Hypersimplicial Subdivisions

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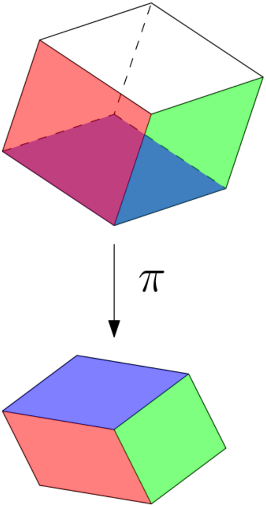
Induced subdivisions

Definition

Let $\pi : P \to Q$ a linear surjective projection between two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^d$. A *π-induced subdivision* S of Q is a polyhedral subdivision such that for every cell $\sigma \in S$ there is a face F of P such that $\pi(F) = \sigma$.



Let A be the image under π of the standard basis.

- When $P = \Delta_n$ is the standard simplex, π-induced subdivisions are just all subdivision on A.
- When $P = [0,1]^n$ is the unit cube, π-induced subdivisions are zonotopal tilings of the zonotope $Z(A)$.
- What if P is a hypersimplex?

Hypersimplicial subdivisions

Let $\Delta_n^{(k)} := [0,1]^n \cap \left\{ \sum_{i=1}^{n} x_i = k \right\}$ be a hypersimplex and let $A^{(k)}$ be the image of the vertices of $\Delta_n^{(k)}$ under π.

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A *hypersimplicial subdivision* of $A^{(k)}$ is a π-induced subdivision for $\pi : \Delta_n^{(k)} \to \operatorname{conv} A^{(k)}$.

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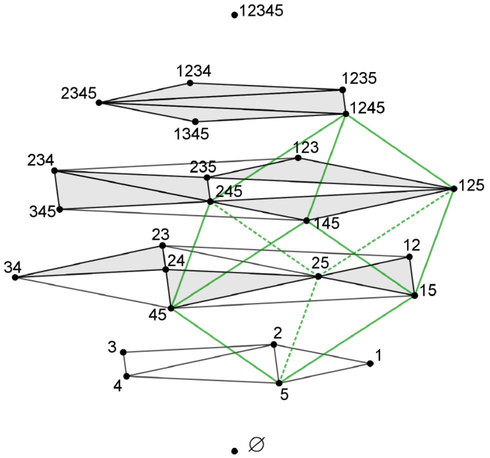
The main motivation to study them is that when A is the set of vertices of a convex polygon, hypersimplicial subdivisions are in bijection with plabic graphs (Galashin 2018).

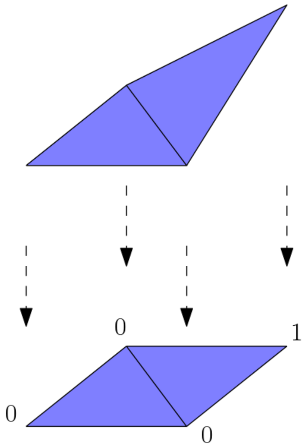
Relationship with zonotopal tilings



Picture taken from *Flip cycles in plabic graphs* by Alexey Balitskiy and Julian Wellman, redrawn from *Plabic graphs and to zonotopal tilings* by Galashin.

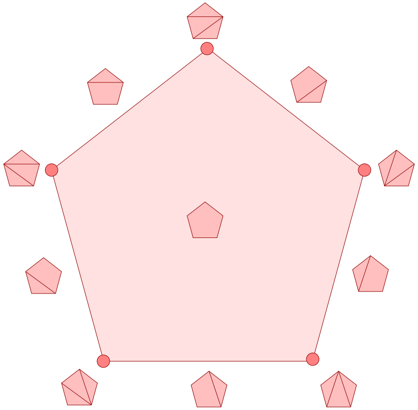
Regular subdivisions

Given a height function $h : A \to \mathbb{R}$, the lower faces of $\text{conv}(\{(a, h(v)) \in \mathbb{R}^{n+1} \mid a \in A\})$ project onto $\text{conv}(A)$ to form a polyhedral subdivision $\text{Sub}_{\text{h}}(\text{A})$. Such subdivisions are called *regular*. This procedure partitions $\mathbb{R}^n$in a fan called the *secondary fan* of A, where two vectors are in the same (relatively open) cone if and only if they produce the same subdivision. This fan is the normal fan of a polytope $\mathcal{F}(A)$ called the secondary polytope.



Example: the associahedron.

If A is the set of vertices of a convex polygon, the secondary polytope is called the associahedron.



Picture taken from the book *Triangulations: Structures for Algorithms and Applications* by De Loera, Rambau and Santos

Coherent subdivisions and fiber polytopes

Coherent subdivisions generalize regular subdivisions.

Definition

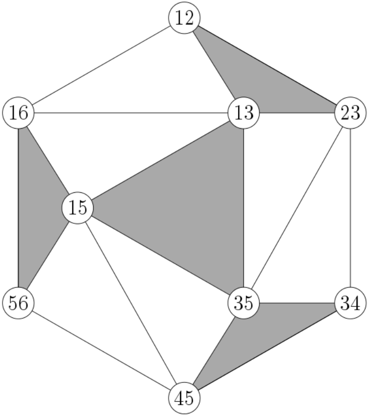
Consider a polytope $P \subset \mathbb{R}^n$ and a projection $\pi : P \to Q$. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a linear function. For each point $q \in Q$, the fiber $f^{-1}(q)$ is a polytope inside P. The function w is minimized in some face F_q of P. The *π-coherent subdivision* given by w consists of $\{\pi(F_q) \mid q \in Q\}$.

The equivalence clases of $\mathbb{R}^n$ according to which π-coherent subdivisions they produce are the cones of the normal fan of a polytope $\mathcal{F}(P \xrightarrow{\pi} Q)$ called the *fiber polytope*.

- When $P = \Delta_n$, the fiber polytope $\mathcal{F}(\Delta_n \xrightarrow{\pi} \mathrm{conv}(A))$ is the secondary polytope of A.
- When $P = [0,1]^n$, the fiber polytope $\mathcal{F}([0,1]^n \xrightarrow{\pi} Z(A))$ is called the secondary zonotope of $Z(A)$.
- When $P = \Delta_n^{(k)}$, we call the fiber polytope $\mathcal{F}(\Delta_n^{(k)} \xrightarrow{\pi} \mathrm{conv}(A^{(k)}))$ the *hypersecondary polytope*.

Example: non coherent subdivision.

Not all hypersimplicial subdivisions are coherent:



Hyperasecondary polytopes

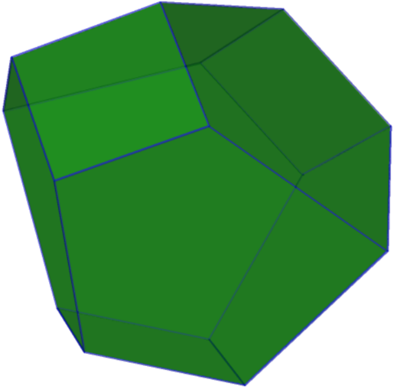
Recall the following:

- The *Minkowski sum* of $A, B \subset \mathbb{R}^n$ is $A + B := \{a + b \mid a \in A \;\; b \in B\}$.
- Two polytopes P and P' are said to be *normally equivalent* if their normal fans are the same.

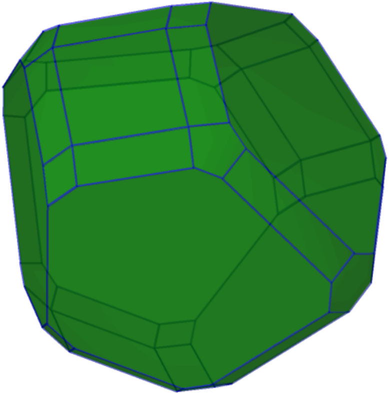
Theorem (O.-Santos)

Let $A \subseteq \mathbb{R}^d$ be a configuration of size n and $1 \leq k \leq d+1$. Let $s = \max(n-k+1, d+2)$. The hypersecondary polytope $\mathcal{F}(\Delta_n^{(k)} \xrightarrow{\pi} A^{(k)})$ is normally equivalent to the Minkowski sum of the secondary polytopes of all subsets of A of size s.

Example: the hyperassociahedron.



The associahedron $\mathcal{F}(\Delta_6 \xrightarrow{\pi} P_6)$.



The second hyperassociahedron $\mathcal{F}(\Delta_6^{(2)} \xrightarrow{\pi} P_6^{(2)})$.

Baues poset

Non trivial π-induced subdivisions form a poset, where the order is given by refinement. This is called the *Baues poset* $\mathcal{B}(P \xrightarrow{\pi} Q)$.

Given a poset $\mathcal{P}$, the *chain complex* $C(\mathcal{P})$ of $\mathcal{P}$ is a simplicial complex where the vertices of $C(\mathcal{P})$ are the elements of $\mathcal{P}$ and the simplices are given by chains of $\mathcal{P}$. The topology of $\mathcal{P}$ is the topology of $C(\mathcal{P})$.

Example

Consider the subposet of $\mathcal{B}(P \xrightarrow{\pi} Q)$ consisting of the **coherent** subdivisions. The chain complex of this poset is the baricentric subdivision of $\mathcal{F}(P \xrightarrow{\pi} Q)$. In particular it has the topology of a sphere.

Generalized Baues Problem

Problem

For which $\pi : P \to Q$ does the Baues poset $\mathcal{B}(P \xrightarrow{\pi} Q)$ retract onto the poset of coherent subdivisions?

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- OPEN: What about $\Delta_n^{(k)} \to A^{(k)}$ where A is the set of vertices of any cyclic polytope? (Postnikov 2018).

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Theorem (O.-Santos 2019+)

Let A be the vertices of a convex polygon. Then the Baues poset $\mathcal{B}(\Delta_n^{(k)} \xrightarrow{\pi} A^{(k)})$ retracts onto the poset of coherent subdivisions. In particular, it has the homotopy of an $n-4$-sphere.

Merci beaucoup!

Hypersimplisical subdivisions, O.-Santos arXiv:1906.05764