Tilings of the Euclidean plane

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Abstract

This document gives a quick overview of a field of mathematics which lies in the intersection of geometry and algebra : tilings.

A few definitions of key notions are presented and a first attempt to classify tilings (according to the structure of their symmetry group) is done. Finally, several more complicated problems arising from tiling theory are mentioned and referenced.

Contents

1 Preliminaries

1.1 First definitions

In the whole paper, we consider the plane $\mathbb{R}^2$ with the natural topology derived from an arbitrarily fixed norm $\|\cdot\|$.

Definition A *general tiling* $\mathscr{T}$ is a countable set $\{T_n \mid n \in \mathbb{N}\}$ whose elements are closed subsets of R^2 such that :

$$\bigcup_{n\in\mathbb{N}} T_n = \mathbb{R}^2 \qquad \text{and} \qquad \forall\, i \neq j,\, \mathring{T}_i \cap \mathring{T}_j = \varnothing.$$

For all $n \in \mathbb{N}$, T_n is called a *tile* of the tiling.

This definition is too general for our study, so we will nextly restrict us to a smaller class of objects, for what we first need more vocabulary.

Definition A *general discrete tiling* is a general tiling $\{T_n \mid n \in \mathbb{N}\}$ such that for all $M > 0$, there is a finite subset $I \subset \mathbb{N}$ satisfying :

$$B(0, M) \subset \bigcup_{i\in I} T_i.$$

where $B(x, R)$ stands for the closed ball of center x and of radius R in the space $(\mathbb{R}^2, \|\cdot\|)$.

Definition A *well-behaved general tiling* is a general tiling whose tiles are all *topological disks*, that is to say homeomorphic images of $B(0, 1)$.

If all tiles of a tiling $\mathscr{T}$ are topological disks, one can consider subsets of $\mathbb{R}^2$ of the form :

$$\bigcap_{i\in I} T_i \qquad \text{for any } I \subset \mathbb{N} \text{ finite.}$$

Any of these subsets can be rewrited as an (eventually empty) union of isolated points and simple closed curves. The isolated points are called the *vertices* of the tiling.

Moreover, for all $n \in \mathbb{N}$, since the border of T_n is a simple closed curve, it is divided into several closed curves by the vertices of the tiling. The elementary closed curves arising are called the *edges* of the tiling.

We can now state what we want to be a tiling.

Definition A *tiling* is a well-behaved discrete general tiling in which each tile contains only a finite number of vertices.

Several examples of general tilings which are not tilings according to our definition can be found in the very beginning of the first chapter of [1].

1.2 Group of the isometries of the plane

Since tilings are geometrical objects, we may consider a group which action on tillings is quite interesting : the group of the isometries of the plane.

Definition An *isometry* is generaly defined in $\mathbb{R}^d$ as a distance-preserving bijection.

Note that the identity, the composition of two isometries, and the inverse map of an isometry are still isometries. Hence, the isometries of form a group for the composition law.

A well-known result states that an isometry can be factorized as a composition of an orthogonal linear transformation and a translation.

Moreover, the reduction of orthogonal linear maps in $\mathbb{R}^2$ yields to the following result.

Theorem *Any isometry of the plane can be factorised into a translation, a rotation and a reflection.*

1.3 Equal tilings

Since we have defined tilings as geometrical objects, we want to provide them some algebraic structure. In fact, we aim at finding invariants or criteria to determine to what extent two given tilings are "similar".

A first attempt is made with the following definition.

Definition Two tilings $\mathscr{T}, \mathscr{T}'$ are said to be *equal* if there is a similarity transformation of the plane s and a permutation $\sigma \in \mathrm{Bij}(\mathbb{N})$ such that :

$$\forall n \in \mathbb{N} \quad s(T_n) = T'_{\sigma(n)}.$$

Unfortunately, this first notion of similarity is too restrictive. We have to introduce new tools to classify tilings in a more appropriate way.

1.4 Symmetry groups

Let $\mathscr{T} = \{T_n \mid n \in \mathbb{N}\}$ be a tiling.

Definition The symmetry group of $\mathscr{T}$ is the set of all isometries of the plane s such that :

$$\exists \sigma \in \mathrm{Bij}(\mathbb{N}) \quad \forall n \in \mathbb{N} \quad s(T_n) = T_{\sigma(n)}.$$

It is denoted by the notation $\mathscr{S}(\mathscr{T})$.

Checking that it is indeed a group for the composition law is left to the reader.

Definition The *diagram* of the symmetry group $\mathscr{S}(\mathscr{T})$ is the representation on the plane obtained by the following rules :

- For each reflection in $\mathscr{S}(\mathscr{T})$, we draw the reflection axis ;
- For each $\theta \in]0, 2\pi[$, we choose a color and draw each center of a θ-rotation belonging to $\mathscr{S}\mathscr{T}$ in this color ;
- For each translation in $\mathscr{S}(\mathscr{T})$, we draw the corresponding free vector ;
- For each free-vector, we choose a color and draw the axis of each glide-reflection in $\mathscr{S}(\mathscr{T})$ with this free-vector in this color.

Definition Two symmetry groups are said to be *isomorphic* if their diagrams are the same up to an affine transformation.

Definition Two tilings $\mathscr{T}, T'$ are said to have the same *symmetry type* if $\mathscr{S}(\mathscr{T})$ and $\mathscr{S}(\mathscr{T}')$ are isomorphic.

2 Classification of tilings according to their symmetry types

The classification stated here has been carefully carried out in [2]. We distinguish between three cases :

- $\mathscr{S}(T)$ contains no translation ;
- There is a vector v such that the translations of $\mathscr{S}(T)$ are exactly the translations corresponding to vectors of $v\mathbb{Z}$;
- There are two independent vectors u, v such that the translations of $\mathscr{S}(T)$ are exactly the translations corresponding to vectors of $u\mathbb{Z} + v\mathbb{Z}$.

In each case, a classification is made so that we finally explicit 26 symmetry types : 2 are in the first case, 7 in the second case (they are the so called *frieze groups*) and 17 in the third case (the so called *wallpaper groups*).

2.1 The n-cyclic group and the n-dihedral group

Let $\mathscr{S}$ be the symmetry group of a fixed tiling, with the hypothesis that $\mathscr{S}$ contains no translation.

Claim *If $\mathscr{S}$ contains two rotations $\rho_{A,\phi}, \rho_{B,\psi}$, then they have the same center : $A = B$.*

Proof Else $\rho_{A,\phi} \circ \rho_{B,\phi} \circ \rho_{A,\phi}^{-1} \circ \rho_{B,\phi}^{-1}$ would be a non-identity translation belonging to $\mathscr{S}$. □

Claim *If $\mathscr{S}$ contains a rotation $\rho_{A,\phi}$, then $\phi \in \pi\mathbb{Q}$.*

Proof Else, the set $E = \phi\mathbb{Z} + 2\pi\mathbb{Z}$ is a non monogenous subgroup of $\mathbb{R}$, so it is dense in $\mathbb{R}$. So, $\mathscr{S}$ contains rotations of center A and of arbitrarily small angles. Taking any tile T of the tiling, one can apply a sufficiently small rotation $\rho_{A,\varepsilon}$ for $\rho_{A,\varepsilon}(\mathring{T})$ to intersect $\mathring{T}$. Contradiction. □

This proof strongly relies on our assumption that tiles are topological disks.

Corollary *If $\mathscr{S}$ contains no reflection, then it is isomorphic to a cyclic group C_n.*

Claim *If $\mathscr{S}$ contains exactly one reflection s_d of axis d, it contains either no rotation, or only a halfturn. It is hence isomorphic to C_2 or C_4.*

Proof For any rotation $\rho_{A,\phi}$, $\rho_{A,\phi}^{-1} \circ s_d \circ \rho_{A,\phi} = s_{\rho_{A,-\phi}(d)}$. So if this rotation belongs to $\mathscr{S}$, the condition $\phi \in \pi\mathbb{Z}$ is required. The claim follows straight-forward. □

Claim *If $\mathscr{S}$ contains at least two reflections, then it is isomorphic to a dihedral group D_n.*

Proof It arises easily when applying both first claims to well chosen composites. □

2.2 The 7 frieze groups

Let $\mathscr{S}$ be the symmetry group of a fixed tiling, with the hypothesis that $\mathscr{S}$ contains at least one translation, but doesn't contain any two independent translations.

Claim *There is a smallest translation vector v such that the translations belonging to $\mathscr{S}$ are exactly the translations of vector in $v\mathbb{Z}$.*

Proof It is still the same density argument as for rotation angles in the previous paragraph. □

Let τ be the tranlation of vector v.

Claim *Each line of direction v is sent on a line of direction v by any element of $\mathscr{S}$.*

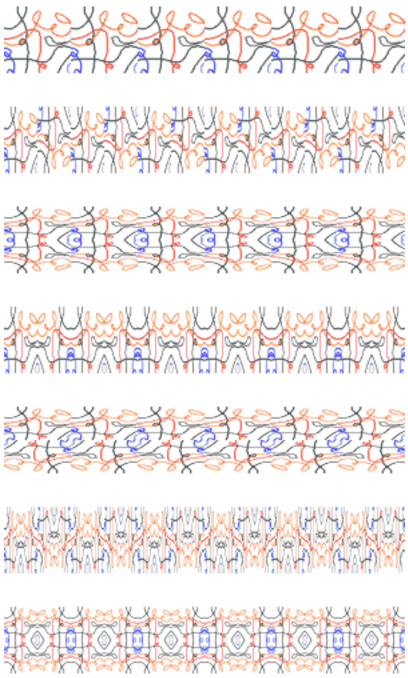
Proof Else we can conjugate τ by such an element of $\mathscr{S}$ and we get an illegal translation in $\mathscr{S}$.□

Corollary *There is a line c which is fixed by any element of $\mathscr{S}$. Moreover, $\mathscr{S}$ contains only half-turns centered on c, reflections of axis c or orthogonal to c and glide-reflections of axis c and of vector in $\frac{v}{2}\mathbb{Z}$.*

We don't give the details of the proof then, but looking carefully at each case yields to the following classification.

- If $\mathscr{S}$ contains a half-turn of center A, it contains all half-turns centered on elements of $A + \frac{v}{2}\mathbb{Z}$:
  - If it contains the reflection of axis c, then it also contains all reflections of axis perpendicular to c and intersecting c in a point of $A + \frac{v}{2}\mathbb{Z}$.
    This is the symmetry type $\mathscr{F}_2^1$, e.g. 7th pattern below.
  - Else :
    - If it contains a reflection, it contains all reflection of axis perpendicular to c and intersecting c in a point of $A + \frac{v}{4} + \frac{v}{2}\mathbb{Z}$.
      This is the symmetry group $\mathscr{F}_2^2$, e.g. 6th pattern below.
    - Else, it also contains no glide-reflection.
      This is the symmetry type $\mathscr{F}_2$, e.g. 5th pattern below.
- Else :
  - If it contains the reflection of axis c,
    This is the symmetry type $\mathscr{F}_1^1$, e.g. 3rd pattern below.
  - Else :
    - If it contains reflections, it contains exactly the reflections of axis perpendicular to c and intersecting c in a point of $A + \frac{v}{2}\mathbb{Z}$.
      This is the symmetry type $\mathscr{F}_1^2$, e.g. 4th pattern below.
    - Else :
      - If it contains the glide reflection of vector $\frac{v}{2}$,it contains exactly the iterates of τ and the glide reflections of vector in $\frac{v}{4} + \frac{v}{2}\mathbb{Z}$.
        This is the symmetry type $\mathscr{F}_1^3$, e.g. 2nd pattern below.
      - Else, it only contains the iterates of τ.
        This is the symmetry type $\mathscr{F}_1$, e.g. 1st pattern below.

Moreover, all cases are effective, as shows the following picture (found on Wikipedia) :



2.3 The 17 wallpaper groups

Lattices and Point Groups If G is a subgroup of $\mathbb{E}(2)$, we write T for the set of translations in G and G_0 is the set

$$\left\{A \in \mathrm{O}_2(\mathbb{R}) \mid (u, A) \in G \text{ for some } u \in \mathbb{R}^2\right\},$$

calling T the transformation subgroup of G and G_0 the point group of G.

Definition A subgroup of $\mathbb{E}(2)$ is called a wallpaper group if its translation subgroup is generated by two independent translations and its point group is finite.

In order to classify the wallpaper groups, we shall precise some properties of its transformation subgroup and point group. From now on G will denote a wallpaper group. First, the elements of T can be viewed as vectors in $\mathbb{R}^2$, hence we obtain a discret subgroup of the plane. Select a non-zero vector a of minimum length in this lattice, then choose a second vector b from the lattice which is skew to a and whose length is as small as possible.

Theorem *The lattice is generated by vectors a and b.*

We shall classify lattices into five different types according to the shape of the basic parallelogram determined by the vectors a and b. Replace b by $-b$ if necessary to ensure that $\|a-b\| \leq \|a+b\|$. With this assumption the different lattices are defined as follows.

1. Oblique $\|a\| < \|b\| < \|a-b\| < \|a+b\|$.
2. Rectangular $\|a\| < \|b\| < \|a-b\| = \|a+b\|$.
3. Centred Rectangular $\|a\| < \|b\| = \|a-b\| < \|a+b\|$.
4. Square $\|a\| = \|b\| < \|a-b\| = \|a+b\|$.
5. Hexagonal $\|a\| = \|b\| = \|a-b\| < \|a+b\|$.

Notice that the point group is a finite subgroup of $\mathrm{O}_2(\mathbb{R})$, thus we have the following important fact.

Theorem *The point group is isomorphic to one of the following ten groups:*

$$\{C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6\}.$$

In fact, we can determine the lattice type of a wallpaper group in terms of its point group by: Take a rotation r in G_0, then the lattice is clairly generated by vectors $a, r(a)$ where a is as above. Since the order of r can only be 2, 3, 4, or 6, the angle between a and $r(a)$ gives the five lattice types.

Theorem *If two wallpaper groups are isomorphic then their point groups are also isomorphic.*

Wallpaper patterns The patterns formed by periodic tilings can be categorized into 17 wallpaper groups. Our goal in this section is to describe the outline of the proof. Here is the list of all wallpaper groups:

Oblique	Rectangular	Centred Rectangular	Square	Hexagonal
p1	pm	cm	p4	p3
p2	pg	cmm	p4m	p3m1
	pmm			p31m
	pmg			p6
	pgg			p6m

Before beginning the classification we add a word or two about notation. First of all, some of these names differ in short and full notation, for exemple p2 is called p211 in its full name. Each wallpaper group has a name made up of several symbols p, c, m, g and the integers 1, 2, 3, 4, 6. The letter p refers to the lattice and stands for the word primitive. When we view a lattice as being made up of primitive cells (copies of the basic parallelogram which do not contain any lattice points in their interiors) we call it a primitive lattice. In one case (the centred rectangular lattice) we take a non-primitive cell together with its centre as the basic building block, and use the letter c to denote the resulting centred lattice. The symbol for a reflection is m (for mirror) and g denotes a glide reflection. Finally, 1 is used for the identity transformation and the numbers 2, 3, 4, 6 indicate rotations of the corresponding order.

We can proceed with case-by-case analysis. In the text, we will only deal with the case that the lattice is oblique, the rest can be treated similarly.

If the lattice is oblique, then the only orthogonal transformations which preserve the lattice are the identity and rotation through π about the origin. Therefore, the point group G_0 is a subgroup of $\{\pm I\}$.

- **(p1)** $G_0 = \{I\}$.

  Then G is the simplest wallpaper group, and it is generated by two independent translations. Its elements have the form $(ma + nb, I)$, where $(m, n) \in \mathbb{Z}^2$.

- **(p2)** $G_0 = \{\pm I\}$.

  In this case G contains a half turn, and we may as well take the fixed point of this half turn as origin, so that $(0, -I)$ belongs to G. It is easy to verify that the elements of G are of the form $(ma + nb, \pm I)$, where $(m, n) \in \mathbb{Z}^2$. In other words, we have all the half turns about the points $\frac{1}{2}ma + \frac{1}{2}nb$.

In the end, we will describe each wallpaper group in terms of generators, in the following a, b are the generators of the lattice defined as above. To simplify notations, we introduce the following elements of $\mathrm{O}_2(R)$:

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Wallpaper group	Generators	Point group
p1	a, b	C_1
p2	$a, b, (0, -I)$	C_2
pm	$a, b, (0, B_2)$	D_1
pg	$a, b, (\frac{1}{2}a, B_2)$	D_1
pmm	$a, b, (0, B_2), (0, -I)$	D_2
pmg	$a, b, (\frac{1}{2}a, B_2), (0, -I)$	D_2
pgg	$a, b, (\frac{1}{2}(a + b), B_2), (0, -I)$	D_2
cm	$a, b, (0, A_2)$	D_1
cmm	$a, b, (0, A_2), (0, -A_2)$	D_2
p4	$a, b, (0, A_4)$	C_4
p4m	$a, b, (0, A_4), (0, B_3)$	D_4
p4g	$a, b, (0, A_4), (\frac{1}{2}(a + b), B_3)$	D_4
p3	$a, b, (0, A_3)$	C_3
p3m1	$a, b, (0, A_3), (0, B_3)$	D_3
p31m	$a, b, (0, A_3), (0, -A_3B_3)$	D_3
p6	$a, b, (0, -A_3^2)$	C_6
p6m	$a, b, (0, -A_3^2), (0, -A_3B_3)$	D_6

3 Overview of other tilings-related problems

Though this classification has been worked out, there are still problems left ! A main challenge is for example to find other invariants than the symmetry group to classify all (or a special class of) tilings.

An obvious instance of such invariant is the *topological type* : two tilings are said to have the same topological type if there is an homeomorphism of the plane which send one of them on the other one.

More interesting examples are to be found in chapters 5 and 6 of [1]. The chapter 5 defines a *pattern type* for which we could also attempt to classify tilings. The chapter 6 provides a classification of the *isohedral* tilings, that are the tilings such that for any pair of vertices (u, v), there is a symmetry of the tilling s satisfying $s(u) = v$.

References

[1] B. Grünbaum and G. C. Shephard. *Tilings and Patterns.* W. H. Freeman and Company, 1987.

[2] G. E. Martin. *Transformation geometry. An Introduction to Symmetry.* Spriger Verlag, 1982.