Geometric and combinatorial aspects of face numbers 6/28/19

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Day 4 — More on centrally symmetric polytopes and spheres

1. (a) Let Δ be a $(d-1)$-dimensional simplicial complex and $r \leq d$. Recall that the $(r-1)$-*skeleton of* Δ, $\mathrm{Skel}_{r-1}(\Delta)$, is the subcomplex of Δ consisting of all the faces of Δ of dimension $\leq r-1$. Show that

$$h(\mathrm{Skel}_{r-1}(\Delta), \lambda) = \mathrm{trunc}_r \left[\frac{h(\Delta, \lambda)}{(1-\lambda)^{d-r}} \right],$$

where the *r-truncation* of a power series is defined by $\mathrm{trunc}_r \left(\sum_{j=0}^{\infty} a_j \lambda^j \right) := \sum_{j=0}^{r} a_j \lambda^j$. (This observation is due to Ron Adin.)
Hint: $h(\Delta, \lambda) = \sum_{j=0}^{d} f_{j-1}(\Delta) \lambda^j (1-\lambda)^{d-j}$.

(b) Use part (a) to obtain expressions for the h-numbers of a cs $(d-1)$-dimensional sphere with $n = 2m$ vertices that is cs-$\lfloor d/2 \rfloor$-neighborly, assuming such a sphere exists.
Hint: What are the h-numbers of the cross-polytope C_m^*?

2. In this exercise we will prove an extension of Adin–Stanley's UBT for cs simplicial spheres to cs odd-dimensional simplicial manifolds. The technique we will use is that of the short simplicial h-vector (see Problem #3 from yesterday's exercises; the proof here is just a slight modification of the proof from yesterday).

(a) Let $\Gamma \supseteq \Delta$ be $(d-1)$-dimensional *Buchsbaum* simplicial complexes. Show that $\widetilde{h}_j(\Gamma) \geq \widetilde{h}_j(\Delta)$ for all $0 \leq j \leq d-1$.

(b) Let Δ be a $(2k-1)$-dimensional simplicial manifold that is a subcomplex of $\mathrm{Skel}_{2k-1}(C_m^*)$ and let S_m be a cs $(2k-1)$-dimensional simplicial sphere with $2m$ vertices that is cs-k-neighborly, assuming such a sphere exists. Use part (a) to show that $\widetilde{h}_j(\Delta) \leq \widetilde{h}_j(S_m)$ for all $j \leq k-1$. Conclude that $\widetilde{h}_j(\Delta) \leq \widetilde{h}_j(S_m)$ for all $j \leq 2k-1$.

(c) Use part (b) to prove the following result: If Δ is a cs $(2k-1)$-dimensional simplicial manifold on $n = 2m$ vertices, then $f_i(\Delta) \leq f_i(S_m)$ for all $1 \leq i \leq 2k-1$.

3. In this exercise we will prove the following theorem:

Theorem: Let P be a cs simplicial d-polytope with vertex set V, $|V| = n$. Then

$$f_1(P) \leq \frac{n^2}{2} \left(1 - 2^{-d} \right).$$

(a) Let u and v (and hence also $-v$) be vertices of P. Show that if the translates $P_u := P + u$ and $P_v := P + v$ of P have intersecting interiors, then u and $-v$ are **not** connected by an edge.

(b) Normalize the Lebesgue measure dx in $\mathbb{R}^d$ in such a way that $\mathrm{Vol}(2P) = 1$. For a subset A of $\mathbb{R}^d$, let $[A] : \mathbb{R}^d \to \mathbb{R}$ denote the indicator function of A. Define $h = \sum_{u \in V} [\mathrm{int}\ P_u]$. Use the Hölder inequality to conclude that

$$\int_{2P} h^2 \, dx \geq n^2 2^{-2d}.$$

(c) On the other hand, use part (a) to show that

$$\int_{2P} h^2(x)\, dx = \sum_{u,v \in V} \mathrm{Vol}\left(P_u \cap P_v\right) \leq n2^{-d} + 2\left(\binom{n}{2} - f_1(P)\right) 2^{-d}.$$

(d) Use the inequalities of parts (b) and (c) to complete the proof of the theorem.

4. (a) Use double counting to show that if Δ is any simplicial complex with n vertices, then for all $j \geq 2$,

$$\frac{f_{j-1}(\Delta)}{\binom{n}{j}} \leq \frac{f_1(\Delta)}{\binom{n}{2}}.$$

(b) Use part (a) and the previous problem to prove the following: if P is a cs simplicial d-polytope with n vertices and $2 \leq j \leq \lfloor d/2 \rfloor$, then

$$f_{j-1}(\Delta) \leq \frac{n}{n-1}\binom{n}{j}\left(1 - 2^{-d}\right).$$