# Topology of tropical varieties

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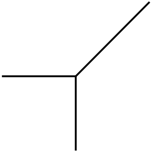
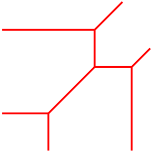
Laboratoire de Mathématiques Jean Leray, Nantes

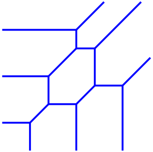
# Exemples of tropical varieties



A line



A conic



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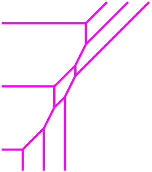
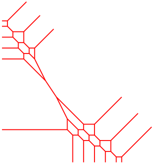
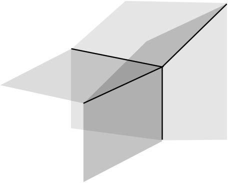


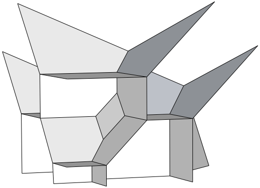
Another cubic



A sextic

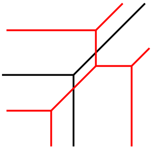


A plane

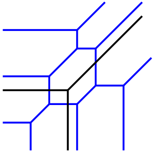


A spatial quadric

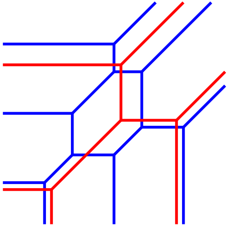
# Tropical intersections



$2 = 1 \times 2$ intersections



$3 = 1 \times 3$ intersections



$6 = 2 \times 3$ intersections

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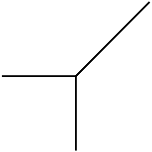

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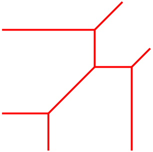

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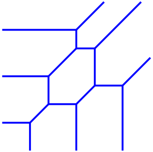

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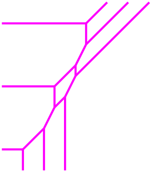

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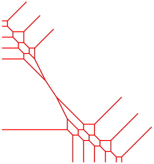

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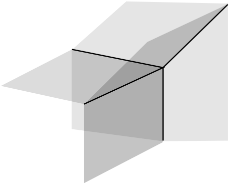

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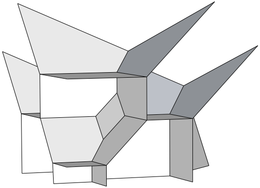
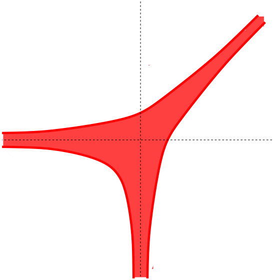


A spatial quadric

# Amoeba of the line $L$ defined by $z - w + 1 = 0$



$Log(L)$

$$Log : \begin{array}{ccc} (\mathbb{C}^*)^2 & \longrightarrow & \mathbb{R}^2 \\ (z, w) & \longmapsto & (\log(|z|), \log(|w|)) \end{array}$$

# Amoeba of the line $L$ defined by $z - w + 1 = 0$

$Log(L)$ $\qquad\qquad$ $Log_t(L)$

$$
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 & (z, w) & \longmapsto & \left(\frac{\log(|z|)}{\log t}, \frac{\log(|w|)}{\log t}\right)
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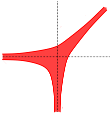

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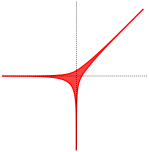

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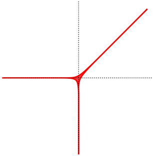

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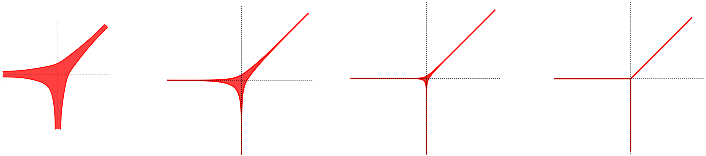

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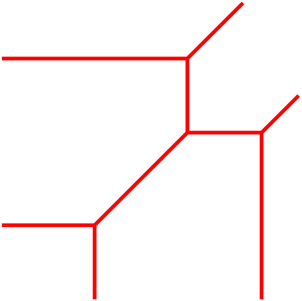
## Amoeba of the conic $C_t$ defined by

$$-1 + z + w - t^{-2}z^2 + t^{-1}zw - t^{-2}w^2 = 0$$

$Log_t(C_t)$

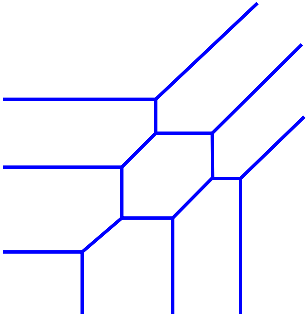


$\lim_{t\to\infty} Log_t(C_t)$

# Amoeba of a cubic $C_t$ defined by

$$-1 + z + w - t^{-2}z^2 + t^{-1}zw - t^{-2}y^2 + t^{-8}z^3 + t^{-5}z^2w + t^{-5}zw^2 + t^{-8}w^3 = 0$$

$Log_t(C_t)$



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# Amoebas and tropical curves

**Theorem (Mikhalkin, Rüllgard)**

$$\{\textit{limits of amoebas of families of algebraic curves in } (\mathbb{C}^*)^2\}$$

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Mikhalkin: There exists a genus one tropical cubic curve in $\mathbb{R}^3$ which is not tropically planar.

# Topology of tropical curves

## Problem

*What is the maximal value of the first Betti number (genus) of a tropical curve of degree $d$ in $\mathbb{R}^n$?*

# Topology of tropical curves

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**Theorem (Mikhalkin-Sturmfels, ∼2000)**

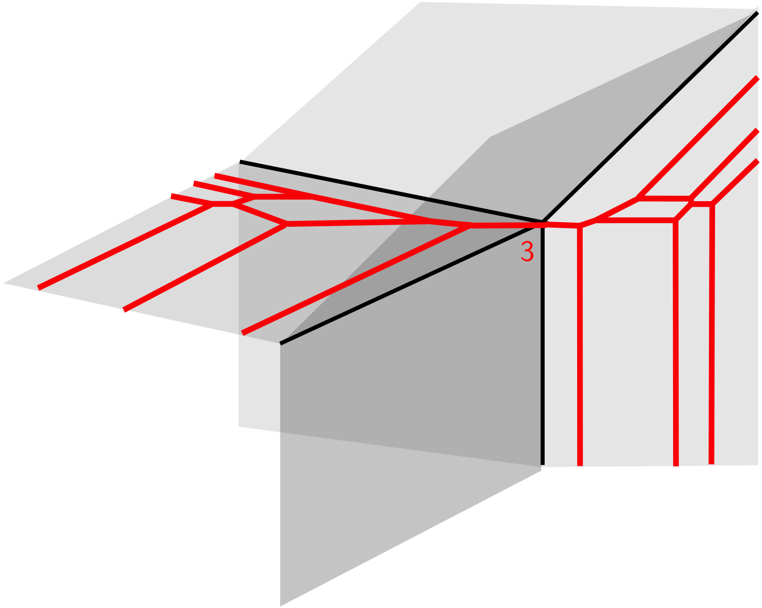
*The maximal genus of a tropical curve of degree $d$ in $\mathbb{R}^2$ is*

$$\frac{(d-1)(d-2)}{2}.$$

**Theorem (Yu Yue, 2014)**

*The genus of a tropical curve of degree $d$ in $\mathbb{R}^n$ is bounded from above by a constant $C(d, n)$.*

# A cubic of genus 2



# Topology of tropical curves

### Theorem (Bertrand-B-López de Medrano)

*There exists a tropical plane $L \subset \mathbb{R}^n$ such that for any $d \geq 1$, $L$ contains a tropical curve $C \subset L$ of degree $d$ with*

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**Theorem (Bertrand-B-López de Medrano)**

*A tropical curve of degree $d$ contained in a tropical plane in $\mathbb{R}^3$ has genus at most $(d-1)(d-2)$.*

# Higher dimensions

Theorem (Bieri-Groves, Mikhalkin, Rüllgard, ...)

$$\left\{ \begin{array}{c} \textit{limits of amoebas of families of algebraic varieties of} \\ \textit{dimension } m \textit{ in } (\mathbb{C}^*)^n \end{array} \right\}$$

$$\cap$$

$$\{\textit{tropical varieties of dimension } m \textit{ in } \mathbb{R}^n\}$$

*Moreover in the case of hypersurfaces ($m = n - 1$), we have equality of the two sets.*

# Higher dimensions

## Theorem (Mikhalkin-Sturmfels)

*A tropical hypersurface of degree $d$ in $\mathbb{R}^n$ satisfies*

$$b_1(X) = \cdots = b_{n-2}(X) = 0, \quad b_{n-1}(X) \leq \binom{d-1}{n}$$

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## Theorem (Bertrand-B-López de Medrano)

1. *$b_i(X)$ for a tropical variety $X$ of dimension $m$ and degree $d$ in $\mathbb{R}^{m+k}$ is bounded from above by a constant $C(m, d, k)$.*
2. *There exists a tropical linear space of dimension $m+1$ in $\mathbb{R}^{m+k}$ such that for any degree $d \geq 1$, there exists a tropical variety $X \subset L$ of dimension $m$ and degree $d$ such that*

$$b_m(X) \geq k \cdot \binom{d-1}{m+1}$$

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Theorem (Mikhalkin-Sturmfels)

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## Theorem (Bertrand-B-López de Medrano)

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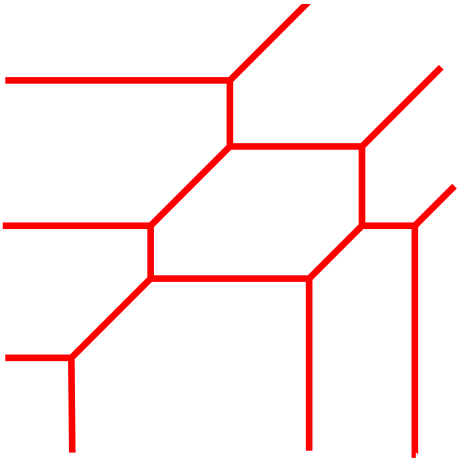
2. *There exists a tropical linear space of dimension $m+1$ in $\mathbb{R}^{m+k}$ such that for any degree $d \geq 1$, there exists a tropical variety $X \subset L$ of dimension $m$ and degree $d$ such that*
$$b_m(X) \geq k \cdot \binom{d-1}{m+1}$$
3. *Analogous statement for all tropical Hodge numbers in the case when $m = 2$.*

# Proofs

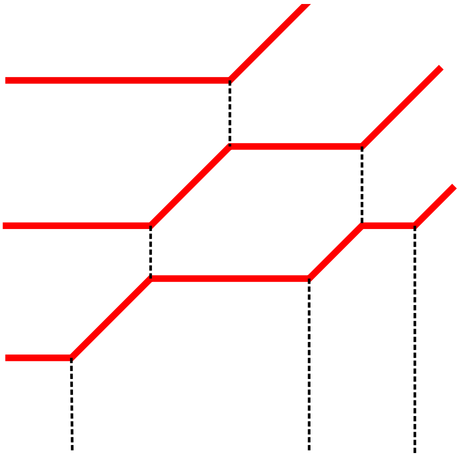
- Upper bounds : projections + tropical intersection theory (Allermann-Rau, Mikhalkin, Shaw)

- Constructions: floor composition

# Floor composition



# Floor composition



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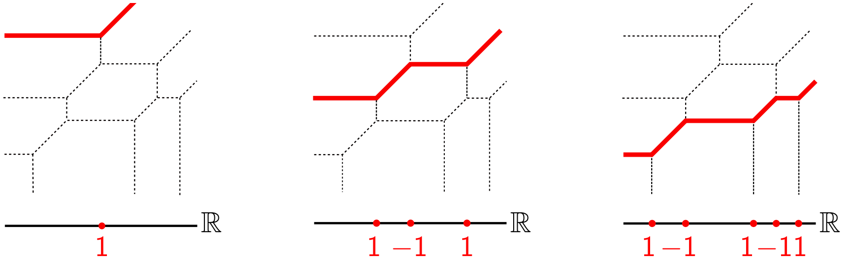

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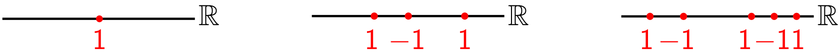

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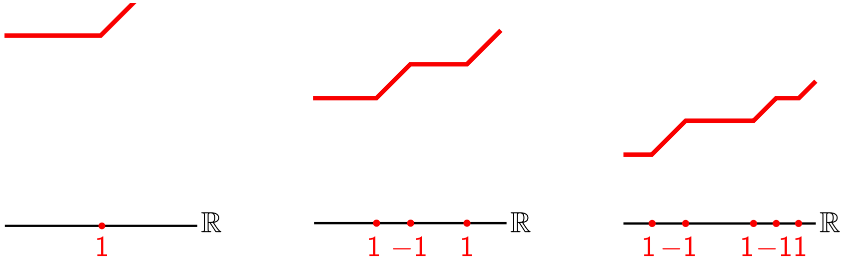

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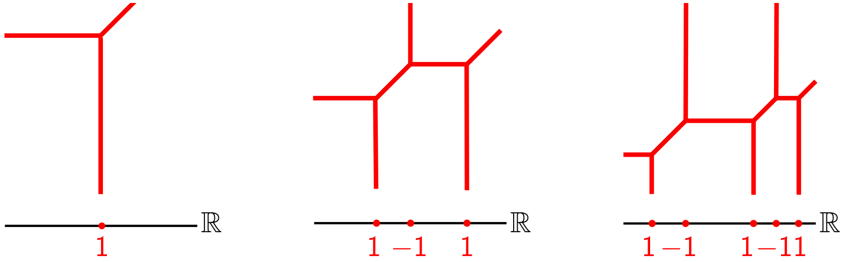

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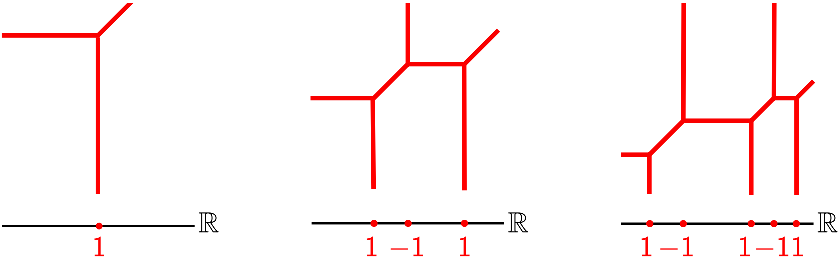


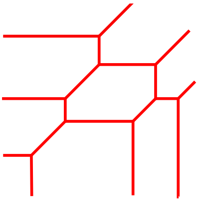

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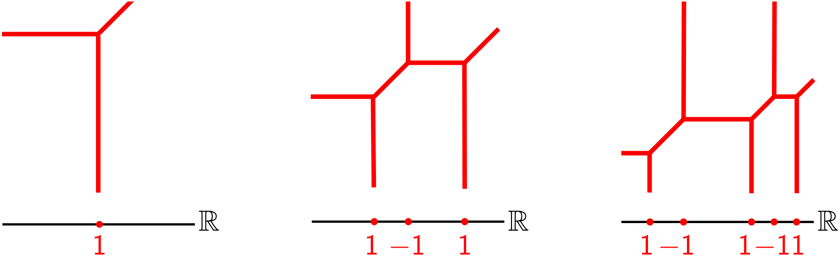


## Proposition

*Let $L$ be a tropical linear space in $\mathbb{R}^n$. Then any effective tropical divisor of degree $d$ is the divisor of a tropical rational function $f : L \to \mathbb{R}$ of degree $d$.*

# Problems

- Find sharp upper bounds on Betti numbers, or more generally on tropical Hodge numbers of tropical varieties.

- What about non-singular tropicall varieties?

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**Proposition (Bertrand-B-López de Medrano)**

*The genus of a tropical cubic curve in $\mathbb{R}^n$ is at most $(n-1)$.*

- What about non-singular tropicall varieties?

**Proposition (Bertrand-B-López de Medrano)**

*There exists a non-singular tropical curve of degree $d$ in $\mathbb{R}^3$ with genus $(d-1)(d-2)$.*