Christopher Borger (joint with G. Balletti)

Families of lattice polytopes of mixed degree one

Summer School on Geometric and Algebraic Combinatorics

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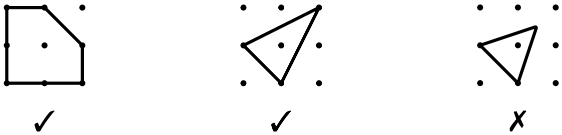
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Lattice Polytopes

Definition (Lattice Polytope)

A lattice polytope is the convex hull of finitely many lattice points $v_1, \ldots, v_k \in \mathbb{Z}^d$.



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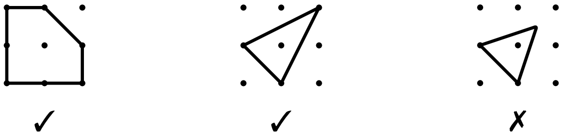

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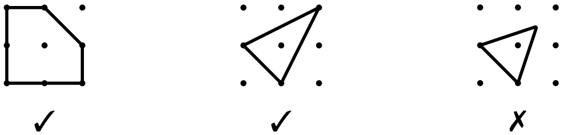

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In this talk: $P \subset \mathbb{R}^d$ always full-dimensional.

A little bit of Ehrhart theory

Consider the integer point counting function:

$$k \mapsto |kP \cap \mathbb{Z}^d|$$

Theorem (Stanley '80)

$$\sum_{k \geqslant 0} |kP \cap \mathbb{Z}^d| t^k = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where h_P^* is a polynomial of degree $\leqslant d$ with coefficients in $\mathbb{Z}_{\geqslant 0}$.

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Definition

The degree of h_P^* is called the degree of P.

The degree of a lattice polytope

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- $d + 1 - \deg(P) = \min\{k \in \mathbb{Z}_{>0} : \mathrm{int}(kP) \cap \mathbb{Z}^d \neq \emptyset\}$

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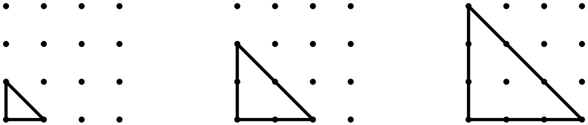

Δ_2	$2\Delta_2$	$3\Delta_2$
codeg $= 3$	codeg $= 2$	codeg $= 1$
deg $= 0$	deg $= 1$	deg $= 2$

Generalizing to tuples: Mixed Degree

Note: $kP = P + \cdots + P = \{p_1 + \cdots + p_k \colon p_i \in P\}$

Definition

Mixed codegree: $\operatorname{mcd}(P_1, \ldots, P_d) =$
$\min\bigl\{k \in \mathbb{Z}_{>0} \colon \exists i_1 < \cdots < i_k \text{ with } \operatorname{int}(P_{i_1} + \cdots + P_{i_k}) \cap \mathbb{Z}^d \neq \varnothing\bigr\}$
(if $P_1 + \cdots + P_d \cap \mathbb{Z}^d = \varnothing$, set $\operatorname{mcd} = d + 1$)

Mixed degree: $\operatorname{md}(P_1, \ldots, P_d) := d + 1 - \operatorname{mcd}(P_1, \ldots, P_d)$.

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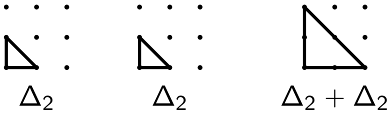
- $\mathrm{md}(P, \ldots, P) = \deg(P)$
- monotone with respect to inclusion
- should measure the complexity of a tuple

The mixed degree: Examples



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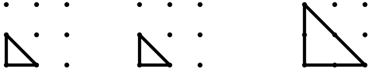

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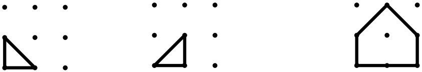

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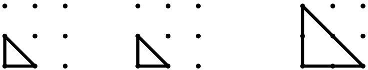

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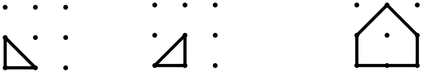

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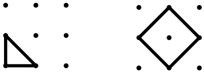

Δ_2 P

$$\mathrm{mcd}(\Delta_2, P) = 1 \Rightarrow \mathrm{md}(\Delta_2, P) = 2$$

Mixed Degree zero

Theorem (Cattani et al. '11, Nill '17)

$P_1, \ldots, P_d \subset \mathbb{R}^d$ full-dimensional:
$\mathrm{md}(P_1, \ldots, P_d) = 0 \Leftrightarrow (P_1, \ldots, P_d) \cong (\Delta_d, \ldots, \Delta_d)$

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Note: $\mathrm{md}(P_1, \ldots, P_d) = 0 \Leftrightarrow P_1 + \cdots + P_d$ is hollow.

Next step: Mixed Degree one

Note: $\mathrm{md}(P_1, \ldots, P_d) \leqslant 1$ iff $P_{i_1} + \cdots + P_{i_{d-1}}$ is hollow for any choice $1 \leqslant i_1 < \cdots < i_{d-1} \leqslant d$.

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Theorem (Soprunov '07, Nill '17)

For $P_1, \ldots, P_d \subset \mathbb{R}^d$ full-dimensional:

$$\mathrm{MV}(P_1, \ldots, P_d) - 1 \leqslant \mathrm{int}(P_1 + \cdots + P_d) \cap \mathbb{Z}^d,$$

with equality iff $\mathrm{md}(P_1, \ldots, P_d) \leqslant 1$.

Soprunov's Question: What are the tuples of lattice polytopes for which the upper bound is attained?

Results for Mixed Degree one

Theorem (Batyrev-Nill '04 (unmixed))

$P \subset \mathbb{R}^d$ *with* $\deg(P) \leqslant 1$. *Then either*

- P *is the* $(d-2)$*-fold pyramid over* $2\Delta_2$, *or*
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Theorem (Balletti-B '19)

$P_1, \ldots, P_d \subset \mathbb{R}^d$ with $\mathrm{md}(P_1, \ldots, P_d) \leqslant 1$ and $d \geqslant 4$. Either

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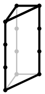
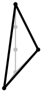
For $d = 3$ there exist infinitely many exceptional families.

Example with projection

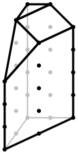


P_1

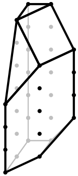
P_2



P_3



$P_1 + P_2$



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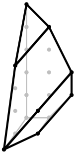
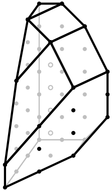
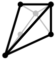


$P_2 + P_3$



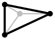
$P_1 + P_2 + P_3$

Example without projection



P_1

P_2



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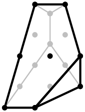

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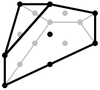

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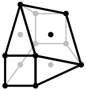

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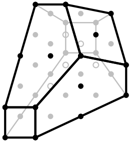

$P_1 + P_2 + P_3$

A bit of proof (and what breaks for $d < 4$)

$\text{md}(P_1, \ldots, P_d) \leqslant 1$ iff $P_{i_1} + \cdots + P_{i_{d-1}}$ is hollow for any choice $1 \leqslant i_1 < \cdots < i_{d-1} \leqslant d$.

Theorem (Nill-Ziegler '11)

Let $P \subset \mathbb{R}^d$ be a hollow lattice polytope. Then either

- *P admits a lattice projection onto a hollow $(d-1)$-polytope, or*
- *P is one of finitely many exceptions.*

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Theorem (Nill-Ziegler '11)

Let $P = P_1 + \cdots + P_{d-1}$ be a hollow d-dimensional lattice polytope. Then either

- *P admits a lattice projection onto $(d-1)\Delta_{d-1}$, or*
- *P is one of finitely many exceptions.*

$\Rightarrow$ leads to finiteness of tuples whenever any sum $P_{i_1} + \cdots + P_{i_{d-1}}$ is exceptional! (there are some things to be shown on the way)

A bit of proof (and what breaks for $d < 4$)

What is left: Any $(d-1)$-subtuple of $P_1, \ldots, P_d$ has a common projection onto Δ_{d-1}.

1. (at least) two of the projections are the same $\Rightarrow$ there exists a common projection for the whole $P_1, \ldots, P_d$
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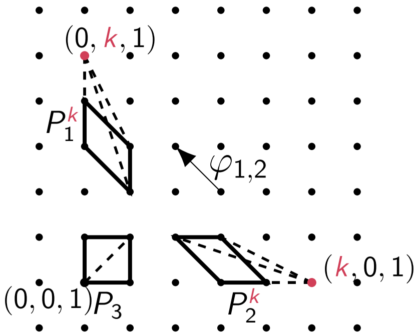
Lemma

For $d \geqslant 5$ the only tuple $P_1, \dots, P_d$ for which all $(d-1)$-subtuples have different common projections onto Δ_{d-1} is $(\Delta_d, \dots, \Delta_d)$.

The case $d = 3$

Theorem (Balletti-B '19)

Let $P_1, P_2, P_3 \subset \mathbb{R}^3$ be an exceptional triple with $\mathrm{md}(P_1, P_2, P_3) = 1$. Then it is equivalent to a triple in a list of 279 triples or it is contained in one of finitely many 1-parameter families of triples.



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Some References

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