Bridges between embedded graphs and the geometry of surfaces

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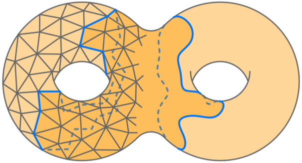


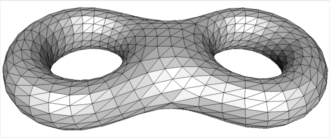
Based on joint works with E. Chambers, G. Chambers, É. Colin de Verdière, A. Hubard, F. Lazarus, T. Ophelders and R. Rotman.

Embedded graphs and surfaces

In this talk, we care about conneected, compact, orientable ***surfaces***, which are classified by their ***genus*** ($\approx$ number of holes).

Embedded graphs

A graph G is ***embedded*** on a surface S if it can be drawn without crossings on S.



It is ***triangulated*** if all the faces have degree 3.

Two (among other) reasons to care about embedded graphs :

- They appear in practice (road networks, computer graphics, CAD...)

- Every graph is embeddable on some surface.
  $\rightarrow$ Very fruitful point of view in graph theory, for example crucial for ***graph minor theory***.

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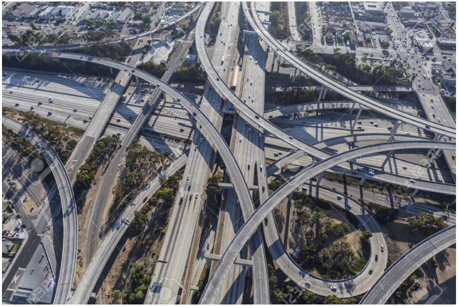

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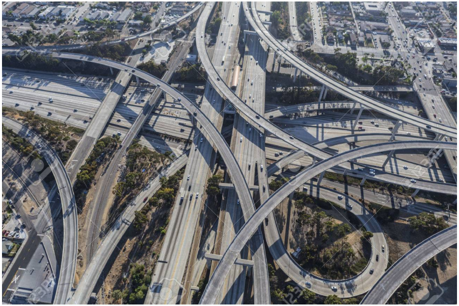

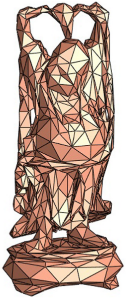

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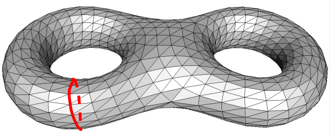
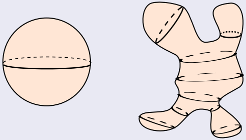
A geometric point of view

- An embedded graph provides a ***discrete metric*** to measure the length of some curves.
- We obtain a ***continuous metric*** by embedding the surface in $\mathbb{R}^3$ and measuring the lengths there.

Discrete metric



Continuous metric

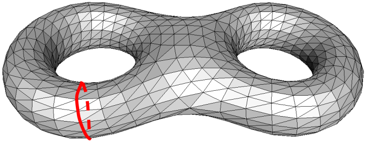


Intrinsic point of view $\Rightarrow$ Riemannian metric.

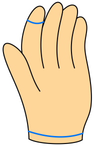
Goal of this talk: Highlight strong interactions between the study of embedded graphs and continuous metrics on surfaces.

Plan

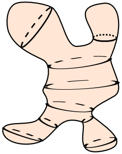
1. Shortest curves : systoles and *edge-width*.



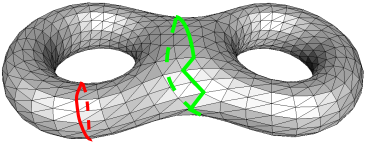
2. Homotopy height and a variant of planar graph searching.



3. Sweep-outs and branch decompositions.

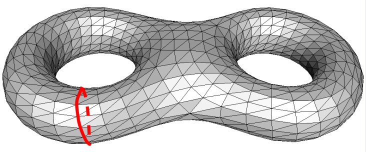


First part: Shortest curves: systoles and edge-width



Shortest non-contractible curves

Discrete setting



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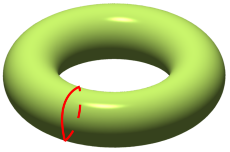
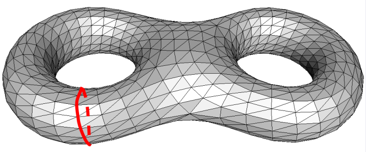


Upper bound on the length of the shortest non-contractible curve ?

Shortest non-contractible curves

Discrete setting



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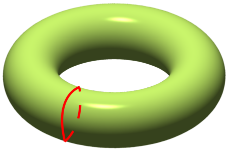
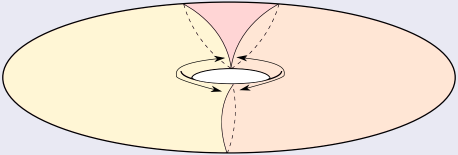


Upper bound on the length of the shortest non-contractible curve ?

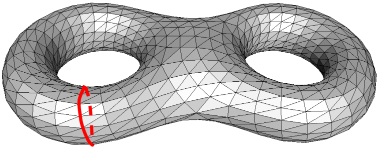
Intuition



It should have length $O(\sqrt{A})$ or $O(\sqrt{n})$, but how does the $O()$ depend on g ?

Discrete setting: topological graph theory

The ***edge-width*** of an embedded graph is the length of the shortest ***non contractible*** cycle.



Theorem (Hutchinson '88)

The edge-width of a triangulated graph with n triangles on a genus g surface is $O(\sqrt{n/g}\log g)$.

- Hutchinson conjectured that the correct bound is $\Theta(\sqrt{n/g})$.
- Disproved by Przytycka et Przytycki '90-97 who obtained lower bounds in $\Omega(\sqrt{n/g}\sqrt{\log g})$, and conjectured $\Theta(\sqrt{n/g}\log g)$.
- What about non-separating curves, or non-contractible but homologically trivial?

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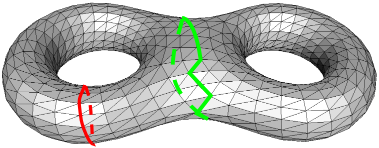

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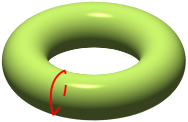
Systolic geometry

The ***systole*** of a Riemannian surface is the length of the shortest ***noncontractible*** cycle.



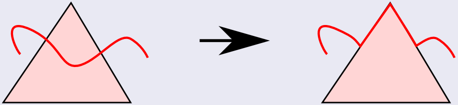
Theorem (Gromov '83, Katz and Sabourau '04)

The systole of a Riemannian surface of genus g and area A is $O(\sqrt{A/g}\log g)$.

- Known variants for non-separating curves and homologically trivial non-contractible [Sabourau '08].
- Buser and Sarnak '94 used ***arithmetic surfaces*** to obtain a matching lower bound: $\Omega(\sqrt{A/g}\log g)$.
- Larry Guth: "Arithmetic hyperbolic surfaces are remarkably hard to picture."

How to go from a discrete metric to a continuous one?

Proof.

- Paste equilateral triangles of area 1 on the triangles.
- Smooth the metric.
- 
- In the worst case, lengths double.

□

Theorem (Colin de Verdière, Hubard, de Mesmay '14)

Let (S, G) be a triangulated surface of genus g, with n triangles. There exists a Riemannian metric m on S with area n such that for every closed curve γ in (S, m) there exists a homotopic closed curve γ' on (S, G) with

$$|\gamma'|_G \leq (1+\delta)\sqrt[4]{3}\,|\gamma|_m$$

for some arbitrarily small δ.

From discrete to continuous

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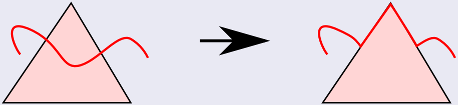
- 
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Corollary

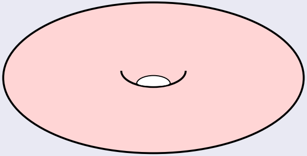
Let (S, G) be a triangulated surface of genus g with n triangles, then there exists a non-contractible/non-separating cycle of length $O(\sqrt{n/g}\log g)$.

Thus Gromov $\Rightarrow$ Hutchinson and we obtain the other variants and improved constants.

From continuous to discrete

How do we switch from a continuous to a discrete metric ?

Proof.



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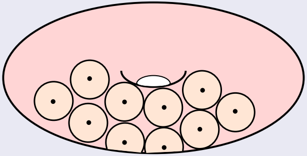
How do we switch from a continuous to a discrete metric ?

Proof.

Take a maximal set of balls of radius ε and perturb them a little.



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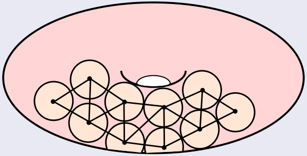

By [Dyer, Zhang and Möller '08], the Delaunay graph is a triangulation for ε small enough.

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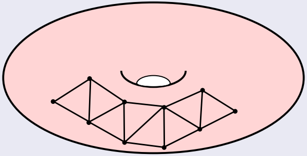
$\Rightarrow$ Delaunay triangulation T



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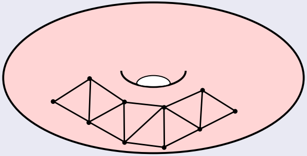

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$$|\gamma|_m \leq 4\varepsilon|\gamma|_G.$$

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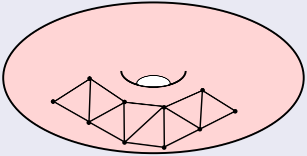

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Each ball has radius $\pi\varepsilon^2 + o(\varepsilon^2)$, and thus $\varepsilon = O(\sqrt{A/n})$.

□

Theorem and Corollaries

Theorem (Colin de Verdière, Hubard, de Mesmay '14)

Let (S, m) be a Riemannian surface of genus g and area A. There exists a triangulated graph G embedded on S with n triangles, such that every closed curve γ in (S, G) satisfies

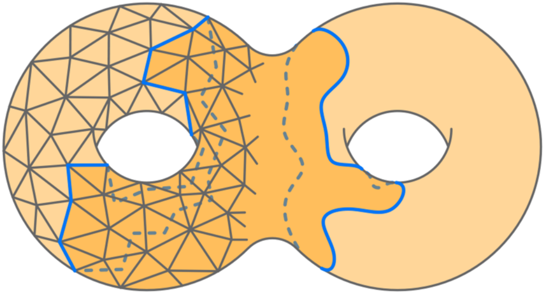
$|\gamma|_m \leq (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{A/n}\ |\gamma|_G$ *for some arbitrarily small δ.*

- This shows that Hutchinson $\Rightarrow$ Gromov.
- Proof of the conjecture of Przytycka and Przytycki:

Corollary

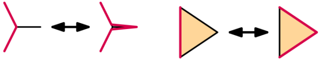
There exist arbitrarily large g and n such that the following holds: There exists a triangulated combinatorial surface of genus g, with n triangles, of edgewidth at least $\frac{1-\delta}{6}\sqrt{n/g}\log g$ for arbitrarily small δ.

Second part: Graph searching and homotopies

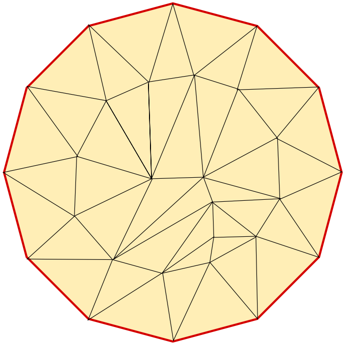


A planar graph searching problem

- Cops are holding hands and want to catch a fugitive on a planar graph.
  → Authorized moves: sequence of ***spikes*** and ***flips***.



- How many cops (= length of the curve) are needed?



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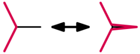

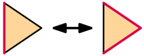

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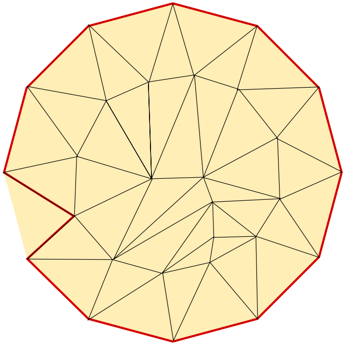

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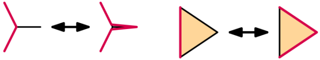

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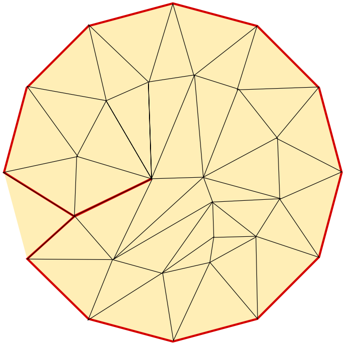

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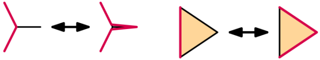

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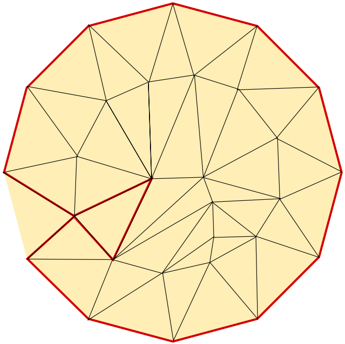

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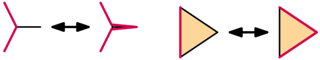

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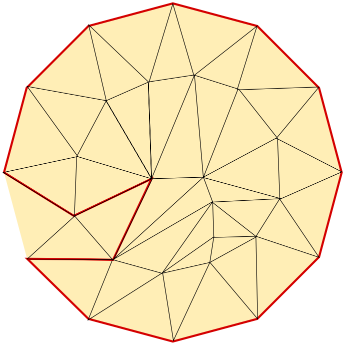

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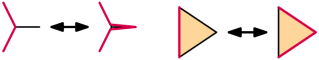

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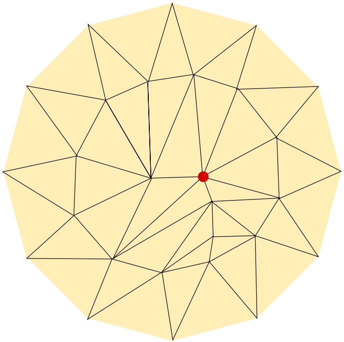

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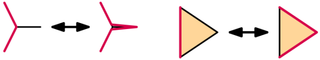
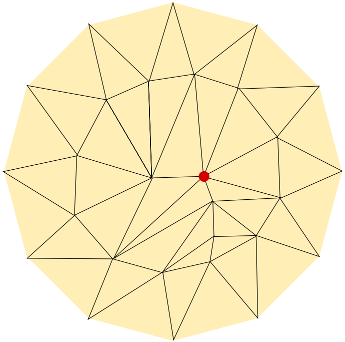
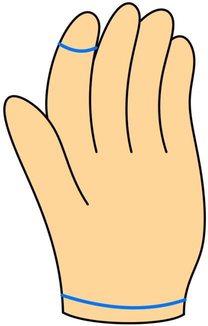


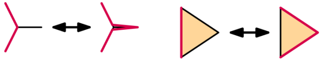
- How many cops (= length of the curve) are needed?
- Alternatively, can I slide a rubber band of fixed maximum length around my wrist?





Homotopy height

- A ***discrete homotopy*** is a sequence of cycles linked by spikes or flips.

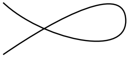


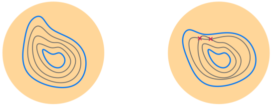
- An ***optimal homotopy*** is a homotopy minimizing the maximum length of intermediary curves (= the ***homotopy height***).

How can on compute an optimal homotopy?

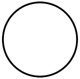
Questions (E.Chambers-Letscher '09)

- Does there exist an optimal homotopy where intermediate cycles do not self-intersect ? (***isotopy***)
- Does there exists an optimal homotopy where pairs of intermediate cycles do not intersect ? (***monotonicity***)





Continuous homotopy: Continuous map h between two curves.



Theorem ([G. Chambers, Liokumovich '14])

Let D be a Riemannian disk, with boundary γ. If there exists a homotopy of height L of γ towards a point, there exists an isotopy of height $L + \varepsilon$ of γ twoards a point, for every $\varepsilon > 0$.

- The proof works verbatim in the discrete case.
- The ε comes from small perturbations which are not necessary in the discrete case.

The very elegant proof analyzes a graph of ***resolutions*** of the intermediate curves.

Monotonicity?

Theorem ([G. Chambers, Rotman '14])

Let D be a Riemannian disk, of boundary γ. If there exists a homotopy of height L from γ towards a point, there exists a monotone isotopy of height $L + \varepsilon$ from γ towards a point, for every $\varepsilon > 0$.

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Theorem (~~[G. Chambers, Rotman '14]~~[E. Chambers, G. Chambers, de Mesmay, Ophelders, Rotman '18])

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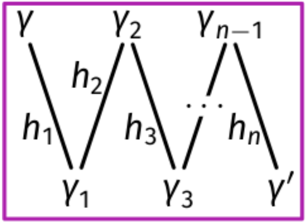

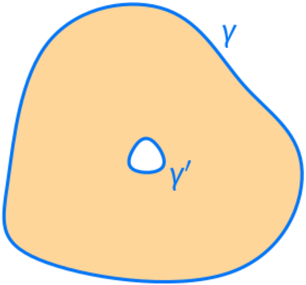

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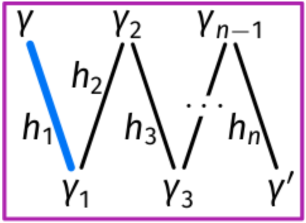

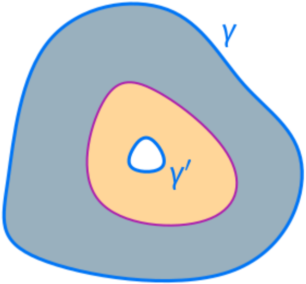

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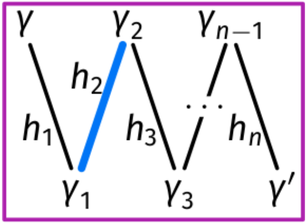

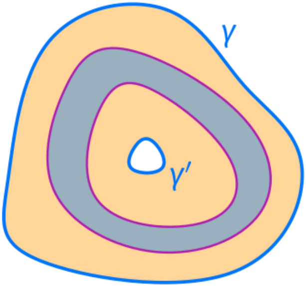

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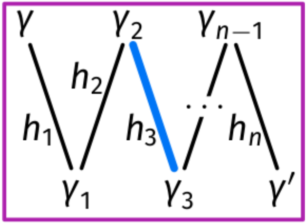

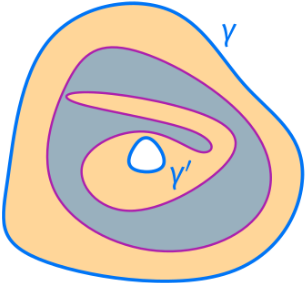

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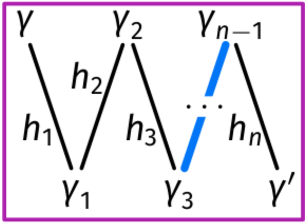

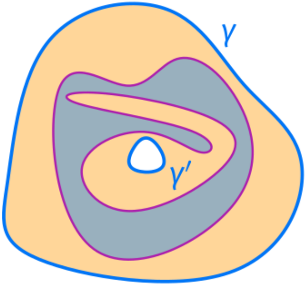

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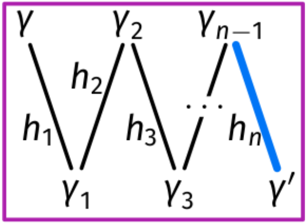

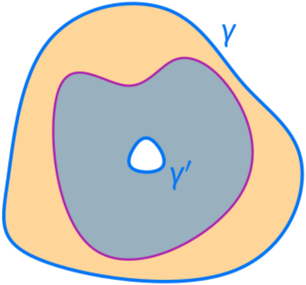

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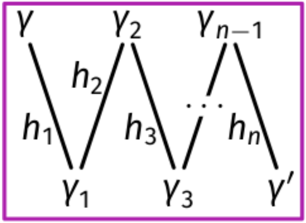

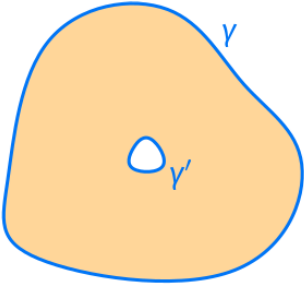

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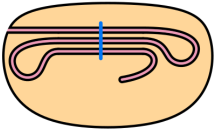

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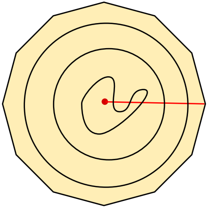
There exists h an optimal monotone contraction of a cycle γ towards a point p, such that each intermediate curve $h(t)$ cuts the shortest path between γ and p exactly once.



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Algorithmic applications

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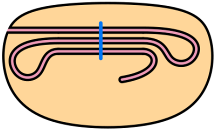

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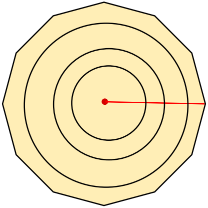

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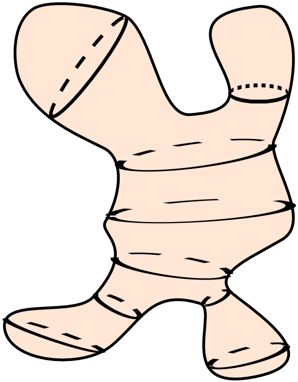
Third part:
Geodesics, sweep-outs and graph decompositions



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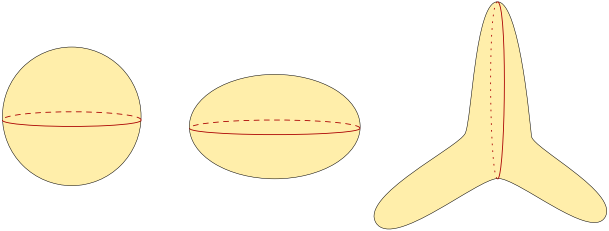

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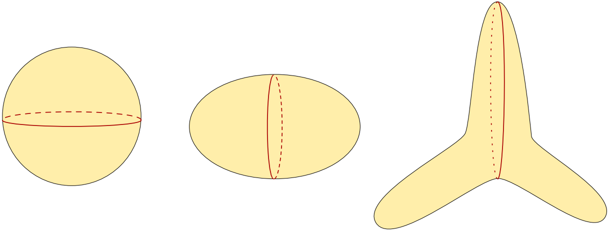

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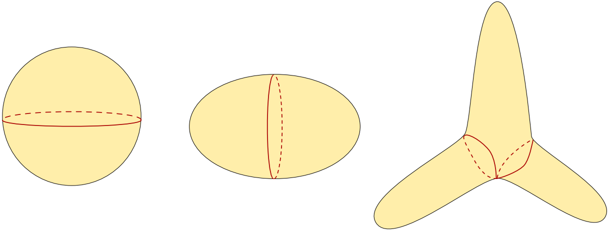

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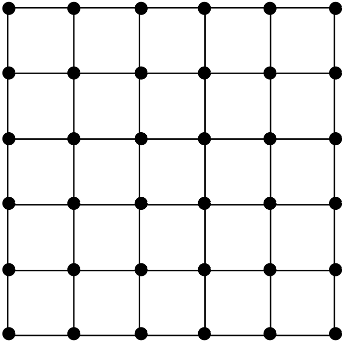
Let G be a triangulated graph with n vertices, then there exists a cycle with at most $2\sqrt{2}\sqrt{n}$ vertices such the the inside and the outside of the cycle contain each at most $2n/3$ vertices.



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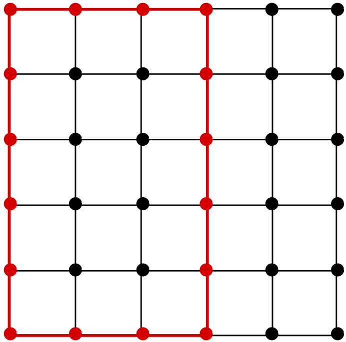

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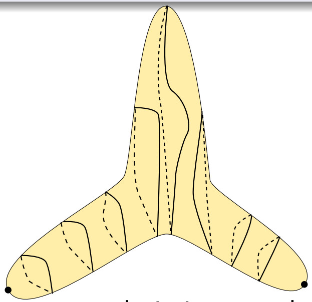
Finding geodesics

How to find a geodesic ? ([Birkhoff '17])

1. Linearly sweep the sphere with curves.
2. Tighten all the curves.
3. Look at the "middle" one.



The length of the shortest geodesic is upper bounded by the ***waist*** of the best sweep-out:

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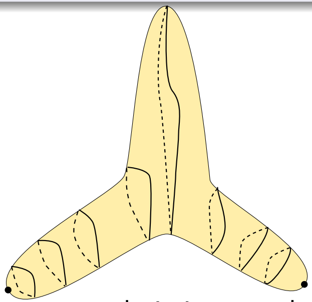

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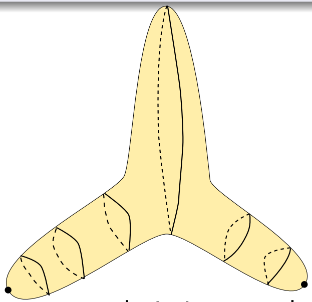

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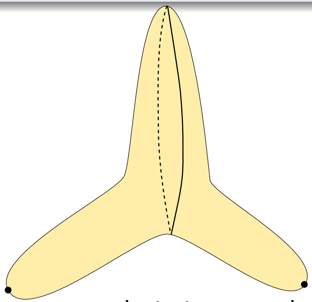

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How to find a geodesic ? ([Calabi-Cao '92]) (sketchy)

1. Sweep the sphere ***in a tree-like fashion*** with curves.
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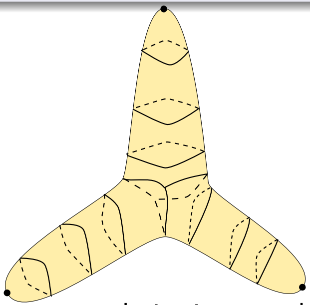

The length of the shortest geodesics is upper bounded by twice the ***waist*** of the best tree-like sweep-out:

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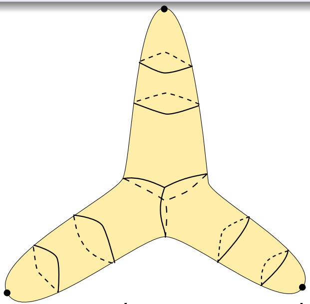

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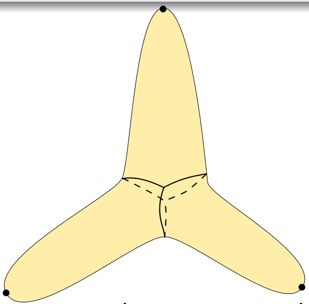

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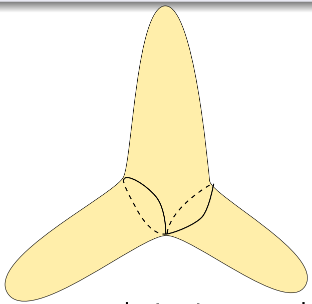

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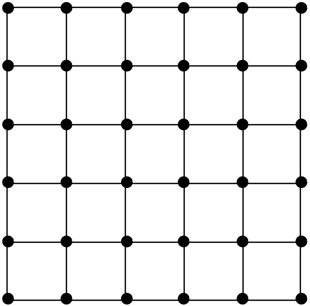
From planar separators to branch-decompositions

- Replace the graph by its radial graph.
- Find separators recursively on both sides.
- This induces a ***branch decomposition*** of the graph.



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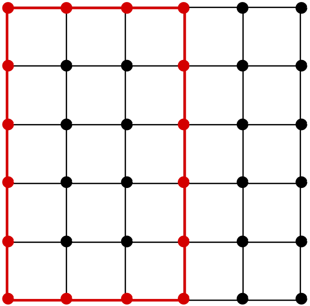

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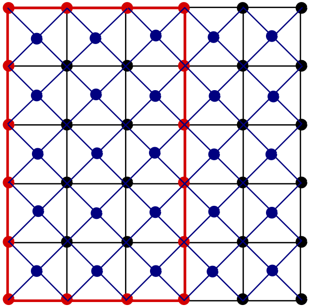

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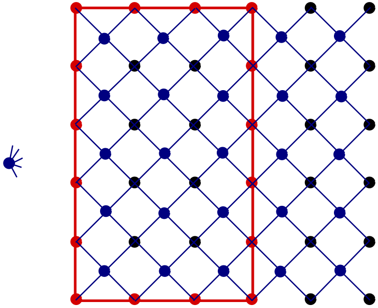

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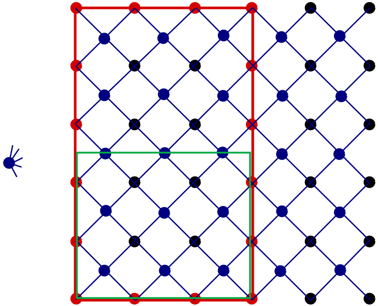

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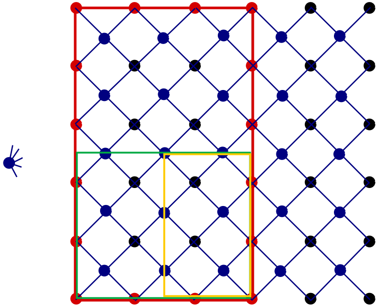

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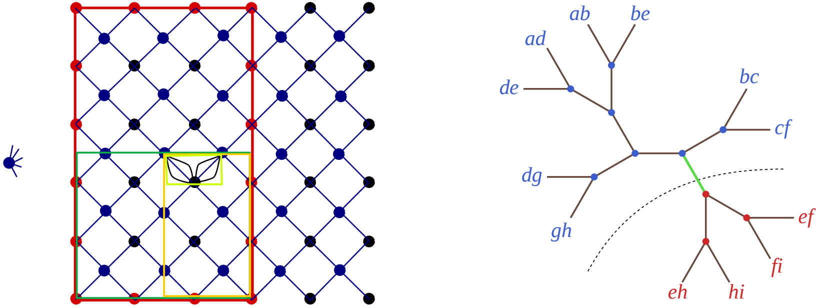

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Harvesting the fruits of this analogy

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Harvesting the fruits of this analogy

- A strong analogy between ***tree-like sweep-outs*** of spheres and ***branch decompositions*** of planar graphs . . .
- . . . than we can exploit.

Theorem (Alon-Seymour-Thomas '94, Fomin-Thilikos '06)

Let G be a planar graph with n vertices, then

- *There exists a cycle with at most $3/2\sqrt{2}\sqrt{n}$ vertices such that the inside and the outside of the cycle contain each at most $2n/3$ vertices,*
- *G has branchwidth at most $3/2\sqrt{2}\sqrt{n}$.*

Improved bounds to sweep spheres

Theorem (Hubard, de Mesmay, Lazarus ['19?])

Let S be a Riemannian sphere of area A.

- *The branchwaist of S satisfies :*

$$branchwaist(S) := \inf_{f:S\to T, T\in\mathcal{T}} \sup_{t\in E(T)} ||f^{-1}(t)|| \leq \sqrt{2\pi}\sqrt{A}$$

- *There exists a closed geodesic of length at most $2\sqrt{2\pi A}$.*

For comparison:

- On the usual round sphere, $A = 4\pi$, $|\gamma| = 2\pi$ and thus $|\gamma| = \sqrt{\pi A}$.
- It is conjectured that the sphere with the longest shortest geodesic is obtained by pasting two equilateral triangles.

Branchwidth of planar graphs can be computed in ***polynomial*** time.

Theorem (Seymour-Thomas '94, relying on Graph Minors XI)

Let G be a planar graph, G has branchwidth at least k if and only if there exists an antipodality of range k.

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Let G be a planar graph, an ***antipodality*** of range k is a map α sending,

- each edge $e \in E(G)$ to a subgraph $\alpha(e)$ in G,
- each face $f \in F(G)$ to a subset $\alpha(f)$ of $V(G)$,

such that

1. For $e \in E(G)$, no endpoint of e belongs to $V(\alpha(e))$,
2. If $e \in E(G), f \in F(G)$ and e is incident to f, then $\alpha(f) \subseteq V(\alpha(e))$ and each component of $\alpha(e)$ has a vertex in $\alpha(f)$,
3. If $e \in E(G), f \in E(\alpha(e))$ then each walk of G^* using e^* and f^* has length at least k.

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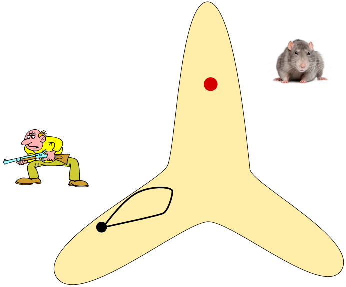
Branchwidth of planar graphs can be computed in ***polynomial*** time.

Theorem (Seymour-Thomas '94, relying on Graph Minors XI)

Let G be a planar graph, G has branchwidth at least k if and only if there exists an antipodality of range k.

An ***antipodality*** of size k is a strategy allowing a rat to escape a ratcatcher having arms of length k.



Continuous version

A (continuous) ***antipodality*** of range k is a continuous mapping $a : S \to S$ such that $x \in S$,

$$d(x, a(x)) \geq k/2.$$

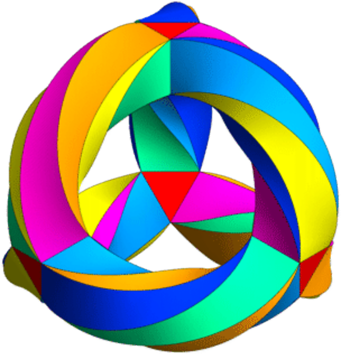
Theorem (Hubard, de Mesmay, Lazarus '19?)

Let S be a Riemannian sphere, then S has branchwaist at least k if and only if there exists an antipodality of range at least $k - \varepsilon$ for any $\varepsilon > 0$, i.e.,

$$\inf_{f:S \to T, t \in T} \sup_{t \in E(T)} ||f^{-1}(t)|| = \sup_{f:S \to S} \inf_{x \in S} 2d(x, a(x))$$

Related to results of Berger (1980) and Gromov (1983).

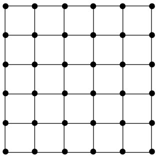
- Natural discretizations of arithmetic surfaces ?



Animation by Greg Egan

A few perspectives II

- Geometric interperpretation for the ***treewidth*** of planar graphs?
- Geometric interpretation of the branchwidth of ***surface-embedded*** graphs?
  $\Rightarrow$ Polynomial-time algorithms?
- More precise connections with ***Finsler*** geometry?



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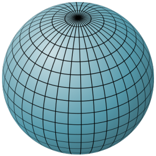

A few perspectives II

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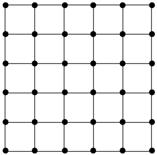

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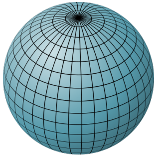

Thank you for your attention!