THE GEOMETRY OF MATROIDS

LECTURES BY FEDERICO ARDILA

Notes taken by Connor Simpson

Notes on the notes. These notes were taken during Federico Ardila's mini-course on matroids at the Summer School on Geometric and Algebraic Combinatorics at Sorbonne Université in June 2019. Any errors are the fault of the note-taker, who can be contacted at csimpson6@wisc.edu. Many thanks to Arnau Padrol, Vincent Pilaud, Alfredo Hubard, and Julienne Belair for organizing the school.

1. Matroid basics (with special attention to geometry)

We'll start with some examples of matroids in order to show some of the many contexts in which they appear. A matroid is a combinatorial object that can be defined by a collection of "independent sets" (to be described in Definition 1.1). All of the examples below have the same collection of independent sets, meaning that they all represent the same matroid. The independent sets are

$$\mathcal{I} = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, abc, abd, abe, acd, ace\}.$$

We will use this matroid as a running example throughout the course.

Linear algebra. Given a finite set E of vectors in $\mathbb{R}^n$, take the collection of independent sets to be the linearly independent subsets of E. One may check that $\mathcal{I}$ is the collection of linearly independent subsets of the set of vectors in $\mathbb{R}^3$ drawn at right. Matroids of this form are **linear** matroids.
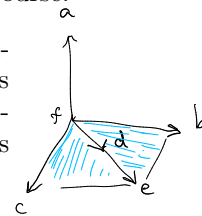
Graph theory. Given a finite graph with edge set E, take the independent subsets of E to be those subsets that do not contain a cycle. One may check that $\mathcal{I}$ is the collection of edge sets of acyclic subgraphs of the graph at right. Matroids of this form are **graphic** matroids.
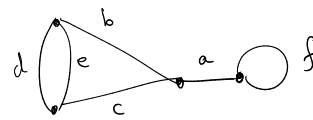
Matching problems. Consider a bipartite graph G with vertex set $V \sqcup E$ and all edges of the form $\{v, e\}$ with $v \in V$ and $e \in E$. A **partial matching** in G is a set S of edges such that each vertex of G is incident to at most one edge in S. We call a subset $I \subset E$ independent if there is a partial matching S of G such that all elements of I are incident to exactly one edge of S, and as in the previous examples, one can check that $\mathcal{I}$ is the collection of all such sets. Matroids from matching problems are called **transversal** matroids.
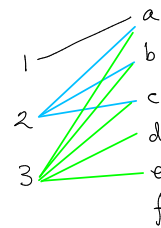
Field extensions. Given a transcendental field extension K/L and generators $\alpha_1, \ldots, \alpha_n$ for K over L, call a subset of the generators independent if it is algebraically independent over L. For example, let $L = \mathbb{C}$ and let

$$a = z^2 \qquad\qquad b = x \qquad\qquad c = y$$

$$d = \frac{1}{xy} \qquad\qquad e = x^2 y^2 \qquad\qquad f = 1$$

be elements of $\mathbb{C}(x, y, z)$. Consider the field extension $\mathbb{C}(a, b, c, d, e, f)/\mathbb{C}$; one can check that the collection of algebraically independent subsets of the generators $\{a, b, c, d, e, f\}$ is $\mathcal{I}$. Matroids of this form are called **algebraic** matroids.

With these examples in mind, we can define matroids.

Definition 1.1. A **matroid** is a pair $(E, \mathcal{I})$ with E a finite set (called the **ground set**) and $\mathcal{I}$ a collection of subsets of E (called **independent**) satisfying

    I1: $\emptyset \in \mathcal{I}$
    I2: Subsets of independent sets are independent
    I3: If $I, J \in \mathcal{I}$ and $\#I > \#J$ then there exists $i \in I \setminus J$ such that $J \cup \{i\}$ is independent.

If you want to think about this geometrically, you can think about the **independence complex**, which is the simplicial complex whose faces are independent sets. The fact that we can think of matroid geometrically immediately raises the question: which simplicial complexes are matroids?

Theorem 1.2. *A simplicial complex Δ on vertex set E is the independence complex of some matroid if and only if the restriction $\Delta|_S := \{I \in \Delta : I \subset S\}$ of Δ to any subset $S \subset E$ of its vertices is pure (a **pure** simplicial complex is one whose maximal faces are all of the same dimension).*

An important characteristic of matroids is that they can be defined with many different kinds of data. Here is another kind that can be used.

Definition 1.3. A **basis** of a matroid is a maximal independent set.

Proposition 1.4. *All bases have the same cardinality.*

In the contexts of our examples at the start, Proposition 1.4 has the following corollaries:

- all bases of a vector space have the same size
- all spanning trees have the same size
- all maximal sets of "fillable jobs" have the same size
- the transcendence degree of an extension is well-defined

As promised, bases yield an equivalent way to define a matroid.

Definition 1.5. A **matroid** is a pair $(E, \mathcal{B})$ with E a finite set and $\mathcal{B}$ a collection of subsets of E called **bases** satisfying

    B1: $\mathcal{B} \neq \emptyset$
    B2: If $A, B \in \mathcal{B}$ and $a \in A - B$ then there is $b \in B - A$ such that $(A - a) \cup b \in \mathcal{B}$.

Axiom B2 is called the **exchange axiom** for bases. Equivalent to B2 is the following **strong exchange axiom**:

    B2*: If $A, B \in \mathcal{B}$ and $a \in A - B$ then there is $b \in B - A$ such that $(A - a) \cup b \in \mathcal{B}$ and $(B - b) \cup a \in \mathcal{B}$.

Remark 1.6. When Federico did his postdoc, he went for a walk with David Eisenbud. Eisenbud asked what he did, and up hearing that Federico worked on matroids, said "I don't like matroids". When asked why, he said that it was because there were some things in his commutative algebra book that he didn't know how to prove without them.
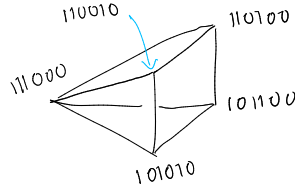
As always, we would like to think of this geometrically

Definition 1.7. The **matroid polytope** of M is

$$P_M := \operatorname{conv}(e_B : B \in \mathcal{B}) \subset \mathbb{R}^E$$

where $e_B = \sum_{i \in B} e_i$ and e_i is the ith standard basis vector.

Example 1.8. Our running example has bases $\{abc, abd, abe, acd, ace\}$, and its matroid polytope is

The modern, geometric theory of matroids begins with the following question: which polytopes are matroid polytopes?

Theorem 1.9 (Gelfand-Goresky-MacPherson-Serganova, Edmonds)**.** *P is the matroid polytope of some matroid if and only if the vertices of P are all 0-1 vectors and all the edges of P are parallel to $e_i - e_j$ for some i and j.*

Proof. We show that a matroid polytope's edges are all parallel to $e_i - e_j$ for some i and j. Suppose that $P = P_M$. If A and B are neighbors, then the edge between them is the set on which a linear functional w is maximized. By the basis exchange axiom, we can fine elements $a \in A$ and $b \in B$ such that $(A - a) \cup b$ and $(B - b) \cup a$ are both bases. We now have a parallelogram
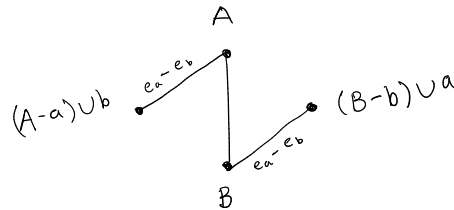
By linearity, $w(A) + w(B) = w((A - a) \cup b) + w((B - b) \cup a)$. But A and B are the only vertices at which w is maximized, so we must have $(A - a) \cup b = B$ and $(B - b) \cup a = A$; therefore, $A - B = e_a - e_b$.

The reverse direction is similar. $\qed$

Remark 1.10. Not much is known about the face lattices of matroid polytopes.

2. Constructions

The geometric perspective afforded by matroid polytopes allow us to quickly construct new matroids from old ones. Some important constructions follow.

Direct sum: Let M_1, M_2 be matroids on disjoint ground sets E_1, E_2. If you take the product of two polytopes, any edge direction in the product is an edge direction of one of the original two polytopes (because any edge of the product is either an edge times a vertex or a vertex times an edge). From this, it follows that $P = P_{M_1} \times P_{M_2}$ is a matroid polytope. The matroid associated to P is the **direct sum** of M_1 and M_2, denoted $M_1 \oplus M_2$.

Duality: Let M be a matroid on E and consider the polytope $-P_M + (1, \ldots, 1) \subset \mathbb{R}^E$. One can check that this has the right edge directions to be a matroid polytope; the associated matroid is M^*, the **dual** of M. From the vertices of $-P_M + (1, \ldots, 1)$, one can see that the bases of M^* are the complements of the bases of M. (Warning: matroid duality does NOT correspond to polytope duality.)

Deletion & contraction: Let M be a matroid on E. Consider the face $F = P_M \cap \{x_e = 0\}$. The edge directions of the face are all edge directions of P_M, so F is a matroid polytope corresponding to the **deletion** of e from M, denoted $M \setminus e$. Likewise, the face $F' = P_M \cap \{x_e = 1\}$ corresponds to a matroid called the **contraction** of M by e, denoted M/e.

Optimization: Let M be a matroid on E and $w : E \to \mathbb{R}$ a weight function. Let $(P_M)_w$ be the face of P_M defined by w, and let M_w be the matroid defined by $(P_M)_w$. The bases of M_w are the bases of M of minimal weight, where the weight of a basis B is defined by $w(B) = \sum_{b \in B} w(b)$.

Evidently, this construction generalizes the previous one. In general, optimization cannot in general be expressed by deletion and contraction alone; however, it can be expressed using deletion, contraction, and direct sums.

This construction also has a number of applications. For example, finding min-cost spanning trees is this problem for (weighted) graphs.

2.1. **What does duality mean?** To get a feel for duality, let's look at duality in the special case of linear matroids.

Definition 2.1. Pick a basis for $\mathbb{R}^n$. If $V \subset \mathbb{R}^n$ is an r-dimensional subspace, the **matroid of** V is defined by

$$B \subset [n] \text{ a basis } \iff V \cap \mathbb{R}^{[n]-B} = 0 \iff V + \mathbb{R}^{[n]-B} = \mathbb{R}^n.$$

Now, let A be an $r \times n$ matrix. There are two different matroids that we can associated to A:

- The matroid of the n columns in $\mathbb{R}^r$
- The matroid of the r-dimensional subspace $V = \text{rowspan}(A) \subset \mathbb{R}^n$.

Proposition 2.2. *The matroid of the columns of A is isomorphic to the matroid of the row-span of A.*

We can realize the dual of the matroid defined by a subspace easily:

Proposition 2.3. *If $V \subset \mathbb{R}^n$ has matroid M, then $V^\perp \subset \mathbb{R}^n$ has matroid M^*.*

Corollary 2.4. *The dual of a linear matroid is linear.*

In the even more special case of planar graphic matroids, one can check that duality corresponds to taking the planar dual of the graph. The dual graph depends on the embedding of the graph; however all dual graphs have the same matroid.

The dual of a transversal matroid is called a **cotransversal** matroid; while a transversal matroid corresponds to a matching problem, a cotransversal matroids corresponds to a routing problem.

Open problem 2.5. *Is the dual of an algebraic matroid algebraic?*

Remark 2.6. Federico holds the opinion that this is probably not true because a duality of this sort would probably have some well-known manifestation in the algebra by now, but does not seem to (Federico also says that you should not take his opinion too seriously).

The essential problem here is that we don't have many tools for showing that a matroid is not algebraic. However, it seems possible that one can just compute some very large example with Sage that wasn't possible in the past and do something specific to that.

To construct a non-linear matroid, one typically builds one whose flats violate some classical geometry theorem (an example is the non-Pappus matroid). To find a non-algebraic matroid, one probably would need to do something similar.

Open problem 2.7. *Why is the matroid polytope of K_4 self-dual? Usually the polar dual of a matroid polytope is not a matroid polytope, but it is in this case... Is there a theory here?*

2.2. **More on optimization and matroids.** Let M be a matroid on E and $w : E \to \mathbb{R}$ a weight function. Recall that the matroid M_w is the matroid whose bases are those of M that have minimal weight, where the weight of a basis B is $w(B) := \sum_{b \in B} w(b)$. We now describe an algorithm for finding a basis of M_w.

Algorithm.
Input: the independent sets of a matroid M and a weight function w
Output: a basis of M_w

(1) Let $I = \emptyset$.
(2) Replace I by $I \cup e$, where $e \in E \setminus I$ is an element of minimal weight such that $I \cup e$ is independent.
(3) Repeat until you have a basis.

Proposition 2.8. *The algorithm above outputs a basis of M that has minimal weight with respect to w.*

Proof. Let $I = \{i_1, i_2, \ldots, i_r\}$ be r elements selected by the greedy algorithm with $w(i_1) \leq w(i_2) \leq \cdots \leq w(i_r)$ and let $J = \{j_1, j_2, \ldots, j_r\}$ be some other independent set with $w(j_1) \leq \cdots \leq w(j_r)$. Suppose towards a contradiction that $w(J) < w(I)$. Then there exists a k such that $w(j_k) < w(i_k)$. Now, the sets $I' = \{i_1, \ldots, i_{k-1}\}$ and $J' = \{j_1, \ldots, j_k\}$ are independent, so there exists $j_\ell \in J' \setminus I'$ such that $I' \cup j_\ell$ is independent. But then $w(j_\ell) \leq w(j_k) < w(i_k)$, contrary to the fact that the greedy algorithm adds an element of minimal weight at each step. $\square$

Corollary 2.9.
(i) *The greedy basis is a w-min basis of M.*
(ii) *$\{w(e) : e \in B\}$ is the same for all min-weight bases B.*
(iii) *If you allow all possible choices of elements, this algorithm will produce all min-weight bases.*

Theorem 2.10. *Let Δ be a simplicial complex. $\Delta = \mathrm{IN}(M)$ for some matroid M if and only if for any $w : E \to \mathbb{R}_{\geq 0}$, the greedy algorithm works.*

Remark 2.11. Suggestion from Federico: Read old optimization papers. They often contain exactly what you need (though perhaps in different language).
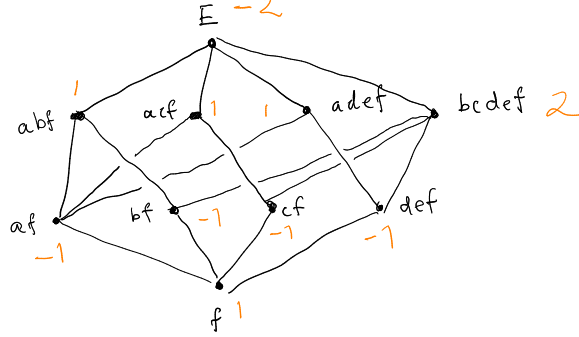
3. The characteristic polynomial

Definition 3.1. The **rank** function of a matroid M on E is $r : 2^E \to \mathbb{N}$ defined by

$$r : A \mapsto \max\{\#I : I \subset A \text{ independent}\}.$$

Definition 3.2. A **flat** of a matroid is $F \subset E$ such that for all $a \in E \setminus F$, $r(F \cup a) > r(F)$. The **lattice of flats**, denoted $\mathcal{L}_M$, is the poset of flats of M ordered by containment.

Definition 3.3. The Möbius function of a finite poset P is $\mu : P \to \mathbb{R}$ defined recursively by $\mu(\hat{0}) := 1$, and $\mu(F) := -\sum_{G < F} \mu(G)$.

Example 3.4. The flats of our running example are: $\{f\}$, $\{a, f\}$, $\{b, f\}$, $\{c, f\}$, $\{d, e, f\}$, $\{a, b, f\}$, $\{a, c, f\}$, $\{b, c, d, e, f\}$, E. Pictorially, the lattice of flats (with values of the Möbius function written in orange beside it) looks like:

Definition 3.5. The **characteristic polynomial** of M is

$$\chi_M(q) := \sum_{F \text{ flat}} \mu(F) q^{r(E)-r(F)} = \sum_{A \subset E} (-1)^{|A|} q^{r(E)-r(A)}$$

and the **reduced characteristic polynomial** is $\bar{\chi}_M(q) := \frac{\chi_M(q)}{1-q}$.

The characteristic polynomial looks arbitrary on the face of it. However, some motivation is provided by the following facts:

- If M is a matroid of a graph with c components, then $q^c \chi_M(q)$ is the number of proper q-colorings of the graph; that is, the number of ways of coloring the vertices using q colors such that the endpoints of every edge have distinct colors.
- Let M be the linear matroid of $A = v_1, \ldots, v_n\} \subset k^n$ for k some field. A determines a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ with $H_i = v_i^{\perp}$.
  - If $k = \mathbb{R}$, then the number of regions of $\mathbb{R}^n \setminus \mathcal{A}$ is equal to $|\chi(-1)|$.
  - If $k = \mathbb{F}_q$, then the number of points of $\mathbb{F}_q^n \setminus \mathcal{A}$ is equal to $\chi(q)$ (Björner and Ekendahl).
  - If $k = \mathbb{C}$, then the topological Betti numbers of $\mathbb{C}^n \setminus \mathcal{A}$ is equal to the absolute values of the coefficients of $\chi(q)$ (Orlik and Solomon).

Example 3.6. Using the values of the Möbius function in Example 3.4, one computes that our running example has characteristic polynomial $\chi(q) = q^3 - 4q^2 + 5q - 2$ and reduced characteristic polynomial $\bar{\chi}(q) = q^2 - 3q + 2$.

A **loop** of a matroid is an element that is in no bases, while a **coloop** is an element that is in every basis.

Proposition 3.7. *We have the following relations:*

$$\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M/e}(q), \qquad e \text{ neither a loop nor a coloop}$$
$$\chi_M(q) = \chi_{M\setminus e}(q), \qquad e \text{ a loop}$$
$$\chi_M(q) = \chi_{M/e}(q), \qquad e \text{ a coloop}$$

Proposition 3.8. *If $\chi_M(q) = \sum_{i=0}^{r} a_i(-1)^{r-i}$, then $a_r, a_{r-1}, \ldots, a_0 \geq 0$.*

The following result was conjectured by Rota in the 1970's. Huh proved it for graphs in 2012 [Huh14], then Huh and Katz did another case [HK12], then Adiprasito-Huh-Katz proved the general case [AHK18].

Theorem 3.9 (Rota's conjecture). *The sequence $(a_0, \ldots, a_r)$ is unimodal and log-concave; that is, $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_r$ and $a_{i-1}a_{i+1} \leq a_i^2$.*

4. The Order Complex and Bergman Fans

The **order complex** of a poset P is the simplicial complex $\Delta(P)$ whose faces are chains in P. For a finite lattice L, write $\bar{L} := L \setminus \{\hat{0}, \hat{1}\}$, where $\hat{0}$ and $\hat{1}$ are the minimal and maximal elements of L, respectively.

Theorem 4.1 (Björner). *The order complex $\Delta(\bar{\mathcal{L}}_M)$ with M a matroid of rank r is a wedge of $|\mu(M)| := |\chi_M(0)|$ spheres of dimension $(r-2)$.*

In essence, this theorem was proved by constructing a shelling (of a kind invented by Bijorner and Wachs) of $\Delta(\bar{\mathcal{L}})$ and tracking when spheres get closed.

Definition 4.2. A **EL-shelling** of a ranked poset P whose Hasse diagram has edge set $E(P)$ is a function $\lambda : E(P) \to \mathbb{Z}$ such that

  (i) for any interval $[x, y]$, there is a unique maximal chain $m_{[x,y]}$ from x to y with increasing labels
  (ii) for any other maximal chain m' from x to y, we have $\lambda(m') > \lambda(M_{[x,y]})$ in lexicographic order.

Proposition 4.3. *The lexicographic order on chains given by an EL-shelling gives a shelling of $\Delta(P - \{\hat{0}, \hat{1}\})$, and this order complex is a wedge of*

$$\#\{\text{weakly decreasing maximal chains from } \hat{0} \text{ to } \hat{1}\}$$

spheres of dimension $\text{height}(P) - 2$.

Example 4.4. Let M be our running example with f removed. Order the ground set of M; in this example, we will use $a < b < c < d < e$, but any order will work. Label an edge from F to G by $\lambda(F, G) := \min(G - F)$. The red labels on the poset in the left-hand side of the figure below illustrate this labelling.
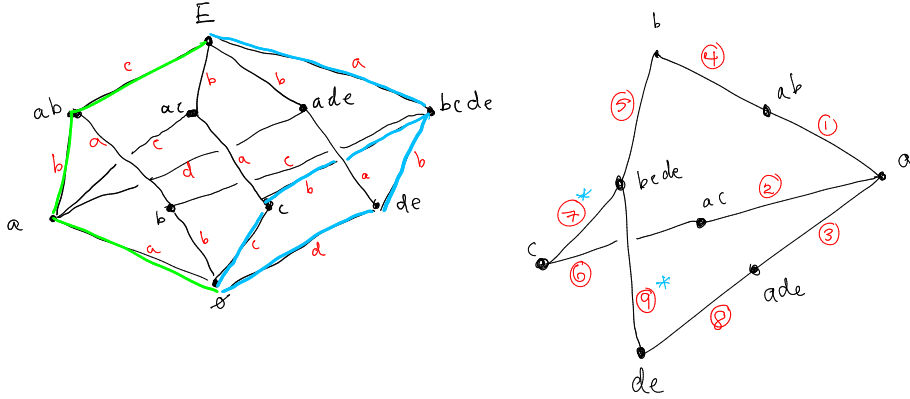
Note that there is a unique maximal increasing path in the labelling, drawn in green, and two weakly decreasing maximal paths, drawn in blue. Using Proposition 4.3 and Theorem 4.1, we can read off properties of the order complex (drawn on the right-hand side above) from $\mathcal{L}_M$ and its labelling: $\mathcal{L}_M$ has height 3, so the order complex should be a wedge of 1-spheres, and the two decreasing paths mean that it should be a wedge of exactly two 1-spheres.

The EL-shelling of $\mathcal{L}_M$ induces an ordering of the facets of $\Delta(\mathcal{L}_M - \{\hat{0}, \hat{1}\})$. For facets F and G, there are unique extensions F', G' of F and G to maximal chains in $\mathcal{L}_M$, and we say that $F < G$ if $F' <_{\text{lex}} G'$, where $<_{\text{lex}}$ is lexicographic order. The facets of the order complex in the figure above have been numbered in increasing order according to the labelling of $\mathcal{L}_M$ that is drawn. The two facets with blue asterisks are facets that close a 1-sphere. They correspond to the two weakly decreasing blue paths in $\mathcal{L}_M$.

Remark 4.5. In general, the existence of an increasing path in an EL-labelling of the lattice of flats follows from the greedy algorithm with weight vector w satisfying $w(a) < w(b) < w(c) < w(d) < w(e)$, as the greedy algorithm builds up a basis by taking elements of minimal weight.

Theorem 4.1 and Proposition 4.3 show that the constant term of the characteristic polynomial is equal to the number of weakly decreasing maximal chains in $\mathcal{L}_M$. The following result generalizes this fact.

Proposition 4.6. *Let M be a matroid and suppose that we have an EL-shelling of $\mathcal{L}_M$.*

(1) *The number of decreasing saturated chains from $\hat{0}$ of length k is equal to the absolute value of the kth coefficient of $\chi_M(q)$.*

(2) *The number of such chains that avoid the minimal label is equal to the absolute value of the kth coefficient of $\bar{\chi}(q) := \frac{\chi(q)}{1-q}$.*

The order complex of the lattice of flats also appears in another context: tropical geometry. Given an algebraic variety V, one can consider its tropicalization, a polyhedral complex that captures some of the data of V.

Theorem 4.7 (Ardila-Klivans 2006, [AK06]). *If V is a linear subspace of $\mathbb{R}^n$, then its tropicalization is a cone over $\Delta(\bar{\mathcal{L}}_{M(V)})$, where $M(V)$ is the matroid of the subspace matroid of V. The tropicalized variety is the **Bergman fan** of $M(V)$.*

Theorem 4.8 (Fink 2010 [Fin13]). *Bergman fans of matroids are the tropical varieties of degree 1.*

This theorem shows that, while some tropical varieties do not come from algebraic ones, they do all come from matroids. Associated to the Bergman fan of a matroid is a toric variety, which in turn has a **Chow ring**. By a theorem of Danilov 1987, the Chow ring is given by the presentation $A^\bullet(M) = \mathbb{C}[x_F : F \text{ proper flat}]/I_1 + I_2$ where

$$I_1 = \langle x_F x_G : F \not\subset G, G \not\subset F \rangle$$
$$I_2 = \langle \sum_{i \in F} x_F - \sum_{j \in G} x_G : i, j \in E \rangle.$$

The great insight of Adiprasito, Huh, and Katz was that the Chow ring behaves like the Chow ring of a smooth projective variety, meaning that it satisfies the Kähler package (keywords that we leave undefined: Poincaré duality, Hard Lefschetz condition, and Hodge-Riemann relations). Without getting into too much detail, $A^\bullet(M)$ is a graded ring such that

- $A^i(M) = 0$ for $i > r(M) - 1$.
- there is an isomorphism $\deg : A^{r(M)-1}(M) \to \mathbb{C}$
- For any ample $a, b \in A^1(M)$, the sequence $\deg(a^{r-1}), \deg(a^{r-2}b), \ldots, \deg(b^{r-1})$ is unimodal and log-concave.
- Let $\alpha = \sum_{i \in F} x_F$ and $\beta = \sum_{i \notin F} x_F$. Then $\deg(\alpha^i \beta^{r-1-i})$ is the number of saturated decreasing chains of length k in the lattice of flats, or equivalently, the kth coefficient of the reduced characteristic polynomial.

The majority of the work done by Adiprasito, Huh, and Katz goes towards establishing the Kähler package for $A(M)$ using an inductive strategy of "flips". Once they have established the Kähler package, they use the properties of $A(M)$ listed above to prove Theorem 3.9.

In [BES19], Backman, Eur, and Simpson use a new presentation of the Chow ring to give a different proof of Poincaré duality, and of both the Hard Leftschetz property and the Hodge-Riemann relation in degree 1, yielding a different proof of Theorem 3.9. Even more recently, the following stronger result was obtained by Ardila, Denham, and Huh.

Theorem 4.9 (Ardila-Denham-Huh 2019+)**.** *The coefficients of $\chi_M(q + 1)$ are log-concave.*

To obtain this result, Ardila, Denham, and Huh study the "cornormal fan", which is different from the Bergman fan.

References

[AHK18] Karim Adiprasito, June Huh, and Eric Katz, *Hodge Theory for combinatorial geometries*, Annals of Mathematics **188** (2018), 381–452.

[AK06] Federico Ardila and Caroline Klivans, *The Bergman complex of a matroid and phylogenetic trees*, Journal of Combinatorial Theory, Series B **96** (2006), no. 1, 38–49.

[BES19] Spencer Backman, Chris Eur, and Connor Simpson, *Simplicial generation of Chow rings of matroids* (2019), available at `arXiv:1905.07114`.

[Fin13] Alex Fink, *Tropical cycles and Chow polytopes*, Beiträge zue Algebra und Geometrie **54** (2013), no. 1, 13–40.

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky, *Chow rings of toric varieties defined by atomic lattices*, Invent. Math. **155** (2004), no. 3, 515–536.

[HK12] June Huh and Eric Katz, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*, Math. Ann. **354** (2012), no. 3, 1103–1116.

[Huh14] June Huh, *Rota's conjecture and positivity of algebraic cycles in permutohedral varieties*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Michigan.