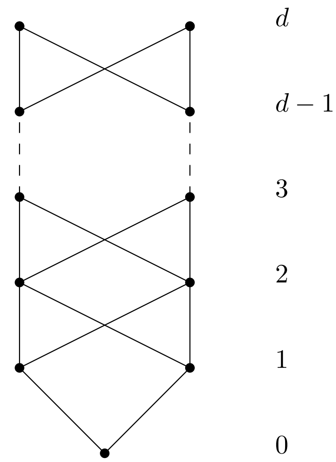
Algebraic and combinatorial aspects of face numbers and Stanley-Reisner rings

Exercise sheet – Day 4

Exercise 1 [Flag f- and h-vectors]

Let Δ be a balanced $(d-1)$-dimensional simplicial complex with coloring κ. We defined the flag f-vector $(\alpha_S(\Delta))_{S\subseteq[d]}$, with $\alpha_S(\Delta) = |\{F \in \Delta : \kappa(F) = S\}|$ and flag h-vector $(\beta_S(\Delta))_{S\subseteq[d]}$, with $\beta_S(\Delta) = \sum_{T\subseteq S}(-1)^{|S\setminus T|}\alpha_S(\Delta)$.

i. Compute the flag f- and h- vector of the order complex of the following poset:



ii. Show that for every balanced $(d-1)$-dimensional simplicial complex Δ we have

$$\alpha_S(\Delta) = \sum_{T\subseteq S} \beta_S(\Delta).$$

iii. Show that

$$\sum_{T\subseteq[d]} \alpha_T(\Delta)\lambda^T(1-\lambda)^{[d]\setminus T} = \sum_{T\subseteq[d]} \beta_T(\Delta)\lambda^T,$$

where $\lambda^T = \prod_{i\in T}\lambda_i$.

iv. Write the $\mathbb{N}^d$-graded Hilbert series $F_{\mathbb{K}[\Delta]}(t_1,\ldots,t_d) = \sum_{a\subseteq\mathbb{N}^d}\dim_{\mathbb{K}}(\mathbb{K}[\Delta]_a)t^a$, with $t^a = \prod_{i=1}^d t_i^{a_i}$ of $\mathbb{K}[\Delta]$ in terms of $\beta_S(\Delta)$.

Exercise 2 [Rank-selected subcomplexes]

Let $\Delta_S = \{F \in \Delta : \kappa(F) \subseteq S\}$ be the rank-selected subcomplex of a balanced $(d-1)$-dimensional simplicial complex with coloring κ.

i. Show that $\mathbb{K}[\Delta_S] \cong \mathbb{K}[\Delta]/I$ for some (very simple) monomial ideal I.

ii. Conclude that if Δ is Cohen-Macaulay, then so is Δ_S.
Hint: Use the fact that a graded $\mathbb{K}$-algebra is Cohen-Macaulay if and only if is a free finitely generated module over an l.s.o.p., and that a balanced simplicial complex has a very special l.s.o.p..

Exercise 3 [Links of balanced simplicial complexes]
Let Δ be a balanced $(d-1)$-dimensional simplicial complex that is pure.

i. Prove that $\mathrm{lk}_\Delta(F)$ is balanced for every $F \in \Delta$.

ii. Show with an example that the previous statement is not necessarily true if Δ is not pure.

Exercise 4 [Balanced spheres]
Let Δ be a $(d-1)$-dimensional balanced simplicial sphere.

i. Show that $f_0(\Delta) \geq 2d$.

ii. Prove that $f_0(\Delta) \neq 2d+1$.

iii. Show that $f_{d-1}(\Delta)$ is even. (For d odd this is true for any simplicial $(d-1)$-sphere).

iv. Describe explicitly the set of h-vectors of balanced simplicial 2-spheres.

Exercise 5 [Cross-polytopal stacked spheres]
Let Δ be a cross-polytopal stacked $(d-1)$-sphere on kd vertices, with $k \geq 2$. Let $v \in \Delta$ be a vertex.

i. Compute $h(\Delta)$.

ii. Show that $\mathrm{lk}_\Delta(v)$ is a cross-polytopal stacked $(d-2)$-sphere.

iii. Can you find a balanced 2-dimensional simplicial complex on 9 vertices homeomorphic to a torus?
Hint: Consider a certain cross-polytopal stacked 2-sphere on 12 vertices.