Semi-Inverted Linear Spaces

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We denote $\mathrm{inv}_I(\mathcal{L})$ the Zariski closure of the image of $\mathcal{L}$ under this map. We obtain an algebraic variety.

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\begin{aligned}
f_C(\mathbf{x}) &= \mathbf{x}^{C \cap I} \cdot \ell_C(\mathrm{inv}_I(\mathbf{x})) \\
&= \sum_{i \in C \cap I} a_i \mathbf{x}^{C \cap I \setminus \{i\}} + \sum_{i \in C \setminus I} a_i \mathbf{x}^{C \cap I \cup \{i\}}.
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Example

$I = \{1, 2, 3\}$

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Then $F \subset \mathcal{I}$ is a universal Gröbner basis for $\mathcal{I}$ if and only if for every $w \in (\mathbb{R}_{\geq 0})^n$, the polynomials $\mathrm{in}_w(F)$ generate $\mathrm{in}_w(\mathcal{I})$

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Broken Circuit Complex

Given a vector $w \in \mathbb{R}^n$ with distinct coordinates, define an ordering on $[n]$, where $i < j$ whenever $w_i < w_j$.

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I-Broken Circuit

For each circuit C of M, define an associated I-broken circuit

$$b_I(C) = \begin{cases} C \setminus \min(C) & \text{if } C \subseteq I \\ (C \cap I) \cup \max(C \setminus I) & \text{if } C \not\subseteq I. \end{cases}$$

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Broken Circuit Complex

Simplicial complex on $[n]$ vertices, whose minimal non-faces are I-broken circuits of M:

$\Delta_w(M, I) = \{\tau \subseteq [n] : \tau \text{ does not contain an } I\text{-broken circuit of } M\}$.

Main Theorem

Recall

Stanley-Reisner ideal of Δ is the square-free monomial ideal

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For $w \in (\mathbb{R}_+)^n$ with distinct coordinates, the initial ideal $\mathrm{in}_w(\mathcal{I})$ is the Stanley-Reisner ideal of the semi-broken circuit complex $\Delta_w(M(\mathcal{L}), I)$.

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[Ref to exercises], the variety of $\langle x_1x_2x_4, x_1x_3x_5, x_2x_3x_5 \rangle$ is the union the seven coordinate linear spaces $\mathrm{span}\{e_i, e_j, e_k\}$ where $\{i,j,k\}$ is a facet of $\Delta_w(M,I)$.

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Simplicial complex on $[n]$ vertices:
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Sketch of proof: Let $\mathcal{J}$ be the homogenization of $\mathcal{I}$ with respect to x_0, the variety of $\mathrm{in}_w(\mathcal{J})$ contains the image in $\mathbb{P}^n$ of both $\{0\} \times \mathrm{inv}_{I\backslash 1}(\mathcal{L}\backslash 1)$ and $\mathbb{A}^1(\mathbb{C}) \times \mathrm{inv}_{I\backslash 1}(\mathcal{L}/1)$.

Recursion on the Degree

$D(\mathcal{L}, I)$ denote the degree of the affine variety $\text{inv}_I(\mathcal{L})$.

Proposition

If $i \in I$ is a loop of $M(\mathcal{L})$, then $\text{inv}_I(\mathcal{L})$ is empty and $D(\mathcal{L}, I) = 0$.

If $i \in I$ is a co-loop of $M(\mathcal{L})$, then $D(\mathcal{L}, I) = D(\mathcal{L}/i, I\backslash i)$.

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Since both these varieties have dimension equal to $\dim(\mathcal{L})$, the degree of the variety of $\text{in}_w(\mathcal{J})$ is at least the sum of their degrees.

Equality of Ideals

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We homogenize and proceed by induction on the size of I. Let Δ_0 be the cone of the broken circuit complex Δ over the vertex 0, Δ_0 has at most $D(\mathcal{L}, I)$ facets.

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Assume, for induction, that $D(\mathcal{L}\backslash i, I\backslash i)$ and $D(\mathcal{L}/i, I\backslash i)$ are the number of facets of $\Delta_w(M\backslash i, I\backslash i)$ and $\Delta_w(M/i, I\backslash i)$, respectively.

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Equality argument

$\mathcal{I}_{\Delta_0} \subseteq in_{0,w}(\mathcal{J}) \subseteq \mathbb{C}[\mathbf{x}]$ are equidimensional homogeneous ideals of dimension d. $\mathcal{I}_{\Delta_0}$ is radical and $\deg(\mathcal{I}_{\Delta_0}) \leq \deg(\mathcal{J}) = D(\mathcal{L}, I)$, therefore $\mathcal{I}_{\Delta_0}$ and $\mathcal{J}$ are equal.

Real Intersections and Hyperplane Arrangements

Proposition

If $\mathcal{L} \subset \mathbb{C}^n$ is invariant under complex conjugation, then for any $u \in \mathbb{R}^n$, all of the intersection points of $\text{inv}_I^-(\mathcal{L})$ with $\mathcal{L}^\perp + u$ are real.

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Proposition

For generic $u \in \mathbb{R}^n$, the intersection points of $\mathrm{inv}_I^-(\mathcal{L})$ with $\mathcal{L}^\perp + u$ are the minima of the function

$$f(x) = \frac{1}{2}\sum_{j \notin I} x_j^2 - \sum_{j \in I} \log|x_j|$$

over the regions in the complement of the (affine) hyperplane arrangement $\{x_i = 0\}_{i \in I}$ in the affine linear space $\mathcal{L}^\perp + u$.

Thank you