New results on arity vs. number of variables

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Abstract

In this paper, definability in existential second-order logic is considered. Our main goal it to study the relationships between the arity of second-order quantified (i.e. guessed) symbols and the number of first-order universal variables in formulas. The key idea behind this work is that bounding the arity of guessed symbols strongly limits the number of first-order universally quantified variables that are really needed to express properties.

We show that, as long as second-order quantifiers of formulas range over function and relation symbols of arity at most one, two first-order variables say no more than one.

We also generalize this result and show that for some varieties of formulas (including positive ones with unary signature) the number of universally quantified first-order variables does not really matter.

These results are first steps towards looking for an arity-based characterization of subclasses of NP and towards proving there exists a strict hierarchy purely based on arity for existential second-order logic, two long standing open questions in finite model theory.

1 Introduction and summary of results

Measuring the expressive power of logical formalisms over finite structures is of great importance mainly because of its connection with database theory and complexity. A well-known and foundational result in this domain is Fagin's theorem [4] which shows that properties decidable in NP are those definable in existential second-order logic (ESO, for short). Such a result (see also [11]) pointed out strong relationships between descriptive and computational complexities of problems and has initiated an important field of research.

The logical characterization of NP has been refined in a number of ways through the years (see [12, 7, 13, 8]). Intuitively, taking the non-deterministic random access machine as computation model, a property is decidable in time $O(n^d)$ if and only if it is definable by some formula of the form

$$\exists f_1 \dots \exists f_k \; \forall \boldsymbol{x} \phi$$

where the f_i are relation or function symbols of arity at most d (the so-called "guessed" predicates), $\boldsymbol{x}$ is a d-tuple of first-order variables and ϕ is a first-order quantifier-free formula. In this result, the computational time bound is in direct relation with the arity of the guessed predicates and the number of universal variables. The question whether there exists a characterization of subclasses of NP based on the arity of guessed predicates only is open (see, for example, Open Problem 7.12 in [10]).

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In the same vein, a long standing open question in finite model theory (see [5, 6] - the original statement concern formulas with relation symbols only) asks whether there exists a strict hierarchy in terms of expressive power based on arity. Considering existential second-order formulas (with no bound on the number of first-order variables),

$$\exists f_1 \dots \exists f_k \ \varphi,$$

is it true that quantifying over $(d+1)$-ary function or relation symbols f_i is *strictly* more expressive than quantifying over d-ary ones ? Over the years, partial answers have been given. For example, for the so-called "sub-diagonal" cases, i.e. as long as the arity of guessed symbols is smaller than the arity of the signature (non-quantified second-order symbols) of formulas, the hierarchy is known to be strict (see [1] and also [9] for a recent study). However, the problem is still open in the general case.
Results of this paper are contributions to the two above-mentioned problems. Looking for an arity-based characterization of subclasses of NP, we focus on the relationships between the arity of guessed symbols in ESO-formulas and the number of universal first-order variables. By transfer and padding arguments, it is easily seen that without loss of generality, one can concentrate on formulas with *unary* function symbols only. The general problem addressed in this paper is the following: given an ESO-formula

$$\Phi \equiv \exists f_1 \dots \exists f_k \ \forall x_1 \dots \forall x_d \ \varphi,$$

where the f_is are unary function symbols, $x_1, \dots, x_d$ are d first-order variables and φ is quantifier-free, does there exist an ESO-formula logically equivalent to Φ with function symbols of arity one also but with *strictly less* universally quantified first-order variables? Or, stronger, with only *one* such variable?
Surprisingly, some partial and positive answers can be given. The first result of this paper (Theorem 8, Section 5) shows that, roughly speaking, two variables say no more than one for ESO-formulas of arity one. In other words, if $d = 2$ in Φ, an equivalent formula with only one variable can be found. In contrast to what happens with purely relational signatures, the expressive power of two-variables functional logic does not suffer intrinsic limitations [1]. This fact gives some relevance to our result. One of its consequences, is that the hierarchy based on arity of guessed predicates for two-variables ESO-formulas is strict between the two first levels and collapses right after (see Corollary 11).
Though it seems not immediate to generalize the result for $d > 2$, we also obtain it for any $d \geq 2$ in the following particular cases (see Section 6):

- when φ is a positive sentence (i.e. without negation);
- when negation appears in φ in a particular way (in, so-called, acyclic formulas).

Finally, in the vein of the results obtained in this paper, we state two realistic conjectures (see Section 7) and show that they imply the existence of an arity-based characterization of subclasses of NP and the strictness of the arity hierarchy as defined in [5, 6].
To our knowledge the method used in this paper is new. It combines logical and combinatorial arguments. In particular, we introduce a notion of (minimal) samples of unary functions, use basic combinatorics to show there are rather few such objects and, knowing this fact, work on the syntactic form of formulas.

Section 2 presents some standard definitions and well-known results. The basic combinatorial tools are described in Section 3 and the method is applied on a very simple example in Section 4. Our main results are then presented along Sections 5 and 6, as mentionned above. Sections 7 and 8 are devoted to some open questions and conclusive remarks.

[1] Recall that even one variable is enough to characterize non-deterministic linear time on RAMs.

2 Basic normalization of formulas

We will often deal with *tuples* of objects. We denote them by bold letters. A *signature* (or *vocabulary*) σ is a finite set of relation and function symbols, each of which has a fixed arity which can be zero (a 0-ary function symbol is a constant symbol and a 0-ary relation symbol is a boolean variable). The *arity* of σ, denoted by arity(σ), is the maximal arity of its symbols. We call *unary* a signature whose arity is at most one. A vocabulary is *functional* (resp. *relational*) if it does not contain any relation (resp. function) symbol. When σ and τ are two disjoint signatures, we often denote by $\sigma\tau$ their union $\sigma \cup \tau$. A (finite) *structure* S of vocabulary σ, or σ-structure, consists of a finite domain D of cardinality $n > 1$, and, for any symbol $s \in \sigma$, an interpretation of s over D, often denoted s, for simplicity. The tuple of the interpretations of the σ-symbols over D is called the *interpretation* of σ over D and, when no confusion results, it is also denoted σ. The *cardinality* of a structure is the cardinality of its domain. For any signature σ, we denote by $\textsc{struc}(\sigma)$ the class of (finite) σ-structures. We denote by $\textsc{models}(\Phi)$ the set of σ-structures which satisfy some fixed formula Φ.

We use the usual definitions and notations in logic and finite model theory (see [3]). We are interested by definability in *Existential Second Order logic* (ESO). Given two disjoint signatures $\sigma = \{s_1, \ldots, s_p\}$ and $\tau = \{t_1, \ldots, t_q\}$ (where the s_i's and the t_i's are relation and function symbols of various arities), we write $\exists\tau\phi(\sigma,\tau)$ for the formula $\exists t_1 \ldots \exists t_q \phi(s_1, \ldots, s_p, t_1, \ldots, t_q)$. For a vocabulary σ, we denote by ESO^σ the class of σ-formulas of the form $\exists\tau\phi(\sigma,\tau)$, where τ is a signature disjoint from σ and ϕ is a first-order sentence of vocabulary $\sigma\tau$. We say that two formulas Φ_1 and Φ_2 of ESO^σ are *equivalent* if they are equivalent over *finite* structures, that is, if the sets $\textsc{models}(\Phi_1)$ and $\textsc{models}(\Phi_2)$ coincide.
In the sequel, we will focus on the following syntactic restriction of ESO^σ:

Definition 1 $\mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\,\ell)$ *is the class of* ESO^σ*-formulas that admit a Skolemized prenex form* $\exists\tau\forall\boldsymbol{x}\phi(\sigma,\tau,\boldsymbol{x})$ *such that:* ϕ *is quantifier-free,* $\boldsymbol{x}$ *is a* d*-tuple of first-order variables and* $\mathit{arity}(\tau) \leq \ell$*. Similarly,* $\mathrm{ESO}^\sigma(\mathit{var}\ d, \mathit{arity}\,\ell)$ *is the class of* ESO^σ*-formulas of the form* $\exists\tau\phi(\sigma,\tau,\boldsymbol{x})$ *where, this time,* ϕ *is a non-necessarily prenex first-order formula with* d *variables. Finally, we set* $\mathrm{ESO}^\sigma(\forall^d) = \bigcup_\ell \mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\,\ell)$*,* $\mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\,\ell) = \bigcup_d \mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\,\ell)$ *and* $\mathrm{ESO}^\sigma(\mathit{arity}\,\ell) = \bigcup_d \mathrm{ESO}^\sigma(\mathit{var}\ d, \mathit{arity}\,\ell)$*.*

The expressive power of these kinds of formulas is quite high. A σ-NRAM is a nondeterministic Random Access Machine that takes σ-structures as input in the following way: for each $s \in \sigma$ of arity ℓ and each ℓ-tuple $\boldsymbol{t} \in D^\ell$, a special register $[s, \boldsymbol{t}]$ contains the value of $s(\boldsymbol{t})$. Let $\mathrm{NTIME}^\sigma(n^d)$ be the class of problems on σ-structures decidable by a σ-NRAM in time $O(n^d)$ where n is the size of the domain D of structures. The following is true.

Theorem 2 ([8]) *For all signature* σ *and all* $d \geq 1$*:*

$$\mathrm{NTIME}^\sigma(n^d) = \mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\, d).$$

Actually, we will mainly deal with the logic $\mathrm{ESO}^\sigma(\forall^d, \mathrm{arity}\,1)$. It is easily seen that the conjunctive normal form of an $\mathrm{ESO}^\sigma(\forall^d, \mathrm{arity}\,1)$-formula also belongs to $\mathrm{ESO}^\sigma(\forall^d, \mathrm{arity}\,1)$. That is, each such formula can be written: $\exists\tau\forall\boldsymbol{x}(C_1 \wedge \ldots \wedge C_p)$, where the C_i's are clauses of signature $\sigma\tau$ over the d-tuple of variables $\boldsymbol{x}$. In the proofs of the forthcoming results, we will need to refer to a more restrictive notion of clause. The end of this section is devoted to the presentation of a normalization of formulas, related to this need.

Definition 3 *A* unary clause *is a quantifier free formula of the form*

$$\bigvee_i u_i(x_i) \neq v_i(y_i) \vee \bigvee_i f_i(a_i) = g_i(b_i) \tag{1}$$

where the u_i, v_i, f_i, g_i are unary function symbols and x_i, y_i, a_i, b_i are first-order variables such that $\forall i$, $x_i \neq y_i$ and $a_i \neq b_i$. We say that C is a τ-unary clause over $\boldsymbol{x}$ if all the unary function symbols of C belong to a signature τ and all its variables are taken from a tuple $\boldsymbol{x}$.

The following lemma provides an easy normal form for formulas in $\mathrm{ESO}^\sigma(\forall^d, \text{arity}\,1)$.

Lemma 4 *Let σ be a unary signature. Any formula Φ in $\mathrm{ESO}^\sigma(\forall^d, arity\,1)$, for $d \geq 2$, is equivalent to a formula Φ' of the form $\exists\tau : \forall x\phi \wedge \forall\boldsymbol{x}(C_1 \wedge \ldots \wedge C_p)$, also in $\mathrm{ESO}^\sigma(\forall^d, arity\,1)$, where:*

- *τ is a unary signature ;*
- *x (resp. $\boldsymbol{x}$) is a first-order variable (resp. a d-tuple of first-order variables) ;*
- *ϕ is a quantifier free formula of signature $\sigma\tau$;*
- *the C_i's are τ-unary clauses over $\boldsymbol{x}$.*

The formula Φ' will be called the clausal normal form *of Φ.*

Proof: The usual conjunctive normal form of Φ gives: $\Phi \equiv \exists\tau\forall\boldsymbol{x}\psi$, where τ is a unary signature, $\boldsymbol{x}$ is a d-tuple of first-order variables and $\psi \equiv (C_1 \wedge \ldots \wedge C_p)$ is a conjunction of $\sigma\tau$-clauses over $\boldsymbol{x}$. For convenience, we will rather write Φ as: $\Phi \equiv \exists\tau : \forall x\phi \wedge \forall\boldsymbol{x}\psi$, where ϕ is initially the boolean constant "true" (or equivalently, the formula $x = x$). Now, the way to pass from this conjunctive normal form to the expected clausal form can be described as a sequence of "instructions":

- Add two new existentially quantified constant symbols $0, 1$ and two new unary function symbols $\mathsf{zero}, \mathsf{one}$ and set that $0 \neq 1$ and that zero (resp. one) maps the domain onto 0 (resp. 1). That is: perform $\tau := \tau \cup \{0, 1, \mathsf{zero}, \mathsf{one}\}$ and $\phi := \phi \wedge (0 \neq 1) \wedge \mathsf{zero}(x) = 0 \wedge \mathsf{one}(x) = 1$.
- For each constant symbol c occurring in ψ, do: $\tau := \tau \cup \{[c]\}$, where $[c]$ is a new unary function symbol; $\phi := \phi \wedge ([c](x) = c)$; replace each occurrence of c in ψ by $[c](y)$, where y is any variable in $\boldsymbol{x}$.
- For each monadic relation symbol U occurring in ψ, do: $\tau := \tau \cup \{[U]\}$, where $[U]$ is a new unary function symbol; $\phi := \phi \wedge ((U(x) \rightarrow [U](x) = 1) \wedge (\neg U(x) \rightarrow [U](x) = 0))$; replace each subformula $U(t)$ (resp. $\neg U(t)$) of ψ, where t is a term, by $[U](t) = \mathsf{one}(t)$ (resp. $[U](t) = \mathsf{zero}(t)$).
- For each tuple $f_1, \ldots, f_i$, $i \geq 0$, of unary function symbols such that a term $f_1 \ldots f_i(y)$, $y \in \boldsymbol{x}$, appears in ψ do: $\tau := \tau \cup \{[f_1 \ldots f_i]\}$, where $[f_1 \ldots f_i]$ is a new unary function symbol; $\phi := \phi \wedge ([f_1 \ldots f_i](x) = f_1 \ldots f_i(x))$; replace each term $f_1 \ldots f_i(y)$ occurring in ψ by $[f_1 \ldots f_i](y)$.

These modifications clearly give rise to a formula equivalent to Φ of the form: $\exists\tau : \forall x\phi \wedge \forall\boldsymbol{x}(C_1 \wedge \ldots \wedge C_p)$, where τ, x,$\boldsymbol{x}$ and ϕ fulfil the requirements of Lemma 4. Furthermore, each C_j is a $\sigma\tau$-clause of the form: $\bigvee_i u_i(x_i) \neq v_i(y_i) \vee \bigvee_i f_i(a_i) = g_i(b_i)$, where the u_i, v_i, f_i, g_i are unary function symbols and x_i, y_i, a_i, b_i are first-order variables. To fit exactly to the statement of the lemma, it remains to write each C_j as a τ-clause and to ensure that for each subformula $u(x) \neq v(y)$ (resp. $f(x) = g(y)$) occurring in C_j, the variables x and y are distinct. This is done through the following instructions:

- For each symbol of function (resp. relation) $s \in \sigma$ occurring in ψ, do: $\tau := \tau \cup \{[s]\}$, where $[s]$ is a new function (resp. relation) symbol of the same arity than s; $\phi := \phi \wedge [s](x) = s(x)$ (resp. $\phi := \phi \wedge [s](x) \leftrightarrow s(x)$); replace each term (resp. subformula) $s(t)$ of ψ by $[s](t)$. After this step, each C_i contains only τ-terms.

- For each pair (a, b) of unary function symbols such that an equality $a(y) = b(y)$ (resp. disequality $a(y) \neq b(y)$) occurs in ψ, do: $\tau := \tau \cup \{[ab]\}$, where $[ab]$ is a new unary function symbol; $\phi := \phi \wedge (a(x) = b(x) \rightarrow [ab](x) = 1) \wedge (a(x) \neq b(x) \rightarrow [ab](x) = 0)$; replace each subformula $a(y) = b(y)$ (resp. $a(y) \neq b(y)$) of ψ by $[ab](y) = \mathsf{one}(z)$ (resp. $[ab](y) = \mathsf{zero}(z)$), where $z \in \boldsymbol{x}$ is a variable distinct from y.

This concludes the expected normalization of Φ. □

Remark 1 *From usual properties of quantifiers and connectives in ESO-formulas, it is clear that a formula* $\Phi \in \mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\,1)$ *can be written in* $\mathrm{ESO}^\sigma(\forall^1, \mathit{arity}\,1)$ *if, and only if, for each unary clause C involved in its clausal normal form, the first-order sentence $\forall \boldsymbol{x} C$ can be written in* $\mathrm{ESO}^\sigma(\forall^1, \mathit{arity}\,1)$*.*

Before concluding this section, we state without proof the following result, which will be used several times along this work:

Proposition 5 ([8]) *For all $\ell \geq d \geq 1$ and all signature σ:*

$$\begin{aligned}\mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\,\ell) &= \mathrm{ESO}^\sigma(\forall^d, \mathit{arity}\,d)\\ &= \mathrm{ESO}^\sigma(\forall^d)\\ &= \mathrm{ESO}^\sigma(\mathit{var}\ d).\end{aligned}$$

Remark 2 *In particular, we will freely use the fact that any* $\mathrm{ESO}^\sigma(\mathit{var}\ 1)$*-formula (with only one first-order variable and with guessed symbols of any arity) is equivalent to an* $\mathrm{ESO}^\sigma(\forall^1, \mathit{arity}\,1)$*-formula.*

3 Combinatorial tools

In this section, some combinatorial objects are defined and some basic properties about them are proved. These objects will play an important role in the rest of the paper. Let $[k]$ denote the interval of integers $\{1, 2, \ldots, k\}$.

Definition 6 *Let E, F be two finite sets and $\boldsymbol{g} = (g_1, \ldots, g_k)$ be a tuple of unary functions from E to F. Let $P \subseteq [k]$ and $(c_i)_{i \in P}$ be a family of elements of F. $(c_i)_{i \in P}$ is said to be a* sample *of $\boldsymbol{g}$ (indexed by P) over E if*

$$E = \bigcup_{i \in P} g_i^{-1}(c_i).$$

A sample is minimal *if, moreover, for all $j \in P$:*

$$E \neq \bigcup_{i \in P \setminus \{j\}} g_i^{-1}(c_i).$$

Finally, if $(c_i)_{i \in P}$ is a sample (resp. minimal sample), the set $(g_i^{-1}(c_i))_{i \in P}$ is called a covering *(resp.* minimal covering*) of E by $\boldsymbol{g}$.*

Remark 3 *Let $P \subseteq [k]$ and $(c_i)_{i \in P}$ be a sample of $\boldsymbol{g}$. Then, there exists $P' \subseteq P$ such that $(c_i)_{i \in P'}$ is a minimal sample of $\boldsymbol{g}$. Informally, it suffices to iteratively eliminate from P the j's such that $E = \bigcup_{i \in P \setminus \{j\}} g_i^{-1}(c_i)$.*

Although the bound on the number of samples for any tuple of functions depends on the size of the domain, the situation is far better for what concerns minimal samples.

Lemma 7 *Let E, F and $\boldsymbol{g}$ be as in Definition 6. The number of minimal samples of $\boldsymbol{g}$ is bounded by $k!$, i.e. depends only on k.*

Proof: The set E is identified with $\{1,\ldots,n\}$. We describe the construction of a tree T of depth n with at most $k!$ leaves of level n. These leaves represent all the minimal samples of $\boldsymbol{g}$. Level i of the tree corresponds to element i of E. Each node x is labelled by a subset $P_x \subseteq [k]$. The root r of the tree is labelled by $P_r = \emptyset$.
Let $i = 1$. There are at most k possibilities to cover element 1 with $g_j(1) = c_j$ $(j = 1,\ldots,k)$. Then, the root of T branches on k nodes x_j each labelled by $P_{x_j} = \{j\}$.
At each level i $(i = 2,\ldots,n)$, the same strategy is used. Let x be a node of level i labelled by P_x. Two cases are possible.

- Either $i \in \bigcup_{j\in P_x} g_j^{-1}(c_j)$ i.e. element i is already covered by $(g_j^{-1}(c_j))_{j\in P_x}$. In this case, node x branches on a unique point x' of level $i+1$ labelled by $P_{x'} = P_x$.
- Or i is not yet covered by $(g_j^{-1}(c_j))_{j\in P_x}$. Two subcases may hold.
  - Either $P_x = [k]$. Then, because $i \notin \bigcup_{j\in[k]} g_j^{-1}(c_j)$, it is not possible to cover element i and the construction fails and stops here for that branch.
  - Or $P_x \subsetneq [k]$. Then, there are $k - card(P_x)$ possible ways to cover element i with $g_j(i) = c_j$ $(j \in [k]\backslash P_x)$. Node x branches on $k - card(P_x)$ nodes x_j each labelled by $P_{x_j} = P_x \cup \{j\}$.

It is easily seen that each leaf of level n corresponds to (at most) one sample of $\boldsymbol{g}$ and that all the minimal samples of $\boldsymbol{g}$ are represented by such leaves. By construction, for every value of $k \geq 2$, there is at most one node of degree k on each branch. All the other nodes of the branch have degree one. Then, the number of leaves of level n is bounded by $k!$ and so is the number of minimal samples. □

4 Some hints of the method

The main goal of this paper is to investigate the possibility to eliminate all but one universal variables in formulas whose relation and function symbols are of arity one. In this section, we informally describe on a very simple example the method which achieves this aim (at least under certain conditions). This should give to the reader a precise idea of the way the proof proceeds in more general statements.
Let us consider the following formula, obtained by fully quantifying a two-variable and positive unary clause:

$$\phi \equiv \forall x \forall y : f_1(x) = g_1(y) \vee \ldots \vee f_k(x) = g_k(y).$$

Clearly, ϕ holds in a finite structure of domain D if, and only if,

$$\forall x \in D : \; D = \bigcup_{i\leq k} g_i^{-1}(f_i(x)). \tag{2}$$

That is, from Definition 6:

$$\forall x \in D, \text{ the tuple } (f_i(x))_{i\in[k]} \text{ is a sample of } \boldsymbol{g} \text{ over } D.$$

Since each sample contains a minimal sample (see Remark 3), we can write (2) as:

$$\begin{array}{c}\forall x \; \exists P \exists (c_i)_{i\in P} : (c_i)_{i\in P} \text{ is a minimal} \\ \text{sample of } \boldsymbol{g} \text{ over } D \text{ and } \bigwedge_{i\in P} f_i(x) = c_i.\end{array} \tag{3}$$

But the number of minimal samples is bounded by $k!$ (Lemma 7). Hence, existential quantification over samples can be replaced by a disjunction over the $k!$ possibilities. Thus, (3) becomes:

$$\begin{array}{ll}\exists (P^1, (c_i^1)_{i\in P^1}), \ldots, (P^{k!}, (c_i^{k!})_{i\in P^{k!}}) : & \\ \quad \bigwedge_{j\leq k!} \forall y \bigvee_{i\in P^j} g_i(y) = c_i^j \; \wedge & (*) \\ \quad \forall x \bigvee_{j\leq k!} \bigwedge_{i\in P^j} f_i(x) = c_i^j & (**)\end{array} \tag{4}$$

Here, $(*)$ asserts that the tuples $(c_i^j)_{i\in P^j}$, for $j \leq k!$, are all samples of $\boldsymbol{g}$ over the domain [2]. Their interpretation can be taken among minimal ones, hence the bound on j. Formula $(**)$ guarantees that for each x, $(f_i(x))_{i\in[k]}$ contains one of these samples.
It remains to formulate (4) in such a way that it fits with our logical framework. The problem lies in the quantifications over the P^j's (which are subsets of $[k]$) and in the way these sets are involved in the definitions of the tuples (c_i^j)'s. In order to "internalize" these quantifications in our logic, we introduce a tuple of boolean variables $(S_i^j)_{i\leq k}$ for each sample $(P^j, (c_i^j)_{i\in P^j})$. The purpose of such a tuple is to describe which i's have to be considered in the tuple (c_i^j). More precisely, it is easily seen that Equation (4) is equivalent to:

$$\begin{array}{c} \exists (S_i^1, c_i^1)_{i\leq k}, \ldots, (S_i^{k!}, c_i^{k!})_{i\leq k} : \\ \bigwedge_{j\leq k!} \forall y \bigvee_{i\leq k} [S_i^j \wedge g_i(y) = c_i^j] \wedge \\ \forall x \bigvee_{j\leq k!} \bigwedge_{i\leq k} [S_i^j \rightarrow f_i(x) = c_i^j] \end{array} \tag{5}$$

Here, the S_i^j's are boolean variables[3] and the c_i^j's are constant symbols. Now, (5) clearly appears as an ESO-formula which can trivially be put in prenex form with only one universal quantifier. We won't give further evidence of the equivalence (2) $\Leftrightarrow$ (5): it will be detailed, in a more general context, in the proof of Theorem 8. Moreover, notice that Formula (5) may also be put in positive form (where no negation occurs) via an easy reformulation that uses a big disjunction instead of introducing explicitly the second-order variables S_i^j.

5 On formulas with two first-order variables

Let us first introduce some notations, in order to simplify the presentation of forthcoming results: if $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ is a tuple of unary functions on a domain D, we denote by $\boldsymbol{u}(x)$ the tuple $(u_1(x), \ldots, u_\ell(x))$ and we abbreviate by $\boldsymbol{u}(x) = \boldsymbol{v}(y)$ the conjunction $\bigwedge_{i\leq \ell} u_i(x) = v_i(y)$. Furthermore, for each $\boldsymbol{\alpha} \in D^\ell$, we denote by $\boldsymbol{u}^{-1}(\boldsymbol{\alpha})$ the set $\{x \in D : \boldsymbol{u}(x) = \boldsymbol{\alpha}\}$. We will use several times the following trivial fact:

Remark 4 *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be two tuples of variables of respective arities p, q and let β be a specified variable in $\boldsymbol{\beta}$. Let furthermore $\boldsymbol{w}$ be a p-tuple of unary function symbols. Then, for any formula $\Phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$, the following formulas are equivalent:*

- $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} : \boldsymbol{w}(\beta) = \boldsymbol{\alpha} \rightarrow \Phi(\boldsymbol{\alpha}, \boldsymbol{\beta})$;
- $\forall \boldsymbol{\beta} : \Phi(\boldsymbol{w}(\beta), \boldsymbol{\beta})$.

Here is our first main result:

Theorem 8 *For every signature σ s.t. arity$(\sigma) \leq 1$:*

$$\mathrm{ESO}^\sigma(\forall^2, \mathit{arity}\,1) = \mathrm{ESO}^\sigma(\forall^1, \mathit{arity}\,1).$$

Proof: The right-to-left inclusion is straightforward. For the other one, we have to prove that each formula of $\mathrm{ESO}^\sigma(\forall^2, \mathrm{arity}\,1)$ can be written in $\mathrm{ESO}^\sigma(\forall^1, \mathrm{arity}\,1)$. Because of Lemma 4 and Remark 1, we can restrict our attention to a formula ϕ obtained by quantifying a single two-variable unary clause. According to Definition 3, we can set: $\phi \equiv \forall x, y : \bigvee_{i\leq \ell} u_i(x) \neq v_i(y) \vee \bigvee_{i\leq k} f_i(x) = g_i(y)$ or, equivalently:

$$\phi \equiv \forall x, y : \bigwedge_{i\leq \ell} u_i(x) = v_i(y) \rightarrow \bigvee_{i\leq k} f_i(x) = g_i(y).$$

[2]Some of them may be identical i.e. there can be repetitions.
[3]The reader who doesn't agree with the use of boolean variables in this framework can replace them by monadic relations and substitute to each occurrence of S_i^j the formula $S_i^j(0)$.

With the notations introduced above, we get:

$$\phi \equiv \forall x, y : \boldsymbol{u}(x) = \boldsymbol{v}(y) \to \bigvee_{i \le k} f_i(x) = g_i(y),$$

or equivalently:

$$\forall x (\forall y \in \boldsymbol{v}^{-1}(\boldsymbol{u}(x))) : \bigvee_{i \le k} f_i(x) = g_i(y),$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ now denote two ℓ-tuples of unary function symbols. With the definition of samples (Definition 6), this exactly means that for each x, $(f_i(x))_{i \le k}$ contains a minimal sample of $\boldsymbol{g}$ over $\boldsymbol{v}^{-1}(\boldsymbol{u}(x))$. We will prefer the more convenient formulation: for each $\boldsymbol{\alpha} \in D^\ell$ and for each $x \in D$: if $\boldsymbol{u}(x) = \boldsymbol{\alpha}$, then $(f_i(x))_{i \le k}$ contains a minimal sample of $\boldsymbol{g}$ over $\boldsymbol{v}^{-1}(\boldsymbol{\alpha})$. Using the fact that the number of such samples is bounded by $k!$ (Lemma 7) and describing samples over $\boldsymbol{v}^{-1}(\boldsymbol{\alpha})$ *via* boolean variables S_i^j (see Section 4), this can be written:

$$\begin{gathered} \forall \boldsymbol{\alpha}\, \exists (S_i^1, c_i^1)_{i \le k}, \ldots, (S_i^{k!}, c_i^{k!})_{i \le k} : \\ (\forall y \in \boldsymbol{v}^{-1}(\boldsymbol{\alpha})) \bigwedge_{j \le k!} \bigvee_{i \le k} [S_i^j \wedge g_i(y) = c_i^j] \wedge \\ (\forall x \in \boldsymbol{u}^{-1}(\boldsymbol{\alpha})) \bigvee_{j \le k!} \bigwedge_{i \le k} [S_i^j \to f_i(x) = c_i^j] \end{gathered}$$

Here, the c_i^j's are constant symbols. The first conjunct asserts that for each $j \le k!$, $(c_i^j)_{\{i \text{ s.t. } S_i^j\}}$ is a sample of $\boldsymbol{g}$ over $\boldsymbol{v}^{-1}(\boldsymbol{\alpha})$, while the second conjunct says that for each $x \in \boldsymbol{u}^{-1}(\boldsymbol{\alpha})$, $(f_i(x))_{i \le k}$ contains one of these samples. Writing this last formula under Skolemized form, we get the following equivalent formula:

$$\begin{gathered} \Theta \equiv \exists (S_i^1, c_i^1)_{i \le k}, \ldots, (S_i^{k!}, c_i^{k!})_{i \le k}\, \forall \boldsymbol{\alpha} : \\ (\forall y \in \boldsymbol{v}^{-1}(\boldsymbol{\alpha})) \bigwedge_{j \le k!} \bigvee_{i \le k} [S_i^j(\boldsymbol{\alpha}) \wedge g_i(y) = c_i^j(\boldsymbol{\alpha})] \\ (\forall x \in \boldsymbol{u}^{-1}(\boldsymbol{\alpha})) \bigvee_{j \le k!} \bigwedge_{i \le k} [S_i^j(\boldsymbol{\alpha}) \to f_i(x) = c_i^j(\boldsymbol{\alpha})] \end{gathered}$$

where the S_i^j (resp. c_i^j) are now ℓ-ary relation (resp. function) symbols (recall that ℓ is both the arity of $\boldsymbol{\alpha}$ and of the tuples of unary functions $\boldsymbol{u}$ and $\boldsymbol{v}$). From Remark 4, Θ, hence ϕ, is finally equivalent to:

$$\begin{gathered} \Phi \equiv \exists (S_i^1, c_i^1)_{i \le k}, \ldots, (S_i^{k!}, c_i^{k!})_{i \le k} : \\ \forall y \bigwedge_{j \le k!} \bigvee_{i \le k} [S_i^j(\boldsymbol{v}(y)) \wedge g_i(y) = c_i^j(\boldsymbol{v}(y))] \\ \forall x \bigvee_{j \le k!} \bigwedge_{i \le k} [S_i^j(\boldsymbol{u}(x)) \to f_i(x) = c_i^j(\boldsymbol{u}(x))], \end{gathered}$$

where the S_i^j (resp. c_i^j) are ℓ-ary relation (resp. function) symbols [4]. This concludes the proof, since Φ can trivially be written in $\mathrm{ESO}(\forall^1, \mathrm{arity}\, \ell)$ and therefore, in $\mathrm{ESO}(\forall^1, \mathrm{arity}\, 1)$ (see Proposition 5). □

Remark 5 *Theorem 8 is tight for σ with arity$(\sigma) \le 1$. It is proved in [8] that if $\sigma = \{E\}$ where E is a binary relation symbol, there exists a graph property (namely : being a complete graph) definable in $\mathrm{ESO}^\sigma(\forall^2, \mathit{arity}\, 1)$ and not in $\mathrm{ESO}^\sigma(\forall^1, \mathit{arity}\, 1) = \mathrm{ESO}^\sigma(\forall^1)$.*

Theorem 8 extends to the case of not necessarily prenex formulas with two variables thanks to the following proposition.

Proposition 9 *For any signature σ and any $d \ge 1$,*

$$\mathrm{ESO}^\sigma(\mathit{var}\ 2, \mathit{arity}\, d) = \mathrm{ESO}^\sigma(\forall^2, \mathit{arity}\, d).$$

[4] But always used with one variable.

Proof: The right-to-left inclusion is evident. For the other direction, we first prove that for any signature τ, every two-variable first-order formula ϕ over τ can be written in $\mathrm{ESO}^\tau(\forall^2, \mathrm{arity}\,1)$. The proof is given by induction on the number n of quantifiers in ϕ. The case $n = 0$ is evident. Suppose now the proposition is true for $n - 1$. Let θ be a sub-formula of ϕ of the form:

$$\theta \equiv Qx\ \lambda(x, y),$$

where $\lambda(x, y)$ is quantifier-free and $Q \in \{\forall, \exists\}$. The intuitive idea is to substitute to the formula $\theta(y)$ a new unary predicate $A(y)$. Let ϕ_0 be the formula obtained after such substitution. Clearly, ϕ is equivalent to:

$$\exists A\ (\phi_0 \wedge \forall y(A(y) \rightarrow Qx\ \lambda(x, y))).$$

Moreover, ϕ_0 has $n-1$ quantifiers (and still at most two variables) and therefore, by the induction hypothesis, it can be written as: $\exists\nu\forall x\forall y\psi_0$, where $\mathrm{arity}(\nu) \leq 1$ and ψ_0 is quantifier-free. Thus, ϕ is equivalent to:

$$\tilde{\phi} \equiv \exists A\exists\nu\ (\forall x\forall y\psi_0 \wedge \forall y(A(y) \rightarrow Qx\ \lambda(x, y))),$$

where $\mathrm{arity}(\nu) \leq 1$ and ψ is quantifier-free. Two cases may now happen:

- $Q = \exists$. By Skolemization, we obtain:

$$\tilde{\phi} \equiv \exists A\exists\nu\exists f\forall x\forall y\ (\psi_0 \wedge (A(y) \rightarrow \lambda(f(y), y)));$$

- $Q = \forall$. The following is obtained:

$$\tilde{\phi} \equiv \exists A\exists\nu\forall x\forall y\ (\psi_0 \wedge (A(y) \rightarrow \lambda(x, y))).$$

In both cases, $\tilde{\phi}$ belongs to $\mathrm{ESO}^\tau(\forall^2, \mathrm{arity}\,1)$. This is the seeked formula.
The left-to-right inclusion of Proposition 9 easily follows: Any formula $\Phi \equiv \exists\tau\phi(\sigma, \tau)$ in $\mathrm{ESO}^\sigma(\mathrm{var}\ 2, \mathrm{arity}\,d)$ is equivalent to $\tilde{\Phi} \equiv \exists\tau\tilde{\phi}(\sigma, \tau)$, where $\tilde{\phi}$ is the $\mathrm{ESO}^{\sigma\tau}(\forall^2, \mathrm{arity}\,1)$-formula equivalent to ϕ. And $\tilde{\Phi}$ clearly belongs to $\mathrm{ESO}^\sigma(\forall^2, \mathrm{arity}\,d)$. □

From Proposition 9 and Theorem 8, the following corollary can be immediately derived.

Corollary 10 *For every unary signature σ:*

$$\mathrm{ESO}^\sigma(\mathit{var}\ 2, \mathit{arity}\,1) = \mathrm{ESO}^\sigma(\forall 1, \mathit{arity}\,1).$$

The results of this section permit to derive a hierarchy based on arity for formulas with two variables (or two universal quantifiers). As stated by the next result, this hierarchy is strict for the two first levels and collapses right after.

Corollary 11 *For every signature σ and every $d \geq 2$,*

$$\begin{array}{rcl} \mathrm{ESO}^\sigma(\forall^2, \mathit{arity}\,1) & \subsetneq & \mathrm{ESO}^\sigma(\forall^2, \mathit{arity}\,2) \\ & = & \mathrm{ESO}^\sigma(\forall^2, \mathit{arity}\,d) \end{array}$$

and

$$\begin{array}{rcl} \mathrm{ESO}^\sigma(\mathit{var}\ 2, \mathit{arity}\,1) & \subsetneq & \mathrm{ESO}^\sigma(\mathit{var}\ 2, \mathit{arity}\,2) \\ & = & \mathrm{ESO}^\sigma(\mathit{var}\ 2, \mathit{arity}\,d) \end{array}$$

Proof: Only the proof of the first hierarchy is given (the other one is obtained similarly). The equality is proved in [8] for every σ. The strict inclusion is proved in two steps depending on the arity of σ.

First, suppose that $\text{arity}(\sigma) \leq 1$. Then,

$$\begin{array}{rcll} \text{ESO}^\sigma(\forall^2, \text{arity}\,1) & = & \text{ESO}^\sigma(\forall^1, \text{arity}\,1) & \text{(Thm.8)} \\ & = & \text{NTIME}^\sigma(n) & \text{(Thm.2)} \\ & \subsetneq & \text{NTIME}^\sigma(n^2) & \text{([2])} \\ & = & \text{ESO}^\sigma(\forall^2, \text{arity}\,2) & \text{(Thm.2)} \end{array}$$

Now, if $\text{arity}(\sigma) \geq 2$, then σ contain at least one binary relation symbol, say E. It is a consequence of a result in [1] that the set of finite graphs with an even number of edges is in $\text{ESO}^{\{E\}}(\forall^2, \text{arity}\,2)$ but not in $\text{ESO}^{\{E\}}(\forall^1, \text{arity}\,1)$. □

6 The case of acyclic or positive formulas

In this section, we generalize Theorem 8 to formulas with any number of universal variables. However, in that context, it seems not immediate to extend this result to the case of formulas with unrestricted negative terms. In the following lines, the notion of *acyclic* formulas is defined that restrict the way negation is used inside clauses.

Definition 12 *Let C be a unary clause over the tuple of variables $\boldsymbol{x}$. The undirected graph $G_C = (V, E)$ is defined by: $V = \{x : x \in \boldsymbol{x}\}$ and $\forall x, y \in V$, $(x, y) \in E$ iff C contains at least one negative atomic formula of the form $u(x) \neq v(y)$.*

Example: If $C \equiv f(x) \neq g(y) \vee u(y) \neq v(z) \vee h(x) \neq k(z) \vee f(t) = h(y)$, then G_C is the clique K_3 over vertices x, y, z.

Definition 13 *A unary clause C is said acyclic if its associated graph G_C is acyclic. An* ESO*-formula Φ is acyclic if all the unary clauses occurring in its clausal normal form*[5] *are acyclic. By $\text{ESO}^\sigma(\forall^d, \mathit{arity}\,1, \mathit{acy})$ we denote the class of acyclic $\text{ESO}^\sigma(\forall^d, \mathit{arity}\,1)$-formulas. Moreover, we denote by $\text{ESO}^\sigma(\forall^d, \mathit{arity}\,1, \mathit{pos})$ the class of $\text{ESO}^\sigma(\forall^d, \mathit{arity}\,1)$-formulas without negation. Finally, we set $\text{ESO}^\sigma(\forall^*, \mathit{arity}\,1, \mathit{acy}) = \bigcup_d \text{ESO}^\sigma(\forall^d, \mathit{arity}\,1, \mathit{acy})$ and $\text{ESO}^\sigma(\forall^*, \mathit{arity}\,1, \mathit{pos}) = \bigcup_d \text{ESO}^\sigma(\forall^d, \mathit{arity}\,1, \mathit{pos})$.*

Note that $\text{ESO}^\sigma(\forall^1, \text{arity}\,1)$-formulas are all acyclic. Note also that the expressive power of $\text{ESO}^\sigma(\forall^*, \text{arity}\,1, \text{pos})$ is quite high since such formulas are able to express some NP-complete problems for $\text{arity}(\sigma) = 1$ (see [14]).
The main result of this section asserts that, provided formulas are acyclic or positive, adding universally quantified variables does not increase the expressive power.

Theorem 14 *For each unary signature σ:*

1. $\text{ESO}^\sigma(\forall^*, \mathit{arity}\,1, \mathit{acy}) = \text{ESO}^\sigma(\forall^1, \mathit{arity}\,1)$;
2. $\text{ESO}^\sigma(\forall^*, \mathit{arity}\,1, \mathit{pos}) = \text{ESO}^\sigma(\forall^1, \mathit{arity}\,1, \mathit{pos})$.

Proof: The right-to-left inclusions are trivial. The proof of the converse inclusions will be done through the following lemma.

Lemma 15 *For each unary signature σ and all $d \geq 2$:*

1. $\text{ESO}^\sigma(\forall^{d+1}, \mathit{arity}\,1, \mathit{acy}) \subseteq \text{ESO}^\sigma(\forall^d, \mathit{arity}\,1, \mathit{acy})$;
2. $\text{ESO}^\sigma(\forall^{d+1}, \mathit{arity}\,1, \mathit{pos}) \subseteq \text{ESO}^\sigma(\forall^d, \mathit{arity}\,1, \mathit{pos})$.

[5] See Lemma 4.

Proof: Only the proof of the first item is given (the proof of the second one is similar but simpler). From Lemma 4 and Definition 13, and since $\mathrm{ESO}^\sigma(\forall^d, \mathrm{arity}\,1, \mathrm{acy})$ is closed under conjunction, the result holds if, and only if, each formula of the form $\forall \boldsymbol{x} C$ - where $\boldsymbol{x}$ is a $(d+1)$-tuple of variables and C is an acyclic unary clause over $\boldsymbol{x}$ - can be written in $\mathrm{ESO}^\sigma(\forall^d, \mathrm{arity}\,1, \mathrm{acy})$. So, let $\phi \equiv \forall \boldsymbol{x} C$ be such a formula. To achieve the proof we need a method to separate the variables of ϕ. For this, we use the well-known fact: *Every nonempty acyclic graph contains at least one vertex of degree bounded by* 1*, that is, a vertex which is either isolated or related to a unique vertex* y. Applied to the acyclic graph G_C, this allows to rewrite ϕ in the following way:

$$\begin{array}{c}\forall x \forall \boldsymbol{y} : \boldsymbol{u}(x) = \boldsymbol{v}(y_1) \rightarrow \\ (\psi(\boldsymbol{y}) \vee \bigvee_{i \leq k} f_i(x) = g_i(y_{p_i}))\end{array} \tag{6}$$

Here, $\boldsymbol{y}$ is a d-tuple $(y_1, \ldots, y_d)$ of variables, x is a variable related to a single variable[6] $y_1 \in \boldsymbol{y}$ in the graph G_C, $\psi(\boldsymbol{y})$ is a unary clause over $\boldsymbol{y}$, and for each $i \leq k$, $y_{p_i} \in \boldsymbol{y}$. The idea is now to treat separately the variable x, in the spirit of the method used so far. For each ℓ-tuple $\boldsymbol{\alpha}$ of variables in a domain D, where ℓ in the number of unary functions occurring in $\boldsymbol{u}$ (resp. $\boldsymbol{v}$), let us denote by $X_{\boldsymbol{\alpha}}$ the set $\{\boldsymbol{y} \in D^{d+1} : \boldsymbol{v}(y_1) = \boldsymbol{\alpha} \wedge \neg\psi(\boldsymbol{y})\}$. Furthermore, let us define a mapping $G_i : D^{d+1} \rightarrow D$, for each $i \leq k$, by: $\forall \boldsymbol{y} \in D^{d+1} : G_i(\boldsymbol{y}) = g_i(y_{p_i})$. Then (6) can be written:

$$\forall x (\forall \boldsymbol{y} \in X_{\boldsymbol{u}(x)}) : \bigvee_{i \leq k} f_i(x) = G_i(\boldsymbol{y})) \tag{7}$$

This means: for each $x \in D$, $(f_i(x))_{i \leq k}$ contains a minimal sample of $\boldsymbol{G} = (G_1, \ldots, G_k)$ over $X_{\boldsymbol{u}(x)}$. The reader should now be familiar with writing this assertion as:

$$\Theta \equiv \exists (S_i^1, c_i^1)_{i \leq k}, \ldots, (S_i^{k!}, c_i^{k!})_{i \leq k} \forall \boldsymbol{\alpha} : (8) \wedge (9)$$

where S_i^j (resp. c_i^j) are new ℓ-ary relation (resp. function) symbols and (8) and (9) are the formulas below:

$$\begin{array}{c}\forall \boldsymbol{y} : (\boldsymbol{v}(y_1) = \boldsymbol{\alpha} \wedge \neg\psi(\boldsymbol{y})) \rightarrow \\ \bigwedge_{j \leq k!} \bigvee_{i \leq k} [S_i^j(\boldsymbol{\alpha}) \wedge G_i(\boldsymbol{y}) = c_i^j(\boldsymbol{\alpha})])\end{array} \tag{8}$$

$$\begin{array}{c}\forall x : \boldsymbol{u}(x) = \boldsymbol{\alpha} \rightarrow \\ \bigvee_{j \leq k!} \bigwedge_{i \leq k} [S_i^j(\alpha) \rightarrow f_i(x) = c_i^j(\alpha)]\end{array} \tag{9}$$

But formula (8) is easily seen equivalent to:

$$\begin{array}{c}\forall \boldsymbol{y} : \boldsymbol{v}(y_1) = \boldsymbol{\alpha} \rightarrow (\psi(\boldsymbol{y}) \vee \\ \bigwedge_{j \leq k!} \bigvee_{i \leq k} [S_i^j(\boldsymbol{\alpha}) \wedge g_i(y_{p_i}) = c_i^j(\boldsymbol{\alpha})])\end{array} \tag{10}$$

Therefore, the formula Θ above is equivalent to:

$$\exists (S_i^1, c_i^1)_{i \leq k}, \ldots, (S_i^{k!}, c_i^{k!})_{i \leq k} \forall \boldsymbol{\alpha} : (10) \wedge (9),$$

which is equivalent, from Remark 4, to:

$$\begin{array}{c}\Phi \equiv \exists (S_i^1, c_i^1)_{i \leq k}, \ldots, (S_i^{k!}, c_i^{k!})_{i \leq k} : \\ \forall \boldsymbol{y} (\psi(\boldsymbol{y}) \vee \\ \bigwedge_{j \leq k!} \bigvee_{i \leq k} [S_i^j(\boldsymbol{v}(y_1)) \wedge g_i(y_{p_i}) = c_i^j(\boldsymbol{v}(y_1))]) \wedge \\ \forall x \bigvee_{j \leq k!} \bigwedge_{i \leq k} [S_i^j(\boldsymbol{u}(x)) \rightarrow f_i(x) = c_i^j(\boldsymbol{u}(x))]\end{array}$$

where S_i^j (resp. c_i^j) are ℓ-ary relation (resp. function) symbols. It should be noted that the ℓ-ary *ESO* symbols S_i^j and c_i^j only occur in terms $t(x)$ or $t(y_1)$ depending on only one variable (x or

[6] In case x is isolated, there is no precondition $\boldsymbol{u}(x) = \boldsymbol{v}(y_1)$ and the proof is even simpler.

y_1). For each t, a unary function $[t]$ is introduced and each term $t(x)$ (resp. $t(y_1)$) is replaced by $[t](x)$ (resp. $[t](y_1)$). Then, an $\mathrm{ESO}^\sigma(\forall^1, \mathrm{arity}\, \ell)$-formula is added that states:

$$\forall x\ t(x) \leftrightarrow [t](x).$$

Applying Remark 2, this formula can be in turn replaced by an equivalent $\mathrm{ESO}^\sigma(\forall^1, \mathrm{arity}\, 1)$-formula. Finally, it is easily seen that Φ can be transformed into a formula in $\mathrm{ESO}^\sigma(\forall^d, \mathrm{arity}\, 1, \mathrm{acy})$. □

The proof of Theorem 14 easily follows by induction from Lemma 15 and Theorem 8 (for the base case). □

Finally, the following separation result means that for positive and universally quantified formulas, binary ESO functions express strictly more than unary ones.

Corollary 16 *For any signature σ:*

$$\mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\, 1, \mathit{pos}) \subsetneq \mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\, 2, \mathit{pos}).$$

Proof: From Theorem 14 and Theorem 2, we get $\mathrm{ESO}^\sigma(\forall^*, \mathrm{arity}\, 1, \mathrm{pos}) \subseteq \mathrm{NTIME}^\sigma(n)$. Moreover, by results of [14], each problem in $\mathrm{NTIME}(n^2)$ can be reduced in linear time to some problem expressible in $\mathrm{ESO}^\sigma(\forall^*, \mathrm{arity}\, 2, \mathrm{pos})$ for some unary signature σ. The result follows for any unary signature σ by the strict inclusion $\mathrm{NTIME}(n) \subsetneq \mathrm{NTIME}(n^2)$ [2]. The result also holds for any signature of arity ≥ 2 since the set of complete graphs belongs to $\mathrm{ESO}^\sigma(\forall^2, \mathrm{arity}\, 2, \mathrm{pos})$ for $\sigma = \{E\}$ but does not belong to $\mathrm{NTIME}^\sigma(n)$ [8]. □

7 Conjectures and consequences on the arity hierarchy

The results given in this paper are modest but real contributions to the study of the trade-off between arity of guessed symbols and number of variables in existential second-order logic. However, we believe that the method used is this paper (that mixes simple combinatorial and logical arguments) is not limited to the case of logic with two variables. We also think it may contribute to prove that there exist strict hierarchies of definability based on the arity of predicates. In this section, we give some plausible (regarding to the results proved so far) conjectures and their consequences on arity hierarchies.

Conjecture 1 *For every signature σ such that $\mathit{arity}(\sigma) \leq 1$,*

$$\mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\, 1) = \mathrm{ESO}^\sigma(\forall^1).$$

So far, only the case of acyclic or positive formulas is solved (see Theorem 14). Notice that this conjecture does not hold if $\mathrm{arity}(\sigma) > 1$. The following proposition can be easily proved by padding arguments.

Proposition 17 *If Conjecture 1 is true then, for every d and every σ such that . $\mathit{arity}(\sigma) \leq d$,*

$$\mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\, d) = \mathrm{ESO}^\sigma(\forall^d).$$

As a corollary, Conjecture 1 implies that the arity hierarchy for universal formulas is true.

Corollary 18 *If Conjecture 1 is true then, for every d and every σ,*

$$\mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\, d) \subsetneq \mathrm{ESO}^\sigma(\forall^*, \mathit{arity}\, (d+1))$$

Proof: There are two cases depending on the relation between arity(σ) and d. If arity(σ) $\leq d$ then:

$$\begin{array}{rcll} \mathrm{ESO}^\sigma(\forall^*, \mathrm{arity}\, d) & = & \mathrm{ESO}^\sigma(\forall^d) & \textit{Conjecture } 1 \\ & = & \mathrm{NTIME}^\sigma(n^d) & (\mathrm{Thm.}2) \\ & \subsetneq & \mathrm{NTIME}^\sigma(n^{d+1}) & [2] \\ & = & \mathrm{ESO}^\sigma(\forall^{d+1}) & \\ & = & \mathrm{ESO}^\sigma(\forall^*, \mathrm{arity}\,(d+1)) & \end{array}$$

If arity(σ) $> d$, the strictness of the inclusion follows from results of [1]. □

Let $\boldsymbol{Q}$ be a quantifier prefix, i.e. a word over the alphabet $\{\forall, \exists\}$. A natural question is whether our result can also be extended to every first-order prefix. We state the following strong conjecture.

Conjecture 2 *For every quantifier prefix $\boldsymbol{Q}$ that contains $\forall$ and every signature σ with arity(σ) $\leq$ 1,*

$$\mathrm{ESO}^\sigma(\boldsymbol{Q}, \mathit{arity}\, 1) = \mathrm{ESO}^\sigma(\forall^1).$$

If Conjecture 2 is true, an analog of Proposition 17 can be proved. More interestingly, we would obtain the following results that would settle two long standing open problems in finite model theory [5, 6].

Corollary 19 *If Conjecture 2 is true then, for every d and every σ such that arity(σ) $\leq d$,*

$$\mathrm{ESO}^\sigma(\mathit{arity}\, d) = \mathrm{ESO}^\sigma(\forall^d).$$

Moreover, for every σ, the arity hierarchy over functions is strict at every level. That is:

$$\mathrm{ESO}^\sigma(\mathit{arity}\, d) \subsetneq \mathrm{ESO}^\sigma(\mathit{arity}\,(d+1))$$

Let $\mathrm{ESO}^\sigma_R(\mathrm{arity}\, d)$ be the restriction of $\mathrm{ESO}^\sigma(\mathrm{arity}\, d)$ to formulas that contain second-order relation symbols (and no function) of arity d.

Corollary 20 *If Conjecture 2 is true, then, for every σ, the arity hierarchy is strict at every level d. That is:*

$$\mathrm{ESO}^\sigma_R(\mathit{arity}\, d) \subsetneq \mathrm{ESO}^\sigma_R(\mathit{arity}\,(d+1))$$

Proof: Corollary 19 and the fact that functions of arity d can be seen as relations of arity $d+1$ imply:

$$\mathrm{ESO}^\sigma_R(\mathrm{arity}\, d) \subsetneq \mathrm{ESO}^\sigma_R(\mathrm{arity}\,(d+2)).$$

Then, by classical transfer results (see [5] and [9]), it can be shown that two consecutive levels cannot collapse. □

When $\sigma = \emptyset$, this hierarchy is originally called the spectrum arity hierarchy (see [5]). From Corollary 20, Conjecture 2 would also imply that the arity hierarchy for spectra is strict at each level.

8 Conclusion

The idea behind this research is that fixing the arity of guessed symbols in ESO-formulas strongly limits the number of first-order variables (or, equivalently, the complexity) that are really needed to express properties. The results proved in this paper are first steps in this program. In particular, we have obtained, for arity(σ) ≤ 1, that:

$$\mathrm{ESO}^\sigma(\forall^2, \mathrm{arity}\, 1) = \mathrm{ESO}^\sigma(\forall^1, \mathrm{arity}\, 1) = \mathrm{NTIME}^\sigma(n)$$

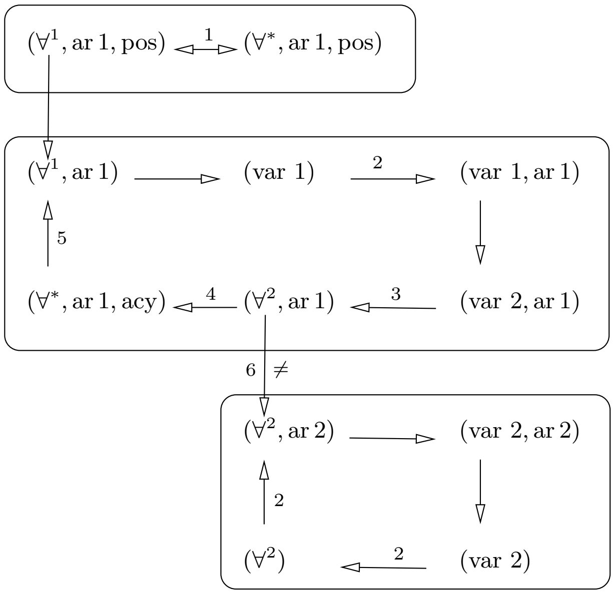

Figure 1: Main inclusions between the logics mentioned in this paper (over unary structures).

$$\text{and } \mathrm{ESO}^\sigma(\forall^*, \mathrm{arity}\,1, \mathrm{pos}) = \mathrm{ESO}^\sigma(\forall^1, \mathrm{arity}\,1, \mathrm{pos}).$$

In view to prove the two conjectures given in Section 7, it would certainly be essential to measure precisely the expressive power of these classes of formulas: $\mathrm{ESO}^\sigma(\forall^3, \mathrm{arity}\,1)$ and $\mathrm{ESO}^\sigma(\forall^2\exists^*, \mathrm{arity}\,1)$.

Figure 1 summarizes the results of this paper, where we denote, for instance, $(\forall^1, \mathrm{ar}\,1)$ for the logic $\mathrm{ESO}(\forall^1, \mathrm{arity}\,1)$. Each arrow corresponds to an inclusion. The classes included in a same frame are equal. Arrows without label correspond to trivial inclusions. Labels refer to the following justifications: (1) Theorem 14. (2) Proposition 5. (3) Proposition 9. (4) Any two-variable clause is acyclic. (5) Theorem 14. (6) Corollary 11.

9 Handling existential quantifiers

In this section, we consider positive prenex formula with two universally quantified variables and (for the moment) one existentially quantified one. Formulas are presented in DNF form:

$$\phi \equiv \forall x_1, x_2 \exists y : \bigvee_{i \leq k} \boldsymbol{u}_i(x_1) = \boldsymbol{v}_i(y) \wedge \boldsymbol{f}_i(x_2) = \boldsymbol{g}_i(y) \wedge \boldsymbol{a}_i(x_1) = \boldsymbol{b}_i(x_2).$$

Obviously, if $\boldsymbol{u}$ is a tuple of functions. Then, $\boldsymbol{u}(x) = \boldsymbol{v}(y)$ stands for $\bigwedge_{j \leq k} u_j(x) = v_j(y)$.
We need first to slightly extend the kind of object on which a (similar) combinatorial lemma can be applied. Let $\boldsymbol{u}$ be a tuple of functions and $\boldsymbol{c}$ be a tuple. Then, we set:

$$\boldsymbol{u}^{-1}(\boldsymbol{c}) = \{y : \boldsymbol{u}(y) = \boldsymbol{c}\}$$

Main definition and lemma of section3 can be adapted to tuples of functions (instead of functions) as follows.

Definition 21 *Let E, F be two finite sets and $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k)$ be a tuple of tuples of unary functions from E to F. Let $P \subseteq [k]$ and $(\boldsymbol{c}_i)_{i \in P}$ be a family of tuples of elements of F. $(\boldsymbol{c}_i)_{i \in P}$ is said to be a* sample *of $\boldsymbol{g}$ (indexed by P) over E if*

$$E = \bigcup_{i \in P} \boldsymbol{g}_i^{-1}(\boldsymbol{c}_i).$$

A sample is minimal *if, moreover, for all $j \in P$:*

$$E \neq \bigcup_{i \in P \setminus \{j\}} \boldsymbol{g}_i^{-1}(\boldsymbol{c}_i).$$

Finally, if $(\boldsymbol{c}_i)_{i \in P}$ is a sample (resp. minimal sample), the set $(\boldsymbol{g}_i^{-1}(\boldsymbol{c}_i))_{i \in P}$ is called a covering *(resp.* minimal covering*) of E by $\boldsymbol{g}$.*

As before, from any sample one can obtain a minimal one.

Lemma 22 *Let E, F and $\boldsymbol{g}$ be as in Definition 6. The number of minimal samples of $\boldsymbol{g}$ is bounded by $k!$, i.e. depends only on k.*

Though the natation are more complicated, the result is simpler to show. Sets of preimages of a unary functions form a partition of the domain while sets of preimages of tuples of unary functions are simply pairwise disjoint sets (which is sufficient in order to prove the lemma).
We now prove the following result:

Theorem 23 *For every signature σ s.t. arity$(\sigma) \leq 1$:*

$$\mathrm{ESO}^{\sigma}(\forall^2\exists^1, \mathit{arity}\,1, \mathit{pos}) \subseteq \mathrm{ESO}^{\sigma}(\forall^2, \mathit{arity}\,1).$$

Proof: Let Φ be a formula in $\mathrm{ESO}^{\sigma}(\forall^2\exists^1, \mathrm{arity}\,1, \mathrm{pos})$whose first-order part is as follows:

$$\phi \equiv \forall x_1, x_2 \exists y : \bigvee_{i \leq k} \boldsymbol{u}_i(x_1) = \boldsymbol{v}_i(y) \wedge \boldsymbol{f}_i(x_2) = \boldsymbol{g}_i(y) \wedge \boldsymbol{a}_i(x_1) = \boldsymbol{b}_i(x_2).$$

The existential quantification can be put inside formulas and the variable renamed:

$$\phi \equiv \forall x_1, x_2 : \bigvee_{i \leq k} \exists y_i \ \boldsymbol{u}_i(x_1) = \boldsymbol{v}_i(y_i) \wedge \boldsymbol{f}_i(x_2) = \boldsymbol{g}_i(y_i) \wedge \boldsymbol{a}_i(x_1) = \boldsymbol{b}_i(x_2).$$

□

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