1}^n b_i$ is even. The weighted count of genus g connected trivalent metric ribbon graphs Γ with integer edges and with n labeled boundary components of lengths $b_1, \ldots, b_n$ is equal to $N_{g,n}(b_1, \ldots, b_n)$ up to the lower order terms:*

$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n) = N_{g,n}(b_1, \ldots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ denote the set of (nonisomorphic) trivalent ribbon graphs Γ of genus g and with n boundary components.

Count of metric ribbon graphs

Theorem (Kontsevich'92; in this stronger form — Norbury'10). *Consider a collection of positive integers $b_1, \ldots, b_n$ such that $\sum_{i=1}^n b_i$ is even. The weighted count of genus g connected trivalent metric ribbon graphs Γ with integer edges and with n labeled boundary components of lengths $b_1, \ldots, b_n$ is equal to $N_{g,n}(b_1, \ldots, b_n)$ up to the lower order terms:*

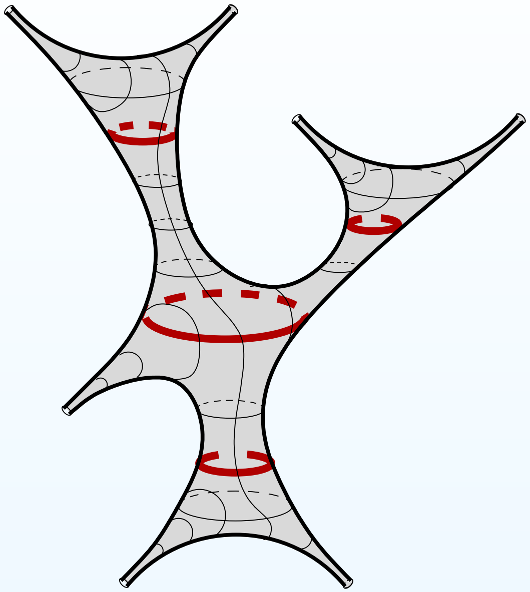
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n) = N_{g,n}(b_1, \ldots, b_n) + \text{lower order terms}\,,$$

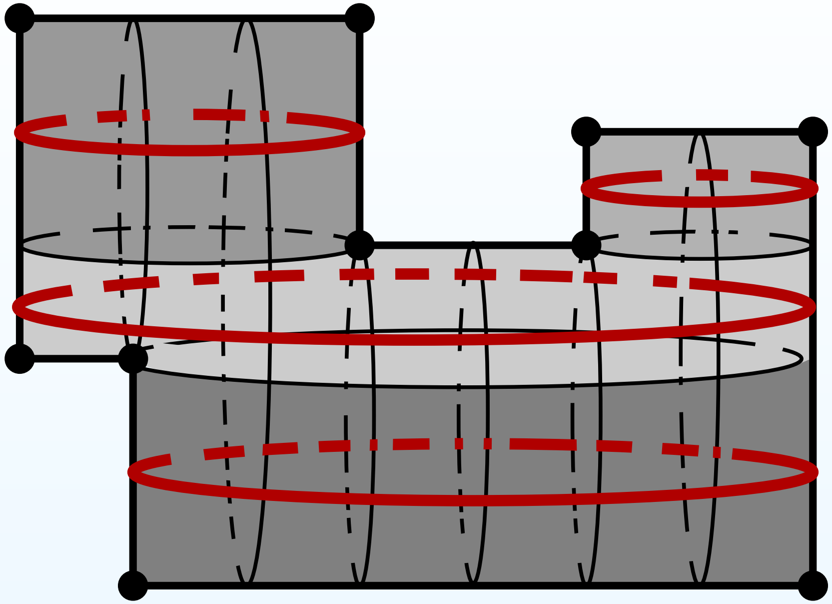
where $\mathcal{R}_{g,n}$ denote the set of (nonisomorphic) trivalent ribbon graphs Γ of genus g and with n boundary components.

(Formal statement justifying the notion of "lower order terms": the right-hand side is a *quasipolynomial* in the integers $b_1, \ldots, b_n$ depending on the number k of odd b_i. The top homogeneous part is zero when k is odd.)

A version of this Theorem is an important part of Kontsevich's proof of Witten's conjecture.

Multicurve associated to a square-tiled surface

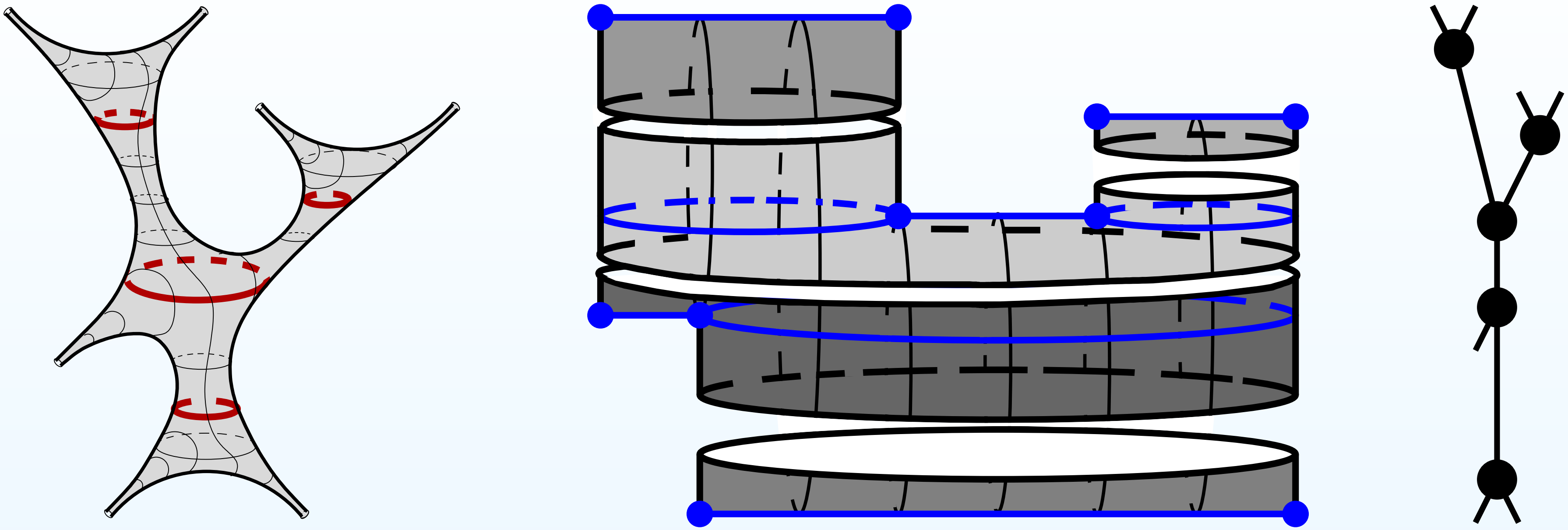




Having a square-tiled surface we associate to it a topological surface S on which we mark all “corners” with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points.

We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. We encode the number of horizontal bands of squares in each cylinder by taking the components of the multicurve with integer weights.

Ribbon graph decomposition of a square-tiled surface



Leaves of the horizontal foliation on the square-tiled surface passing through singular points (in blue) are called *critical*. Considering tubular neighborhoods of these critical leaves we get metric ribbon graphs. Cylinders (represented by the multicurve in red) are joining boundary components of these ribbon graphs.

A dual graph to the multicurve is called *stable graph* Γ. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by “legs” — half-edges of the dual graph.

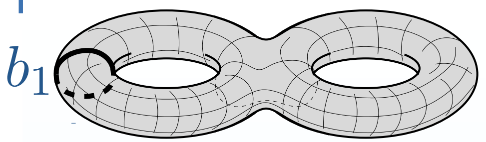
Count of square-tiled surfaces: an algorithm

1. Fix a genus g and a number n of corners (conical points) of angle π.
2. Consider a finite collection of *stable graphs* encoding all possible admissible decompositions of a hyperbolic surface of genus g with n cusps (equivalently, all complex stable curves of genus g with n marked points).
3. For each stable graph with k edges associate formal variables $b_1, \ldots, b_k$ to its edges and associate metric ribbon graphs to the vertices.
4. Using the Kontsevich–Norbury count of metric ribbon graphs, count the number of ways to join them by square-tiled cylinders.

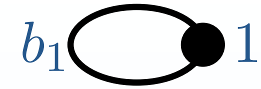
Count of square-tiled surfaces: an algorithm

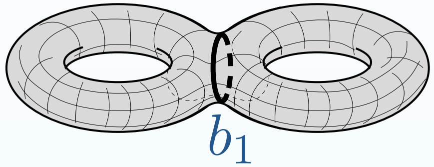
1. Fix a genus g and a number n of corners (conical points) of angle π.
2. Consider a finite collection of *stable graphs* encoding all possible admissible decompositions of a hyperbolic surface of genus g with n cusps (equivalently, all complex stable curves of genus g with n marked points).
3. For each stable graph with k edges associate formal variables $b_1, \ldots, b_k$ to its edges and associate metric ribbon graphs to the vertices.
4. Using the Kontsevich–Norbury count of metric ribbon graphs, count the number of ways to join them by square-tiled cylinders.

Masur–Veech volume of the moduli space of quadratic differentials. This moduli space is the total space of the cotangent bundle to the moduli space $\mathcal{M}_{g,n}$ of complex curves with n marked points and, hence, has a canonical symplectic structure, and a volume element. Square-tiled surfaces represent integer points in this space. Those, which are tiled with at most N squares, are integer points in a “bundle of balls of radius N” over $\mathcal{M}_{g,n}$. Thus, asymptotics of the number of square-tiled surfaces of genus g with n conical points of angle π tiled with at most $N \to +\infty$ squares gives us the Masur–Veech volume.

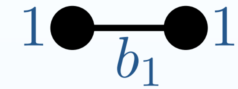


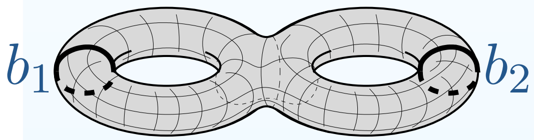
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



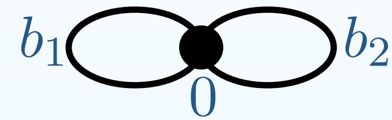


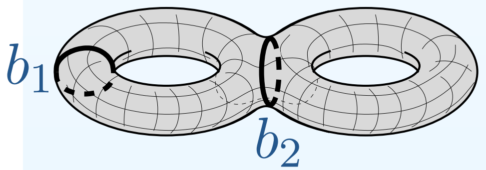
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



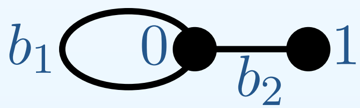


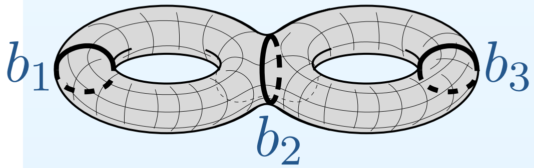
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



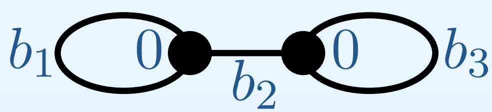


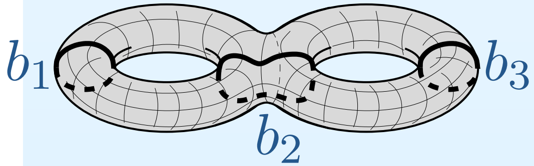
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



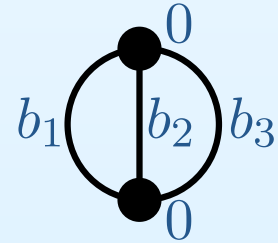


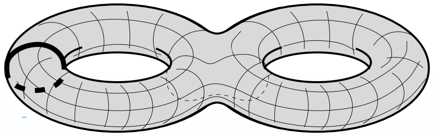
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



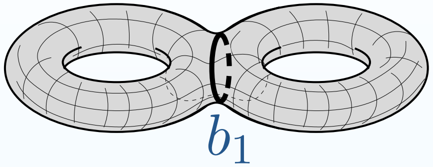


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$

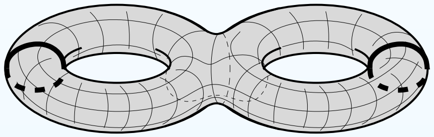




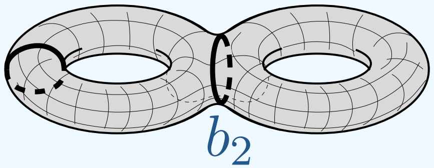
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) = \frac{1}{2} \cdot b_1 \left(\frac{1}{384}(2b_1^2)(2b_1^2)\right)$$



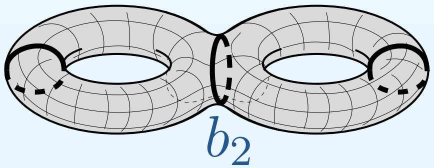
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2\right) \left(\frac{1}{48} b_1^2\right)$$



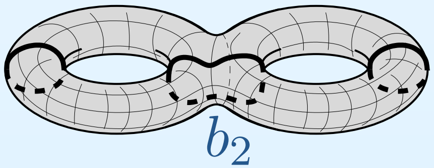
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4}(2b_1^2 + 2b_2^2)\right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2\right)$$

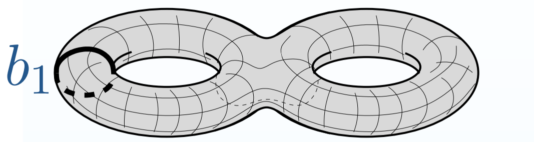


$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

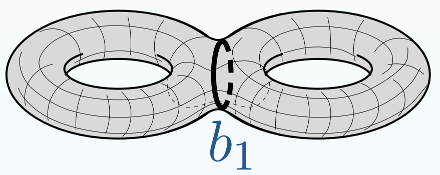


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

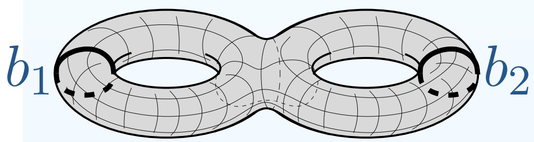
Volume of $\mathcal{Q}_2$



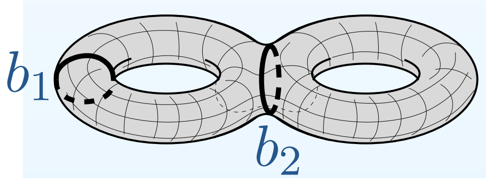
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



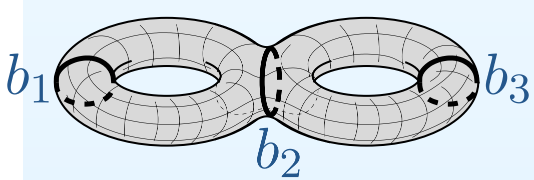
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



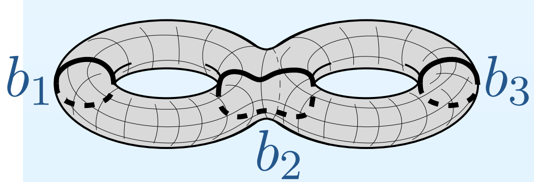
$$\frac{1}{16}(b_1^3b_2 + b_1b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16}b_1b_2b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$

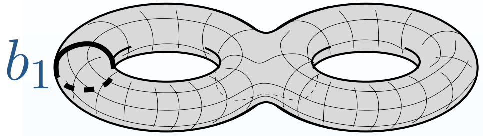


$$\frac{1}{24}b_1b_2b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

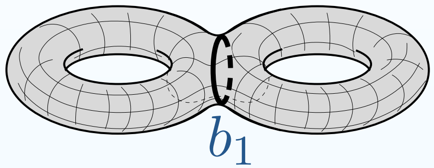
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

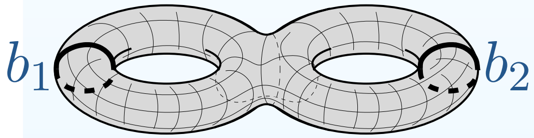
Volume of $\mathcal{Q}_2$



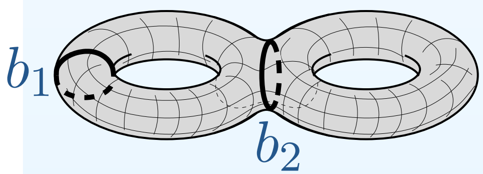
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



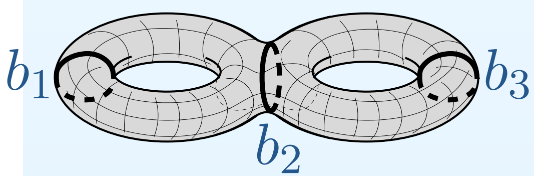
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



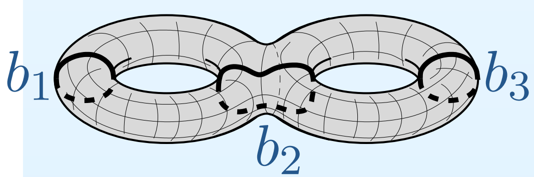
$$\frac{1}{16}(b_1^3 b_2 + +b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix–Goujard–Zograf–Zorich'21). *The Masur–Veech volume $\operatorname{Vol}\mathcal{Q}_{g,n}$ of the moduli space of meromorphic quadratic differentials with n simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k of edges gives the contribution of k-cylinder square-tiled surfaces.

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix–Goujard–Zograf–Zorich'21). *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot\sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right),$$

Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

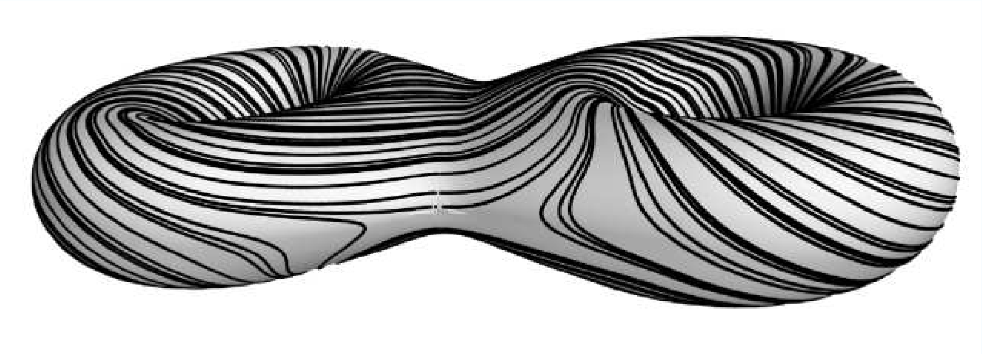
Mirzakhani’s count of simple closed geodesics

Picture taken by F. Labourie at CIRM, Luminy

Simple closed multicurve, its topological type and underlying primitive multicurve

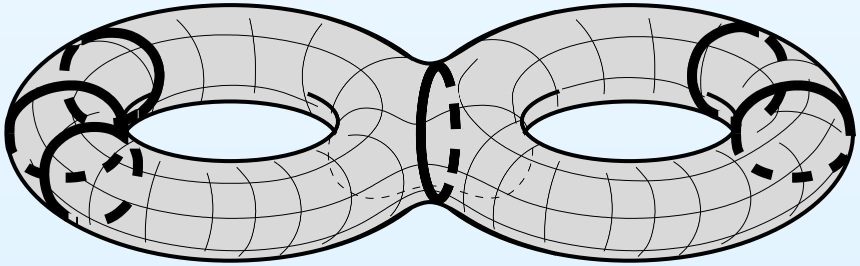
Having an arbitrary collection of complicated non self-intersecting and non pairwise intersecting curves (called a *multicurve*), one can apply an appropriate diffeomorphism of the surface which "unwraps" the multicurve to a simple canonical representative.

A general multicurve ρ:

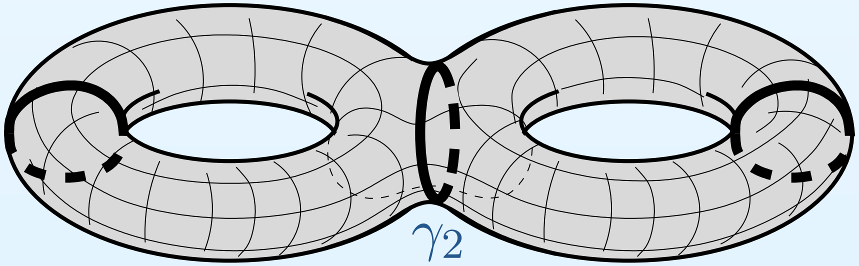


the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

$$\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3.$$



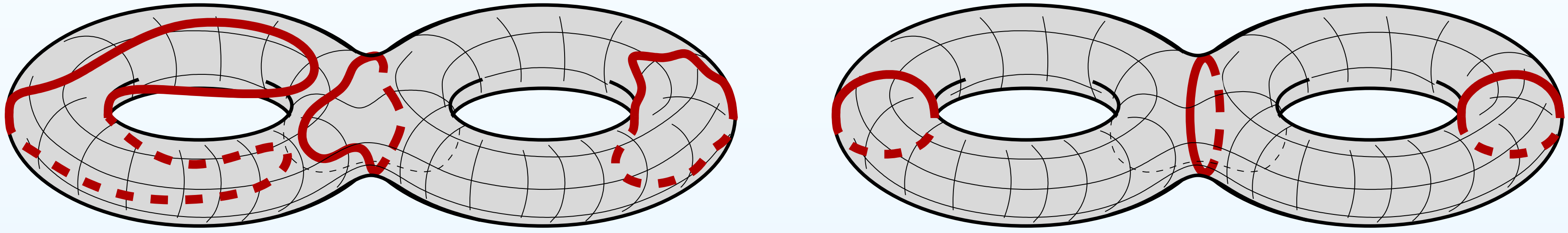
$$\gamma_{reduced} = \gamma_1 + \gamma_2 + \gamma_3$$



(You can practice in unwinding curves at `https://aharalab.sakura.ne.jp/teruaki.html`)

Geodesic representatives of multicurves

Consider several pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. In the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves $\gamma_1, \ldots, \gamma_k$ contract to simple closed geodesics.



Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

We define the hyperbolic length of a multicurve $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

Frequencies of multicurves

Theorem (Mirzakhani'08). *For any integral multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *the number* $s_X(L,\gamma)$ *of simple closed geodesic multicurves on* X *of topological type* $[\gamma]$ *and of hyperbolic length at most* L *has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

Frequencies of multicurves

Theorem (Mirzakhani'08). *For any integral multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *the number* $s_X(L,\gamma)$ *of simple closed geodesic multicurves on* X *of topological type* $[\gamma]$ *and of hyperbolic length at most* L *has the following asymptotics:*

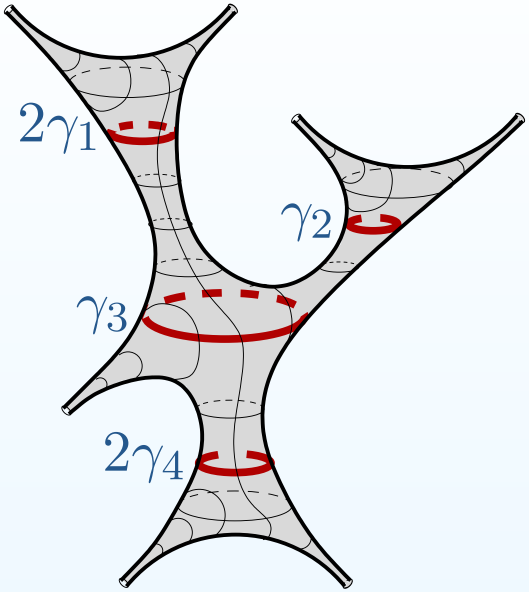
$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

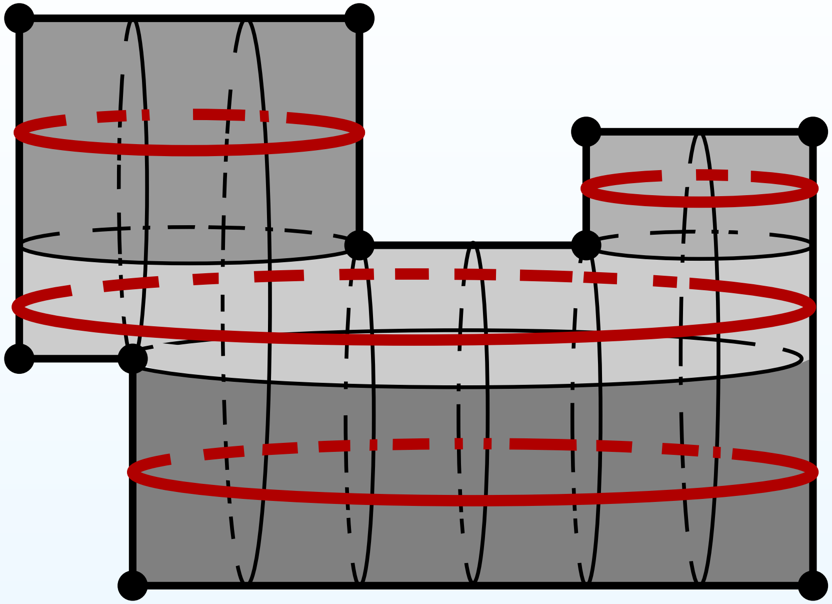
Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

Corollary (Mirzakhani'08). *For any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$*, and any two rational multicurves* γ_1, γ_2 *on a smooth surface* $S_{g,n}$ *considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

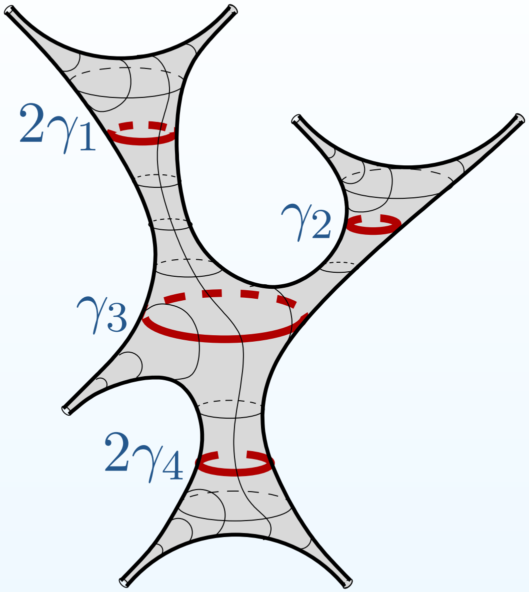
Hyperbolic and flat geodesic multicurves

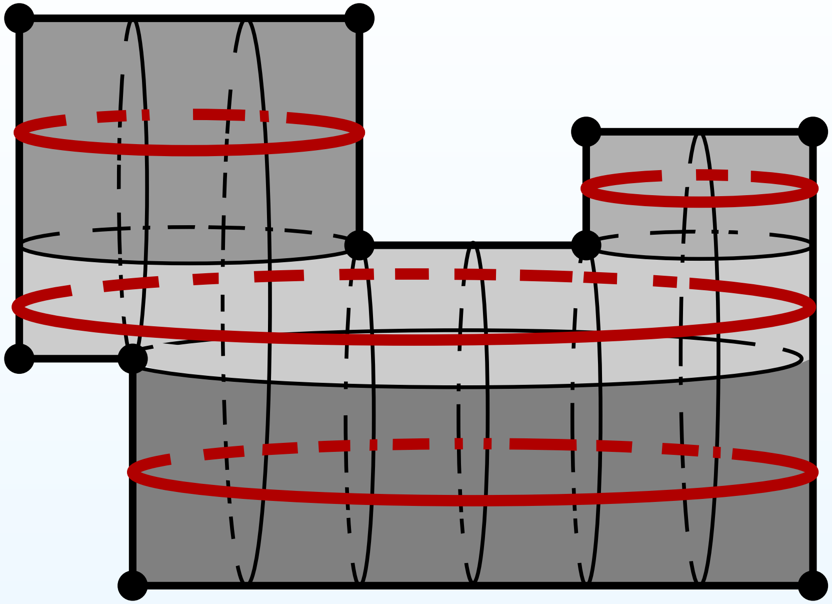




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves



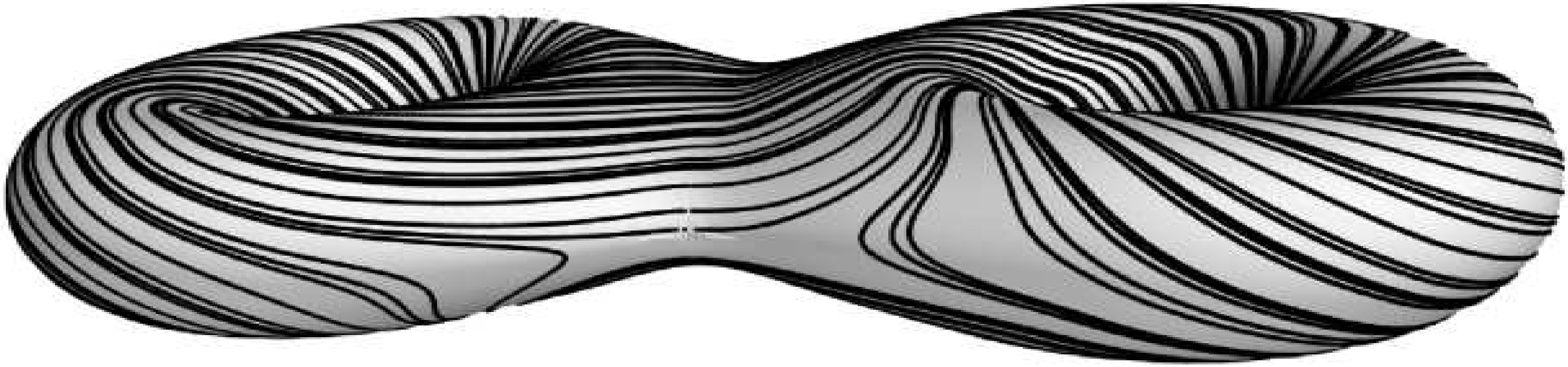


Theorem (Delecroix–Goujard–Zograf–Zorich'21). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of **hyperbolic** multicurves coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera has found an alternative proof of this result. His proof uses more geometric approach.

Shape of a random multicurve on a surface of genus two

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

• *Which simple closed geodesics are more frequent: separating or non-separating?*

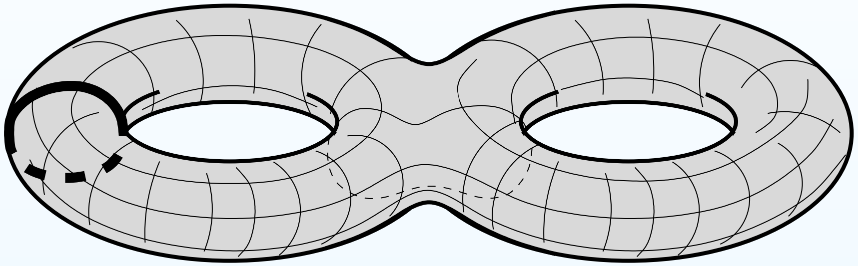
Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

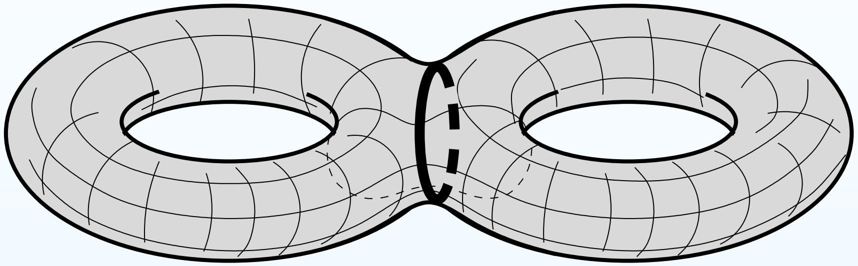
• *What is the probability that $\gamma_{reduced}$ separates S into distinct connected components?*

• *What are the probabilities that $\gamma_{reduced}$ has $k = 1, 2, 3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$

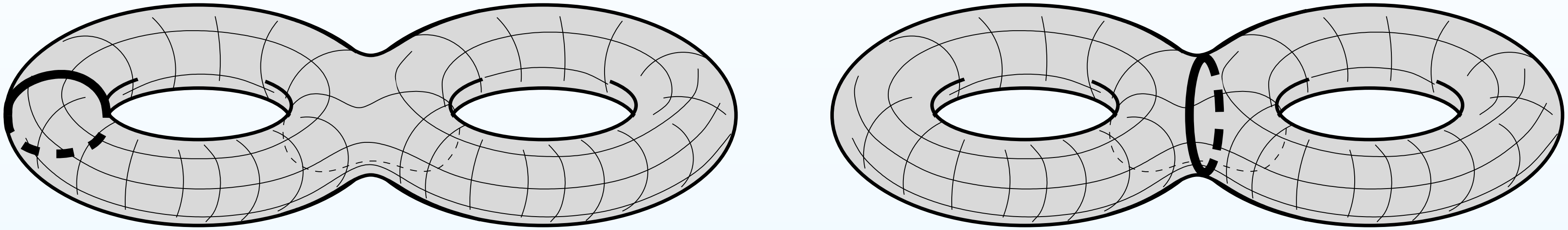




$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$

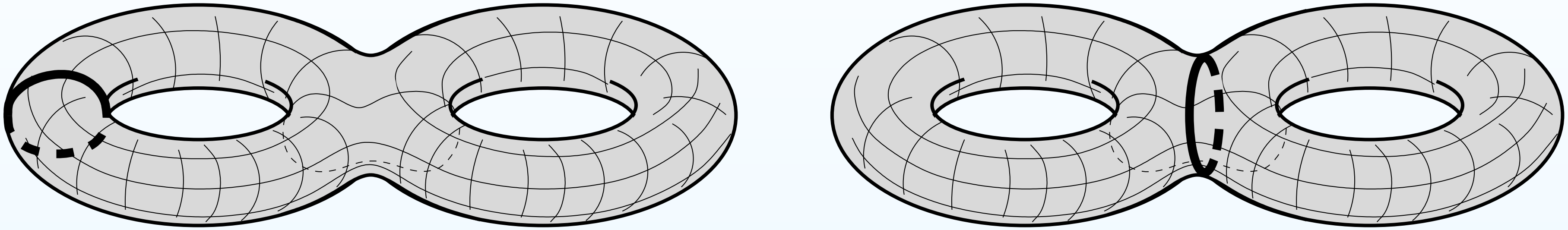


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Mirzakhani's calculation.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$

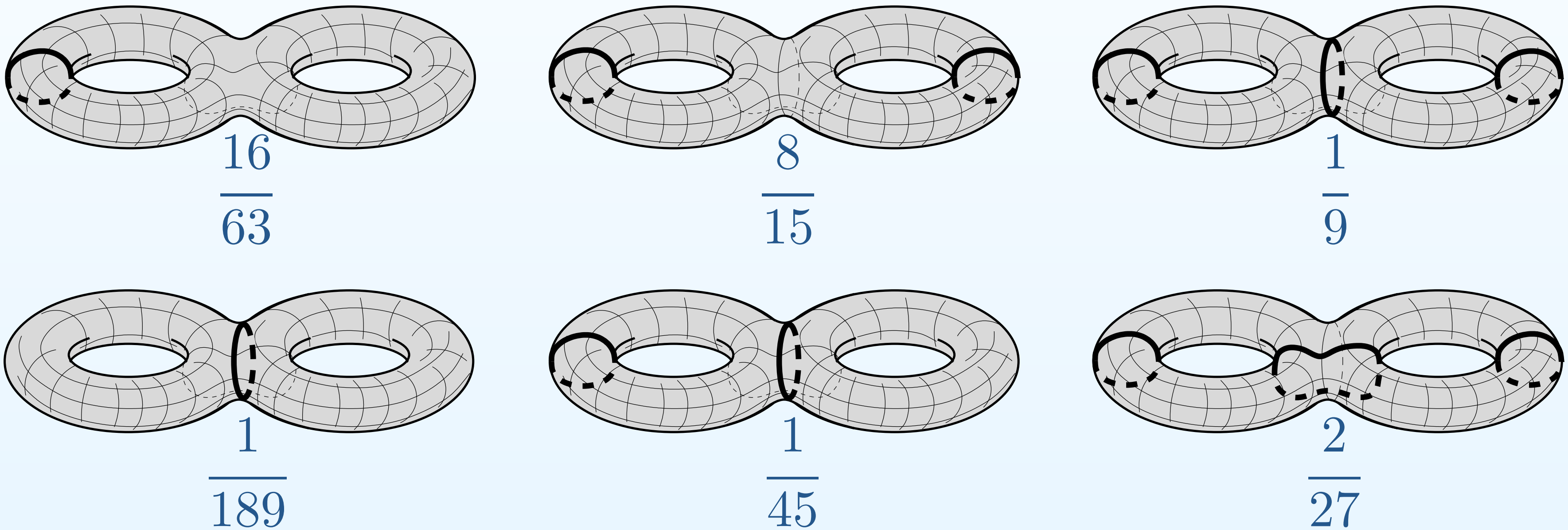


$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell. Most recently it was independently confirmed by V. Erlandsson, K. Rafi, J. Souto and by A. Wright by methods independent of ours.

Multicurves on a surface of genus two and their frequencies

The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Shape of a random multicurve on a surface of large genus. Shape of a random square-tiled surface of large genus.

Statistics of prime decompositions: random integer numbers

The Prime Number Theorem states that an integer number n taken randomly in a large interval $[1, N]$ is prime with asymptotic probability $\frac{\log N}{N}$.

Actually, one can tell much more about prime decomposition of a large random integer. Denote by $\omega(n)$ the number of prime divisors of an integer n counted without multiplicities. In other words, if n has prime decomposition $n = p_1^{m_1} \dots p_k^{m_k}$, let $\omega(n) = k$. By the Erdős–Kac theorem, the centered and rescaled distribution prescribed by the counting function $\omega(n)$ tends to the normal distribution:

Erdős–Kac Theorem (1939)

$$\lim_{N \to +\infty} \frac{1}{N} \operatorname{card} \left\{ n \leq N \,\middle|\, \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt\,.$$

The subsequent results of A. Selberg (1954) and of A. Rényi and P. Turán (1958) describe the rate of convergence.

Statistics of prime decompositions: random permutations

Denote by $\mathrm{K}_n(\sigma)$ the number of disjoint cycles in the cycle decomposition of a permutation σ in the symmetric group S_n. Consider the uniform probability measure on S_n. A random permutation σ of n elements has exactly k cycles in its cyclic decomposition with probability $\mathbb{P}\big(\mathrm{K}_n(\sigma) = k\big) = \frac{s(n,k)}{n!}$, where $s(n,k)$ is the unsigned Stirling number of the first kind. It is immediate to see that $\mathbb{P}\big(\mathrm{K}_n(\sigma) = 1\big) = \frac{1}{n}$. V. L. Goncharov computed the expected value and the variance of K_n as $n \to +\infty$:

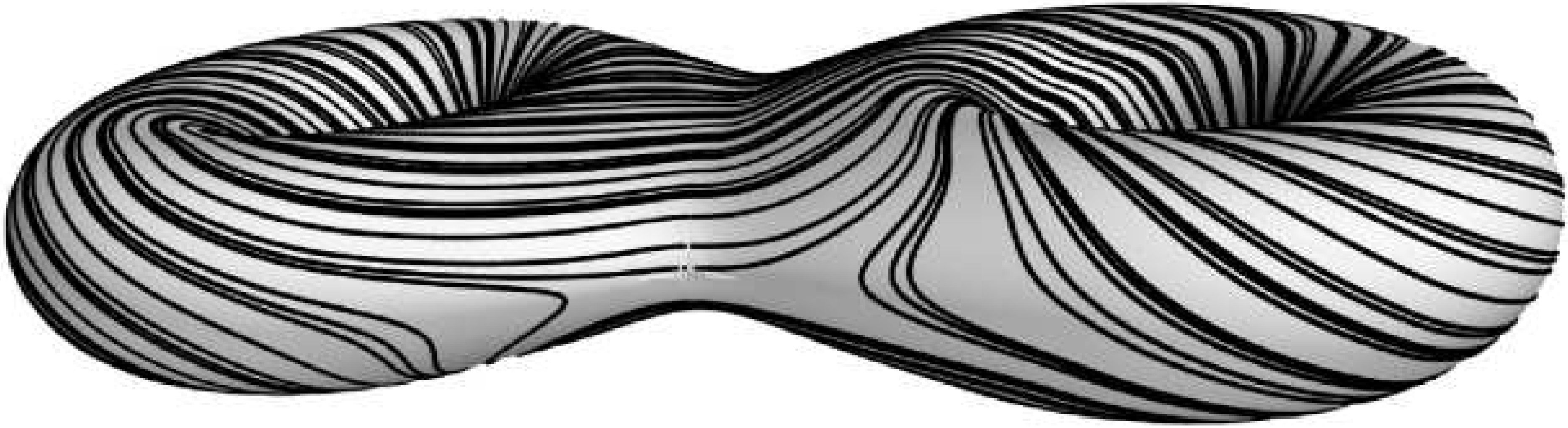
$$\mathbb{E}(\mathrm{K}_n) = \log n + \gamma + o(1)\,, \qquad \mathbb{V}(\mathrm{K}_n) = \log n + \gamma - \zeta(2) + o(1)\,,$$

and proved the following central limit theorem:

Theorem (V. L. Goncharov, 1944)

$$\lim_{n \to +\infty} \frac{1}{n!}\,\mathrm{card}\left\{\sigma \in S_n \,\middle|\, \frac{\mathrm{K}_n(\sigma) - \log n}{\sqrt{\log n}} \le x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt\,.$$

What shape has a random simple closed multicurve on a surface of large genus?



Picture from a book of Danny Calegari

Questions.

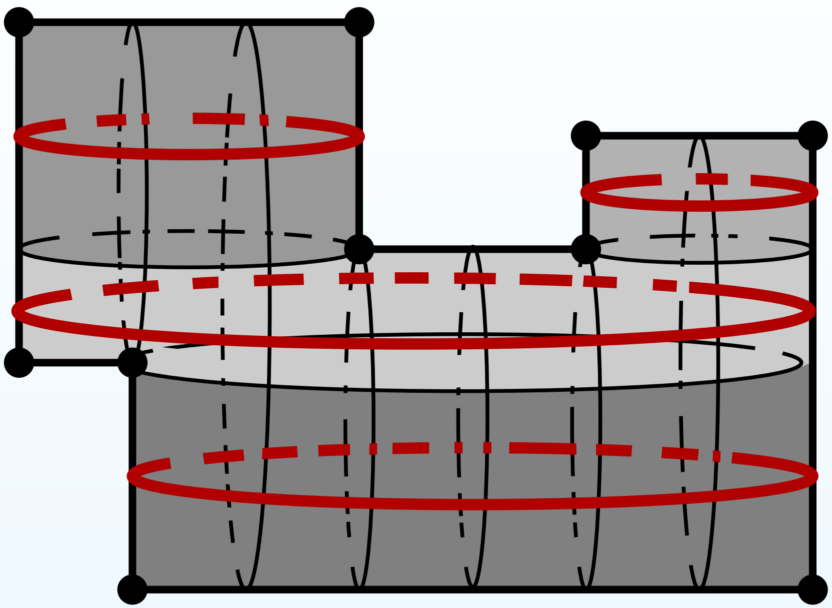
• *With what probability a random primitive multicurve* γ *on a surface of genus* g *slices the surface into* $1, 2, 3, \dots$ *connected components?*

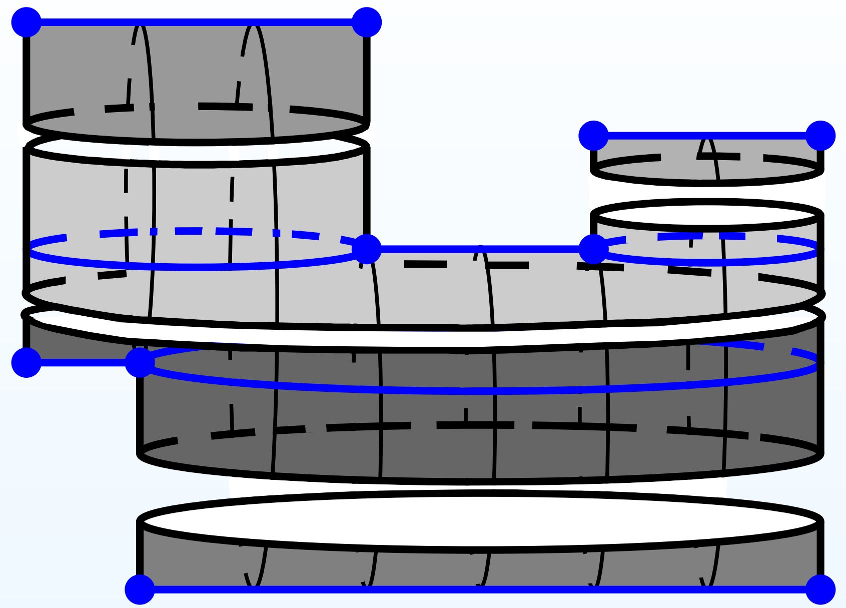
• *With what probability a random multicurve* $m_1\gamma_1 + m_2\gamma_2 + \dots + m_k\gamma_k$ *has* $k = 1, 2, \dots, 3g-3$ *primitive connected components* $\gamma_1, \dots, \gamma_k$*?*

• *What are the typical weights* $m_1, \dots, m_k$*?*

• *What is the shape of a random multicurve on a surface of large genus?*

Shape of a random square-tiled surface of large genus





Questions.

● *How many singular horizontal leaves (in blue on the right picture) has a random square-tiled surface of genus* g*?*

● *Find the probability distribution for the number* $K_g(S) = 1, 2, 3, \ldots, 3g-3$ *of maximal horizontal cylinders (represented by red waist curves on the left picture)*

● *What are the typical heights* $h_1, \ldots, h_k$ *of the cylinders?*

● *What is the shape of a random square-tiled surface of large genus?*

Random multicurves and random square-tiled surfaces

Denote by $K_{g,n}(\gamma)$ the number k of components of a multicurve $\gamma = \sum_{i=1}^{k} m_i\gamma_i$ (counted *without* multiplicities m_i) on a surface of genus g with n cusps.

Denote by $K_{g,n}(S)$ the number of maximal horizontal cylinders in the cylinder decomposition of a square-tiled surface S of genus g with n cone-angles π

Theorem (Delecroix–Goujard–Zograf–Zorich'21.). *For any genus $g \geq 2$ and for any $k \in \mathbb{N}$, the probability $p_g(k)$ that a random multicurve γ on a surface of genus g has exactly k components counted without multiplicities coincides with the probability that a random square-tiled surface S of genus g has exactly k maximal horizontal cylinders:*

$$\mathbb{P}\big(K_{g,n}(\gamma) = k\big) = \mathbb{P}\big(K_{g,n}(S) = k\big)\,.$$

In other words, $K_{g,n}(\gamma)$ and $K_{g,n}(S)$, considered as random variables, determine the same probability distribution for any g, n, $3g + n \geq 4$.

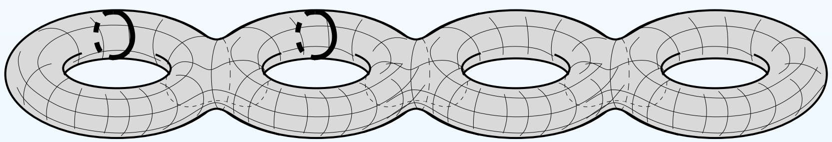
From now on we consider only hyperbolic surfaces without cusps and only square-tiled surfaces without cone-angles π (i.e. the ones corresponding to *holomorphic* quadratic differentials).

Shape of a random multicurve (random square-tiled surface) on a surface of large genus in simple words

Theorem (Delecroix–Goujard–Zograf–Zorich'20.) *With probability which tends to 1 as $g \to \infty$,*

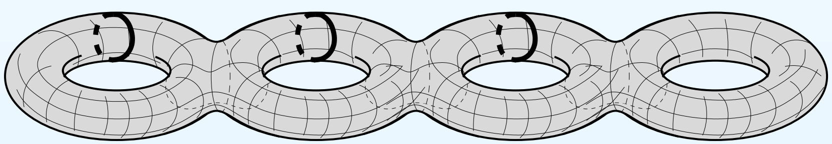
• The reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ associated to a random integral multicurve $\gamma = m_1\gamma_1 + \dots m_k\gamma_k$ does not separate the surface;

• $\gamma_{reduced}$ has about $(\log g)/2$ components and has one of the following types:



$0.09 \log(g)$ components

$\dots$ $\dots$ $\dots$ $\dots$



$0.62 \log(g)$ components

$$\mathbb{P}\Big(0.09 \log g < K_g(\gamma) < 0.62 \log g\Big) = 1 - O\left((\log g)^{24} g^{-1/4}\right) .$$

A random square-tiled surface (without conical points of angle π) of large genus has about $\frac{\log(g)}{2}$ cylinders, and ***all conical points sit at the same horizontal and the same vertical level with probability which tends to 1 as $g \to \infty$.***

Heights of cylinders of a ransom square-tiled surface

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *If we fix any k and consider only k-cylinder square-tiled surfaces, then a (conditional) probability that every horizontal cylinder is composed of a single band of squares tends to 1 as $g \to +\infty$.*

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *If we do not fix the number of horizontal cylinders, then the probability that every horizontal cylinder of a random square-tiled surface is composed of a single band of squares tends to $\frac{\sqrt{2}}{2}$ as genus grows. More generally, each of the heights $m_1, \ldots m_k$ of horizontal cylinders of a random square-tiled surface is bounded from above by an integer m with probability which tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

However, the mean value of $m_1 + \ldots + m_k$ is infinite in any genus g.

Number of cycles in a random permutation

Given a permutation $\sigma \in S_n$ of cycle type $(1^{\mu_1} 2^{\mu_2} \ldots n^{\mu_n})$ define its *weight* as

$$w_\theta(\sigma) := \theta_1^{\mu_1} \theta_2^{\mu_2} \cdots \theta_n^{\mu_n} \,,$$

where $\theta_j = \frac{\zeta(2j)}{2}$, $j \in \mathbb{N}$. Define a probability measure on S_n by setting

$$\mathbb{P}_\theta(\sigma) := \frac{w_\theta(\sigma)}{W_\theta} \,, \qquad \text{where} \qquad W_\theta := \sum_{\sigma \in S_n} w_\theta(\sigma).$$

Measures with $\theta_k = const$, $k \in \mathbb{N}$, are called *Ewens measures*; for $const = 1$ we get the uniform measure on S_n.

Number of cycles in a random permutation

Given a permutation $\sigma \in S_n$ of cycle type $(1^{\mu_1} 2^{\mu_2} \dots n^{\mu_n})$ define its *weight* as

$$w_\theta(\sigma) := \theta_1^{\mu_1} \theta_2^{\mu_2} \cdots \theta_n^{\mu_n} \,,$$

where $\theta_j = \frac{\zeta(2j)}{2}$, $j \in \mathbb{N}$. Define a probability measure on S_n by setting

$$\mathbb{P}_\theta(\sigma) := \frac{w_\theta(\sigma)}{W_\theta}\,, \qquad \text{where} \qquad W_\theta := \sum_{\sigma \in S_n} w_\theta(\sigma).$$

Measures with $\theta_k = const$, $k \in \mathbb{N}$, are called *Ewens measures*; for $const = 1$ we get the uniform measure on S_n.

The random variable $K(\sigma)$ counting the number of disjoint cycles in the cyclic decomposition of a random permutation is very well studied (Goncharov'44, ... Hwang'94–95, ... Kowalski–Nikeghbali'10,...). The corresponding probability distribution is given by the Poisson distribution with parameter depending on n, corrected by a convolution with certain explicit function independent of n.

Using this *Mod-Poisson convergence* technique we also get a very precise description of the law for the number of cycles $K(\sigma)$ in a random permutation for our nonuniform Ewens-like measure $\mathbb{P}_\theta$.

Main Theorem (informally)

Main Theorem (Delecroix–Goujard–Zograf–Zorich'20). *As g grows, the probability distribution $\mathbb{P}(K_g = k)$ rapidly becomes, basically, indistinguishable from the distribution of the number $K_{3g-3}(\sigma)$ of disjoint cycles in a random permutation σ of $3g-3$ elements (with respect to some explicit nonuniform probability measure on the symmetric group). In particular, for any $j \in \mathbb{N}$ the difference of the j-th moments of the two distributions is of the order $O(g^{-1})$.*

We have an explicit asymptotic formula for all cumulants. It gives

$$\mathbb{E}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 + o(1)\,,$$

$$\mathbb{V}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1)\,,$$

where $\gamma = 0.5772\dots$ denotes the Euler–Mascheroni constant.

In practice, already for $g = 12$ the match of the graphs of the distributions is such that they are visually indistinguishable.

Idea of the proof and further conjectures

Schematic idea of the proof

- Observe that square-tiled surfaces corresponding to stable graphs with more than one vertex taken together contribute only $O\left(\frac{1}{g}\right)$ to the count of all square-tiled surfaces of genus g (this conjecture of ours was proved by A. Aggarwal).

- Using large genus asymptotics for the Witten–Kontsevich correlators (conjectured by us and proved by A. Aggarwal) compute the contribution of square-tiled surfaces of genus g represented by the stable graph with exactly one vertex and with k loops. Recognize in the resulting expression the multivariate harmonic sum as in the above Lemma corresponding to parameters $\theta_j = \zeta(2j)/2$, where $j = 1, 2, \dots$.

- Apply the analytic technique developed by H. Hwang for random permutations to prove mod-Poisson convergence of the resulting distribution of the number of cycles $\mathrm{K}_n(\sigma)$ of a random permutation σ, where “randomness” is defined using parameters $\theta_j = \zeta(2j)/2$, where $j = 1, 2, \dots$.

Keystone underlying result

Our results use the Delecroix–Goujard–Zograf–Zorich'19 conjecture proved in

Theorem (Aggarwal'21). *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol} \mathcal{Q}_g \sim \frac{4}{\pi} \cdot \left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g \to +\infty\,.$$

Keystone underlying result

Our results use the Delecroix–Goujard–Zograf–Zorich'19 conjecture proved in

Theorem (Aggarwal'21). *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol} \mathcal{Q}_g \sim \frac{4}{\pi} \cdot \left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g \to +\infty\,.$$

The similar conjecture of Eskin–Zorich'03 on the large genus asymptotics of Masur–Veech volumes of individual strata of *Abelian* differentials is recently proved by Aggarwal'19 and by Chen–Möller–Sauvaget–Zagier'20. The analogous conjecture for *quadratic* differentials still resists:

Conjecture (ADGZZ'20). *The Masur–Veech volume of any stratum of meromorphic quadratic differentials with at most simple poles has the following large genus asymptotics (with the error term uniformly small for all partitions* $\boldsymbol{d}$*):*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_n) \stackrel{?}{\sim} \frac{4}{\pi} \cdot \prod_{i=1}^{n} \frac{2^{d_i+2}}{d_i+2} \quad \text{as } g \to +\infty\,,$$

under assumption that the number of simple poles is bounded or grows much slower than the genus.

Combinatorial formulation of Witten's conjecture

Initial data: $\langle\tau_0^3\rangle = 1, \qquad \langle\tau_1\rangle = \frac{1}{24}\,.$

String equation:

$$\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = \langle\tau_{d_1-1}\ldots\tau_{d_n}\rangle_{g,n} + \cdots + \langle\tau_{d_1}\ldots\tau_{d_n-1}\rangle_{g,n}\,.$$

Dilaton equation:

$$\langle\tau_1\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = (2g-2+n)\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n}\,.$$

Virasoro constraints (in Dijkgraaf–Verlinde–Verlinde form; $k \geq 1$):

$$\langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\Bigg[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\sum_{\{1,\ldots,n\}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg]\,.$$

Keystone underlying result

We also strongly use the uniform large genus asymptotics of ψ-classes, which we conjectured in 2019. We proved it for 2-correlators; a general formula was proved by A. Aggarwal:

Theorem (Aggarwal'21). *The following* ***uniform*** *asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1 + \dots + d_n = 3g-3+n$*, as* $g \to +\infty$*.*

Keystone underlying result

We also strongly use the uniform large genus asymptotics of ψ-classes, which we conjectured in 2019. We proved it for 2-correlators; a general formula was proved by A. Aggarwal:

Theorem (Aggarwal'21). *The following **uniform** asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)!\cdots(2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1+\frac{(n+\log g)^2}{g}\right)$ ***uniformly** for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1+\dots+d_n = 3g-3+n$*, as* $g \to +\infty$*.*

Conjecture* **(Delecroix–Goujard–Zograf–Zorich).** *The distribution of the number of maximal horizontal cylinders in a random Abelian square-tiled surfaces of genus* g *gets very well approximated by the distribution of the number of disjoint cycles in a uniformly random permutation of* $4g-3$ *elements as* $g \to \infty$*.*

* About 2 years of CPU-time of two independent computer experiments with strata of genera from 40 to 10 000. Proved for $k=1$.

Distribution of lengths of components of a random multicurve

Distribution of lengths of components of a random multicurve on a surface of large genus

Consider a random multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$ on a hyperbolic surface $X \in \mathcal{M}_g$ and rearrange the components of the vector of weighted lengths $(m_1\ell_X(\gamma_1), \ldots, m_k\ell_X(\gamma_k))$ in a decreasing order to produce a vector $\ell_X^{\downarrow}(\gamma)$. Normalize $\ell_X^{\downarrow}(\gamma)$ by $\ell_X(\gamma) = m_1\ell_X(\gamma_1) + \cdots + m_k\ell_X(\gamma_k))$.

Theorem (V. Delecroix, M. Liu, 2022). *For any X in $\mathcal{M}_g$ and for any $j \in \mathbb{N}$ the average of $\frac{(\ell_X^{\downarrow})_j}{\ell_X}$ over multicurves of bounded length gives in the limit a well-defined random variable $L_j^{(g)\downarrow}$ which depends only on j and g.*

When $g \to +\infty$, the probability distribution of $L_j^{(g)\downarrow}$ weakly converges to a limiting probability distribution V_j. The distribution V_j coincides with the limiting distribution of the normalized length of the j-th longest cycle of a non-uniformly random permutation with respect to the Evens measure with parameter $\theta = \frac{1}{2}$ on S_n as $n \to +\infty$. It is the distribution of the Poisson–Dirichlet process with parameter $\theta = \frac{1}{2}$. In particular,

$$\mathbb{E}(V_1) \approx 0.758, \qquad \mathbb{E}(V_2) \approx 0.171, \qquad \mathbb{E}(V_3) \approx 0.049\,.$$

Poisson–Dirichlet process

Stick breaking process. Let $U_1, U_2, \ldots,$ be i.i.d. random variables supported on $[0,1]$ with density $\theta(1-x)^{\theta-1}$. Take a stick of length one and chop a piece of length U_1 out of it. Chop a piece of length proportional to U_2 out of the remaining part, etc. We get a random vector

$$V = (U_1, (1-U_1)U_2, (1-U_1)(1-U_2)U_3, \ldots)\,.$$

The law of V is the *Griffiths–Engen–McCloskey distribution with parameter* θ. The *Poisson–Dirichlet distribution with parameter* θ is the distribution of $V^{\downarrow}$, obtained from V by rearranging its components in the decreasing order. Both distributions are very well studied. In particular,

$$\mathbb{E}(V_j^{\downarrow}) = \int_0^{+\infty} \frac{\big(\theta E_1(x)\big)^{j-1}}{(j-1)!} e^{-x-\theta E_1(x)}\, dx\,,$$

where $E_1(x) = \int_x^{+\infty} \frac{e^{-y}}{y}\, dy$.

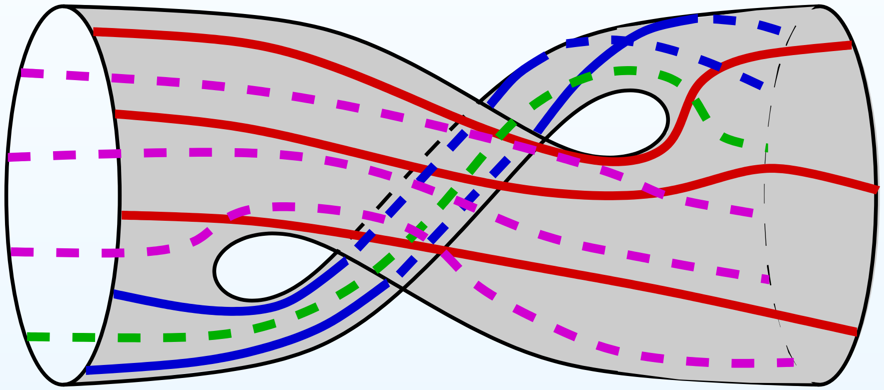
Equivalent statement for random square-tiled surfaces

Theorem (V. Delecroix, M. Liu, 2022). *Consider the decomposition of a random square-tiled surface of genus g into maximal horizontal cylinders. Consider the vector of normalized areas of these cylinders and rearrange its components in the decreasing order. The probability distribution of the resulting random vector weakly converges to the distribution of the Poisson–Dirichlet process with parameter $\theta = \frac{1}{2}$ as g tends to ∞.*

Restricting consideration to those random square-tiled surfaces, for which each cylinder contains at most m horizontal bands of squares, where $m = 1, 2, \ldots$, one gets in the limit the very same distribution of the Poisson–Dirichlet process with parameter $\theta = \frac{1}{2}$ as g tends to ∞.

Arnold's problem (2002-8)

Glue randomly two boundary components of a braid with a large number N of strands on a surface of genus $g-1$ so that the endpoints fit.



Theorem. *The probability p_g to get a single connected curve upon a random gluing of a random braid is*

$$p_g = \frac{1}{(4g-2)2^{2g-4}\,\mathrm{Vol}\,\mathcal{H}_g} \to \frac{1}{4g} + o\left(\frac{1}{g}\right) \text{ as } g \to +\infty\,.$$

Examples: $p_1 = \frac{6}{\pi^2}$, $p_2 = \frac{45}{2\pi^4}$, $p_3 = \frac{243}{2\pi^6}$.


2 ARRT
RUE DES
PETITS CARREAUX