geometry of matroids – exercise session three

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computational exercises

1. The root system A_3 – or the complete graph K_4.
   Let $A_3 = \{e_i - e_j \,:\, 1 \leq i < j \leq 4\}$.
   (a) Describe all the flats of the matroid of A_3.
   (b) Draw the lattice of flats.
   (c) Find the Möbius function.
   (d) Compute the characteristic polynomial.
2. Combinatorial interpretations of the characteristic polynomial.
   Consider our running example, which has $\chi(q) = q^3 - 4q^2 + 5q - 2$. Verify that
   (a) the number of proper q-colorings of the graph is $\chi(q)$,
   (b) the number of regions of the real arrangement is $|\chi(-1)|$, and
   (c) the number of points not on the $\mathbb{F}_q$–arrangement is $\chi(q)$.
3. The root system A_{n-1} – or the complete graph K_n.
   Compute the characteristic polynomial of $A_{n-1} = \{e_i - e_j \,:\, 1 \leq i < j \leq n\}$.
4. The root system D_n.
   Compute the characteristic polynomial of $D_n = \{e_i - e_j, e_i + e_j \,:\, 1 \leq i < j \leq n\}$.
5. Generic arrangements.
   Consider N hyperplanes in general position in $\mathbb{R}^n$. How many regions do they form in $\mathbb{R}^n$?

conceptual exercises

1. Counting proper colorings and acyclic orientations of graphs. Let G be a graph and M be its graphical matroid. Let c be the number of connected components of G.
   (a) Let q be a positive integer. Prove that $q^c \chi_M(q)$ is the number of ways of coloring the vertices of G with q given colors in such a way that neighboring vertices have different colors.
   (b) Prove that $|\chi_M(-1)|$ is the number of ways of orienting each edge of G in such a way that no directed cycles are formed.
2. The Tutte polynomial
   The *Tutte polynomial* of a matroid M is $T_M(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}$.
   (a) Prove that $M \mapsto T_M(x,y)$ is the unique function from the set of all matroids to $\mathbb{Z}[x,y]$ satisfying all of the following conditions:
      i. $M \cong N \Longrightarrow T_M(x,y) = T_N(x,y)$.
      ii. $T_{coloop}(x,y) = x$ and $T_{loop}(x,y) = y$.
      iii. $T_M(x,y) = T_{M\backslash e}(x,y) + T_{M/e}(x,y)$ if e is not a loop or a coloop.
      iv. $T_M(x,y) = T_e(x,y) T_{M\backslash e}(x,y)$ if e is a loop or a coloop.

      (A function satisfying i, iii, and iv is called a Tutte-Grothendieck invariant. One can show that all Tutte-Grothendieck invariants are specializations of the Tutte polynomial.)
   (b) Show that the characteristic polynomial of a matroid is a specialization of the Tutte polynomial.