Algebraic and combinatorial aspects of face numbers and Stanley-Reisner rings

Exercise sheet – Day 2

Exercise 1 [Kind-Kleinschmidt criterion for an l.s.o.p.]
Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$. For a linear form $y = \sum_{i=1}^{n} c_i x_i \in \mathbb{K}[x_1, \ldots, x_n]$ and a face $F \in \Delta$ we define the restriction of y to F to be $y|_F = \sum_{i \in F} c_i x_i$.

i. Prove that linear forms $y_1, \ldots, y_d \in \mathbb{K}[x_1, \ldots, x_n]$ are an l.s.o.p. for $\mathbb{K}[\Delta]$ if and only if $y_1|_F, \ldots, y_d|_F$ span an $|F|$-dimensional vector space for every face $F \in \Delta$.

ii. Use the criterion above to show that there is no l.s.o.p. for the boundary complex of a d-simplex (that is, $\partial\Delta_d = 2^{d+1} \setminus [d+1]$) with coefficients in $\{0, 1\}$.

Exercise 2 [Prime decomposition and Alexander duality]
Let Δ be a simplicial complex on $[n]$.

i. Show that
$$I_\Delta = \bigcap_{S \in \mathcal{F}(\Delta)} \langle x_i : i \in [n] \setminus S \rangle ,$$
where $\mathcal{F}(\Delta)$ is the set of facets of Δ.

ii. Assume $I_\Delta = \langle \mathrm{x}^{\alpha_1}, \ldots, \mathrm{x}^{\alpha_g} \rangle$, where $\mathrm{x}^{\alpha_i} = \prod_{j=1}^{n} x^{\alpha_{i,j}}$ for some $\alpha_i \in \{0,1\}^n$. We define the *Alexander dual* of I_Δ to be
$$I_\Delta^* = \bigcap_{i=1}^{g} \langle x_j : \alpha_{i,j} = 1 \rangle .$$
Observe that I_Δ^* is again a squarefree monomial ideal and describe its minimal generating set.

iii. Prove that $(I_\Delta^*)^* = I_\Delta$.

iv. Denote by Δ^* the simplicial complex such that $I_{\Delta^*} = I_\Delta^*$. Find a 2-dimensional simplicial complex Δ such that $\Delta \cong \Delta^*$.

Exercise 3 [Hilbert series of Stanley-Reisner rings]
Let Δ be a $(d-1)$-dimensional simplicial complex.

i. Show that
$$F_{\mathbb{K}[\Delta]}(t) = \sum_{i=0}^{d} f_{i-1}(\Delta) \frac{t^i}{(1-t)^i}.$$

ii. Is $F(t) = (1 + 2t - t^2)/(1 - t)^2$ the Hilbert series of a Stanley-Reisner ring? Is it the Hilbert series of a Stanley-Reisner ring of a CM complex?

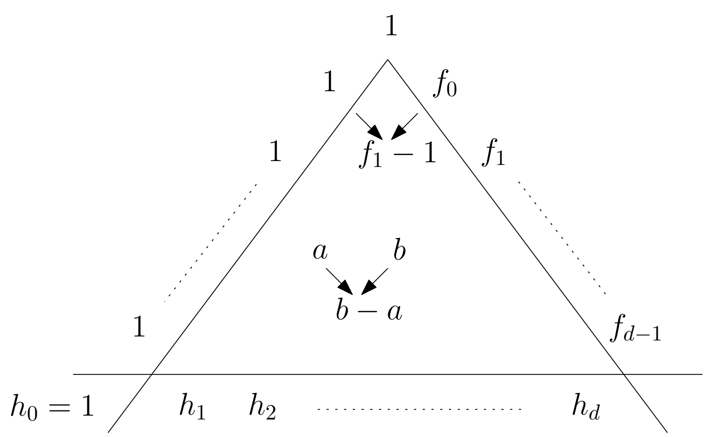
Exercise 4 [Stanley trick]

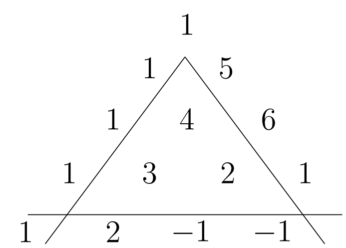
Let $f(\Delta)$ be the f-vector of a simplicial complex Δ.

i. Prove that

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta) \text{ and } f_{i-1}(\Delta) = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j(\Delta).$$

ii. Show that $h(\Delta)$ can be computed via the following Pascal-like triangle:





In words, the j-th entry in the i-th row of the triangle is given by the difference between the j-th and the $(j-1)$-th entries in the $(i-1)$-th row.

Exercise 5 [Injectivity of the multiplication map]

Let A be a standard graded $\mathbb{K}$-algebra and let $y \in A_1$ be such that the multiplication map $\cdot y : A_i \to A_{i+1}$ is injective.

i. Show that $F_{A/yA}(t) = (1 - t)F_A(t)$.

ii. Conclude that if $y_1, \ldots, y_r$ is an l.s.o.p. for $\mathbb{K}[\Delta]$ then A is Cohen-Macaulay if and only if $F_{A/(y_1,\ldots,y_r)A}(t) = (1 - t)^r F_A(t)$.

Exercise 6 [Shellability]

Let Δ be a pure simplicial complex, where pure means that all facets have the same dimension.

i. Show that the following conditions are equivalent: There exists a linear ordering $F_1, \ldots, F_m$ of the facets of Δ such that

(a) $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper faces of $\langle F_i \rangle$ for all $2 \leq i \leq m$. (Here, $\langle F_1, \ldots, F_{i-1} \rangle$ denotes the smallest simplicial complex containing $F_1, \ldots, F_{i-1}$.)

(b) the set $\{F \ : \ F \in \langle F_1, \ldots, F_i \rangle, \ F \notin \langle F_1, \ldots, F_{i-1} \rangle\}$ has a unique minimal element $R(F_i)$ for all $2 \leq i \leq m$.

(c) for all i, j, $1 \leq j < i \leq m$, there exist some $v \in F_i \setminus F_j$ and some $k \in \{1, 2, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

If one resp. all of these conditions hold, then Δ is called *shellable*.

ii. Show that if Δ is shellable, then so is $\mathrm{lk}_\Delta(F)$ for all $F \in \Delta$.

iii. Show that a shellable simplicial complex is Cohen-Macaulay over any field. You can use (without proof) that any shellable simplicial complex Δ is homotopy equivalent to a wedge of $h_{\dim \Delta + 1}(\Delta)$ many $\dim \Delta$-dimensional spheres.

iv. Show that for a shellable simplicial complex Δ we have

$$h_i(\Delta) = \#\{F \in \Delta \ : \ \#R(F) = i\} \quad \text{for } i \geq 1.$$

Exercise 7 [Noether normalization]

Let A be a standard graded $\mathbb{K}$-algebra. Prove that there exists a number $r \geq 0$ and r homogeneous elements $y_1, \ldots, y_r$ such that:

i. $y_1, \ldots, y_r$ are algebraically independent over $\mathbb{K}$, i.e., if $f(y_1, \ldots, y_r)$ for some $f \in \mathbb{K}[x_1, \ldots, x_r]$ then $f = 0$.

ii. A is a finitely generated $\mathbb{K}[y_1, \ldots, y_r]$-module, i.e., there exists a finite set B of homogeneous elements of A such that $A = \sum_{b \in B} b \cdot \mathbb{K}[y_1, \ldots, y_r]$.