Homotopy

Remark. *We restraint ourselves to the arcwise connected topological spaces. Otherwise, the arcwise connected components don't have anything to do with each other.*

1. Definitions

Definition. *Let X be an arcwise connected topological space. A* loop *of* base point *x is a continuous map $c : [0,1] \to X$ such that $c(0) = c(1) = x$.*
c_x denotes the constant loop always equal to x.
$\bar{c}$ denotes the loop $t \mapsto c(1-t)$
cc' denotes the loop

$$t \mapsto \begin{cases} c(2t) & \text{if } t \leqslant \frac{1}{2} \\ c'(2t-1) & \text{if } t \geqslant \frac{1}{2} \end{cases}$$

Definition. *Two paths c, c' are said to be* homotopic *whenever there exists a continuous map $H[0,1]^2 \to X$ such that*

$$\begin{cases} H(\cdot, 0) = c \\ H(\cdot, 1) = c' \\ H(0, \cdot) = H(1, \cdot) = x \end{cases}$$

We then denote $c \sim c'$.

Lemma. *$\sim$ is an equivalence relation. We denote the class of c by $[c]$.*

Proof. Transitivity is obtained through concatenation. □

Definition. *The set of all* homotopy classes *is the* fundamental group *$\pi_1(X,x)$. $\forall x, x' \in X, \pi_1(X,x) \equiv \pi_1(X,x')$, so we denote $\pi_1(X)$ the fundamental group.*

Proof. Let $\gamma \sim c$ and $\gamma' \sim c'$ and H, H' be continuous maps as in the definition. Then $\bar{c} \sim \bar{\gamma}$ through $\bar{H}(t,s) = H(1-t,s)$. $cc' \sim \gamma\gamma'$ through $HH'(t,s) = \begin{cases} H(2t,s) & \text{if } t \leqslant \frac{1}{2} \\ H'(2t-1,s) & \text{if } t \geqslant \frac{1}{2} \end{cases}$. This proves that the group law is well defined. Moreover if c_1, c_2, c_3 are loops, $(c_1c_2)(c_3) = (c_1)(c_2c_3)$ through

$$H(t,s) = \begin{cases} c_1(\frac{4t}{1+s}) \text{ if } t \leqslant \frac{1+s}{4} \\ c_2(4t-s-1) \text{ if } \frac{1+s}{4} \leqslant t \leqslant \frac{2+s}{4} \\ c_3(\frac{4t-s-2}{2-s}) \text{ if } \frac{2+s}{4} \leqslant t \end{cases}$$

$[c_x]$ is a neutral element because $cc_x \sim c_xc \sim c$ (left to the reader). $c\bar{c} \sim c_x$ so $[\bar{c}]$ is the inverse of $[c]$ for the law group.

Finally, let γ be a path from x to x', $[c] \mapsto [\gamma c \bar{\gamma}]$ is an isomorphism between $\pi_1(X,x)$ and $\pi_1(X,x')$. □

Definition. *A topological connected set is said to be* simply connected *if its fundamental group is reduced to the neutral element.*

Lemma. *$\mathbb{R}^n$ is simply connected through $H(t,s) = s\,c(t)$ for all loop c.*

Let f be a continuous map between two arcwise connected topological spaces X and Y then if $c \sim c'$, $f \circ c \sim f \circ c'$ and $f \circ cc' \sim (f \circ c)(f \circ c')$. Thus f induces a map $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$ such that $f_*(cc') = f_*(c)f_*(c')$. Thus, we have a morphism between $\pi_1(X,x)$ and $\pi_1(Y,f(x))$. If f is a homeomorphism, f_* is an isomorphism of inverse $(f^{-1})_*$.

2. Some fundamental groups

Lemma. *Let X and Y be two arcwise connected topological spaces. Then, if p_1 and p_2 are the projection from $X \times Y$, then $(p_1)_* \times (p_2)_*$ is an isomorphism between $\pi_1(X \times Y, (x, y))$ and $\pi_1(X, x) \times \pi_1(Y, y)$.*

Proof. It is a morphism as a product of two morphisms. It is injective by taking the product of two homotopies and surjective because the product of two loops of X and Y is a loop of $X \times Y$. □

Example. *The* torus $\mathbb{T}^2$ *is defined as $\mathbb{S}^1 \times \mathbb{S}^1$. Therefore its fundamental group is $\mathbb{Z}^2$*

Remark. *If Y is simply connected, $X \times Y$ has the same fundamental group as X, and if X and Y are both simply connected, so is $X \times Y$.*

Definition. *A subset Y of an arcwise connected topological space X is said to be a* deformation retract *if there exists continuous applications $r : X \mapsto Y$ and $H : X \times [0, 1] \mapsto X$ such that:*

- $\forall y \in Y, r(y) = y$.
- $\forall x \in X, H(x, 0) = x$.
- $\forall x \in X, H(x, 1) = r(x)$.
- $\forall y \in Y, \forall t \in [0, 1], H(y, t) = y$.

Example. *$\mathbb{S}^{m-1}$ is a deformation retract of $\mathbb{R}^m \setminus \{0\}$ through $r(x) = \frac{x}{||x||}$ and $H(x, t) = t\frac{x}{||x||} + (1 - t)x$.*

If Y is a deformation retract of X, i the injection of Y into X and $x \in Y$, the iduced morphism $i_* : \pi_1(Y, x) \to \pi_1(X, x)$ is an isomorphism. Indeed, surjectivity only has to be proved. If $c \in \pi_1(X, x)$ then $r \circ c \in \pi_1(Y, x)$ and $i_*(r \circ c) \sim c$ (through the H of the definition).

Theorem. *Let $p : \mathbb{R} \mapsto \mathbb{S}^1$ be the projection $t \mapsto e^{2i\pi t}$, and for all $n \in \mathbb{Z}$, γ_n the loop $t \mapsto p(nt)$. Then the application $n \mapsto [\gamma_n]$ is an isomorphism from $\mathbb{Z}$ to $\pi_1(\mathbb{S}^1)$. Thus $\pi_1(\mathbb{S}^1) \equiv \mathbb{Z}$.*

Lemma. *If c is a loop of base point 1 in $\mathbb{S}^1$, then there exists a path $\tilde{c}$ of origin 0 in $\mathbb{R}$ which is uniquely determined by $p \circ \tilde{c} = c$. A such path is called a* lifting *of c.*

Proof. If $a \in \mathbb{R}$, p induces an isomorphism between $]a - \frac{1}{2}, a + \frac{1}{2}[$ and $\mathbb{S}^1 \setminus \{-p(a)\}$. Consequently if we denote, p_a^{-1} the inverse of this induced homeomorphism, and if $\tilde{c_1}$ and $\tilde{c_2}$ are both as in the lemma, then $\tilde{c_1} = \tilde{c_2} = c \circ p_a^{-1}$ on $]a - \frac{1}{2}, a + \frac{1}{2}[$ and thus $\tilde{c_1} = \tilde{c_2}$.

Furthermore, for c is continuous on $[0, 1]$, it is uniformly continuous and there exists $n \in \mathbb{N}$ such that $\forall t, t' \in [0, 1]$, if $|t - t'| \leqslant \frac{1}{n}, ||c(t) - c(t')|| \leqslant 1$, so $c(t)\overline{c(t')} \neq -1$. Hence, if $|t - t'| \leqslant \frac{1}{n}$, $p_0^{-1}(c(t)\overline{c(t')})$ is well defined. Thus, we can define

$$\tilde{c}(t) = p_0^{-1}\left(c(t)\overline{c\left(\frac{j}{n}\right)}\right) + \sum_{i=1}^{j} p_0^{-1}\left(c\left(\frac{i}{n}\right)\overline{c\left(\frac{i-1}{n}\right)}\right)$$

for $\frac{j}{n} \leqslant t \leqslant \frac{j+1}{n}$, which works. □

Lemma. *If H is a continuous map from $[0, 1]^2$ to $\mathbb{S}^1$ such that $H(0, 0) = 1$, then there exists a continuous map $\tilde{H}$ which is uniquely determined by $p \circ \tilde{H} = H$ and $\tilde{H}(0, 0) = 0$.*

Lemma. *Two loops c and γ are homotopic iff $\tilde{c}(1) = \tilde{\gamma}(1)$ whenever $\tilde{c}$ and $\tilde{\gamma}$ are liftings.*

Proof. c and γ are homotopic through $H(t, s) = p((1 - s)\tilde{c}(t) + s\tilde{\gamma}(t))$. Conversly if c and γ are homotopic, let $\tilde{H}$ be a lifting of H, then $\tilde{c} = \tilde{H}(\cdot, 0)$ and $\tilde{\gamma} = \tilde{H}(\cdot, 1)$ are liftings of c and γ. But, then, since $\tilde{H}(1, \cdot)$ is continuous and maps into $\mathbb{Z}$, it is constant and $\tilde{c}(1) = \tilde{H}(1, 0) = \tilde{H}(1, 1) = \tilde{\gamma}(1)$. □

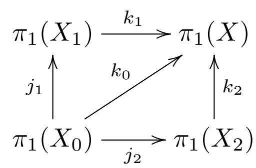
Proof. (of the theorem) We conclude by letting $\phi([c]) = \tilde{c}(1)$, which is well defined. Moreover if c and γ are loops, and $\tilde{c}$ and $\tilde{\gamma}$ are liftings, then $a(t) = \tilde{c}(2t)$ if $t \leqslant \frac{1}{2}$, and $\tilde{\gamma}(2t - 1) + \tilde{c}(1)$ otherwise, is a lifting of $c\gamma$. Thus ϕ is an isomorphism which inverse is $n \mapsto [\gamma_n]$. □

Remark. *The fundamental group of $\mathbb{R}^2 \setminus \{0\}$ is $\mathbb{Z}$, because it is a deformation retract of $\mathbb{S}^1$.*

3. Van Kampen's theorem

Let X be an arcwise connected space, X_1 and X_2 two arcwise connected open subspaces of X such that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$ is a non empty arwise connected subspace of X.

Let $x \in X_0$. We denote $\pi_1(X_i) = \pi_1(X_i, x)$ for $i = 0, 1, 2$, and $\pi_1(X) = \pi_1(X, x)$. Let j_i be the morphism from $\pi_1(X_0)$ to $\pi_1(X_i)$ induced by the injection from X_0 into X_i for $i = 1, 2$, and k_i the morphism from $\pi_1(X_i)$ to $\pi_1(X)$ induced by the injection from X_i into X.



Lemma. *$\pi_1(X)$ is generated by k_1's and k_2's ranges.*

Proof. Let γ be a loop in X. $[0,1] = \gamma^{-1}(X_1) \cup \gamma^{-1}(X_2)$ is an open covering. Thus, using Lebesgue's lemma, we get an integer n such that, $\forall k \in [\![0, n-1]\!]$, $\left[\frac{k}{n}, \frac{k+1}{n}\right] \subset X_i$ for a given i.

If $\gamma\left(\frac{k}{n}\right) \in X_0$ (respectively $X_i \setminus X_0$), let α_k be a path from $\gamma\left(\frac{k}{n}\right)$ to the base point in X_0 (X_i).

Let $\gamma_k(t) = \gamma(\frac{k+t}{n})$.

Then, $\gamma \sim (\gamma_0\alpha_0)(\overline{\alpha_0}\gamma_1\alpha_1)\dots(\overline{\alpha_{n-2}}\gamma_{n-1})$, which ends the proof since each $\overline{\alpha_{i-1}}\gamma_i\alpha_i$ is a loop in X_1 or X_2. □

Corollary. *If X_1 and X_2 are simply connected, X is simply connected.*

Corollary. *If $m \geqslant 2$, $\mathbb{S}^m$ is simply connected.*

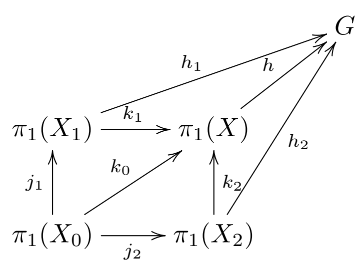
Proof. Let a be the point $(0, \dots, 0, 1)$ of $X = \mathbb{S}^m$, and $X_1 = X \setminus \{a\}$, $X_2 = X \setminus \{-a\}$. X_i is homeomorphic to $\mathbb{R}^{m-1}$ and thus simply connected. Indeed, the map

$$\mathbb{S}^m \setminus \{a\} \to \mathbb{R}^{m-1}$$
$$(x_1, \dots, x_m) \mapsto \left(\frac{x_1}{1 - x_m}, \dots, \frac{x_{m-1}}{1 - x_m}\right)$$

is an homeomorphism of inverse : $(y_1, \dots, y_{m-1}) \mapsto \left(\frac{2y_1}{1+\sum\limits_{j=1}^{m-1} y_j^2}, \dots, \frac{2y_{m-1}}{1+\sum\limits_{j=1}^{m-1} y_j^2}, 1 - \frac{2}{1+\sum\limits_{j=1}^{m-1} y_j^2}\right)$.

Moreover, $X_0 = X_1 \cap X_2$ is arcwise connected : if $x_0, x_1 \in X_0$, there is a circle containing those two points which doesn't touch a and $-a$. Using the lemma $\mathbb{S}^m$ is simply connected. □

Theorem. (Van Kampen) *If h_i is a morphism from $\pi_1(X_i)$ to a group G for $i = 1, 2$ and if $h_1 \circ j_1 = h_2 \circ j_2$, then there is a morphism h from $\pi_1(X)$ to G which is uniquely determined by $h \circ k_i = h_i$ for $i = 1, 2$. In other words the following diagram is commutative:*



Proof. Firstly, h is uniquely determined : if h is as in the theorem, and $[x] \in \pi_1(X)$, using the lemma, $[x] = [x_1] \cdots [x_r]$ where $[x_i] = k_{j(i)}([t_{j(i)}])$ and $[t_{j(i)}] \in \pi_1(X_{j(i)})$, $h([x]) = h_{j(1)}([t_{j(1)}]) \cdots h_{j(r)}([t_{j(r)}])$ and therefore, h is uniquely determined. Our goal is then to prove that if we assert $h([x]) = h_{j(1)}([t_{j(i)}]) \cdots h_{j(r)}([t_{j(r)}])$, it defines correctly an element of G.

If $[x_1]\dots[x_r] = [y_1]\dots[y_s]$, where $[x_k],[y_j] \in k_i(\pi_1(X_i))$, let H be an homotopy between $x_1\cdots x_r$ and $y_1\cdots y_r$. As $[0,1]^2 = H^{-1}(X_1)\cup H^{-1}(X_2)$, using LEBESGUE's lemma, we get $n\in\mathbb{N}^*$ such that

$$\forall k,l, \left[\frac{k}{n},\frac{k+1}{n}\right]\times\left[\frac{l}{n},\frac{l+1}{n}\right]\subset H^{-1}(X_i)$$

for a given i.

For $0\leqslant k\leqslant n$, $1\leqslant l\leqslant n-1$, let $p_{k,l}$ be a path from the base point to $H(\frac{k}{n},\frac{l}{n})$ in the same X_i than $H(\frac{k}{n},\frac{l}{n})$, and let $f(x) = \begin{cases} 2\,x+\frac{1}{2} & \text{if } x\leqslant -\frac{1}{4} \\ 0 & \text{if } -\frac{1}{4}\leqslant x\leqslant\frac{1}{4} \\ 2\,x-\frac{1}{2} & \text{if } \frac{1}{4}\leqslant x \end{cases}$.

Then

$$H'\left(\frac{k+u}{n},\frac{l+v}{n}\right) = \begin{cases} H\left(\frac{k+u}{n},\frac{l+v}{n}\right) & \text{if } l=0 \text{ or } l=n \\ p_{k,l}\left(4\max\left(|u|,|v|\right)\right) & \text{if } \max\left(|u|,|v|\right)\leqslant\frac{1}{4} \\ H\left(\frac{k+f(u)}{n},\frac{l+f(v)}{n}\right) & \text{else} \end{cases}$$

with $u,v\in[-\frac{1}{2},\frac{1}{2}]$ and $0\leqslant k,l\leqslant n$, is a homotopy between the two loops and $H'\left(\frac{k}{n},\frac{l}{n}\right)$ is the base point if $l\neq 0,n$.

Let $\alpha_{k,l}(t) = H\left(\frac{k+t}{n},\frac{l}{n}\right)$, $\beta_{k,l}(t) = H\left(\frac{k+1}{n},\frac{l+t}{n}\right)$, $\gamma_{k,l}(t) = H\left(\frac{k+1-t}{n},\frac{l+1}{n}\right)$ and $\delta_{k,l}(t) = H\left(\frac{k}{n},\frac{l+1-t}{n}\right)$ the paths around the square $[\frac{k}{n},\frac{k+1}{n}]\times[\frac{l}{n},\frac{l+1}{n}]$ induced by H.

Let $a_{k,l} = h_i([\alpha_{k,l}])$, with i such that $[\alpha_{k,l}]\in\pi_1(X_i)$. We define $b_{k,l}$, $c_{k,l}$ and $d_{k,l}$ similarly.

Then $a_{k,l+1} = c_{k,l}^{-1}$ and $b_{k,l} = d_{k+1,l}^{-1}$ wherever it is defined.

Let's finish the proof by induction on l: as $h_i(x_1)\dots h_j(x_r) = a_{0,0}\dots a_{n-1,0}$ and $h_p(x_1)\dots h_q(x_r) = a_{0,n-1}\dots a_{n-1,n-1}$, we just need to prove that $\forall l\geqslant 0, a_{0,l}\dots a_{n-1,l} = a_{0,l+1}\dots a_{n-1,l+1}$.

If $l=0$,

$$x_1 = \alpha_{0,0}\dots\alpha_{k_1,0}\sim\delta_{0,0}^{-1}\gamma_{0,0}^{-1}\dots\gamma_{k_1,0}^{-1}\beta_{k_1,0}^{-1},$$
$$\dots$$
$$x_r = \alpha_{k_{r-1}+1,0}\dots\alpha_{n-1,0}\sim\delta_{k_{r-1}+1,0}^{-1}\gamma_{k_{r-1}+1,0}^{-1}\dots\gamma_{n-1,0}^{-1}\beta_{n-1,0}^{-1},$$

and all these homotopies live either in X_1 or in X_2.
Thus $a_{0,0}\dots a_{n-1,0} = d_{0,0}^{-1}c_{0,0}^{-1}\dots c_{k_1,0}^{-1}b_{k_1,0}^{-1}\cdots d_{k_{r-1}+1,0}^{-1}c_{k_{r-1}+1,0}^{-1}\dots c_{n-1,0}^{-1}b_{n_1,0}^{-1} = a_{0,1}\dots a_{n-1,1}$.

If $0<l<n-1$, $a_{0,l}\dots a_{n-1,l} = d_{0,l}^{-1}c_{0,l}^{-1}b_{0,l}^{-1}\dots d_{n-1,l}^{-1}c_{n-1,l}^{-1}b_{n-1,l}^{-1} = a_{0,l+1}\dots a_{n-1,l+1}$.

If $l=n-1$, we use the same method than for $l=0$. ☐

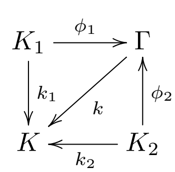
Definition. *Let K_1 and K_2 be groups. The* free product $\Gamma = K_1\star K_2$ *of K_1 and K_2 is the group of words on the alphabet $K_1\cup K_2$.*

Lemma. *There are morphisms $\phi_1 : K_1\to\Gamma$ and $\phi_2 : K_2\to\Gamma$ such that*

(i) $Im(\phi_1)\cup Im(\phi_2)$ generates Γ.

(ii) If K is a group and $k_i : K_i\to K$ are morphisms, then there is a morphism $k:\Gamma\to K$ which is uniquely determined by $k_i = k\circ\phi_i$

In other words the following diagramm is commutative :



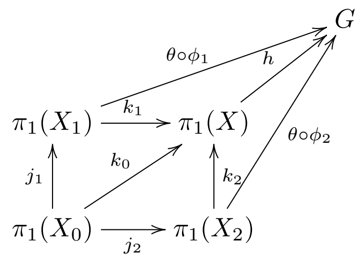
Proof. Since K is a group of words without any relation we can define unambigously the image of any word as the product of the images of its letters. ☐

We let $K_1 = \pi_1(X_1)$ and $K_2 = \pi_1(X_2)$ in the following propositions.

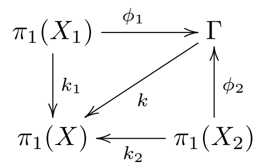
Definition. *Let* $G = \pi_1(X_1) \star_{\pi_1(X_0)} \pi_1(X_2) := \Gamma / < \phi_1(j_1(g))\phi_2(j_2(g^{-1})), g \in \pi_1(X_0) >$. *$G$ is called the amalgamated product.*

Lemma. *We have* $\pi_1(X) \approxeq G$.

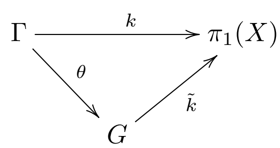
Proof. Let ϕ_i be the injection $\pi_1(X_i) \to \Gamma$, and θ the projection $\Gamma \to G$. The following diagrams are all commutative :



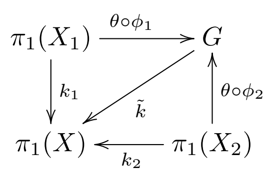
(Using Van Kampen's theorem).



(By definition of Γ).



(By definition of G. Indeed, $\forall g \in \pi_1(X_0)$, $k(\phi_1(j_1(g))\phi_2(j_2(g^{-1}))) = k_1(j_1(g))k_2(j_2(g^{-1})) = c_x$.)



Therefore $\tilde{k} \circ h \circ k_i = k_i$ so (as $\pi_1(X)$ is generated by the images of k_i) $\tilde{k} \circ h = Id$. Moreover, $h \circ \tilde{k} \circ \theta \circ \phi_i = \theta \circ \phi_i$ so $h \circ \tilde{k} = Id$. Therefore $\pi_1(X) \approxeq G$. $\square$

4. Examples

- The fundamental group of $\mathbb{C} \setminus \{a_1, \ldots a_n\}$ is the free group generated by n elements $\mathcal{L}(a_1, \ldots, a_n)$. We prove it for two elements. The result follows by induction on n. We may very well suppose that $a_1 = 0$ and $a_2 = 1$. Let $X_0 = \{\Re z < \frac{3}{4}\}$ and $X_1 = \{\Re z > \frac{1}{4}\}$ Then $X_0 = \{\frac{1}{4} < \Re z < \frac{3}{4}\}$ is homeomorphic to $\mathbb{C}$ and hence simply connected. $\pi_1(X_i) \approxeq \mathbb{Z}$. Thus $\pi_1(X) \approxeq \mathbb{Z} \star \mathbb{Z}$ which is exactly the free group generated by two elements.

- The fundamental group of the torus $\mathbb{T}^2$ deprived of one dot x is $\mathcal{L}(a, b)$.
  Indeed, if the loops a and b generate $\pi_1(\mathbb{T}^2)$, consider two strips S_1, S_2 containing a and b. You can transform the dot into $S_1 \cap S_2$ through a continuous deformation. Call $X_1 = \mathbb{T}^2 \setminus S_1$ and $X_2 = \mathbb{T}^2 \setminus S_2$. Using Van Kampen's theorem, we get $\pi_1(\mathbb{T}^2 \setminus \{x\}) = \mathcal{L}(a) \star_{\{1\}} \mathcal{L}(b) = \mathcal{L}(a, b)$.

- The fundamental group of the torus $\mathbb{T}^2$ deprived of n dots $x_1, \ldots, x_n$ is $\mathcal{L}(a_1, \ldots, a_{n+1})$.
  Surround every dot x_i by a loop c_i. The loop $c = c_1 \ldots c_n$ surrounds all the dots. Consider a strip S containing c and no dot. Call I the interior of c, $X_1 = S \cup I$ and $X_2 = S \cup I^c$. As $c = [a, b]$, we get $\pi_1(\mathbb{T}^2 \setminus \{x_1, \ldots, x_n\}) = \mathcal{L}(a, b) \star_{\mathcal{L}(c)} \mathcal{L}(c_1, \ldots, c_n) = \mathcal{L}(a, b, c_1, \ldots, c_{n-1})$.

- The fundamental group of the torus $T_{k,n}$ with k holes deprived of n dots $x_1, \ldots, x_k$ is $\mathcal{L}(d_1, \ldots, d_{2k+n-1})$.
  Group all the dots on the first torus through deformation retracts and proceed as previously. By induction on k, we get $\pi_1(T_{k,n}) = \mathcal{L}(a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_n)/\{[a_1, b_1] \ldots [a_k, b_k] c_1 \ldots c_n\} = \mathcal{L}(d_1, \ldots, d_{2k+n-1})$.