The Shapes of Level Curves of Real Polynomials Near Strict Local Minima

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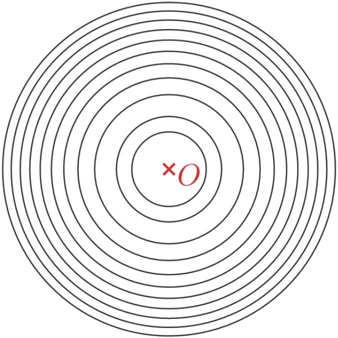
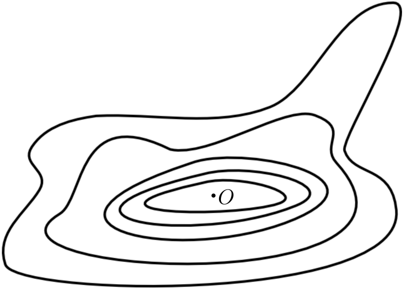
Goals

- **objects** : polynomial functions $f : \mathbb{R}^2 \to \mathbb{R}$, $f(0,0) = 0$ such that O is a strict local minimum ;
- **goal** : study the real Milnor fibres of the polynomial (i.e. the level curves $(f(x,y) = \varepsilon)$, for $0 < \varepsilon \ll 1$, in a small enough neighbourhood of the origin).



$f(x,y) = x^2 + y^2$

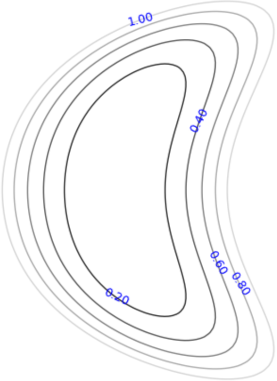


Whenever the origin is a **Morse** strict local minimum the **small enough** level curves are boundaries of **convex** topological disks.

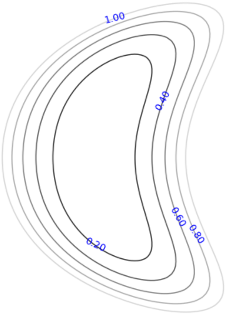
Question (Giroux asked Popescu-Pampu, 2004)

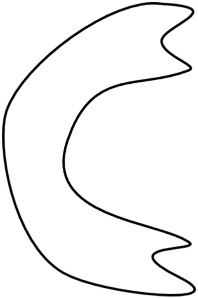
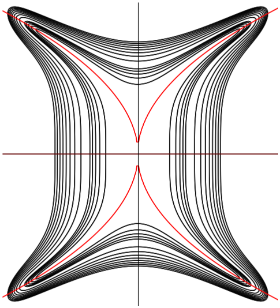
*Are the small enough level curves of f near strict local minima always boundaries of **convex** disks?*

Counterexample by M. Coste : $f(x, y) = x^2 + (y^2 - x)^2$.



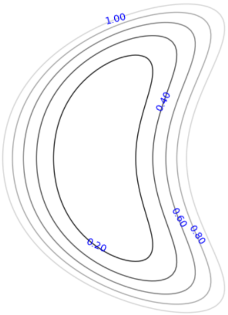
- Problem : understand these phenomena of non-convexity.
- Subproblem : construct non-Morse strict local minima whose nearby small levels are far from being convex.

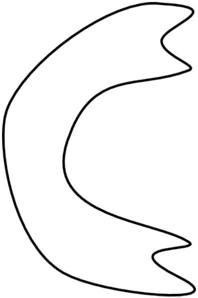


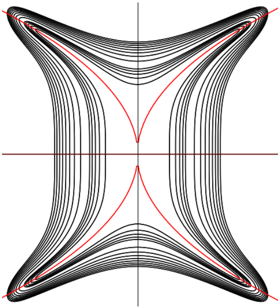



Question

What ***combinatorial object*** *can encode the shape by* ***measuring the non-convexity*** *of a smooth and compact connected component of an algebraic curve in* $\mathbb{R}^2$ *?*





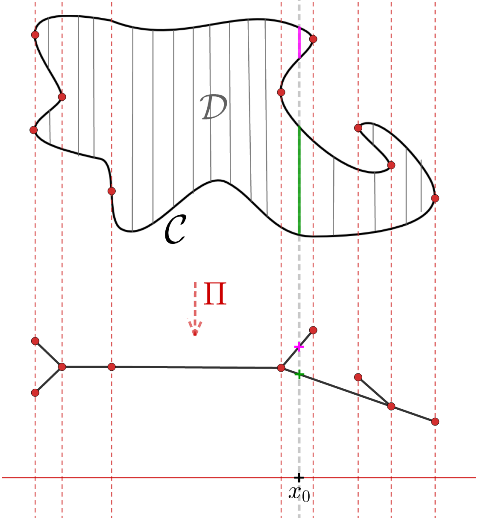


The Poincaré-Reeb graph

associated to a curve and to a direction x

Definition

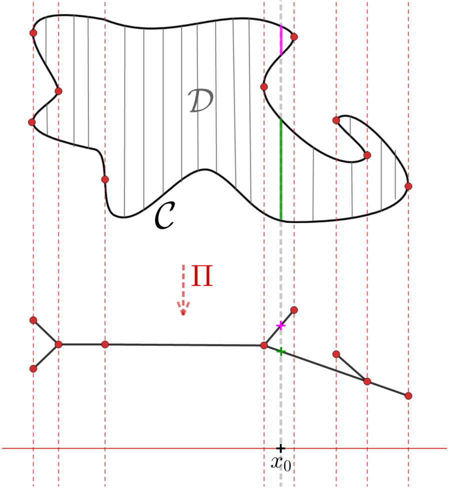
Two points of $\mathcal{D}$ are equivalent if they belong to the same connected component of a fibre of the projection $\Pi : \mathbb{R}^2 \to \mathbb{R}$, $\Pi(x, y) := x$.



The Poincaré-Reeb tree

Theorem

The Poincaré-Reeb graph is a ***transversal tree*** *: it is a* ***plane tree*** *whose open edges are* ***transverse to the foliation*** *induced by the function x ; its vertices are endowed with a* ***total preorder*** *relation induced by the function x.*



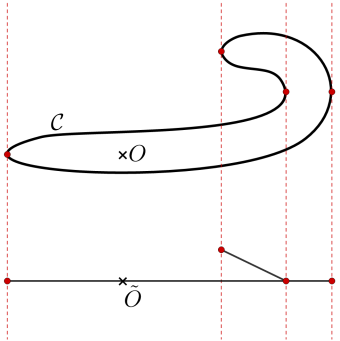
The asymptotic Poincaré-Reeb tree

-small enough level curves ;
-near a strict local minimum.

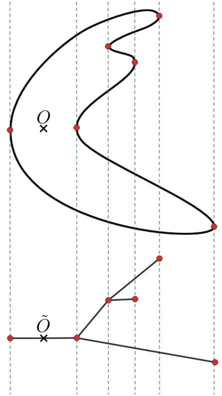
Theorem

The asymptotic Poincaré-Reeb tree ***stabilises****. It is a* ***rooted*** *tree ; the total preorder relation on its vertices is* ***strictly monotone*** *on each geodesic starting from the root.*

Impossible asymptotic configuration :



- Characterise all possible topological types of asymptotic Poincaré-Reeb trees.
- Construct a family of polynomials realising a large class of transversal trees as their Poincaré-Reeb trees.



Main result

- introduction of new combinatorial objects ;
- polar curve, discriminant curve ;
- genericity hypotheses $(x > 0)$;
- univariate case : explicit construction of separable snakes ;
- a result of realisation of a large class of Poincaré-Reeb trees.

Main result

- introduction of new combinatorial objects ;
- polar curve, discriminant curve ;
- genericity hypotheses $(x > 0)$;
- univariate case : explicit construction of separable snakes ;
- a result of realisation of a large class of Poincaré-Reeb trees.

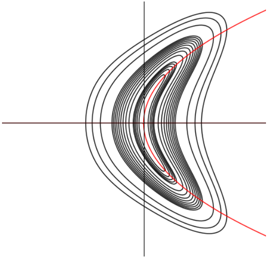
Theorem

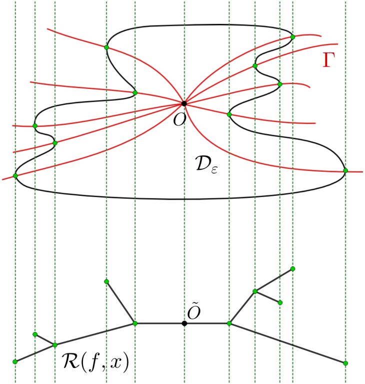
*Given any **separable positive generic rooted transversal tree**, we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree.*

Tool 1 : The polar curve

$$\Gamma(f,x) := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{\partial f}{\partial y}(x,y) = 0 \right\}$$

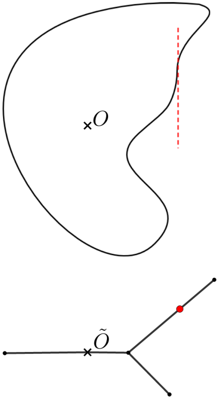
It is the set of points where the tangent to a level curve is vertical.



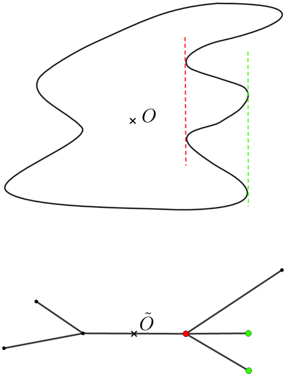


Tool 2 : Choosing a generic projection

Avoid vertical inflections :



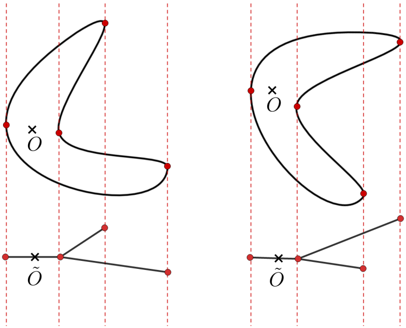
Avoid vertical bitangents :



The generic asymptotic Poincaré-Reeb tree

Theorem

In the asymptotic case, if the direction x is generic, then we have a ***total order*** *relation and a* ***complete binary*** *tree.*



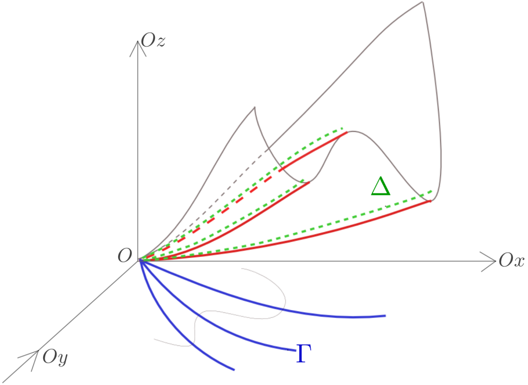
Two inequivalent trees

Tool 3 : The discriminant locus

$$\Phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{x,z}, \Phi(x,y) = \Big(x, f(x,y)\Big).$$

The critical locus of Φ is the polar curve $\Gamma(f,x)$.

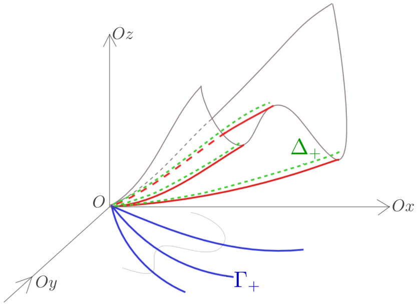
The discriminant locus of Φ is the critical image $\Delta = \Phi(\Gamma)$.



Genericity hypotheses

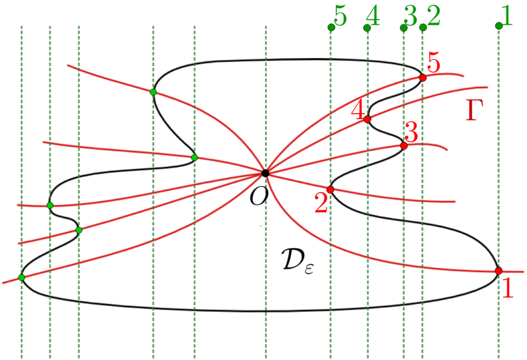
The family of polynomials that we construct satisfies the following two genericity hypotheses :

- the curve Γ_+ is **reduced** ;
- the map $\Phi_{|\Gamma_+} : \Gamma_+ \to \Delta_+$ is a **homeomorphism**.



1. Positive asymptotic snake

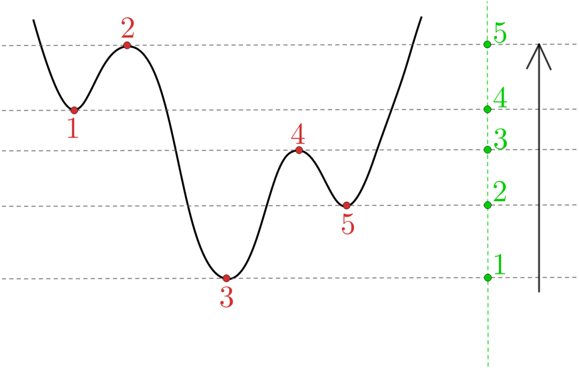
To any positive (i.e. for $x > 0$) generic asymptotic Poincaré-Reeb tree we can associate a permutation σ, called the **positive asymptotic snake**.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$

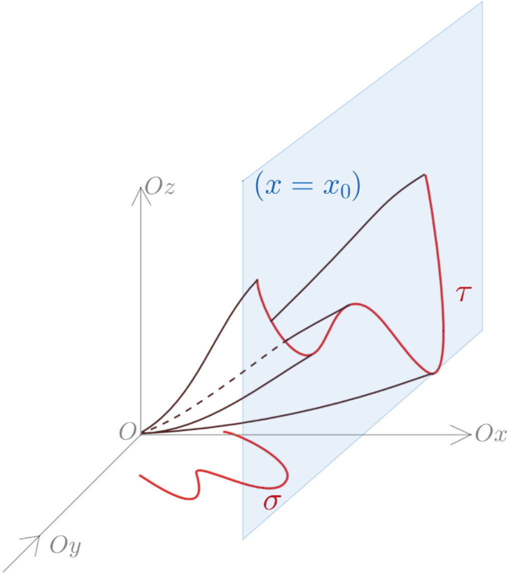
2. Arnold's snake (one variable)

One can associate a permutation to a **Morse polynomial**, by considering two total order relations on the set of its critical points : **Arnold's snake**.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

2. Arnold's snake (one variable)



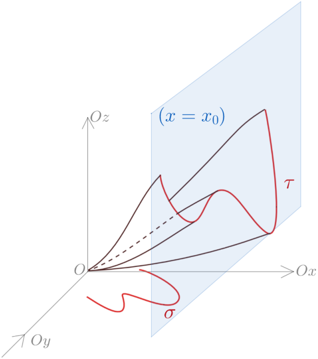
The study of asymptotic forms of **the graphs of one variate polynomials** $f(x_0, y)$, for x_0 tending to zero.

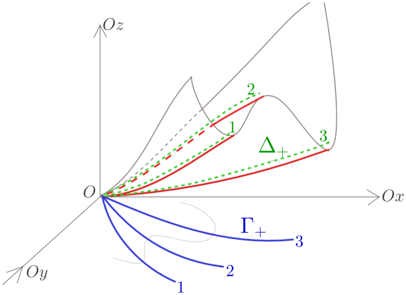
Theorem

$$\sigma = \tau.$$

Proof

The interplay between the polar curve and the discriminant curve :





$$\sigma = \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

The construction

Subquestion

Given a generic rooted transversal tree, can we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree ?

The construction

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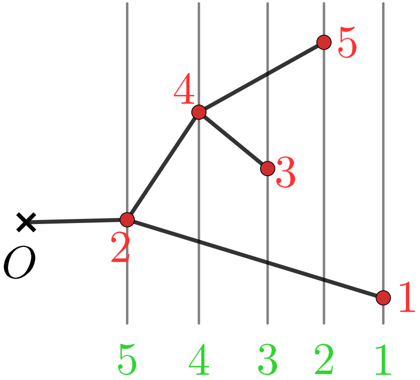
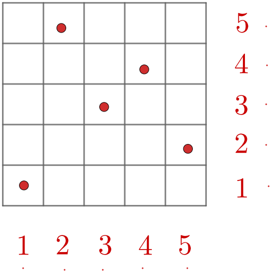
Given a generic rooted transversal tree, can we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree ?

Theorem

*We give a **positive constructive answer** : we construct a family of polynomials that realise all **separable** positive generic rooted transversal trees.*



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$



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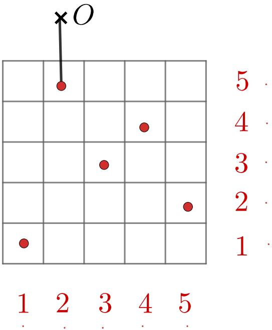

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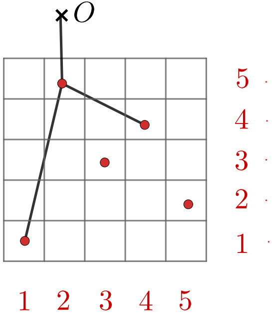

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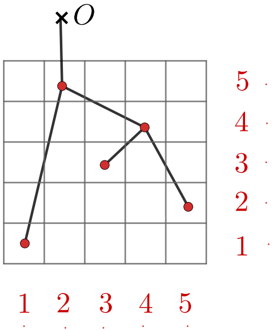

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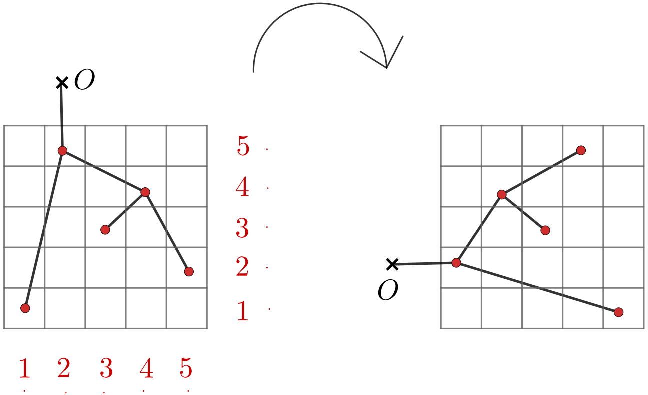

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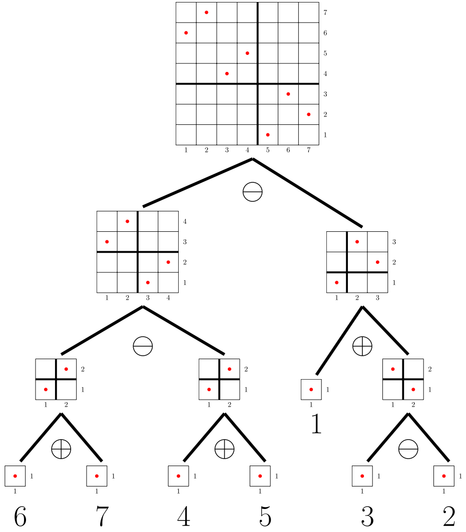
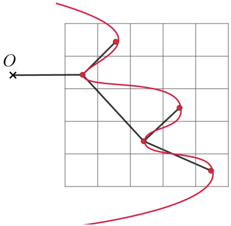
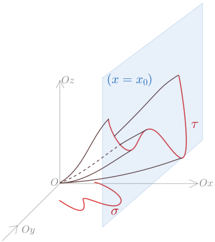
Separable permutations

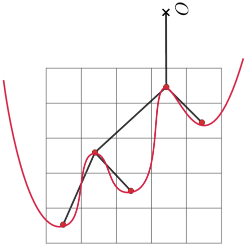


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 1 & 3 & 2 \end{pmatrix} = ((\boxdot \oplus \boxdot) \ominus (\boxdot \oplus \boxdot)) \ominus (\boxdot \oplus (\boxdot \ominus \boxdot)).$$

Passing to the univariate case



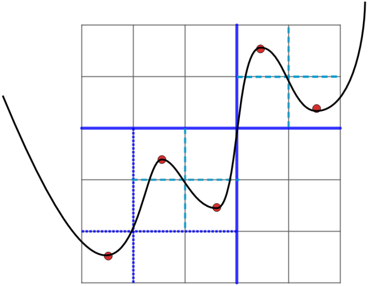


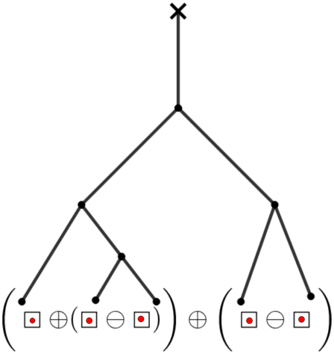


Question

Given a separable snake σ, is it possible to construct a Morse polynomial $Q : \mathbb{R} \to \mathbb{R}$ that realises σ ?

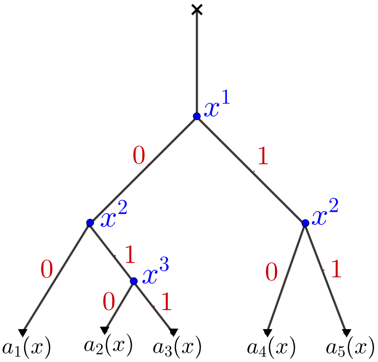
Example





$$= \left(\boxdot \oplus (\boxdot \ominus \boxdot) \right) \oplus (\boxdot \ominus \boxdot) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}.$$

The contact tree



$a_1(x) = 0,$
$a_2(x) = x^2,$
$a_3(x) = x^2 + x^3,$
$a_4(x) = x^1,$
$a_5(x) = x^1 + x^2.$

Answer in the univariate case

Theorem

Consider $m \in \mathbb{N}$ and fix a separable $(m+1)$-snake $\sigma : \{1, 2, \dots, m+1\} \to \{1, 2, \dots, m+1\}$ such that $\sigma(m) > \sigma(m+1)$. Construct the polynomials $a_i(x) \in \mathbb{R}[x]$ such that their contact tree is one of the binary separating trees of σ. Let $Q_x(y) \in \mathbb{R}[x][y]$ be

$$Q_x(y) := \int_0^y \prod_{i=1}^{m+1} \big(t - a_i(x)\big)\mathrm{d}t.$$

Then $Q_x(y)$ is a one variable Morse polynomial and for sufficiently small $x > 0$, the Arnold snake associated to $Q_x(y)$ is σ.

Construction of the desired bivariate polynomial f

Theorem

Let σ be a separable $(m+1)$-snake, with m an even integer, $\sigma(m) > \sigma(m+1)$. Let $f \in \mathbb{R}[x,y]$ be constructed as follows :
(a) construct $Q_x(y) \in \mathbb{R}[x][y]$,

$$Q_x(y) := \int_0^y \prod_{i=1}^{m+1} \big(t - a_i(x)\big)\mathrm{d}t,$$

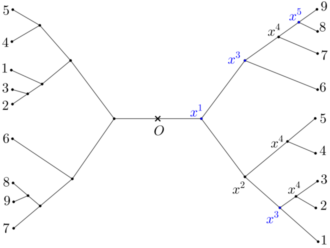
by choosing the polynomials $a_i(x) \in \mathbb{R}[x]$ such that their contact tree is one of the binary separating trees of σ.

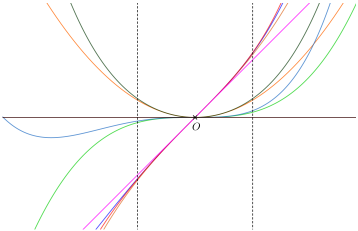
(b) take $f(x,y) := x^2 + Q_x(y)$.

Then f has a strict local minimum at the origin and the positive asymptotic snake of f is the given σ.

Positive-negative contact trees (one variable) [1]

Pairwise distinct polynomials $a_i(x) \in \mathbb{R}[x]$ that pass through a common zero at the origin





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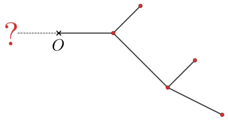
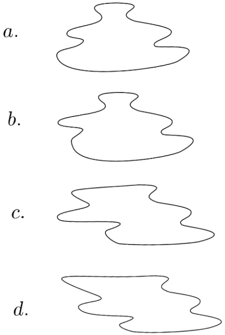
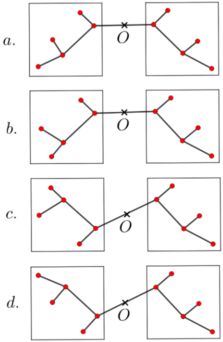
Algorithm flip-flop







Summing-up

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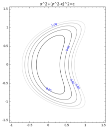

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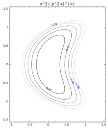

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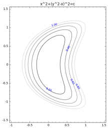

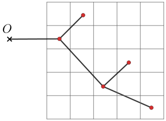

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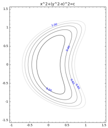

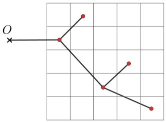

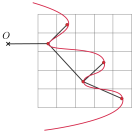

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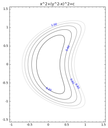
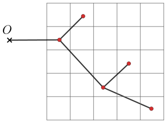
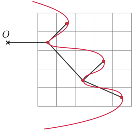
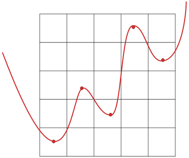


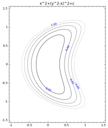


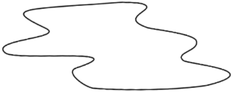


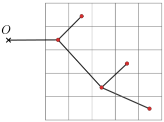


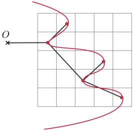
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$$

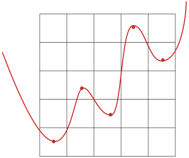
Summing-up



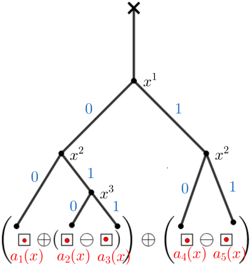




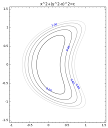


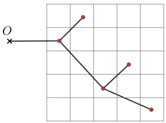


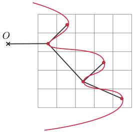
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$$

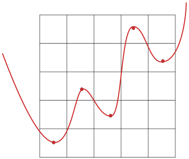


Summing-up

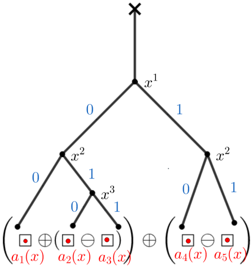


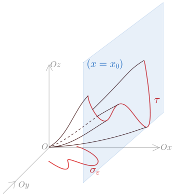




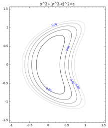


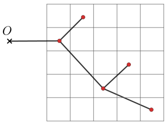
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$$

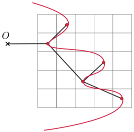


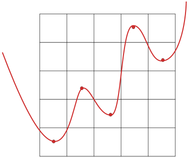


Summing-up

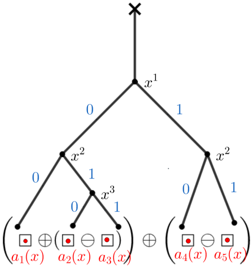


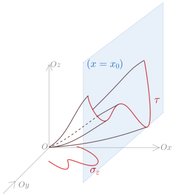






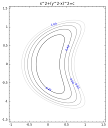
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$$

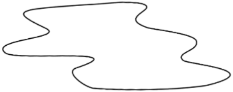


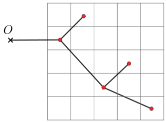


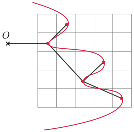
$$f(x,y) := x^2 + \int_0^y \prod_{i=1}^5 \left(t - a_i(x)\right) \mathrm{d}t.$$

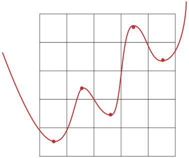
Summing-up



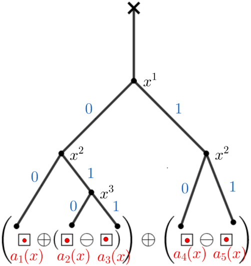


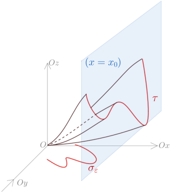




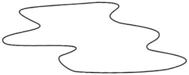


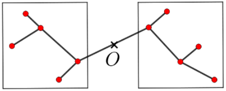
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$$





$$f(x, y) := x^2 + \int_0^y \prod_{i=1}^5 \big(t - a_i(x)\big)\mathrm{d}t.$$





Thank you !