Geometry and dynamics in moduli spaces

Homework assignment 1

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Pillowcase and its double cover

Pillowcase

Consider a quadratic differential

$$q = \frac{dz^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

on $\mathbb{C}\mathrm{P}^1$, where we assume that $z_i \neq z_j$ for $i \neq j$. Clearly, q has simple poles at z_i, $i = 1, \ldots, 4$. Note that the sum of degrees of zeroes and poles of a meromorphic quadratic differential on a surface of genus g (where we count degrees of poles with sign minus) equals $4g-4$. Thus, q should have a regular point at ∞. This can be verified by a direct computation. Let $u = \frac{1}{z}$ be a local coordinate in a neighborhood of $\infty \in \mathbb{C}\mathrm{P}^1$. Then

$$q = \frac{\left(d\left(\frac{1}{u}\right)\right)^2}{\left(\frac{1}{u}-z_1\right)\left(\frac{1}{u}-z_2\right)\left(\frac{1}{u}-z_3\right)\left(\frac{1}{u}-z_4\right)}$$

Pillowcase

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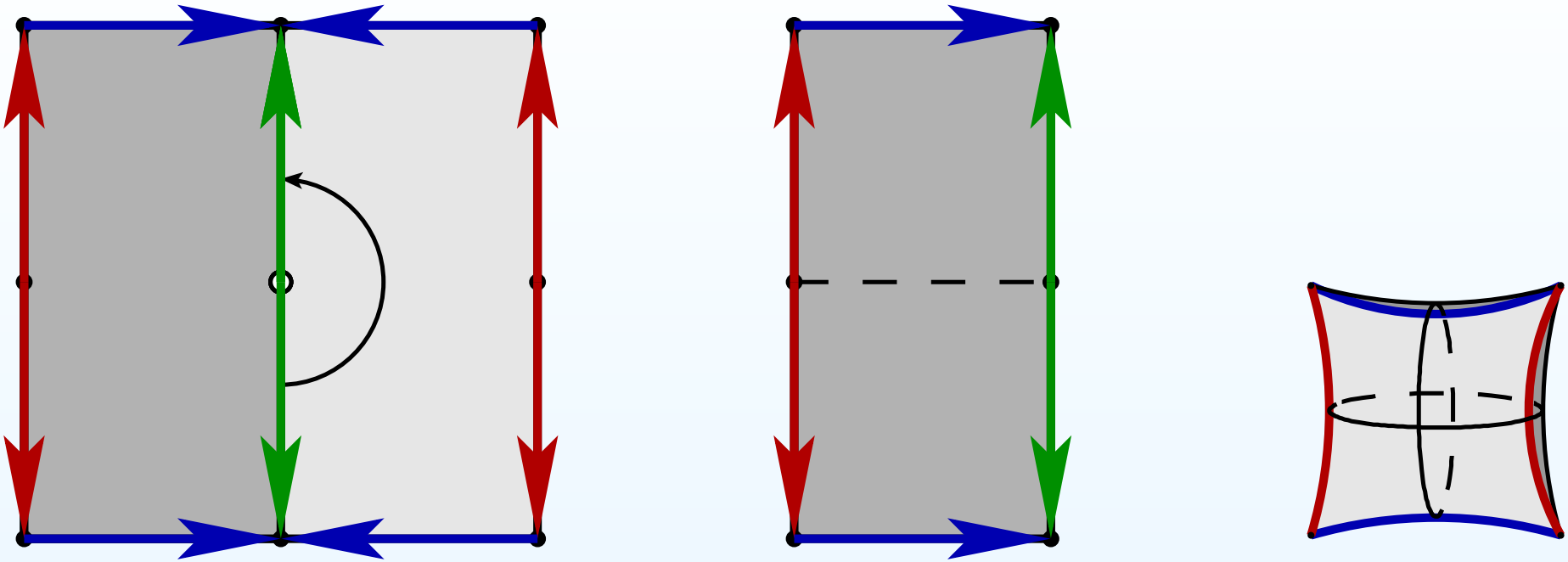
We see that $u = 0$ is a regular point of q.

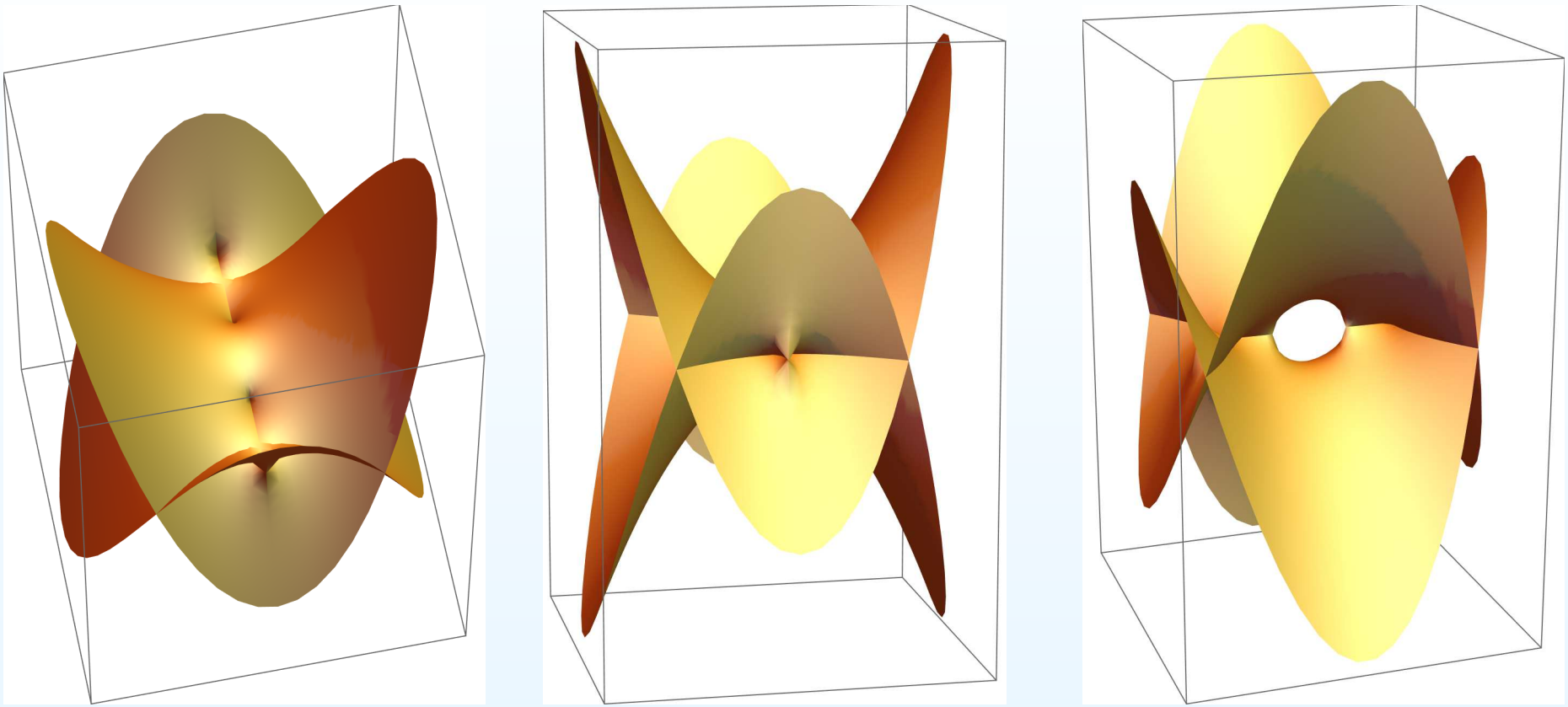
The canonical double cover $p : \hat{X} \to \mathbb{C}\mathrm{P}^1$ is ramified at $z_1, \ldots, z_4$. The associated holomorphic form $\hat{\omega}$ on the torus $\hat{X}$ is everywhere nonzero.

Torus as a ramified double cover



Consider a torus $\mathbb{T}^2$ glued from a unit square by identification of opposite sides by parallel translations. A central symmetry of the square acts as an involution τ of $\mathbb{T}^2$. The left shaded region of the square provides a fundamental domain of the involution τ. The quotient of $\mathbb{T}^2$ by τ can be represented by identifications of sides of the fundamental domain as indicated in the picture; they correspond to folding the vertical shaded rectangle with respect to horizontal axes followed by identification of the boundary. We get a "pillow" $\mathbb{C}\mathrm{P}^1$ as on the right picture and a double cover of $\mathbb{C}\mathrm{P}^1$ by an elliptic curve $\mathbb{T}^2$ ramified at four points.

Elliptic surface



A schematic graphical representation of an elliptic surface

$$w^2 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)\,,$$

with real z_i. Pictures represent the same surface seen from different angles. It is not instant to recognize a topological torus in these pictures!

Problem 1. Pillow metric on $\mathbb{C}\mathrm{P}^1$

Describe the flat structure on $\mathbb{C}\mathrm{P}^1$ associated to the quadratic differential

$$q_1(z) = \frac{(dz)^2}{z(z^2-1)}\,.$$

Draw schematically associated horizontal and vertical foliations in coordiante z on $\mathbb{C}$. (Warning: exceptionally z *does not* denote the flat coordinate!)

Same questions for $\quad q_2(z) = -\dfrac{(dz)^2}{z(z^2-1)}\,.$

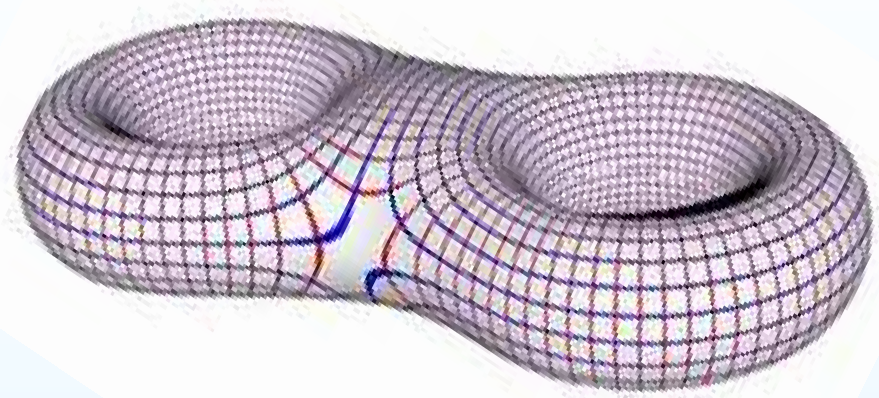
What conical points has the flat metric defined by the quadratic differential

$$q(z) = \frac{(dz)^2}{z(z-1)(z-a)}$$

on the Riemann sphere when $a \notin \{0,1\}$? Is there a value of $a \in \mathbb{C}\setminus\{0,1\}$ for which this flat sphere takes a shape of a square pillow? If so, give this value. Draw schematically horizontal and vertical foliations in coordiante z on $\mathbb{C}$.

Ambient stratum and separatrix diagrams

Problem 2. Which stratum?

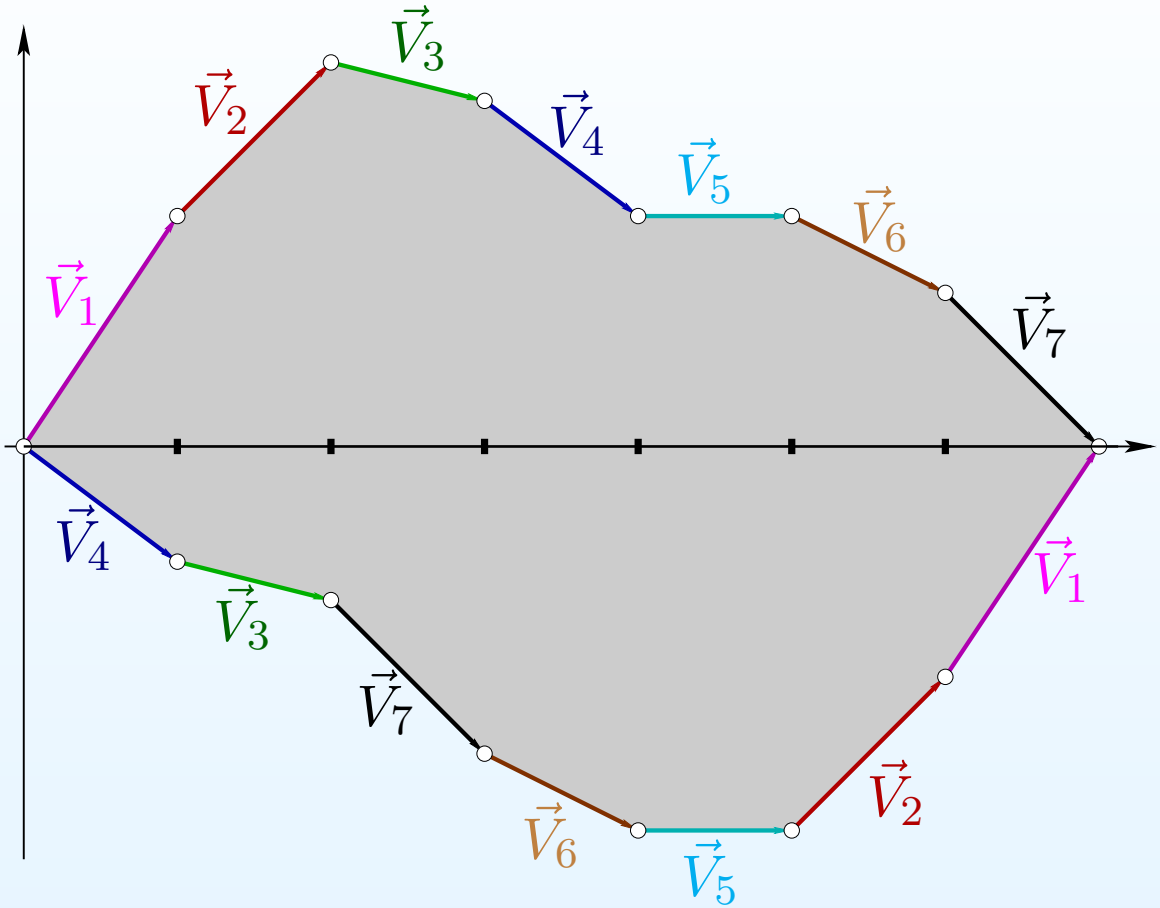


Picture created by Jian Jiang

Questions.

- *To what stratum belongs this square-tiled surface?*
- *Find all realizabe separatrix diagrams for this stratum.*
- *To which of the found diagrams corresponds the square-tiled surface from the picture?*

Problem 3. Determine the ambient stratum?

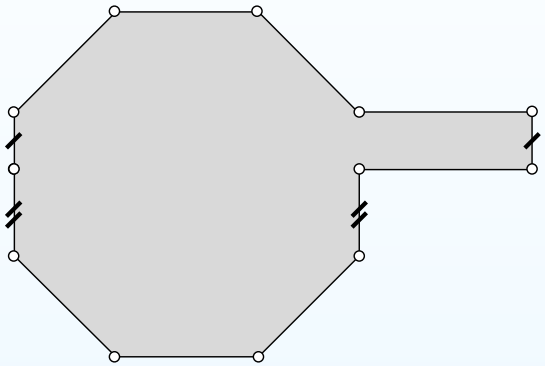


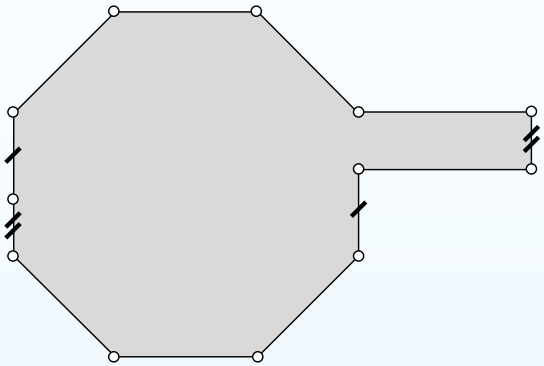
What is the ambient stratum $\mathcal{H}(d_1, \ldots, d_n)$ for the translation surface obtained by identifying the pairs of sides corresponding to the same vectors $\vec{V}_1, \ldots, \vec{V}_7$ by parallel translations?

Hyperelliptic involution and Weierstrass points

Problem 4. Hyperelliptic involution

- Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.





A *hyperelliptic involution* is a holomorphic involution of a Riemann surface such that the quotient over the involution is a Riemann sphere.

Determine which of the two surfaces (if any) is hyperelliptic and find the hyperelliptic involution in geometric terms. Find all *Weierstrass points* (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.