Algebraic and combinatorial aspects of face numbers and Stanley-Reisner rings

Exercise sheet – Day 3

Exercise 1 [k-binomial representation of a positive integer]

Let $m, k \in \mathbb{Z}_{\geq 0}$.

i. Prove there exists a unique expression of m as

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

with $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$.

Let J_k be the set of all k-subsets of $\mathbb{Z}_{\geq 0}$ and let $F_m = \{a_1 < \cdots < a_k\}$ be the $(m+1)$-th smallest element of J_k in the rev-lex order.

ii. Show that $m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1}$, with $\binom{a_i}{i} = 0$ if $i > a_i$.

Let $\mathcal{F}$ be an initial segment of J_k, i.e., $\mathcal{F}$ contains the $|\mathcal{F}|$ smallest elements of J_k in the rev-lex order. Set $m + 1 = |\mathcal{F}|$.

iii. Show that $|\partial\mathcal{F}| = \partial_k(m)$.

iv. Show that $\partial\mathcal{F}$ is an initial segment of J_{k-1}.

Exercise 2 [Shifting]

i. Show with simple (even 1-dimensional) example that the operation $S_j(\Delta)$, introduced in the lecture, depends on the order of facets $F_1, \ldots, F_M$ of Δ in which it is applied. Do Δ and $S_j(\Delta)$ have the same simplicial homology?

Recall that a simplicial complex Δ is *shifted* if for every $F \in \Delta$ and $j < i$ it follows that $(F \setminus \{i\}) \cup \{j\} \in \Delta$.

ii. Prove that a pure shifted simplicial complex is shellable.
Hint: Consider the lexicographic order on the facets.

Exercise 3 [h-vectors of simplicial polytopes (or not?)]

Decide which of the following integer vectors are h-vectors of simplicial 6-polytopes.

i. $v_1 = (1, 6, 18, 16, 16, 18, 6, 1)$.

ii. $v_2 = (1, 6, 15, 20, 20, 15, 6, 1)$.

iii. $v_3 = (1, 8, 9, 9, 9, 9, 8, 1)$.

iv. $v_4 = (1, 8, 9, 10, 10, 9, 8, 1)$.

v. $v_5 = (1, 8, 38, 100, 100, 38, 8, 1)$.

Exercise 4 [Monomial $\mathbb{K}$-basis]

Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an homogenous ideal.

i. Show that there exists a $\mathbb{K}$-basis B_I for the vector space $\mathbb{K}[x_1, \ldots, x_n]/I$ consisting of monomials and show that B_I is a multicomplex.

ii. Conclude that there exists a monomial ideal $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ such that $\mathbb{K}[x_1, \ldots, x_n]/I$ and $\mathbb{K}[x_1, \ldots, x_n]/J$ have the same Hilbert function.

Exercise 5 [Special algebras from special polytopes]

Exhibit 0-dimensional standard graded algebras of the form R/I with I a monomial ideal, whose Hilbert function is the h-vector of the boundary complex of:

i. The 6-simplex.

ii. The 6-dimensional cross-polytope.

iii. The cyclic polytope $C(10, 6)$.

Hint: The h-vectors of the three polytopes above are rather special. Can you think of a corresponding multicomplex and then mod out the complement?