On the Counting Complexity of Propositional Circumscription

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Abstract

Propositional circumscription, asking for the minimal models of a Boolean formula, is an important problem in artificial intelligence, in data mining, in coding theory, and in the model checking based procedures in automated reasoning. We consider the counting problems of propositional circumscription for several subclasses with respect to the structure of the formula. We prove that the counting problem of propositional circumscription for dual Horn, bijunctive, and affine formulas is #P-complete for a particular case of Turing reduction, whereas for Horn and 2affine formulas it is in FP. As a corollary, we obtain also the #P-completeness result for the counting problem of hypergraph transversal.

1 Introduction

Circumscription is a well-developed formalism of common-sense reasoning introduced by McCarthy [11] and extensively studied by the artificial intelligence community. The key idea behind circumscription is that one is interested in the *minimal models* of formulas, since they are the ones that have as few "exceptions" as possible and, therefore, embody common sense. In the context of Boolean logic, circumscription amounts to the study of satisfying assignments of Boolean formulas that are *minimal* with respect to the *pointwise partial order* on truth assignments.

The propositional circumscription problem and related problems find many applications in model building and model checking based procedures in automated reasoning [13], in data mining [6], and in coding theory when we are interested in generating minimal-inclusion or minimal-weight codewords [1], where the latter makes the affine case especially interesting. The special case of propositional circumscription for positive formulas, better known under the name of hypergraph transversal, was extensively studied in combinatorics and graph theory [4]. In the model checking environment it is sufficient to test a given logical property only for the minimal models if the property is monotone and compatible with the pointwise ordering on models. Since there often exists an exponential difference between the cardinality of the set of all models and the set of minimal models of a given formula (especially in the affine case), a model checking procedure that tests a logical property only on the minimal models can be much more efficient than a

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standard model checking procedure, provided that we can produce the minimal models rapidly. The efficiency question immediately motivates the complexity analysis and classification for the counting problems concerning the minimal models of propositional formulas.

The complexity of the decision problem for propositional circumscription, formulated as "Given a propositional formula φ in conjunctive normal form and a model m, is m a minimal model of φ?", was first considered by Cadoli [2], who proved the problem to be coNP-complete. Kirousis and Kolaitis [9] presented a dichotomy theorem for the model checking problem of propositional circumscription that provides a classification between coNP-complete and polynomial cases depending on the structure of the considered formula. The special case of propositional circumscription for positive formulas, which is equivalent to the hypergraph transversal problem, was extensively studied in combinatorics and graph theory, although the exact complexity of this problem was not yet pinned down. Eiter and Gottlob [4] performed an exhaustive study of this problem and identified many polynomial-time cases. Fredman and Khachiyan [5] presented an algorithm for the general case of hypergraph transversal running in quasi-polynomial time. This constitutes a strong indication that the hypergraph transversal problem is tractable and the corresponding decision problem of monotone Boolean duality is unlikely to be NP-hard.

In this paper, we focus on the counting complexity of propositional circumscription. Durand, Hermann, and Kolaitis [3] proved that the counting problem for propositional circumscription, asking for the number of minimal models of a Boolean formula, is #·coNP-complete, i.e. highly intractable since it is complete for a counting class that is two stages higher than #P. Finding complete problems for counting classes in this setting is a challenging task. Unless $\Sigma_2\mathrm{P} = \mathrm{NP}$, there is no parsimonious reduction that maps accepting paths of a $\Sigma_2\mathrm{P}$ machine to minimal models of a propositional formula in a one-to-one manner [15, 18]. Similarly, it is well known that counting classes are not closed under classical Turing reductions. Hence, there is a need for reductions methods suitable for counting problems. In [3] a new reduction is introduced, the so called *subtractive reduction* (cf. Section 2) which meets the following requirements: (1) #P and higher complexity classes are closed under it and (2) the reduction does not establish a direct bijection between solution sets of problems. Hence, it allows us to reduce counting problems whose underlying decision problems are of different complexity.

In this note, we pursue on this line of research and show that subtractive reductions are suitable to handle semantic conditions on models inherent to circumscription and can be used to show the #P-hardness of natural particular cases of propositional circumscription counting problems. Horn, dual Horn, bijunctive, and affine formulas constitute the most studied classes of Boolean formulas with respect to several decision and counting problems. Their importance stems from the fact that the corresponding problems restricted to these classes often have lower complexity than the general case. Moreover, several database, logic programming, and graph-theoretic tasks reduce to decision or counting problems upon these restricted classes of Boolean formulas. We perform a more fine-grained complexity analysis of the counting problem for circumscription, based on the structure of the considered Boolean formulas. Knowing the complexity of these natural cases seem to be an obliged step towards a complete classification. In addition, we also prove a completeness result for the hypergraph transversal counting problem.

2 Preliminaries

We suppose that the reader is familiar with the basic notions and definitions in computational complexity. Additional material on this topic can be found in the monograph [12]. The research in counting complexity has been started by Valiant [16, 17] and is now a well-established part of the complexity theory, where the most known class is #P. Many counting variants of decision problems have been proved #P-complete. Higher counting complexity classes do exist, but they are not commonly known. A counting equivalent of the polynomial hierarchy was defined by Hemaspaandra and Vollmer [7], whereas generic complete problems for these counting hierarchy classes were presented in [3].

Formally, a *counting problem* is presented using a *witness* function which for every input x returns a set of *witnesses* for x. A *witness* function is a function $w\colon \Sigma^* \to \mathcal{P}^{<\omega}(\Gamma^*)$, where Σ and Γ are two alphabets, and $\mathcal{P}^{<\omega}(\Gamma^*)$ is the collection of all finite subsets of Γ^*. Every such witness function gives rise to the following *counting problem*: given a string $x \in \Sigma^*$, find the cardinality $|w(x)|$ of the *witness* set $w(x)$. According to [7], if $\mathcal{C}$ is a complexity class of decision problems, we define $\#{\cdot}\mathcal{C}$ to be the class of all counting problems whose witness function w satisfies the following conditions.

1. There is a polynomial $p(n)$ such that for every $x \in \Sigma^*$ and every $y \in w(x)$ we have $|y| \leq p(|x|)$;

2. The problem "given x and y, is $y \in w(x)$?" is in $\mathcal{C}$.

It is easy to verify that $\#\mathrm{P} = \#{\cdot}\mathrm{P}$. The counting hierarchy is ordered by linear inclusion [7]. In particular, we have that $\#\mathrm{P} \subseteq \#{\cdot}\mathrm{NP} \subseteq \#{\cdot}\mathrm{coNP} \subseteq \#{\cdot}\Sigma_2\mathrm{P} \subseteq \#{\cdot}\Pi_2\mathrm{P} \subseteq \#{\cdot}\Sigma_3\mathrm{P} \subseteq \#{\cdot}\Pi_3\mathrm{P}$, etc.

Completeness of counting problems in #P is usually proved by means of Turing reductions. However, the counting classes $\#{\cdot}\Pi_k\mathrm{P}$ are not closed under these reductions. It is therefore better to use *subtractive reductions* [3], a particular case of Turing reductions, which preserve membership in the aforementioned counting classes. Let Σ, Γ be two alphabets and let $R \subseteq \Sigma^* \times \Gamma^*$ be a binary relation between strings such that, for each $x \in \Sigma^*$, the set $R(x) = \{y \in \Gamma^* \mid R(x, y)\}$ is finite. We write $\#{\cdot}R$ to denote the following counting problem: given a string $x \in \Sigma^*$, find the cardinality of the witness set $R(x)$ associated with x. It is easy to see that every counting problem is of the form $\#{\cdot}R$ for some R. Let $\#{\cdot}A$ and $\#{\cdot}B$ be two counting problems determined by the binary relations A and B between strings from Σ and Γ. We say that the counting problem $\#{\cdot}A$ reduces to the counting problem $\#{\cdot}B$ via a *strong subtractive reduction* if there exist two polynomial-time computable functions f, g, such that for every string $x \in \Sigma^*$ the following conditions hold:

$$B(f(x)) \subseteq B(g(x))$$
and
$$|A(x)| = |B(g(x))| - |B(f(x))|\,.$$

A strong subtractive reduction with $B(f(x)) = \emptyset$ is called *parsimonious*. The counting problem $\#{\cdot}A$ reduces to the counting problem $\#{\cdot}B$ via a *subtractive reduction* if there exists a positive integer n and a sequence of counting problems $\#{\cdot}A_1, \ldots, \#{\cdot}A_n$ such that $\#{\cdot}A = \#{\cdot}A_1$, $\#{\cdot}B = \#{\cdot}A_n$, and $\#{\cdot}A_i$ reduces to $\#{\cdot}A_{i+1}$ via a strong subtractive reduction, for each $i = 1, \ldots, n-1$. I.e., a *subtractive reduction* is a transitive closure of strong subtractive reductions.

We consider the counting problem of propositional circumscription for Boolean formulas in conjunctive normal form. The special cases are classified according to the structure of the Boolean formula, where HORN formulas contain at most one positive literal per clause, DUAL HORN formulas contain at most one negative literal per clause, BIJUNCTIVE formulas contain at most two literals per clause, and AFFINE formulas are equivalent to a linear system of equations $S\colon Ax = b$ over the field $\mathbb{Z}_2$, with A being a $k \times n$ Boolean matrix, x being a variable vector, and b being a Boolean vector. This structural classification of Boolean formulas, encompassing Horn, dual Horn, bijunctive, and affine formulas, is called the *Schaefer's class*.

A *hypergraph* is a combinatorial structure (V, H), where V is a finite set of vertices and H is a subset of the powerset $\mathcal{P}(V)$ called the hyperedges. A hypergraph with k vertices and n hyperedges can be identified by means of the vertex-hyperedge incidence matrix A, where A is a Boolean $k \times n$ matrix, such that $A(i,j) = 1$ if the vertex i is included in the hyperedge j, and $A(i,j) = 0$ otherwise. The columns of the matrix A can be also interpreted as the Boolean variables $x_1, \ldots, x_n$, whereas the rows of A represent the clauses $c_1, \ldots, c_k$. In this case $A(i,j) = 1$ if and only if the positive literal x_j occurs in the clause c_i. This establishes the equivalence between hypergraphs and positive Boolean formulas in conjunctive normal form. A similar construction can be performed for a loop-free graph $G = (V, E)$. We form the corresponding vertex-edge incidence matrix A, where A is a Boolean matrix, such that $A(i,j) = 1$ if the vertex $v_i \in V$ is adjacent to the edge $e_j \in E$. The rows of A correspond to the vertices V, whereas the columns of A correspond to the edges E. It is clear that every column of A contains exactly two occurrences of 1, the other positions are occupied by 0s.

If φ is a formula and x is a variable occurring in φ, we denote by $\varphi[x/b]$ the substitution of the Boolean constant $b \in \mathbb{Z}_2$ for the variable x in φ. A *model* m of a propositional formula $\varphi(x_1, \ldots, x_n)$ with n variables is a satisfying truth assignment of φ considered as a Boolean vector from $\mathbb{Z}_2^n$. If $m = (m_1, \ldots, m_n)$ is a model and x_i is the ith variable of φ, we denote by $m(x_i) = m_i$ the ith coordinate of the model m. If the formula φ is affine, it is equivalent to a linear system of equations $S\colon Ax = b$ over $\mathbb{Z}_2$, and the model m is a solution of S. The *pointwise ordering* on the models from $\mathbb{Z}_2^n$, denoted by $<_{pt}$, is defined by $(m_1, \ldots, m_n) <_{pt} (m'_1, \ldots, m'_n)$, if $(m_1, \ldots, m_n) \neq (m'_1, \ldots, m'_n)$ and $m_i \leq m'_i$ holds for each $i = 1, \ldots, n$. We say that m is a *minimal model* of φ if m satisfies φ and there is no satisfying truth assignment m' of φ such that $m' <_{pt} m$.

3 Counting Problems for Propositional Circumscription

The concept of minimality of models with respect to the partial ordering $<_{pt}$ gives rise to the following natural counting problem.

Problem: #CIRC
Input: A Boolean formula $\varphi(x_1, \ldots, x_n)$ in conjunctive normal form.
Output: Number of minimal models of $\varphi(x_1, \ldots, x_n)$.

We denote by #CIRC(HORN), #CIRC(DUAL HORN), #CIRC(BIJUNCTIVE), and #CIRC(AFFINE) the restriction of the previous counting problem to Horn, dual Horn, bijunctive, and affine formulas φ, respectively. These classes follow the usual major classes of propositional formulas studied in logic and algorithmics. The special case #CIRC(POSITIVE SAT) is better known as the counting problem for hypergraph transversal.

Problem: #HYPERGRAPH TRANSVERSAL
Input: A Boolean formula $\varphi(x_1, \ldots, x_n)$ in conjunctive normal form, containing only positive literals.
Output: Number of minimal models of $\varphi(x_1, \ldots, x_n)$.

The counting problem #CIRC has been proved #·coNP-complete via subtractive reductions in [3], but no further analysis was performed. We investigate here the counting complexity of the propositional circumscription according to the structure of the considered formula in Schaefer's class. It is common folklore that #CIRC(HORN) is in FP, since the models of Horn formulas are closed under conjunction. Therefore a Horn formula φ has either one minimal model or none, depending on the satisfiability of φ. Let us investigate the counting problems #CIRC(DUAL HORN) and #CIRC(BIJUNCTIVE). We prove the lower bound of these two counting problems by a subtractive reduction from the following problem.

Problem: #DNF
Input: A Boolean formula φ in disjunctive normal form.
Output: Number of truth assignments that satisfy φ.

It has been known for a long time that the counting problem #DNF is #P-complete. The proof of this #P-completeness result by means of subtractive reductions can be found in [3]. #DNF can be easily proved #P-complete by a subtractive reduction from the counting problem #SAT, the problem of counting the satisfying truth assignments of a propositional CNF-formula, proved #P-complete by Valiant [17]. Indeed, the number of satisfying assignments of $\varphi(x_1, \ldots, x_n)$, denoted by $|\varphi|$, is equal to $|\varphi| = 2^n - |\neg\varphi|$. The formula $\neg\varphi$ can be produced from φ by de Morgan laws in polynomial time. The value 2^n can be produced as the number of satisfying assignments of a tautology over n variables.
The following proposition presents the necessary #P-membership proof for the four special subcases of circumscription we are interested in. This result was already present in a different wording in [2] and reused in [9].

Proposition 1 ([2, 9]) *For each class of formulas $F \in \{$HORN, DUAL HORN, BIJUNCTIVE, AFFINE$\}$ #CIRC(F) is in #P.*

Proof: We will show that the problem of testing whether a solution m is minimal for a formula φ can be reduced to the satisfiability problem. Note first that all four considered classes of formulas are hereditary, i.e., if φ is Horn, dual Horn, bijunctive, or affine, then both $\varphi[x/0]$ and $\varphi[x/1]$ (even after a possible simplification) remain Horn, dual Horn, bijunctive, or affine, respectively. This means that the formula remains in the same class after the substitution. Since the satisfiability problem is decidable in polynomial time for all four mentioned classes, this property is maintained for them after substitution by constants.

Let $\varphi(x_1, \ldots, x_k)$ be a formula over the variables $x_1, \ldots, x_k$ belonging to the class F. Given a Boolean vector $m = (m_1, \ldots, m_k)$, we can check in polynomial time if m is a minimal solution of φ as follows. For each i such that $m(x_i) = 0$ holds, substitute the value 0 to φ, obtaining a new formula φ', where each variable x satisfies the identity $m(x) = 1$. Repeat the following operation for each variable x in φ'. Substitute 0 for x and check whether $\varphi'[x/0]$ is satisfiable. If there exists a variable x for which $\varphi'[x/0]$ is satisfiable, then m cannot be a minimal solution of φ. If the satisfiability of φ can be tested in polynomial time, the same also holds for φ'. Hence, we can decide in polynomial time whether m is a witness of the instance φ of #CIRC(F), provided that F is one of the aforementioned classes of formulas. □

Theorem 2 *The problems #CIRC(BIJUNCTIVE) and #CIRC(DUAL HORN) are #P-complete via subtractive reductions.*

Proof: Both problems #CIRC(BIJUNCTIVE) and #CIRC(DUAL HORN) are in #P according to Proposition 1.

For the lower bound, we present a subtractive reduction from #DNF. Let $\varphi(x_1, \ldots, x_k) = d_1 \vee \cdots \vee d_n$ be a propositional formula in disjunctive normal form where $d_i = p_1^i \wedge p_2^i \wedge p_3^i$. For each conjunct d_i we construct a new formula $c_i = (y_i \vee p_1^i) \wedge (y_i \vee p_2^i) \wedge (y_i \vee p_3^i)$, where y_i is a new variable associated with the conjunct d_i. Furthermore we construct the formula

$$\psi(x, x', y, z) \quad = \quad \bigwedge_{i=1}^{k} (x_i \vee x'_i) \wedge \bigwedge_{i=1}^{n} c_i \wedge \bigwedge_{i=1}^{n} (y_i \vee z),$$

where x' is a new variable vector corresponding to x, and z is a new variable. Note that ψ is both a bijunctive and a dual Horn formula. Let $B(\psi)$ be the set of minimal models of ψ. The subset A of minimal models of ψ, where at least one variable y_i satisfies the condition $y_i = 0$, are the models with $z = 1$ and $p_1^i \wedge p_2^i \wedge p_3^i = 1$. Hence they extend some model of φ. Conversely, the subformula $(x_1 \vee x'_1) \wedge \cdots \wedge (x_k \vee x'_k)$ forces different models of φ to be mapped to incomparable, hence minimal models of ψ. The remaining minimal models of ψ all satisfy $y_i = 1$ for all $i \leq n$ and $z = 0$. So, they are the minimal models of formula

$$\psi'(x, x', y, z) \quad = \quad \bigwedge_{i=1}^{k} (x_i \vee x'_i) \wedge \bigwedge_{i=1}^{n} y_i \wedge \neg z.$$

It is clear that the minimal models of ψ' are included in the set of minimal models of ψ. The number of models of φ is equal to the number of minimal models of ψ minus the number of minimal models of ψ'. This concludes a subtractive reductions from #DNF to both target problems. □

An adaptation of the previous proof makes it possible to prove also a completeness result for the hypergraph transversal counting problem.

Corollary 3 #HYPERGRAPH TRANSVERSAL *is #P-complete via subtractive reductions.*

Proof: The membership in #P is obvious. The proof of the lower bound is obtained by a reduction from #CIRC(BIJUNCTIVE). It uses a trick due to Kavvadias, Sideri, and Stavropoulos [8] that allows us to eliminate negative literals from Boolean formulas while preserving the set of minimal models. Applied to bijunctive formulas, this method is polynomial-time computable. For the sake of completeness, we present it briefly in this proof.
Let φ be a dual Horn formula with at most two literals per clause. Let C be the set of clauses of φ and $N(x)$ (respectively $P(x)$), be the set of clauses of φ with a negative (respectively positive) occurrence of the variable x. We assume without loss of generality that the formula φ is simplified, i.e. that it does not contain tautology clauses of the type $x \vee \neg x$. For each couple (c, d) of a clause $c = \neg x \vee y$ from $N(x)$ and a clause $d = x \vee z$ from $P(x)$ we form the resolvent clause $r = y \vee z$. Let $R(x)$ be the set of all resolvent clauses obtained for a variable x. Consider the formula ψ formed as the conjunction of the clauses $(C \smallsetminus N(x)) \cup R(x)$. The formula ψ has no negative occurrences of the variable x and, moreover, it has the same set of minimal models as φ. This process is iterated for each variable until no negative variable occurrences remain. Finally, in case there exists a variable x with only negative occurrences in φ, then it is clear that each minimal model m of φ satisfies $m(x) = 0$. Therefore in this case we can eliminate the variable x from φ, knowing that there is a one-to-one correspondence between the minimal models of the new formula $\varphi' = \varphi[x/0]$ and those of the original formula φ. Since φ is a bijunctive, the whole resolution process requires only polynomial time. □

We need the following problem from [17] to analyze the counting complexity of affine circumscription.

Problem: #s-t PATHS
Input: Graph $G = (V, E)$ and two vertices $s, t \in V$.
Output: Number of paths from s to t that visit every vertex at most once.

Valiant proved that #s-t PATHS is #P-complete by a Turing reduction from #HAMILTONIAN PATHS. The reductions from #3SAT to the problem #HAMILTONIAN PATHS is parsimonious, since already the reduction between the decision variants of these problems has this property. Observe that #3SAT is #P-complete via parsimonious reductions (see e.g. [10]). Hence #s-t PATHS is #P-complete via Turing reductions.

Theorem 4 #CIRC(AFFINE) *is #P-complete via Turing reductions. It remains #P-complete even if the number of occurrences of each variable is restricted to two.*

Proof: The #P-membership is clear from the fact that it can be checked in polynomial time whether a Boolean vector m is a minimal model of an affine formula according to Proposition 1, since the satisfiability problem for affine formulas is in P [14].
For the lower bound we perform a parsimonious reduction from the problem #s-t PATHS. For the graph $G = (V, E)$, where $|V| = k$ and $|E| = n$, we construct the corresponding vertex-edge $k \times n$ incidence matrix A. The rows of A represent the vertices V, whereas the

columns of A represent the edges E. Without loss of generality we assume that the first row corresponds to s and the last row to t. We set the vector b equal to $(1, 0, \ldots, 0, 1)$, i.e., $b_1 = b_k = 1$ and $b_j = 0$ for each $j = 2, \ldots, k-1$. This concludes the construction of the affine system $Ax = b$. Note that since variables correspond to edges, each variable has exactly two occurences in the system.

We need to show a one-to-one correspondence between the s-t paths of G and the minimal solutions of the system $Ax = b$. Let $e_1 = (s, v_1)$, $e_2 = (v_1, v_2)$, $\ldots$, $e_l = (v_{l-1}, v_l)$, $e_{l+1} = (v_l, t)$ be an s-t path $\bar{p}$ in G passing through the vertices v_1, $\ldots$, v_l. Let x_i be the variable corresponding to the edge e_i for $i = 1, \ldots, l+1$. Let $p = (p_1, \ldots, p_n)$ be a Boolean vector, such that $p(x_i) = 1$ for each $i = 1, \ldots, l+1$ and $p(x_j) = 0$ for each $e_j \in E \smallsetminus \{e_1, \ldots, e_{l+1}\}$. The extremal vertices (i.e., s and t) appear only once in the path $\bar{p}$: they are incident to only one edge. Hence, equations of rows 1 and k which sum up to 1 are satisfied. Similarly, internal vertices of the path $\bar{p}$ are incident to exactly two edges. Then the respective equations sum up to 0 and are satisfied. Finally, every other vertex of the graph is incident to no edge of the path. Hence only variables x_j for $e_j \in E \smallsetminus \{e_1, \ldots, e_{l+1}\}$ appear in their respective equations which also sum up to 0. Clearly, the Boolean vector $p = (p_1, ..., p_n)$ is a solution of the system $Ax = b$.

Suppose that there exists a solution $p' <_{pt} p$. Then the vector $p^* = p' + p$ has a Hamming weight different from 0, but we have $Ap^* = 0$. Hence, the edges e_i corresponding to the variables x_i with $p^*(x_i) = 1$ form a cycle. Since $p' <_{pt} p$ holds, we also have $p^* <_{pt} p$. This implies that the initial s-t path $\bar{p}$ contains a cycle, constituting a contradiction.

Conversely, let p be a minimal solution of the system $Ax = b$ and let $\bar{p}$ be the corresponding subgraph of G. Observe first that s and t are of odd degree in $\bar{p}$, since $b_1 = b_k = 1$. Suppose that there exists another vertex v_j of odd degree in $\bar{p}$. Then the row j in $Ax = b$ cannot be satisfied, since $b_j = 0$. Suppose now that $\bar{p}$ contains a cycle $\bar{c}$ consisting of edges $e_1, \ldots, e_l$. Let x_i be the variable corresponding to the edge e_i for $i = 1, \ldots, l$. Let $c = (c_1, \ldots, c_n)$ be a Boolean vector corresponding to $\bar{c}$, such that $c(x_i) = 1$ for each $i = 1, \ldots, l$ and $c(x_j) = 0$ for each $e_j \in E \smallsetminus \{e_1, \ldots, e_l\}$. It is clear that c is a solution of $Ax = 0$ and that $c <_{pt} p$. Construct the subgraph $\bar{q} = \bar{p} \smallsetminus \bar{c}$ with the corresponding Boolean vector $q = p + c$. Then q is a solution of $Ax = b$ and $q <_{pt} p$ holds, contradicting the assumption that p is minimal. Then $\bar{p}$ is a subgraph containing no cycle, where s and t are of odd degree and where all other vertices are of even degree. Hence it is a path.

It is clear that different minimal solutions of $Ax = b$ correspond to different s-t paths in G. This concludes the proof of a parsimonious reduction from the counting problem #s-t PATHS. □

It is clear that #CIRC(AFFINE) remains #P-complete if we restrict each row of an affine system $Ax = b$ to *three* variables. A natural question arises about the counting complexity of #CIRC(AFFINE) restricted to *two* variables in $Ax = b$. In fact, this amounts to investigate the counting complexity of the circumscription problem for formulas both affine and bijunctive. We call these formulas 2affine.

Theorem 5 *#CIRC(2AFFINE) is in* FP*.*

Proof: The clauses of a 2affine formula are equations of the form $x + y = 0$ and $x + y = 1$. The equation of the type $x + y = 0$ means that the variables x and y are equivalent in

formula	complexity
general	#·coNP-complete
Horn	in FP
dual Horn	#P-complete
bijunctive	#P-complete
affine	#P-complete
2affine	in FP

Figure 1: Complexity of propositional circumscription

the affine system $Ax = b$. From two equations $x + y = 1$ and $y + z = 1$ we can deduce by gaussian elimination the equation $x + z = 0$. Compute all these consequences from the equations in $Ax = b$ and add them to the system. For each equation of the type $x + y = 0$, replace both variables x and y in $Ax = b$ by a new variable x' and discard the equation $x' + x' = 0$. If the equality $0 = 1$ is deduced then there is no solution of the affine system. Otherwise, after the aforementioned transformations, there are only equations of the type $x + y = 1$ in the affine system and every variable occurs exactly once. This implies that for each equation $x + y = 1$ there are exactly two solutions, both incomparable, hence minimal. If there are q equations in the transformed system, then there are 2^q minimal solutions. Since the transformations preserved the number of minimal solutions, this is also the number of solutions of the affine system $Ax = b$. □

4 Concluding Remarks

We proved that the counting complexity of the propositional circumscription problem for dual Horn, bijunctive, and affine formulas is #P-complete, what constitutes a more fine grained analysis compared to the #·coNP-completeness result for the general propositional circumscription counting problem from [3]. The lower bounds of all these completeness results were proved by means of subtractive reductions, what shows their power and usefulness. As a byproduct, we also obtained the #P-completeness result of the hypergraph transversal counting problem. On the other hand, the counting complexity of circumscription for Horn and 2affine formulas is in FP. Figure 1 summarizes the complexity results for the counting problems of propositional circumscription.

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