THE TEN MARTINI PROBLEM

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Abstract. We prove the conjecture (known as the "Ten Martini Problem" after Kac and Simon) that the spectrum of the almost Mathieu operator is a Cantor set for all non-zero values of the coupling and all irrational frequencies.

1. Introduction

The almost Mathieu operator is the Schrödinger operator on $\ell^2(\mathbb{Z})$,

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(\theta + n\alpha)u_n, \tag{1.1}$$

where $\lambda, \alpha, \theta \in \mathbb{R}$ are parameters (called the *coupling*, *frequency*, and *phase*, respectively), and one assumes that $\lambda \neq 0$. The interest in this particular model is motivated both by its connections to physics and by a remarkable richness of the related spectral theory. This has made the latter a subject of intense research in the last three decades (see [L2] for a recent historical account and for the physics background). Here we are concerned with the topological structure of the spectrum.

If $\alpha = \frac{p}{q}$ is rational, it is well known that the spectrum consists of the union of q intervals called *bands*, possibly touching at the endpoints. In the case of irrational α, the spectrum $\Sigma_{\lambda,\alpha}$ (which in this case does not depend on θ) has been conjectured for a long time to be a Cantor set (see a 1964 paper of Azbel [Az]). To prove this conjecture has been dubbed The Ten Martini Problem by Barry Simon, after an offer of Mark Kac in 1981, see Problem 4 in [Sim1]. For a history of this problem see [L2]. Earlier partial results include [BS], [Sin], [HS], [CEY], [L1], and recent advances include [P] and [AK1]. In this paper, we solve the Ten Martini Problem as stated in [Sim1].

Main Theorem. *The spectrum of the almost Mathieu operator is a Cantor set for all irrational α and for all $\lambda \neq 0$.*

It is important to emphasize that the previous results mentioned above covered a large set of parameters (λ, α), which is both topologically generic ([BS]), and of full Lebesgue measure ([P]). As it often happens in the analysis of quasiperiodic systems, the "topologically generic" behavior is quite distinct from the "full Lebesgue measure" behavior, and the narrow set of parameters left behind does indeed lie in the interface of two distinct regimes. Furthermore, our analysis seems to indicate an interesting characteristic of the Ten Martini Problem, that the two regimes do not cover nicely the parameter space and hence there is a non-empty "critical region" of parameters in between (see Remarks 1.1, 5.1, 5.2 and the comments after Theorem 8.2).

This is to some degree reflected in the structure of the proof. While the reasoning outside of the critical region can be made quite effective, in the sense that one essentially identifies specific gaps in the spectrum[1], in order to be able to cover the critical region we make use of very indirect arguments. As an example, we show that absence of Cantor spectrum enables us to "analytically

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[1]Related either to gaps of periodic approximations or to eigenvalues of a dual almost Mathieu operator.

continuate" effective solutions of a small divisor problem, and it is the non-effective solutions thus obtained that can be related to gaps in the spectrum.

This paper builds on a large theory. Especially important for us are [CEY], [J], [P], whose methods we improve, but several other ingredients are needed (such as Kotani Theory [Sim2], the recent estimates on Lyapunov exponents of [BJ]). An important new ingredient is the use of analytic continuation techniques in the study of m-functions and in extending the reach of the analysis of Anderson localization.

1.1. **Strategy.** In this problem, arithmetics of α rules the game. When α is not very Liouville, it is reasonable to try to deal with the small divisors. When α is not very Diophantine, this does not work and we deal instead with rational approximation arguments. Let $\frac{p_n}{q_n}$ be the approximants of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$\beta = \beta(\alpha) = \limsup \frac{\ln q_{n+1}}{q_n}. \tag{1.2}$$

The relation between e^β and λ will play an important role in our argument, and will decide whether we approach the problem from the Diophantine side or from the Liouvillian side. As discussed before, our analysis indicates that there are parameters that can not be effectively described from either side, and it is only through the use of indirect arguments that we can enlarge artificially the Diophantine and Liouville regimes to cover all parameters. It should be noted that even with such tricks, both sides will just about meet in the middle.

Since $\Sigma_{\lambda,\alpha} = \Sigma_{-\lambda,\alpha}$, it is enough to assume $\lambda > 0$. It is known that the behavior of the almost Mathieu operator changes drastically at $\lambda = 1$ ("metal-insulator" transition [J]). Aubry duality shows that $\Sigma_{\lambda,\alpha} = \lambda \Sigma_{\lambda^{-1},\alpha}$. So each $\lambda \neq 1$ admits two lines of attack, and this will be determinant in what follows. The case $\lambda = 1$ was settled in [AK1] (after several partial results [AvMS], [HS], [L1]), but it is also recovered in our approach.

We will work on $\lambda < 1$ when approaching from the Liouville side. The approach from the Diophantine side is more delicate. There are actually two classical small divisor problems that apply to the study of the almost Mathieu operator, corresponding to Floquet reducibility (for $\lambda < 1$) and Anderson localization (for $\lambda > 1$). An important point is to attack both problems simultaneously, mixing the best of each problem ("soft" analysis in one case, "hard" analysis in the other).

A key idea in this paper is that absence of Cantor spectrum implies improved regularity of m-functions in the regime $0 < \lambda \leq 1$. This is proved by analytic continuation techniques. The improved regularity of m-functions (which is fictitious, since we will prove Cantor spectrum) will be used both in the Liouville side and in the Diophantine side. In the Liouville side, it will give improved estimates for the continuity of the spectrum with respect to the frequency. In the Diophantine side, it will allow us to use (again) analytic continuation techniques to solve some small divisor problems in some situations which are beyond what is expected to be possible.

Remark 1.1. Since our approach, designed to overcome the difficulties in the interface of the Diophantine and Liouville regimes, works equally well for other ranges of parameters, it will not be necessary in the proof to precisely delimitate a critical region. For the reasons discussed in Remarks 5.1, 5.2 and in the comments after Theorem 8.2, the critical region is believed to contain the parameters such that $\beta > 0$ and $\beta \leq |\ln \lambda| \leq 2\beta$, the parameters such that $\beta = |\ln \lambda|$ (respectively, $2\beta = |\ln \lambda|$) being seemingly inaccessible (even after artificial extension) by the Diophantine method (respectively, Liouville method). It is reasonable to expect that something should be different in the indicated critical region. For instance, it is the natural place to look for possible counterexamples to the "Dry Ten Martini" conjecture (for a precise formulation see Section 8).

2. Background

2.1. Cocycles, Lyapunov exponents, fibered rotation number. A (one-dimensional quasiperiodic $\mathrm{SL}(2,\mathbb{R})$) *cocycle* is a pair $(\alpha, A) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, understood as a *linear skew-product*:

$$\begin{aligned} (\alpha, A) : &\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \\ &(x, w) \mapsto (x+\alpha, A(x) \cdot w). \end{aligned} \tag{2.1}$$

For $n \geq 1$, we let

$$A_n(x) = A(x+(n-1)\alpha) \cdots A(x) \tag{2.2}$$

(α is implicit in this notation).

Given two cocycles (α, A) and (α, A'), a *conjugacy* between them is a continuous $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ such that

$$B(x+\alpha)A(x)B(x)^{-1} = A'(x). \tag{2.3}$$

The *Lyapunov exponent* is defined by

$$\lim \frac{1}{n} \int \ln \|A_n(x)\| dx, \tag{2.4}$$

so $L(\alpha, A) \geq 0$. It is invariant under conjugacy.

Assume now that $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is homotopic to the identity. Then there exists $\psi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and $u : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+$ such that

$$A(x) \cdot \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = u(x,y) \begin{pmatrix} \cos 2\pi(y+\psi(x,y)) \\ \sin 2\pi(y+\psi(x,y)) \end{pmatrix}. \tag{2.5}$$

The function ψ is called a *lift* of A. Let μ be any probability on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ which is invariant by the continuous map $T : (x,y) \mapsto (x+\alpha, y+\psi(x,y))$, projecting over Lebesgue measure on the first coordinate (for instance, take μ as any accumulation point of $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \nu$ where ν is Lebesgue measure on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$). Then the number

$$\rho(\alpha, A) = \int \psi d\mu \bmod \mathbb{Z} \tag{2.6}$$

does not depend on the choices of ψ and μ, and is called the *fibered rotation number* of (α, A), see [JM] and [H]. It is invariant under conjugacies homotopic to the identity. It immediately follows from the definitions that the fibered rotation number is a continuous function of (α, A).

Notice that if $A, A' : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ and $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ are continuous such that A is homotopic to the identity and $B(x+\alpha)A(x)B(x)^{-1} = A'(x)$, then $\rho(\alpha, A) = \rho(\alpha, A') - k\alpha$, where k is such that $x \mapsto B(x)$ is homotopic to $x \mapsto R_{kx}$, where

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{2.7}$$

2.2. Almost Mathieu cocycles, integrated density of states, spectrum. Let

$$S_{\lambda,E} = \begin{pmatrix} E - 2\lambda\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

We call $(\alpha, S_{\lambda,E})$, $\lambda, \alpha, E \in \mathbb{R}$, $\lambda \neq 0$ *almost Mathieu cocycles*. A sequence $(u_n)_{n\in\mathbb{Z}}$ is a formal solution of the eigenvalue equation $H_{\lambda,\alpha,\theta}u = Eu$ if and only if $S_{\lambda,E}(\theta+n\alpha) \cdot \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$.

Let

$$L_{\lambda,\alpha}(E) = L(\alpha, S_{\lambda,E}). \tag{2.9}$$

It is easy to see that $\rho(\alpha, S_{\lambda,E})$ admits a determination $\rho_{\lambda,\alpha}(E) \in [0, 1/2]$. We let

$$N_{\lambda,\alpha}(E) = 1 - 2\rho_{\lambda,\alpha}(E) \in [0, 1]. \tag{2.10}$$

It follows that $E \mapsto N_{\lambda,\alpha}(E)$ is a continuous non-decreasing function. The function N is the usually defined *integrated density of states* of $H_{\lambda,\alpha,\theta}$ if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (for $\alpha \in \mathbb{Q}$, N is the integral of the density of states over different θ), see [AS] and [JM]. Thus defining

$$\Sigma_{\lambda,\alpha} = \{E \in \mathbb{R},\ N_{\lambda,\alpha} \text{ is not constant in a neighborhood of } E\}, \tag{2.11}$$

we see that (consistently with the introduction) $\Sigma_{\lambda,\alpha}$ is the spectrum of $H_{\lambda,\alpha,\theta}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (in this case the spectrum does not depend on θ), while for $\alpha \in \mathbb{Q}$, $\Sigma_{\lambda,\alpha}$ is the union of the spectra of $H_{\lambda,\alpha,\theta}$, $\theta \in \mathbb{R}$. One also has

$$\Sigma_{\lambda,\alpha} \subset [-2 - 2|\lambda|, 2 + 2|\lambda|]. \tag{2.12}$$

Continuity of the fibered rotation number implies that $N_{\lambda,\alpha}$ depends continuously on (λ, α) on $L^\infty(\mathbb{R})$.

It turns out that there is a relation between N and L, the *Thouless formula*, see [AS]

$$L(E) = \int \ln |E - E'| dN(E'). \tag{2.13}$$

By the Schwarz reflection principle, if $J \subset \mathbb{R}$ is an open interval where the Lyapunov exponent vanishes, then $E \mapsto N_{\lambda,\alpha}(E)$ is an increasing analytic function of $E \in J$[2] (and obviously $J \subset \Sigma_{\lambda,\alpha}$).

We will use several times the following result [BJ].

Theorem 2.1 ([BJ], Corollary 2)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda \neq 0$. If $E \in \Sigma_{\lambda,\alpha}$ then*

$$L_{\lambda,\alpha}(E) = \max\{0, \ln |\lambda|\}. \tag{2.14}$$

This result will be mostly important for us for what it says about the range $0 < \lambda \leq 1$ (zero Lyapunov exponent on the spectrum). It will be also very minorly used in our proof of localization when $\lambda > 1$.

2.3. **Kotani theory.** Recall the usual action of $\mathrm{SL}(2, \mathbb{C})$ on the Riemann sphere $\overline{\mathbb{C}}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. We can of course define $\mathrm{SL}(2, \mathbb{C})$ cocycles as pairs $(\alpha, A) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$, but it is convenient to view a $\mathrm{SL}(2, \mathbb{C})$ cocycle as acting by Moebius transformations:

$$\begin{aligned} (\alpha, A) :& \mathbb{R}/\mathbb{Z} \times \overline{\mathbb{C}} \to \mathbb{R}/\mathbb{Z} \times \overline{\mathbb{C}} \\ & (x, z) \mapsto (x + \alpha, A(x) \cdot z). \end{aligned} \tag{2.15}$$

If one lets E become a complex number in the definition of the almost Mathieu cocycle, we get a $\mathrm{SL}(2, \mathbb{C})$ cocycle.

Let $\mathbb{H}$ be the upper half plane. Fix (λ, α). It is well known that there exists a continuous function $m = m_{\lambda,\alpha} : \mathbb{H} \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ such that $S_{\lambda,E}(x) \cdot m(E, x) = m(E, x + \alpha)$, thus defining an invariant section for the cocycle $(\alpha, S_{\lambda,E})$:

$$(\alpha, S_{\lambda,E})(x, m(E, x)) = (x + \alpha, m(E, x + \alpha)). \tag{2.16}$$

Moreover, $E \mapsto m(E, x)$ is holomorphic on $\mathbb{H}$.

[2]Since $N + \frac{iL}{\pi}$ is holomorphic in upper half plane and real on J. This can also be obtained from the Thouless formula.

Remark 2.1. In the litterature (for instance, in [Sim1]), it is more common to find the definition of a pair of m-functions, $m_\pm(x,E)$, which is given in terms of non-zero solutions $(u_\pm(n))_{n\in\mathbb{Z}}$ of $H_{\lambda,\alpha,x}u = Eu$ which are ℓ^2 at $\pm\infty$: $m_\pm(x,E) = -\frac{u_\pm(\pm 1)}{u_\pm(0)}$. In this notation we have $m(x,E) = -\frac{1}{m_-(x,E)}$ (the relation $S_{\lambda,E}(x)\cdot m(E,x) = m(E,x+\alpha)$ is an immediate consequence of the definition of $m_-(x,E)$).

The following result of Kotani theory [Sim1] will be important in two key parts of this paper.

Theorem 2.2. *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and assume that $L(\alpha,E) = 0$ in an open interval $J \subset \mathbb{R}$. Then for every $x \in \mathbb{R}/\mathbb{Z}$, the functions $E \mapsto m(E,x)$ admit a holomorphic extension to $\mathbb{C}\setminus(\mathbb{R}\setminus J)$, with values in $\mathbb{H}$. The function $m : \mathbb{C}\setminus(\mathbb{R}\setminus J)\times\mathbb{R}/\mathbb{Z} \to \mathbb{H}$ is continuous in both variables.*

2.4. **Polar sets.** Recall one of the possible definitions of a polar set in $\mathbb{C}$: it is a set of zero *logarithmic capacity*. We will need only some properties of polar sets in $\mathbb{C}$ (see for instance [Ho]):

(1) A countable union of polar sets is polar,
(2) The image of a polar set by a non-constant holomorphic function (defined in some domain of $\mathbb{C}$) is a polar set,
(3) Polar sets have Hausdorff dimension zero, thus their intersections with $\mathbb{R}$ have zero Lebesque measure,
(4) Let $U \subset \mathbb{C}$ be a domain and let $f_n : U \to \mathbb{R}$ be a sequence of subharmonic functions which is uniformly bounded in compacts of U. Then $f : U \to \mathbb{R}$ given by $f = \limsup f_n$ coincides with its (subharmonic) upper regularization $f^* : U \to \mathbb{R}$ (given by $f^*(z) = \limsup_{w\to z} f(w)$) outside a polar set.

We will say that a subset of $\mathbb{R}$ is polar if it is polar as a subset of $\mathbb{C}$.

The following result on analytic continuation is well known. We will quickly go through the proof, since a similar idea will play a role later in a small divisor problem.

Lemma 2.3. *Let $W \subset \mathbb{C}$ be a domain and let $f : W\times\mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a continuous function. If $z \mapsto f(z,w)$ is holomorphic for all $w \in \mathbb{R}/\mathbb{Z}$ and $w \mapsto f(z,w)$ is analytic for some non-polar set of $z \in W$ then f is analytic.*

Proof. We may assume that $|f(z,w)| < 1$, $(z,w) \in W\times\mathbb{R}/\mathbb{Z}$. Let

$$f(z,w) = \sum \hat{f}_z(k)e^{2\pi ikw}. \tag{2.17}$$

Then $z \mapsto \hat{f}_z(k)$ is holomorphic and $|\hat{f}_z(k)| < 1$. Using property (1) of polar sets, we obtain that there exists a non-polar set $\Delta \subset W$, $\epsilon > 0$, and $k > 0$ such that $|\hat{f}_z(n)| \le e^{-\epsilon|n|}$ for $z \in \Delta$ and $|n| > k$. Let

$$h(z) = \sup_{|n|>k} \frac{1}{|n|}\ln|\hat{f}_z(n)|. \tag{2.18}$$

Then, by property (4) of polar sets, h^* is a non-positive subharmonic function satisfying $h^*(z) \le -\epsilon$, $z \in \Delta\setminus X$, where X is polar. Since Δ is non-polar, we conclude that h^* is not identically 0 in W. It follows from the maximum principle that $h^*(z) < 0$, $z \in W$. Thus for any domain $U \subset W$ compactly contained in W, there exists $\delta = \delta(U) > 0$ such that $h(z) \le -\delta$, $z \in U$. We conclude that

$$\frac{1}{|n|}\ln|\hat{f}_z(n)| \le -\delta, \quad |n| > k, z \in U, \tag{2.19}$$

which implies that (2.17) converges uniformly on compacts of $W\times\{w \in \mathbb{C}/\mathbb{Z},\ 2\pi|\Im w| < \delta\}$. □

3. Regularity of the m-functions

Theorem 3.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda > 0$. Let $m = m_{\lambda,\alpha} : \mathbb{H} \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ be as in §2.3. Then m is analytic.*

Proof. Let us show that m has a holomorphic extension to

$$\Omega_\lambda = \{(E,x),\, \Im E > 0,\, 2\lambda \sinh|2\pi \Im x| < \Im E\}. \tag{3.1}$$

We have

$$S_{\lambda,E}(x) \cdot z = E - 2\lambda\cos(2\pi x) - \frac{1}{z}. \tag{3.2}$$

For (E,t) satisfying

$$\Im E > 0, \quad 2\lambda \sinh|2\pi t| < \Im E, \tag{3.3}$$

define the half-plane

$$K^1_{\lambda,E,t} = \{z,\, \Im z > \Im E - 2\lambda\sinh|2\pi t|\} \subset \mathbb{H}, \tag{3.4}$$

and the disk

$$K^2_{\lambda,E,t} = \left\{|z| < |E| + 2\lambda\cosh|2\pi t| + \frac{1}{\Im E - 2\lambda\sinh|2\pi t|}\right\}, \tag{3.5}$$

and let

$$K_{\lambda,E,t} = K^1_{\lambda,E,t} \cap K^2_{\lambda,E,t}, \tag{3.6}$$

which is a domain compactly contained in $\mathbb{H}$ depending continuously on (E,t) satisfying (3.3). If $(E,x) \in \Omega_\lambda$ then $(E, \Im x)$ satisfies (3.3) and one checks directly that

$$S_{\lambda,E}(x) \cdot \mathbb{H} \subset K^1_{\lambda,E,\Im x}, \tag{3.7}$$

$$S_{\lambda,E}(x) \cdot K^1_{\lambda,E,\Im x} \subset K^2_{\lambda,E,\Im x}. \tag{3.8}$$

Since $\Im x = \Im x + \alpha$, we have

$$S_{\lambda,E}(x+\alpha) \cdot S_{\lambda,E}(x) \cdot \mathbb{H} \subset K_{\lambda,E,\Im x}. \tag{3.9}$$

Thus, by the Schwarz Lemma applied to $\mathbb{H}$, for every $(E,x) \in \Omega_\lambda$,

$$S_{\lambda,E}(x-\alpha) \cdots S_{\lambda,E}(x-n\alpha) \cdot \overline{\mathbb{H}} \tag{3.10}$$

is a sequence of nested compact sets shrinking to a single point $\hat{m}(E,x)$. This implies that $\hat{m}(E,x)$ is the unique solution to (2.16) in $\mathbb{H}$. Since $m : \mathbb{H} \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ is a continuous function satisfying (2.16), we must have $\hat{m}(E,x) = m(E,x)$ for $(E,x) \in \mathbb{H} \times \mathbb{R}/\mathbb{Z}$.

Since holomorphic functions $m^n : \Omega_\lambda \to \mathbb{H}$ given by

$$m^n(E,x) = S_{\lambda,E}(x-\alpha) \cdots S_{\lambda,E}(x-n\alpha) \cdot i, \tag{3.11}$$

take values in $\mathbb{H}$, the sequence m^n is normal. Since it converges pointwise to $\hat{m}$, we conclude that $\hat{m}$ is holomorphic. □

Theorem 3.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and let $0 < \lambda \leq 1$. Let $m = m_{\lambda,\alpha} : \mathbb{H} \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ be as in §2.3. If $J \subset \Sigma_{\lambda,\alpha}$ is an open interval then m admits an analytic extension $m : \mathbb{C} \setminus (\mathbb{R} \setminus J) \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$.*

Proof. By Theorems 2.1 and 2.2, there exists a continuous extension $m : \mathbb{C} \setminus (\mathbb{R} \setminus J) \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ which is analytic in E. By Theorem 3.1, m is also analytic in x for $E \in \mathbb{H}$. Analiticity in (E,x) then follows by Lemma 2.3. □

Remark 3.1. Notice that the proof of Theorem 3.1 uses strongly that the dynamics in the basis of the almost Mathieu cocycle is a rotation (and not, say, a hyperbolic toral automorphism or the skew shift). But one may still get weaker results on smoothness of m-functions (in the line of Theorem 3.2) for those dynamics (via estimates in the line of [AK2]).

4. Analytic continuation

Lemma 4.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and let $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be analytic, and let $\theta = \int_{\mathbb{R}/\mathbb{Z}} \phi(x)dx$. The following are equivalent:*

(1) *There exists an analytic function $O : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, homotopic to the identity, such that*

$$O(x+\alpha)R_{\phi(x)}O(x)^{-1} = R_\theta, \tag{4.1}$$

(2) *There exists an analytic function $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that*

$$\phi(x) - \theta = \psi(x+\alpha) - \psi(x). \tag{4.2}$$

Proof. Obviously (2) implies (1): it is enough to take $O(x) = R_{-\psi(x)}$.

Let us show that (1) implies (2). If $O(x) \cdot i = i$ for all x then $O(x) \in \mathrm{SO}(2,\mathbb{R})$ for all x and since O is homotopic to the identity we have $O(x) = R_{-\psi(x)}$ for some analytic function $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ which has to satisfy $\phi(x) - \theta = \psi(x+\alpha) - \psi(x)$.

Thus we may assume that $O(x_0) \cdot i \neq i$ for some x_0. Notice that

$$O(x_0 + n\alpha) \cdot i = R_{n\theta}O(x_0) \cdot i. \tag{4.3}$$

It follows that if $n_k\alpha \to 0$ in $\mathbb{R}/\mathbb{Z}$ then $2n_k\theta \to 0$ in $\mathbb{R}/\mathbb{Z}$. This implies that $\theta = \frac{l}{2}\alpha$ for some $l \in \mathbb{Z}$. We have

$$O(x+\alpha)R_{\phi(x)}O(x)^{-1} = R_{\frac{l}{2}(x+\alpha)}R_{-\frac{l}{2}x}, \tag{4.4}$$

which implies

$$R_{-\frac{l}{2}(x+\alpha)}O(x+\alpha)R_{\phi(x)} = R_{-\frac{l}{2}x}O(x), \tag{4.5}$$

and we get

$$R_{-\frac{l}{2}(x+\alpha)}O(x+\alpha) \cdot i = R_{-\frac{l}{2}x}O(x) \cdot i. \tag{4.6}$$

It follows that $R_{-\frac{l}{2}x}O(x) \cdot i = z$ does not depend on x. Let $Q \in \mathrm{SL}(2,\mathbb{R})$ be such that $Q \cdot z = i$, and set

$$S(x) = R_{\frac{l}{2}x}QR_{-\frac{l}{2}x}O(x). \tag{4.7}$$

Since $O, Q : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, where $Q(x) = Q$ are homotopic to the identity, we have that $S : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is homotopic to the identity and, using that $\theta = \frac{l}{2}\alpha$, we have

$$S(x+\alpha)R_{\phi(x)}S(x)^{-1} = R_\theta. \tag{4.8}$$

Moreover, $S(x) \cdot i = i$, so $S(x) \in \mathrm{SO}(2,\mathbb{R})$ and we have $S(x) = R_{-\psi(x)}$, $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. It follows that ψ satisfies (4.2). □

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $0 < \lambda \leq 1$, let $\Lambda_{\lambda,\alpha}$ be the set of E such that there exists an analytic function $B_E : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$, homotopic to the identity, and $\theta(E) \in \mathbb{R}$, such that

$$B_E(x+\alpha)S_{\lambda,E}(x)B_E(x)^{-1} = R_{\theta(E)}. \tag{4.9}$$

Theorem 4.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $0 < \lambda \leq 1$. Let $J \subset \Sigma_{\lambda,\alpha}$ be an open interval. Then*

(1) *If $\beta = 0$ then $\Lambda_{\lambda,\alpha} \cap J = J$,*
(2) *If $\beta < \infty$ then either $\Lambda_{\lambda,\alpha} \cap J$ is polar or $\mathrm{int}\,\Lambda_{\lambda,\alpha} \cap J \neq \emptyset$.*

Proof. Assume that $J \subset \Sigma_{\lambda,\alpha}$ is an open interval. Let $m = m_{\lambda,\alpha}$ be given by Theorem 3.2, so that $m : \mathbb{C} \setminus (\mathbb{R} \setminus J) \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}$ is continuous, $E \mapsto m(E,x)$ is holomorphic and

$$S_{\lambda,E} \cdot m(E,x) = m(E,x+\alpha). \tag{4.10}$$

Let

$$C_E(x) = \begin{pmatrix} \frac{\Re m(E,x)}{|m(E,x)|(\Im m(E,x))^{1/2}} & -\frac{|m(E,x)|}{(\Im m(E,x))^{1/2}} \\ \frac{(\Im m(E,x))^{1/2}}{|m(E,x)|} & 0 \end{pmatrix}. \tag{4.11}$$

Then

$$C_E(x+\alpha) S_{\lambda,E}(x) C_E(x)^{-1} \in \mathrm{SO}(2,\mathbb{R}) \tag{4.12}$$

for $E \in J$, $x \in \mathbb{R}/\mathbb{Z}$. Since $x \mapsto C_E(x)$ is easily verified to be homotopic to the identity for $E \in J$, we have

$$C_E(x+\alpha) S_{\lambda,E}(x) C_E(x)^{-1} = R_{\phi(E,x)} \tag{4.13}$$

for some real-analytic function $\phi : J \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. It follows that ϕ has a holomorphic extension $\phi : Z \to \mathbb{C}$ where $Z \subset \mathbb{C} \times \mathbb{C}/\mathbb{Z}$ is some domain containing $J \times \mathbb{R}/\mathbb{Z}$. So there exists a domain $\Delta \subset \mathbb{C}$ such that $J \subset \Delta$ and $\Delta \times \mathbb{R}/\mathbb{Z} \subset Z$. For $E \in \Delta$, let

$$\phi(E,x) = \sum \hat{\phi}_E(k) e^{2\pi i k x}. \tag{4.14}$$

Let $E \in J$ be such that there exists an analytic function $\psi_E : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}/\mathbb{Z}} \psi_E(x) dx = 0, \tag{4.15}$$

$$\phi(E,x) - \int_{\mathbb{R}/\mathbb{Z}} \phi(E,x) dx = \psi_E(x+\alpha) - \psi_E(x). \tag{4.16}$$

Then

$$\psi_E(x) = \sum \hat{\psi}_E(k) e^{2\pi i k x} \tag{4.17}$$

where

$$\hat{\psi}_E(k) = \frac{\hat{\phi}_E(k)}{e^{2\pi i k \alpha} - 1}, \quad k \neq 0, \tag{4.18}$$

$$\hat{\psi}_E(0) = 0. \tag{4.19}$$

We can then define an analytic function $B_E : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ by

$$B_E(x) = R_{-\psi_E(x)} C_E(x), \tag{4.20}$$

which satisfies

$$B_E(x+\alpha) S_{\lambda,E}(x) B_E(x)^{-1} = R_{\theta(E)}, \quad \theta(E) = \int_{\mathbb{R}/\mathbb{Z}} \phi(E,x) dx. \tag{4.21}$$

Reciprocally, if there exists an analytic function $B_E : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ homotopic to the identity such that $B_E(x+\alpha) S_{\lambda,E}(x) B_E(x)^{-1} = R_{\theta(E)}$ for some $\theta(E) \in \mathbb{R}$, then we can write

$$O_E(x+\alpha) R_{\phi(x)} O_E(x)^{-1} = R_{\theta(E)}, \tag{4.22}$$

where

$$O_E(x) = B_E(x) C_E(x)^{-1}. \tag{4.23}$$

By the previous lemma, there exists an analytic function (having average 0) $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfying $\phi(x) - \int_{\mathbb{R}/\mathbb{Z}} \phi(x)dx = \psi(x+\alpha) - \psi(x)$.

Notice that

$$\limsup_{|k|\to\infty} \frac{1}{|k|} \ln \frac{1}{|e^{2\pi i k\alpha} - 1|} = \beta, \tag{4.24}$$

so that if $\beta = 0$ then (4.17), (4.18), and (4.19) really define an analytic function for any $E \in J$, so (1) follows.

Let $a : \Delta \to [-\infty, \beta]$ be given by

$$a(E) = \limsup_{|k|\to\infty} \frac{1}{|k|} \ln \left| \frac{\hat{\phi}_E(k)}{e^{2\pi i k\alpha} - 1} \right|. \tag{4.25}$$

By the previous discussion, $\Lambda_{\lambda,\alpha} = \{E \in J,\, a(E) < 0\}$. If $\beta < \infty$ then a is lim sup of a sequence of subharmonic functions which are uniformly bounded on compacts of Δ. It follows that a coincides with its upper regularization

$$a^*(E) = \limsup_{E' \to E} a(E') \tag{4.26}$$

for E outside some exceptional set which is polar. Thus the set $\{E \in J,\, a(E) < 0\}$ is either polar (contained in the exceptional set) or it has non-empty interior. □

Lemma 4.3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda > 0$. Then $\Lambda_{\lambda,\alpha}$ has empty interior.*

Proof. We may assume that $0 < \lambda \le 1$ (otherwise the Lyapunov exponent is positive on $\Sigma_{\lambda,\alpha}$ which easily implies that $\Lambda_{\lambda,\alpha} = \emptyset$). Assume that $J \subset \Lambda_{\lambda,\alpha}$ is an open interval. Then $J \subset \Sigma_{\lambda,\alpha}$ (since $L_{\lambda,\alpha}(E) = 0$ for $E \in J$). Let B_E be as in the definition of $\Lambda_{\lambda,\alpha}$. Then the definition of fibered rotation number (see §2.1) implies

$$\rho_{\lambda,\alpha}(E) = \theta(E) (\mathrm{mod}\, \mathbb{Z}). \tag{4.27}$$

By the analyticity of ρ on J there exists $E \in J$, $l \in \mathbb{Z}$, such that $\theta(E) = l\alpha (\mathrm{mod}\, \mathbb{Z})$. Let $T_E : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be given by

$$T_E(x) = R_{-lx} B_E(x). \tag{4.28}$$

Then

$$T_E(x+\alpha) S_{\lambda,E}(x) T_E(x)^{-1} = \mathrm{id}\,. \tag{4.29}$$

The conclusion is as in [P]. For $v \in \mathbb{R}^2$,

$$S_{\lambda,E}(x) T_E(x)^{-1} v = T_E(x+\alpha)^{-1} v. \tag{4.30}$$

So by (2.8) there exists an analytic $U_v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that

$$T_E(x+\alpha)^{-1} \cdot v = \begin{pmatrix} U_v(x) \\ U_v(x-\alpha) \end{pmatrix}. \tag{4.31}$$

Let

$$U_v(x) = \sum u_n^v e^{2\pi i n x}. \tag{4.32}$$

It is a standard Aubry duality argument (and can be checked by direct calculation) that $u_n^v \in \ell^2(\mathbb{Z})$ is an eigenvector of $H_{\lambda^{-1},\alpha,0}$ with eigenvalue $\lambda^{-1}E$. The fact that we get such an eigenvector for every $v \in \mathbb{R}^2$ contradicts the simplicity of the point spectrum. □

Remark 4.1. Notice that Lemma 4.3 and item (1) of Theorem 4.2 already imply the Ten Martini Problem in the case $\beta = 0$, and we did not need any localization result (the only recent result we used was Theorem 2.1).

5. Localization and Cantor spectrum

We say that the operator $H_{\lambda,\alpha,\theta}$ displays *Anderson localization* if it has pure point spectrum with exponentially decaying eigenvectors. This requires $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and implies that eigenvalues are dense in $\Sigma_{\lambda,\alpha}$.

Theorem 5.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and let $\lambda \geq 1$. Assume that $\beta < \infty$. If $H_{\lambda,\alpha,\theta}$ displays Anderson localization for a non-polar set of $\theta \in \mathbb{R}$, then $\Sigma_{\lambda,\alpha}$ is a Cantor set.*

Proof. Let Θ be the set of θ such that $H_{\lambda,\alpha,\theta}$ displays Anderson localization. If $\theta \in \Theta$, and E is an eigenvalue for $H_{\lambda,\alpha,\theta}$, let $(u_n)_{n\in\mathbb{Z}}$ be a non-zero eigenvector. Then

$$S_{\lambda^{-1},\lambda^{-1}E} \cdot W(x) = e^{2\pi i\theta} W(x+\alpha), \tag{5.1}$$

where

$$W(x) = \begin{pmatrix} U(x)e^{2\pi i\theta} \\ U(x-\alpha) \end{pmatrix}, \tag{5.2}$$

and

$$U(x) = \sum u_n e^{2\pi i n x}. \tag{5.3}$$

Let $M(x)$ be the matrix with columns $W(x)$ and $\overline{W(x)}$. Then

$$S_{\lambda^{-1},\lambda^{-1}E}(x) \cdot M(x) = M(x+\alpha) \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix}. \tag{5.4}$$

This implies that $\det M(x)$ is independent of x, so $\det M(x) = ci$ for some $c \in \mathbb{R}$. Notice that if $c = 0$ then

$$V(x+\alpha) = e^{-4\pi i\theta} V(x), \tag{5.5}$$

with

$$V(x) = \frac{U(x)}{\overline{U(x)}} \tag{5.6}$$

(notice that $U(x) \neq 0$ except at finitely many x since $U(x)$ is a non-constant analytic function) and in particular, if $n_k\alpha \to 0$ then $2n_k\theta \to 0$. So $2\theta = k\alpha + l$ for some $k, l \in \mathbb{Z}$. If $c > 0$, we have

$$S_{\lambda^{-1},\lambda^{-1}E}(x) = Q(x+\alpha) R_\theta Q(x)^{-1} \tag{5.7}$$

where $Q : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is given by

$$Q(x) = \frac{1}{(2c)^{1/2}} M(x) \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{5.8}$$

and if $c < 0$, we have

$$S_{\lambda^{-1},\lambda^{-1}E}(x) = Q(x+\alpha) R_{-\theta} Q(x)^{-1} \tag{5.9}$$

where $Q : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is given by

$$Q(x) = \frac{1}{(-2c)^{1/2}} M(x) \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.10}$$

It follows that in either case $\lambda^{-1}E \in \Lambda_{\lambda^{-1},\alpha}$ and moreover,

$$\rho_{\lambda^{-1},\alpha}(\lambda^{-1}E) = \pm\theta + k\alpha (\mathrm{mod}\, \mathbb{Z}) \tag{5.11}$$

for some $k \in \mathbb{Z}$.

Let $\Theta' \subset \Theta$ be the set of all θ such that $2\theta \neq k\alpha + l$ for all $k, l \in \mathbb{Z}$. Let $J \subset \Sigma_{\lambda,\alpha}$ be an open interval. Then for any $\theta \in \Theta'$, there exists some $E \in J$ such that E is an eigenvalue for $H_{\lambda,\alpha,\theta}$, and by the previous discussion any such E satisfies

$$N_{\lambda^{-1},\alpha}(\lambda^{-1}E) = 1 - 2\rho_{\lambda^{-1},\alpha}(\lambda^{-1}E) = 1 - 2(\varepsilon\theta + k\alpha + l), \quad \text{for some } k, l \in \mathbb{Z}, \varepsilon \in \{1, -1\}, \tag{5.12}$$

$$\lambda^{-1}E \in \Lambda_{\lambda^{-1},\alpha}. \tag{5.13}$$

It follows that

$$\Theta' \subset \left\{\varepsilon \frac{1 - N_{\lambda^{-1},\alpha}(\Lambda_{\lambda^{-1},\alpha} \cap \lambda^{-1}J)}{2} - k\alpha - l, \quad k, l \in \mathbb{Z}, \varepsilon \in \{1, -1\}\right\}. \tag{5.14}$$

By item (2) of Theorem 4.2 and Lemma 4.3, $\Lambda_{\lambda^{-1},\alpha} \cap \lambda^{-1}J$ is polar. Since $N_{\lambda^{-1},\alpha}$ is a non-constant analytic function on $\lambda^{-1}J$, it follows that Θ' is also polar. Thus $\Theta \subset \Theta' \cup \left\{\frac{1}{2}(k\alpha + l),\ k, l \in \mathbb{Z}\right\}$ is polar. □

Remark 5.1. In [P], it is shown that if $\alpha \in DC$ then Anderson localization of $H_{\lambda,\alpha,0}$ implies Cantor spectrum. We can not however use the argument of Puig (based on analytic reducibility) to conclude Cantor spectrum in the generality we need. Indeed, we are not able to conclude analytic reducibility from localization of $H_{\lambda,\alpha,0}$ in our setting (in a sense, we spend all our regularity to take care of small divisors in the localization result, which is half of analytic reducibility, and there is nothing left for the other half). Though this can be bypassed (using Kotani theory to conclude continuous reducibility under the assumption of non-Cantor spectrum), there is a much more serious difficulty in this approach, see Remark 5.2.

The next result gives us a large range of λ and α where Theorem 5.1 can be applied.

Theorem 5.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be such that $\beta = \beta(\alpha) < \infty$, and let $\lambda > e^{\frac{16}{9}\beta}$. Then $H_{\lambda,\alpha,\theta}$ displays Anderson localization for almost every θ.*

This result improves on [J], where Anderson localization was proved under the assumption that α is Diophantine. Recall that α is said to satisfy a Diophantine condition (briefly, $\alpha \in DC$) if

$$\ln q_{n+1} = O(\ln q_n) \tag{5.15}$$

where $\frac{p_n}{q_n}$ are the rational approximations of α. In particular $\alpha \in DC$ implies (but is strictly stronger than) $\beta(\alpha) = 0$. The proof in [J] with some modifications can be extended to the case $\beta(\alpha) = 0$ but not to the case $\beta(\alpha) > 0$.

The proof of Theorem 5.2 is the most technical part of this paper, and the considerations involved are independent from our other arguments. We will thus postpone its proof to § 9.

Remark 5.2. We expect that the operator $H_{\lambda,\alpha,0}$ does not display Anderson localization for $1 < \lambda \leq e^{2\beta}$. The key reason is that in this regime 0 is a very resonant phase, and since α is Diophantine only in a very weak sense, the compound effect on the small divisors can not be compensated by the Lyapunov exponent. See also Remark 9.1.

6. Fictitious results on continuity of the spectrum

The spectrum $\Sigma_{\lambda,\alpha}$ is a continuous function of α in the Hausdorff topology. There are several results in the literature about quantitative continuity. The best general result is due to [AvMS], 1/2-Hölder continuity. Better estimates can be obtained for α not very Liouville in the region of positive Lyapunov exponent [JK]. None of those results are enough for our purposes.

The results described above have something in common: they deal with something that actually happens, and it is not clear if it is possible to improve them sufficiently (to the level we need).

Thus we will argue by contradiction: assuming the spectrum is not Cantor, we will get very good continuity estimates. This will allow us to proceed the argument, but obviously, since we will eventually conclude that the spectrum is a Cantor set, estimates in this section are not valid for any existing almost Mathieu operator. Those estimates might be useful also when analyzing more general Schrödinger operators.

Theorem 6.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $0 < \lambda \leq 1$. Let $J \subset \mathbb{R}$ be an open interval such that $\overline{J} \subset \operatorname{int} \Sigma_{\lambda,\alpha}$. There exists $K > 0$ such that*

$$|N_{\lambda,\alpha}(E) - N_{\lambda,\alpha'}(E)| \leq K|\alpha - \alpha'|, \quad E \in J. \tag{6.1}$$

Proof. Let $m = m_{\lambda,\alpha}$ be as in Theorem 3.2. Define $x \mapsto C_E(x)$ by (4.11). Then, as discussed in the proof of Theorem 4.2, $C_E : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is homotopic to the identity and satisfies $C_E(x+\alpha)S_{\lambda,E}(x)C_E(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ so

$$C_E(x+\alpha)S_{\lambda,E}(x)C_E(x)^{-1} = R_{\phi_E(x)}, \tag{6.2}$$

where $\phi_E : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is analytic. Recall the definition of the fibered rotation number §2.1. Then

$$\rho(\alpha, S_{\lambda,E}(x)) = \rho(\alpha, R_{\phi(x)}). \tag{6.3}$$

In this case we can take as lift of $R_{\phi_E(x)}$ the function $\psi(x,y) = \phi(x)$.

Write

$$\rho(\alpha', S_{\lambda,E}) = \rho(\alpha', C_E(x+\alpha')S_{\lambda,E}(x)C_E(x)^{-1}) = \rho(\alpha', C_E(x+\alpha')C_E(x+\alpha)^{-1}R_{\phi_E(x)}). \tag{6.4}$$

Since m is analytic in x, we can take as lift of $C_E(x+\alpha')C_E(x+\alpha)^{-1}R_{\phi_E(x)}$ a function $\tilde{\psi}(x,y)$ satisfying $|\tilde{\psi}(x,y) - \phi(x)| \leq K|\alpha - \alpha'|$. Thus

$$\|\rho(\alpha, S_{\lambda,E}) - \rho(\alpha', S_{\lambda,E})\|_{\mathbb{R}/\mathbb{Z}} \leq \int \sup_y |\phi(x) - \tilde{\psi}(x,y)| dx \leq K|\alpha - \alpha'|. \tag{6.5}$$

The result now follows, since $N = 1 - 2\rho$ (see §2.2) for the determination of ρ in $[0, 1/2]$. □

Remark 6.1. Clearly we also get the fictitious estimate

$$|L_{\lambda,\alpha'}(E) - L_{\lambda,\alpha}(E)| \leq K|\alpha - \alpha'|, \quad E \in J. \tag{6.6}$$

7. Gaps for rational approximants

It is well known that for any $\lambda \neq 0$ if $\frac{p}{q}$ is a minimal denomination of a rational number then $\Sigma_{\lambda,\frac{p}{q}}$ consists of q bands with disjoint interior. All those bands are actually disjoint, except if q is even when there are two bands touching at 0 [vM], [CEY]. The variation of $N_{\lambda,\frac{p}{q}}$ in each band is precisely $1/q$. The connected components of $\mathbb{R} \setminus \Sigma_{\lambda,\frac{p}{q}}$ are called gaps. Let $M(\lambda, \frac{p}{q})$ be the maximum size of the bands of $\Sigma_{\lambda,\frac{p}{q}}$.

The following result is well known.

Lemma 7.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda \neq 0$. If $\frac{p_n}{q_n} \to \alpha$, then $M(\lambda, p_n/q_n) \to 0$. In particular (since $N_{\lambda,p_n/q_n} \to N_{\lambda,\alpha}$ uniformly), if one selects a point $a_{n,i}$ in each band of $\Sigma_{\lambda,p_n/q_n}$ then*

$$\frac{1}{q_n} \sum_i \delta a_{n,i} \to dN_{\lambda,\alpha} \quad \textit{in the weak}^* \textit{ topology.} \tag{7.1}$$

In [CEY], a lower bound for the size of gaps of $\Sigma_{\lambda,\frac{p}{q}}$ is derived of the form $C(\lambda)^{-q}$, where, for instance, $C(1) = 8$. We will need the following sharpening of this estimate, in the case where $\frac{p}{q}$ are close to a given irrational number.

Theorem 7.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $0 < \lambda \leq 1$. Let $\frac{p_n}{q_n} \to \alpha$. For every $\epsilon > 0$, for every n sufficiently large, all gaps of $\Sigma_{\lambda, \frac{p_n}{q_n}}$ have size ar least $e^{-\epsilon q_n} \lambda^{q_n/2}$.*

Proof. It is known (see the proof of [CEY] Theorem 3.3 for the case $\lambda = 1$, the general case being obtained as described in the proof of [CEY] Corollary 3.4) that for any bounded gap G of $\Sigma_{\lambda, \frac{p}{q}}$, one can find a sequence a_i, $1 \leq i \leq q$, with one a_i in each band of $\Sigma_{\lambda, \frac{p}{q}}$, such that $G = (a_i, a_{i+1})$ and

$$\prod_{j \neq i} |a_j - a_i| \geq \lambda^m, \tag{7.2}$$

where $q = 2m + 1$ or $q = 2m + 2$.

Let G_n be a bounded gap of $\Sigma_{\lambda, \frac{p_n}{q_n}}$ of minimal size. Then

$$|G_n| \geq \lambda^{q_n/2} \prod_{j \neq i_n, i_n+1} |a_{n,j} - a_{n,i_n}|^{-1}, \tag{7.3}$$

where the $a_{n,i}$ satisfy the hypothesis of the previous lemma. Passing to a subsequence, we may assume that $a_{n,i_n} \to E \in \Sigma_{\lambda,\alpha}$ and $|G_n| \to 0$ (otherwise the result is obvious). By the previous lemma, we get the estimate for $0 < \delta < 1$ and for n large

$$\frac{1}{q_n} \ln(|G_n| \lambda^{-q_n/2}) \geq -\frac{1}{q_n} \sum_{j \neq i_n, i_n+1} \ln |a_{n,j} - a_{n,i_n}| \geq -\frac{1}{q_n} \sum_{|a_{n,j} - a_{n,i_n}| > \delta} \ln |a_{n,j} - a_{n,i_n}|, \tag{7.4}$$

which implies by (7.1) and the definition of the weak* topology that

$$\liminf \frac{1}{q_n} \ln(|G_n| \lambda^{-q_n/2}) \geq -\int_{|E' - E| > \delta} \ln |E - E'| dN_{\lambda,\alpha}(E'). \tag{7.5}$$

Thus

$$\liminf \frac{1}{q_n} \ln(|G_n| \lambda^{-q_n/2}) \geq -\int \ln |E - E'| dN_{\lambda,\alpha}(E'). \tag{7.6}$$

By the Thouless formula and Theorem 2.1, this gives $\liminf \frac{1}{q_n} \ln(|G_n| \lambda^{-q_n/2}) \geq -L_{\lambda,\alpha}(E) = 0$. $\square$

Remark 7.1. It is possible to get an estimate on the convergence rate on Lemma 7.1 using [AvMS]. This implies an estimate on the rate of convergence in Theorem 7.2.

8. Proof of the Main Theorem

We now put together the results of the previous sections. Recall that it is enough to consider $\lambda > 0$, and that the case $\lambda = 1$ follows from Theorem 1.5 of [AK1]. Moreover, Cantor spectrum for λ implies Cantor spectrum for $\frac{1}{\lambda}$. Let $\beta = \beta(\alpha)$. The Main Theorem follows then from the following.

Theorem 8.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then*

1. *If $\beta < \infty$ and $\lambda > e^{\frac{16}{9}\beta}$, $\Sigma_{\lambda,\alpha}$ is a Cantor set,*
2. *If $\beta = \infty$ or if $0 < \beta < \infty$ and $e^{-2\beta} < \lambda \leq 1$, $\Sigma_{\lambda,\alpha}$ is a Cantor set.*

Proof. Item (1) follows from Theorems 5.2 and 5.1.

To get item (2), we argue by contradiction. Let $J \subset \operatorname{int} \Sigma$ be a compact interval. Then the density of states satisfies $\frac{dN}{dE} \geq c > 0$ for $E \in J$ [AS]. Let $\frac{p}{q}$ be close to α such that $\frac{1}{q} \ln |\alpha - \frac{p}{q}|$ is close to $-\beta$. By Lemma 7.1 and Theorem 7.2, $J \setminus \Sigma_{p/q}$ contains an interval $G = (a, b)$ of size $e^{-\epsilon q} \lambda^{q/2}$. Notice that $N_{\lambda, \frac{p}{q}}(a) = N_{\lambda, \frac{p}{q}}(b)$. Theorem 6.1 implies

$$|N_{\lambda,\alpha}(a) - N_{\lambda,\alpha}(b)| \leq K \left| \alpha - \frac{p}{q} \right| \leq e^{\epsilon q} e^{-\beta q}. \tag{8.1}$$

Thus

$$c \leq \frac{N_{\lambda,\alpha}(a) - N_{\lambda,\alpha}(b)}{a-b} \leq e^{2\epsilon q} e^{-\beta q} \lambda^{-q/2}. \tag{8.2}$$

By taking $\epsilon \to 0$, we conclude that $\lambda \leq e^{-2\beta}$. □

Let us point out that 1/2-Hölder continuity of the spectrum [AvMS] (which holds for every α and λ) together with Theorem 7.2 implies the following improvement of [CEY]. Let us say that all gaps of $\Sigma_{\lambda,\alpha}$ are open if whenever $E \in \Sigma_{\lambda,\alpha}$ is such that $N_{\lambda,\alpha}(E) = k\alpha + l$ for some $k \in \mathbb{Z} \setminus \{0\}$, $l \in \mathbb{Z}$ then E is the endpoint of some bounded gap (this obviously implies Cantor spectrum). The conjecture that $\Sigma_{\lambda,\alpha}$ has all gaps open for all $\lambda \neq 0$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is sometimes called the "dry" version of the Ten Martini Problem.

Theorem 8.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $\beta = \beta(\alpha)$. If $\beta = \infty$ or if $0 < \beta < \infty$ and $e^{-\beta} < \lambda < e^{\beta}$, $\Sigma_{\lambda,\alpha}$ has all gaps open.*

The conclusion from Theorem 8.2 appears to be the natural boundary of what can be taken honestly from the Liouvillian method: our computations indicate that although one can get improved estimates on continuity of the spectrum for $\lambda > e^{\beta}$ (following [JK]), things seem to break up at the precise parameter $\lambda = e^{\beta}$. Notice that $\lambda = e^{\beta}$ is the expected threshold for localization (for almost every phase)[3] and falls short of the expected threshold for localization with phase $\theta = 0$, $\lambda = e^{2\beta}$. Thus the use of fictitious estimates does not seem to be an artifact of our estimates, but a rather essential aspect of an approach that tries to cover all parameters with Diophantine and Liouvillian techniques.

Remark 8.1. Notice that we do not actually need the measure-theoretical result of [AK1] to obtain Cantor spectrum for $|\lambda| = 1$. Indeed, Lemma 4.3 and item (1) of Theorem 4.2 imply Cantor spectrum for $\beta = 0$ (any $\lambda \neq 0$; see Remark 4.1) and item (2) of Theorem 8.1 implies Cantor spectrum for $\beta > 0$ (if $|\lambda| = 1$).

9. Proof of Theorem 5.2

We will actually prove a slightly more precise version of Theorem 5.2. Let

$$\Theta = \{\theta : \ |\sin 2\pi(\theta + (k/2)\alpha)| < k^{-2} \text{ holds for infinitely many } k'\text{s}\} \cup \{\frac{s\pi\alpha}{2},\ s \in \mathbb{Z}\}. \tag{9.1}$$

Θ is easily seen to have zero Lebesgue measure by the Borel-Cantelli Lemma.

Theorem 9.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\beta = \beta(\alpha) < \infty$, and let $\lambda > e^{\frac{16\beta}{9}}$. Then for $\theta \notin \Theta$, $H_{\lambda,\alpha,\theta}$ displays Anderson localization.*

Remark 9.1. For $\beta = 0$ the theorem holds as well for $\theta = \frac{s\pi\alpha}{2}$, however the proof as presented here will not work. See [JKS] for the detail of the argument needed for this case. In general, we believe that for θ of the form $\frac{s\pi\alpha}{2}$, $s \in \mathbb{Z}$ the localization would only hold for $\lambda > e^{2\beta}$.

Remark 9.2. We believe that for $\theta \notin \Theta$, localization should hold for $\lambda > e^{\beta}$. The proof of this fact would require some additional arguments. Moreover, for $\lambda \leq e^{\beta}$, we do not expect any exponentially decaying eigenvectors.

Remark 9.3. The bound k^{-2} in (9.1) can be replaced by any other sub-exponential function without significant changes in the proof.

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[3] In particular, by the Gordon's argument enhanced with the Theorem 2.1, $H_{\lambda,\alpha,\theta}$ has no eigenvalues for $\lambda < e^{\beta}$., and no localized eigenfunctions for $\lambda = e^{\beta}$.

We will use the general setup of [J], however our key technical procedure will have to be quite different.

A formal solution $\Psi_E(x)$ of the equation $H_{\lambda,\alpha,\theta}\Psi_E = E\Psi_E$ will be called *a generalized eigenfunction* if

$$|\Psi_E(x)| \leq C(1+|x|) \tag{9.2}$$

for some $C = C(\Psi_E) < \infty$. The corresponding E is called a *generalized eigenvalue.* It is well known that to prove Theorem 9.1 it suffices to prove that generalized eigenfunctions decay exponentially [Be].

We will use the notation $G_{[x_1,x_2]}(x,y)$ for matrix elements of the Green's function $(H-E)^{-1}$ of the operator $H_{\lambda,\alpha,\theta}$ restricted to the interval $[x_1,x_2]$ with zero boundary conditions at x_1-1 and x_2+1. We now fix λ,α as in Theorem 9.1.

Fix a generalized eigenvalue E, and let Ψ be the corresponding generalized eigenfunction. Then

$$L(E) = \ln\lambda > 0. \tag{9.3}$$

λ will enter into our analysis through L only and it will be convenient to use L instead. To simplify notations, in some cases the E,λ,α-dependence of various quantities will be omitted.

Fix $m>0$. A point $y \in \mathbb{Z}$ will be called (m,k)*-regular* if there exists an interval $[x_1,x_2]$, $x_2 = x_1+k-1$, containing y, such that

$$|G_{[x_1,x_2]}(y,x_i)| < e^{-m|y-x_i|}, \text{ and } \operatorname{dist}(y,x_i) \geq \frac{1}{40}k;\ i=1,2.$$

Otherwise, y will be called (m,k)*-singular.*

It is well known and can be checked easily that values of any formal solution Ψ of the equation $H\Psi = E\Psi$ at a point $x \in I = [x_1,x_2] \subset \mathbb{Z}$ can be reconstructed from the boundary values via

$$\Psi(x) = -G_I(x,x_1)\Psi(x_1-1) - G_I(x,x_2)\Psi(x_2+1). \tag{9.4}$$

This implies that if Ψ_E is a generalized eigenfunction, then every point $y \in \mathbb{Z}$ with $\Psi_E(y) \neq 0$ is (m,k)-singular for k sufficiently large: $k > k_1(E,m,\theta,y)$. We assume without loss of generality that $\Psi(0) \neq 0$ and normalize Ψ so that $\Psi(0)=1$. Our strategy will be to show first that every sufficiently large y is $(m,\ell(y))$-regular for appropriate (m,ℓ). While ℓ will vary with y, m will have a uniform lower bound. This will be shown in subsections 9.4 and 9.3. Exponential decay will be derived out of this property via a "patching argument" in subsection 9.1.

Let us denote

$$P_k(\theta) = \det\left[(H_{\lambda,\alpha,\theta}-E)\Big|_{[0,k-1]}\right].$$

Then the $k-$step transfer-matrix $A_n(\theta)$ (which is the k-th iterate of the Almost Mathieu cocycle, $A_k(\theta) = S_{\lambda,E}(\theta+(k-1)\alpha)\cdots S_{\lambda,E}(\theta))$ can be written as

$$A_k(\theta) = \begin{pmatrix} P_k(\theta) & -P_{k-1}(\theta+\alpha) \\ P_{k-1}(\theta) & -P_{k-2}(\theta+\alpha)\end{pmatrix}. \tag{9.5}$$

Herman's subharmonicity trick [H] yields $\int_0^1 \ln|P_k(\theta)|d\theta \geq k\ln\lambda$; together with (9.3) this implies that there exists $\theta \in [0,1]$ with $|P_k(\theta)| \geq e^{kL(E)}$. Note that this is the only place in the proof of localization where we have used (9.3). While this is not really necessary (the rest of the proof can proceed, with only minor technical changes, under the assumption of the lower bound on only one of the four matrix elements, which follows immediately from the positivity of $L(E)$) it simplifies certain arguments in what follows.

By an application of the Cramer's rule we have that for any $x_1,\ x_2 = x_1 + k - 1,\ x_1 \le y \le x_2$,

$$\begin{aligned} |G_{[x_1,x_2]}(x_1,y)| &= \left|\tfrac{P_{x_2-y}(\theta+(y+1)\alpha)}{P_k(\theta+x_1\alpha)}\right|, \\ |G_{[x_1,x_2]}(y,x_2)| &= \left|\tfrac{P_{y-x_1}(\theta+x_1\alpha)}{P_k(\theta+x_1\alpha)}\right|. \end{aligned} \tag{9.6}$$

The numerators in (9.6) can be bounded uniformly in θ [J, Fur]. Namely, for every $E \in \mathbb{R}$, $\epsilon > 0$, there exists $k_2(\epsilon, E, \alpha)$ such that

$$|P_n(\theta)| < e^{(L(E)+\epsilon)n} \tag{9.7}$$

for all $n > k_2(\epsilon, E, \alpha)$, all θ.

$P_k(\theta)$ is an even function of $\theta + \frac{k-1}{2}\alpha$ and can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{k-1}{2}\alpha)$:

$$P_k(\theta) = \sum_{j=0}^{k} c_j \cos^j 2\pi(\theta + \frac{k-1}{2}\alpha) \stackrel{\text{def}}{=} Q_k(\cos 2\pi(\theta + \frac{k-1}{2}\alpha)).$$

Let $A_{k,r} = \{\theta \in \mathbb{R}, |Q_k(\cos 2\pi\theta)| \le e^{(k+1)r}\}$. The next lemma shows that every singular point "produces" a long piece of the trajectory of the rotation consisting of points belonging to an appropriate $A_{k,r}$.

Lemma 9.2. *Suppose $y \in \mathbb{Z}$ is $(L-\epsilon,k)$-singular, $0 < \epsilon < L$. Then for any $\epsilon_1 > 0$, $\frac{1}{40} \le \delta < 1/2$, for sufficiently large $k > k(\epsilon, E, \alpha, \epsilon_1, \delta)$, and for any $x \in \mathbb{Z}$ such that $y - (1-\delta)k \le x \le y - \delta k$, we have that $\theta + (x + \frac{k-1}{2})\alpha$ belongs to $A_{k,L-\epsilon\delta+\epsilon_1}$.*

Proof. Follows immediately from the definition of regularity, (9.6), and (9.7). ☐

The idea now is to show that $A_{k,r}$ cannot contain $k+1$ uniformly distributed points. In order to quantify this concept of uniformity we introduce the following.

Definition 9.1. We will say that the set $\{\theta_1, \ldots, \theta_{k+1}\}$ is ϵ-*uniform* if

$$\max_{z\in[-1,1]} \max_{j=1,\ldots,k+1} \prod_{\substack{\ell=1 \\ \ell\ne j}}^{k+1} \frac{|z - \cos 2\pi\theta_\ell)|}{|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)|} < e^{k\epsilon} \tag{9.8}$$

Note that we will use this terminology with "large" values of ϵ as well. ϵ-uniformity (the smaller ϵ the better) involves uniformity along with certain cumulative repulsion of $\pm\theta_i(\text{mod } 1)$'s.

Lemma 9.3. *Let $\epsilon_1 < \epsilon$. If $\theta_1, \ldots, \theta_{k+1} \in A_{k,L-\epsilon}$ and $k > k(\epsilon, \epsilon_1)$ is sufficiently large, then $\{\theta_1, \ldots, \theta_{k+1}\}$ is not ϵ_1-uniform.*

Proof. Write polynomial $Q_k(z)$ in the Lagrange interpolation form using $\cos 2\pi\theta_1, \ldots, \cos 2\pi\theta_{k+1}$:

$$|Q_k(z)| = \left|\sum_{j=1}^{k+1} Q_k(\cos 2\pi\theta_j) \frac{\prod_{\ell\ne j}(z - \cos 2\pi\theta_\ell)}{\prod_{\ell\ne j}(\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)}\right| \tag{9.9}$$

Let θ_0 be such that $|P_k(\theta_0)| \ge \exp(kL)$. The lemma now follows immediately from (9.9) with $z = \cos 2\pi(\theta_0 + \frac{k-1}{2}\alpha)$. ☐

Suppose we can find two intervals, I_1 around 0 and I_2 around y, of combined length $|I_1| + |I_2| = k+1$,[4] such that we can establish the uniformity of $\{\theta_i\}$ where $\theta_i = \theta + (x + \frac{k-1}{2})\alpha,\ i = 1, \ldots, k+1$, for

[4] Here and in what follows, the "length" $|I|$ of an interval $I = [a, b] \subset \mathbb{Z}$ denotes cardinality, $|I| = b - a + 1$.

x ranging through $I_1 \cup I_2$. Then we can apply Lemma 9.2 and Lemma 9.3 to show regularity of y. This is roughly going to be the framework for our strategy to establish regularity. The implementation will depend highly on the position of k with respect to the sequence of denominators q_n.

Assume without loss of generality that $k > 0$. Define $b_n = \max\{q_n^{8/9}, \frac{1}{20}q_{n-1}\}$ Find n such that $b_n < k \leq b_{n+1}$. We will distinguish between the two cases:

(1) **Resonant:** meaning $|k - \ell q_n| \leq b_n$ for some $\ell \geq 1$ and
(2) **Non-resonant:** meaning $|k - \ell q_n| > b_n$ for all $\ell \geq 0$.

We will prove the following estimates.

Lemma 9.4. *Assume $\theta \notin \Theta$. Suppose k is non-resonant. Let $s \in \mathbb{N} \cup \{0\}$ be the largest number such that $sq_{n-1} \leq \operatorname{dist}(k, \{\ell q_n\}_{\ell \geq 0}) \equiv k_0$. Then for any $\epsilon > 0$ for sufficiently large n,*

(1) *If $s \geq 1$ and $L > \beta$, k is $(L - \frac{\ln q_n}{q_{n-1}} - \epsilon, 2sq_{n-1} - 1)$-regular.*
(2) *If $s = 0$ then k is either $(L - \epsilon, 2[\frac{q_{n-1}}{2}] - 1)$ or $(L - \epsilon, 2[\frac{q_n}{2}] - 1)$ or $(L - \epsilon, 2q_{n-1} - 1)$-regular.*

Lemma 9.5. *Let in addition $L > \frac{16}{9}\beta$. Then for sufficiently large n, every resonant k is $(\frac{L}{50}, 2q_n - 1)$-regular.*

We will prove Lemma 9.4 in Subsection 9.3 and Lemma 9.5 in Subsection 9.4. Note that these two subsections are not independent: the proof of Lemma 9.5 uses a corollary of the proof of Lemma 9.4 as an important ingredient. As our proofs rely on establishing ϵ-uniformity of certain quasiperiodic sequences, we will use repeatedly estimates on trigonometric products that we prove in Subsection 9.2.

Theorem 9.1 can be immediately derived from Lemmas 9.4, 9.5 via a "patching argument" which we describe now. (A patching argument will also be used in one step of the proof of Lemma 9.5.)

9.1. **Patching. Proof of Theorem 9.1 assuming Lemmas 9.4 and 9.5.** It is an important technical ansatz of the multiscale analysis that the exponential decay of a Green's function at a scale k under certain conditions generates exponential decay with the same rate at a larger scale. The proof is usually done using block-resolvent expansion, with the combinatorial factor being killed by the growth of scales. The proof of Theorem 9.1 will consist, roughly, of adapting this type of argument to our situation.

Fix a generalized eigenvalue E of $H_{\lambda,\alpha,\theta}$, and let Ψ be the corresponding generalized eigenfunction.

Assume without loss of generality that k is positive. Find n so that $k > q_n$. We assume that n is sufficiently large. Let $L_1 = \frac{L}{50} \leq L - \beta$. By Lemmas 9.4 and 9.5 and definition of regularity, for any $y > b_n$ there exists an interval $y \in I(y) = [x_1, x_2] \subset \mathbb{Z}$ such that

$$\operatorname{dist}(y, \partial I(y)) > \frac{1}{40}|I(y)|, \tag{9.10}$$

$$|I(y)| \geq q_n^{8/9} - 2, \tag{9.11}$$

$$G_{I(y)}(k, x_i) < e^{-L_1|k - x_i|}, \ i = 1, 2. \tag{9.12}$$

In addition, if $b_j < y \leq b_{j+1}$ we have

$$|I(y)| \leq 2q_j. \tag{9.13}$$

We denote the boundary of the interval $I(y)$, the set $\{x_1, x_2\}$, by $\partial I(y)$. For $z \in \partial I(y)$ we let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

We now expand $\Psi(x_2 + 1)$ in (9.4) iterating (9.4) with $I = I(x_2 + 1)$. In case $q_n^{8/9} < x_1 - 1$ we also expand $\Psi(x_1 - 1)$ using (9.4) with $I = I(x_1 - 1)$. We continue to expand each term of the form $\Psi(z)$

in the same fashion until we arrive to z such that either $z \le b_n,\ z > k^2$ or the number of G_I terms in the product becomes $[\frac{40k}{q_n^{8/9}}]$, whichever comes first. We then obtain an expression of the form

$$\Psi(k) = \sum_{s;z_{i+1}\in\partial I(z_i')} G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\Psi(z_{s+1}'). \tag{9.14}$$

where in each term of the summation we have $z_i > b_n$, $i = 1,\dots,s$, and either $0 < z_{s+1}' \le b_n$, $s \le \frac{40k}{q_n^{8/9}}$ or $z_{s+1}' > k^2$, $s \le \frac{40k}{q_n^{8/9}}$ or $s+1 = [40\frac{k}{q_n^{8/9}}]$. By construction, for each z_i', $i \le s$, we have that $I(z_i')$ is well-defined and satisfies (9.12), (9.13). We now consider the three cases, $0 < z_{s+1}' \le b_n$, $z_{s+1}' > k^2$ and $s+1 = [\frac{40k}{q_n^{8/9}}]$ separately. If $0 < z_{s+1}' \le b_n$ we have, by (9.12) and (9.2),

$$\begin{aligned} &|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\Psi(z_{s+1}')| \\ &\qquad \le Ce^{-L_1(|k-z_1|+\sum_{i=1}^s|z_i'-z_{i+1}|)}(1+b_n) \\ &\qquad \le Ce^{-L_1(|k-z_{s+1}|-(s+1))}(1+b_n) \le Ce^{-L_1(k-b_n-\frac{40k}{q_n^{8/9}})}(1+b_n). \end{aligned} \tag{9.15}$$

Similarly, if $z_{s+1}' > k^2$, we use (9.12) and (9.13) to get

$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\Psi(z_{s+1}')| \le Ce^{-L_1(k^2-k-\frac{40k}{q_n^{8/9}})}(1+3k^{9/4}).$$

Finally, if $s+1 = [\frac{40k}{q_n^{8/9}}]$, using again (9.2), (9.12), and also (9.10) we can estimate

$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\Psi(z_{s+1}')| \le Ce^{-L_1\frac{1}{40}q_n^{8/9}[\frac{40k}{q_n^{8/9}}]}(1+k^2).$$

In either case,

$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\Psi(z_{s+1}')| \le e^{-\frac{9L_1}{10}k} \tag{9.16}$$

for k sufficiently large. Finally, we observe that the total number of terms in (9.14) is bounded above by $2^{[\frac{40k}{q_n^{8/9}}]}$. Combining it with (9.14), (9.16) we obtain

$$|\Psi(k)| \le 2^{[\frac{40k}{q_n^{8/9}}]}e^{-\frac{9L_1}{10}k} < e^{-\frac{4L_1}{5}k}$$

for large k. □

9.2. **Estimates on trigonometric products.** We will use the notation $\|z\|_{\mathbb{R}/\mathbb{Z}}$ for the distance to the nearest integer.

Lemma 9.6. *Let p, q be relatively prime. We have*

(1) *Let $1 \le k_0 \le q$ be such that $|\sin 2\pi(x+\frac{k_0p}{2q})| = \min_{1\le k\le q}|\sin 2\pi(x+\frac{kp}{2q})|$. Then*

$$\ln q + \ln\frac{2}{\pi} < \sum_{\substack{k=1\\k\ne k_0}}^{q} \ln|\sin 2\pi(x+\frac{kp}{2q})| + (q-1)\ln 2 \le \ln q, \tag{9.17}$$

(2)

$$\sum_{k=1}^{q-1}\ln|\sin\frac{\pi kp}{q}| = -(q-1)\ln 2 + \ln q. \tag{9.18}$$

Proof. We use that

$$\ln|\sin\frac{x}{2}| = -\ln 2 - \frac{1}{2}\sum_{k\neq 0}\frac{e^{ikx}}{|k|}. \tag{9.19}$$

Thus, for $x \neq \frac{k\pi}{2q}$,

$$\begin{aligned}\sum_{j=1}^{q}\ln|\sin 2\pi(x+\frac{jp}{2q})| &= -q\ln 2 - \frac{1}{2}\sum_{k\neq 0}\frac{1}{|k|}\sum_{j=1}^{q}e^{2\pi ik(2x+\frac{jp}{q})} \\ &= -q\ln 2 - \frac{1}{2}\sum_{k\neq 0}\frac{1}{|k|}e^{4\pi ikqx} = -q\ln 2+\ln 2+\ln|\sin 2\pi qx|.\end{aligned} \tag{9.20}$$

Thus

$$\sum_{\substack{k=1\\k\neq k_0}}^{q}\ln|\sin 2\pi(x+\frac{kp}{2q})| + (q-1)\ln 2 = \ln\frac{|\sin 2\pi q(x+\frac{k_0p}{2q})|}{|\sin 2\pi(x+\frac{k_0p}{2q})|} \tag{9.21}$$

It is easily checked that if $0<qx\le\frac{\pi}{2}$, then $\frac{2q}{\pi}<\frac{\sin qx}{\sin x}<q$. Since $\|2x+\frac{k_0p}{q}\|_{\mathbb{R}/\mathbb{Z}}\le\frac{1}{2q}$, (9.21) implies (9.17). (9.18) follows by taking the limit in (9.21). □

For $\alpha\notin\mathbb{Q}$ let $\frac{p_n}{q_n}$ be its continued fraction approximants. Set $\Delta_n = |q_n\alpha - p_n|$. We recall the following basic estimates

$$\frac{1}{q_n} > \Delta_{n-1} > \frac{1}{q_n+q_{n-1}}, \tag{9.22}$$

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \Delta_{n-1}, \quad q_{n-1}+1\le k\le q_n-1. \tag{9.23}$$

Notice that if $z,w\in\mathbb{R}$ are such that $\cos(z-w)\ge 0$ then

$$\left|\frac{\sin z}{\sin w}-1\right| \le \left|\cos(z-w)-1+\frac{\cos w}{\sin w}\sin(z-w)\right| \le \left|2\frac{\sin(z-w)}{\sin w}\right|. \tag{9.24}$$

Lemma 9.7. *Let $1\le k_0\le q_n$ be such that $|\sin 2\pi(x+\frac{k_0\alpha}{2})| = \min_{1\le k\le q_n}|\sin 2\pi(x+\frac{k\alpha}{2})|$. Then*

$$\left|\sum_{\substack{k=1\\k\neq k_0}}^{q_n}\ln|\sin 2\pi(x+\frac{k\alpha}{2})| + (q_n-1)\ln 2\right| < C\ln q_n. \tag{9.25}$$

Proof. Let $1\le k_1\le q_n$ be such that

$$|\sin 2\pi(x+\frac{k_1p_n}{2q_n})| = \min_{1\le k\le q_n}|\sin 2\pi(x+\frac{kp_n}{2q_n})|. \tag{9.26}$$

We first remark that, by (9.23)

$$\|(2x+k\alpha)-(2x+k'\alpha)\|_{\mathbb{R}/\mathbb{Z}} = \|(k-k')\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \Delta_{n-1}, \quad 1\le k,k'\le q_n, k\neq k'. \tag{9.27}$$

Applying this to the case $k'=k_0$, we get, by (9.22), $|\ln|\sin 2\pi(x+\frac{k\alpha}{2})|| < C\ln q_n$, $k\neq k_0$. An even simpler argument

$$\|(2x+k\frac{p_n}{q_n})-(2x+k'\frac{p_n}{q_n})\|_{\mathbb{R}/\mathbb{Z}} = \|(k-k')\frac{p_n}{q_n}\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{1}{q_n}, \quad 1\le k,k'\le q_n, k\neq k', \tag{9.28}$$

also gives that if $k \neq k_1$ then $|\ln|\sin 2\pi(x+\frac{kp_n}{2q_n})|| < C\ln q_n$. This and (9.17) show that it is enough to get the estimate

$$\sum_{\substack{k=1\\k\neq k_0,k_1}}^{q_n} \ln\left|\frac{\sin 2\pi(x+\frac{k\alpha}{2})}{\sin 2\pi(x+\frac{kp_n}{2q_n})}\right| < C\ln q_n. \tag{9.29}$$

By (9.24),

$$\left|\frac{\sin 2\pi(x+\frac{k\alpha}{2})}{\sin 2\pi(x+\frac{kp_n}{2q_n})}-1\right| < \frac{C_0\Delta_n}{|\sin 2\pi(x+\frac{kp_n}{2q_n})|}, \tag{9.30}$$

so we have

$$\ln\left|\frac{\sin 2\pi(x+\frac{k\alpha}{2})}{\sin 2\pi(x+\frac{kp_n}{2q_n})}\right| < \frac{C\Delta_n}{|\sin 2\pi(x+\frac{kp_n}{2q_n})|}, \tag{9.31}$$

provided that $C_0\Delta_n < \frac{1}{4}|\sin 2\pi(x+\frac{kp_n}{2q_n})|$. Let $s_1,...,s_r$ be an enumeration of $\{1 \leq k \leq q_n,\ k \neq k_0, k_1\}$ in non-decreasing order of $|\sin 2\pi(x+\frac{kp_n}{2q_n})|$ (so $r = q_n - 1$ or $r = q_n - 2$). By (9.28), we have $|\sin 2\pi(x+\frac{s_j p_n}{2q_n})| > C_1\frac{j}{q_n}$. Then

$$\begin{aligned}\sum_{\substack{k=1\\k\neq k_0,k_1}}^{q_n} \ln\left|\frac{\sin 2\pi(x+\frac{k\alpha}{2})}{\sin 2\pi(x+\frac{kp_n}{2q_n})}\right| &= \sum_{1\leq j\leq 4\frac{C_0}{C_1}} \ln\left|\frac{\sin 2\pi(x+\frac{s_j\alpha}{2})}{\sin 2\pi(x+\frac{s_jp_n}{2q_n})}\right| + \sum_{4\frac{C_0}{C_1}<j\leq r} \ln\left|\frac{\sin 2\pi(x+\frac{s_j\alpha}{2})}{\sin 2\pi(x+\frac{s_jp_n}{2q_n})}\right| \\ &\leq C\ln q_n + \sum_{4\frac{C_0}{C_1}<j\leq r} \frac{Cq_n\Delta_n}{j} \leq C\ln q_n,\end{aligned} \tag{9.32}$$

which is (9.29). □

Lemma 9.8. *Let $\ell \in \mathbb{N}$ be such that $\ell < \frac{q_{r+1}}{10q_n}$, where $r \geq n$. Given a sequence $|\ell_k| \leq \ell - 1$, $k = 1,\ldots,q_n$, let $1 \leq k_0 \leq q_n$ be such that*

$$|\sin 2\pi(x+\frac{(k_0+\ell_{k_0}q_r)\alpha}{2})| = \min_{1\leq k\leq q_n} |\sin 2\pi(x+\frac{(k+\ell_kq_r)\alpha}{2})|. \tag{9.33}$$

Then

$$\left|\sum_{\substack{k=1\\k\neq k_0}}^{q_n} \ln|\sin 2\pi(x+\frac{(k+\ell_kq_r)\alpha}{2})| + (q_n-1)\ln 2\right| < \ln q_n + C(\Delta_n + (\ell-1)\Delta_r)q_n\ln q_n. \tag{9.34}$$

Proof. Notice that $\|(k-k')\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \Delta_{n-1} \geq \frac{1}{2q_n}$, while $\|(\ell_k - \ell_{k'})q_r\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{q_{r+1}}{5q_n}\Delta_r < \frac{1}{5q_n}$. This implies

$$\|(2x+(k+\ell_kq_r)\alpha) - (2x+(k'+\ell_{k'}q_r)\alpha)\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{5q_n}, \quad 1 \leq k,k' \leq q_n, k \neq k'. \tag{9.35}$$

By (9.24), we have

$$\left|\frac{\sin 2\pi(x+\frac{(k+\ell_kq_r)\alpha}{2})}{\sin 2\pi(x+\frac{kp_n}{2q_n})}-1\right| \leq \frac{C_0(\Delta_n+(\ell-1)\Delta_r)}{|\sin 2\pi(x+\frac{kp_n}{2q_n})|}. \tag{9.36}$$

We now argue as in the previous lemma, using (9.35) and (9.36) instead of (9.27) and (9.30). □

9.3. **Non-resonant case. Proof of Lemma 9.4.** In the arguments that follow, we will actually consider a slightly larger range of k, by assuming a weaker upper bound $k \le \max\{\frac{q_n}{20}, 50q_{n+1}^{8/9}\}$. The fact that the estimates hold for this larger range will be useful later (when dealing with the resonant case).

We start with the proof of the first part. Let $k = mq_n \pm (sq_{n-1} + r) = mq_n \pm k_0,\ s \ge 1,\ 0 \le r < q_{n-1},\ k_0 \le \frac{q_n}{2}$, be non-resonant. Notice that $2sq_{n-1} < q_n$. Assume without loss of generality that $k = mq_n + k_0$, the other case being treated similarly.

Notice that if $m \ge 1$ then $k > \frac{q_n}{20}$ which implies that $k \le 50q_{n+1}^{8/9}$, and we have

$$m \le \frac{50q_{n+1}^{8/9}}{q_n} \tag{9.37}$$

(which is also obviously satisfied if $m = 0$).

Set $I_1 = [-[\frac{sq_{n-1}}{2}], sq_{n-1} - [\frac{sq_{n-1}}{2}] - 1]$ and $I_2 = [mq_n + k_0 - [\frac{sq_{n-1}}{2}], mq_n + k_0 + sq_{n-1} - [\frac{sq_{n-1}}{2}] - 1]$. Set $\theta_j = \theta + j\alpha, j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j \in I_1 \cup I_2}$ consists of $2sq_{n-1}$ elements.

Lemma 9.9. *For any $\epsilon > 0$ and sufficiently large n, set $\{\theta_j\}_{j\in I_1\cup I_2}$ is $\frac{-2\ln\frac{s}{q_n}}{q_{n-1}} + \epsilon$-uniform.*

Proof. We will first estimate the numerator in (9.8). We have

$$\begin{aligned} \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} &\ln|\cos 2\pi a - \cos 2\pi\theta_j| \\ &= \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin 2\pi\frac{a+\theta_j}{2}| + \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin 2\pi\frac{a-\theta_j}{2}| + (2sq_{n-1}-1)\ln 2 \\ &= \Sigma_+ + \Sigma_- + (2sq_{n-1}-1)\ln 2. \end{aligned} \tag{9.38}$$

Both Σ_+ and Σ_- consist of $2s$ terms of the form of (9.25) plus $2s$ terms of the form

$$\ln \min_{j=1,\dots,q_{n-1}} |\sin(2\pi(x + \frac{j\alpha}{2}))|, \tag{9.39}$$

minus $\ln|\sin\frac{a\pm\theta_i}{2}|$. Therefore, by (9.25)

$$\sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \le -2sq_{n-1}\ln 2 + Cs\ln q_{n-1}. \tag{9.40}$$

To estimate the denominator of (9.8) we represent it again in the form (9.38) with $a = \theta_i$. Assume that $i = j_0 q_{n-1} + i_0 \in I_1,\ 0 \le i_0 < q_{n-1}$, the other case being treated similarly. Then

$$\Sigma_- = \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin\pi(i-j)\alpha|. \tag{9.41}$$

On each interval $I \subset I_1$ of length q_{n-1}, minimum over $t \in I$ of $|\sin\pi(t-i)\alpha|$ is achieved at $t - i$ of the form jq_{n-1} for some j. This follows from the fact that if $0 < |z| < q_{n-1}$ and $2|j|q_{n-1} < q_n$ then $\|(jq_{n-1}+z)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \|jq_{n-1}\alpha\|_{\mathbb{R}/\mathbb{Z}}$, since $\|z\alpha\| \ge \Delta_{n-2}$ and $\|jq_{n-1}\alpha\| < \Delta_{n-2}/2$. The possible values of j form an interval $[j_-^0, j_+^0]$ of size s containing j_0.

Let now T be an arbitrary interval of length q_{n-1} contained in I_2. Notice that T is contained in $[i + mq_n + 1, i + (m+1)q_n - 1]$. The minimum over $t \in T$ of $|\sin\pi(t-i)\alpha|$ is achieved at $t - i$ of either the form $mq_n + jq_{n-1}$ or the form $(m+1)q_n - jq_{n-1}$ for some $j \in \mathbb{N}$.[5] For $u \in \{0,1\}$, let $t_u \in T$ be (the unique number) of the form $t_u = i + (m+u)q_n + (-1)^u j_u q_{n-1}$ for some $j_u \in \mathbb{N}$. Since $|t_u - t_{1-u}| < q_{n-1}$ it follows that

$$0 \le j_{1-u} + j_u - [\frac{q_n}{q_{n-1}}] \le 1. \tag{9.42}$$

For all $j \in [1, [\frac{q_n}{q_{n-1}}]]$, we have the lower bound

$$\|((-1)^u jq_{n-1} + (m+u)q_n)\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \Delta_{n-1}/2. \tag{9.43}$$

Indeed, by (9.37), if $m \ge 1$ then $(m+u)\Delta_n \le 100\frac{q_{n+1}^{8/9}}{q_n}\Delta_n \le \Delta_{n-1}/2$, while $\|jq_{n-1}\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \Delta_{n-1}$. If $m = 0$ then $(m+u)q_n + (-1)^u jq_{n-1} \in [1, q_n - 1]$, and we get the lower bound Δ_{n-1}. Those considerations also give the upper bound

$$(m+u)\Delta_n \le \max\{\Delta_{n-1}/2, \Delta_n\}. \tag{9.44}$$

This gives the estimate, for all $j \in [1, [\frac{q_n}{q_{n-1}}]]$,

$$\|((-1)^u jq_{n-1} + (m+u)q_n)\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge j\Delta_{n-1}/C. \tag{9.45}$$

Let T now run through the set of disjoint segments T^p, each of length q_{n-1}, such that $I_2 = \cup_{p=1}^s T^p$. It is not difficult to see that there exists u (possibly both $u = 0,1$) such that for all p corresponding j_u satisfy $j_u \le \frac{3}{4}[\frac{q_n}{q_{n-1}}]$.[6] We now fix $u \in \{0,1\}$ with this property. Then $\|((m+u)q_n + (-1)^u j_u q_{n-1})\alpha\|_{\mathbb{R}/\mathbb{Z}} \le \frac{3}{4}[\frac{q_n}{q_{n-1}}]\Delta_{n-1} + |m+u|\Delta_n \le (\frac{3}{4}[\frac{q_n}{q_{n-1}}] + 1)\Delta_{n-1}$. Then, by (9.42), $j_{1-u} \ge \frac{1}{4}[\frac{q_n}{q_{n-1}}]$ and by (9.45), $\|((-1)^{1-u} j_{1-u} q_{n-1} + (m+1-u)q_n)\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{1}{4C}[\frac{q_n}{q_{n-1}}]\Delta_{n-1}$. Thus $\|(t_{1-u} - i)\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \frac{\|(t_u - i)\alpha\|_{\mathbb{R}/\mathbb{Z}}}{7C}$.

Notice that the j_u form an interval $[j_-, j_+]$ of length s, contained in $[1, [\frac{3q_n}{4q_{n-1}}]]$.

Splitting again Σ_- into $2s$ sums of length q_{n-1} and applying (9.25) on each we obtain

$$\Sigma_- > -2sq_{n-1}\ln 2 + \sum_{\substack{j_-^0 \le j \le j_+^0 \\ j \ne j_0}} \ln|\sin \pi(j - j_0)q_{n-1}\alpha| \tag{9.46}$$
$$+ \sum_{j=j_-}^{j_+} \ln|\sin \pi((-1)^u jq_{n-1} + (m+u)q_n)\alpha| - Cs - Cs\ln q_{n-1}.$$

Denote the sums in (9.46) by Σ_1 and Σ_2. Since $j_-^0 \le j_0 \le j_+^0$ and $|[j_-^0, j_+^0]| = s$ we have that

$$\Sigma_1 > 2\sum_{j=1}^{[\frac{s}{2}]} \ln\sin|\pi jq_{n-1}\alpha| > 2\sum_{j=1}^{[\frac{s}{2}]} \ln 2j\Delta_{n-1} > s(\ln\frac{s}{q_n} - C). \tag{9.47}$$

For $j \in [j_-, j_+]$ we use (9.45) to obtain

$$\Sigma_2 > \sum_{j=1}^{s} \ln j\Delta_{n-1} - Cs > s\ln\frac{s}{q_n} - Cs. \tag{9.48}$$

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[5]Let $t \in T$ minimize $\|(t-i)\alpha\|_{\mathbb{R}/\mathbb{Z}}$, and let j_u, $u \in \{0,1\}$ be such that $t_u = (m+u)q_n + (-1)^u j_u q_{n-1} + i \in T$. If $t \ne t_0$ and $t \ne t_1$ then $\|(t - t_u)\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge \Delta_{n-2}$ as above. Since the $(t_u - i)\alpha$ minus nearest integer are on opposite sides of 0, this implies that $\|(t_0 - t_1)\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge 2\Delta_{n-2}$. But one easily checks that $\|(t_0 - t_1)\alpha\|_{\mathbb{R}/\mathbb{Z}}$ is either equal to Δ_{n-2} (if $t_1 > t_0$) or to $\Delta_{n-1} + \Delta_{n-2}$ (if $t_1 \le t_0$).

[6]For $u = 0,1$ the j_u form an interval $[j_-^u, j_+^u]$ of length s contained in $[1, [\frac{q_n}{q_{n-1}}]]$. If $j_+^u > \frac{3}{4}[\frac{q_n}{q_{n-1}}]$, then, since $s \le [\frac{q_n}{2q_{n-1}}]$, we have that $j_-^u > \frac{1}{4}[\frac{q_n}{q_{n-1}}] + 1$. Then, by (9.42), $j_+^{1-u} < \frac{3}{4}[\frac{q_n}{q_{n-1}}]$.

Therefore,

$$\Sigma_- > -2sq_{n-1}\ln 2 + 2s(\ln\frac{s}{q_n} - C\ln q_{n-1}). \tag{9.49}$$

Σ_+ is estimated in a similar way. Set $J_1 = [-[\frac{s+1}{2}], s-[\frac{s+1}{2}]-1]$ and $J_2 = [[s/2], s+[s/2]-1]$, which are two adjacent disjoint intervals of length s. Then $I_1 \cup I_2$ can be represented as a disjoint union of segments B_j, $j \in J_1 \cup J_2$, each of length q_{n-1}. Applying (9.25) on each B_j we obtain

$$\Sigma_+ > -2sq_{n-1}\ln 2 + \sum_{j\in J_1\cup J_2} \ln|\sin 2\pi\hat{\theta}_j| - Cs\ln q_{n-1} - \ln|\sin 2\pi(\theta+i\alpha)| \tag{9.50}$$

where

$$|\sin 2\pi\hat{\theta}_j| = \min_{\ell\in B_j} |\sin 2\pi(\theta + \frac{(i+\ell)\alpha}{2})|. \tag{9.51}$$

Let $\tilde{\theta}_j = \hat{\theta}_j$, $j \in J_1$, and $\tilde{\theta}_j = \hat{\theta}_j - \frac{mq_n\alpha}{2}$, $j \in J_2$. Since $\theta \notin \Theta$, for sufficiently large n, we have that

$$\min_{j\in J_1\cup J_2} |\sin 2\pi\tilde{\theta}_j| > \frac{1}{9s^2q_{n-1}^2}.$$

To estimate $|\sin 2\pi\hat{\theta}_j|$, $j \in J_2$, we distinguish the two cases:

(1) If $q_{n+1} > (20s^2q_{n-1}^2)^9$, we write

$$|\sin 2\pi\hat{\theta}_j| \geq |\sin 2\pi\tilde{\theta}_j \cos\pi m\Delta_n| - |\cos 2\pi\tilde{\theta}_j \sin\pi m\Delta_n| > \frac{1}{10s^2q_{n-1}^2} - \frac{1}{q_{n+1}^{1/9}} > \frac{1}{20s^2q_{n-1}^2}. \tag{9.52}$$

(2) If $q_{n+1} \leq (20s^2q_{n-1}^2)^9$ we use that since $\theta \notin \Theta$, for large n,

$$\min_{j\in J_2} |\sin 2\pi\hat{\theta}_j| > ((2m+2)q_n)^{-2} > (4q_{n+1})^{-2} > (20sq_{n-1})^{-36} \tag{9.53}$$

In either case,

$$\ln\min_{j\in J_2} |\sin 2\pi\hat{\theta}_j| > -C\ln sq_{n-1}$$

Let $J = J_1$ or $J = J_2$ and assume that $\hat{\theta}_{j+1} = \hat{\theta}_j + \frac{q_{n-1}}{2}\alpha$ for every $j, j+1 \in J$. Applying again the Stirling formula we obtain

$$\sum_{j\in J} \ln|\sin 2\pi\hat{\theta}_j| > -C\ln sq_{n-1} + \sum_{j=1}^{s} \ln\frac{j\Delta_{n-1}}{C} > s\ln\frac{s}{q_n} - C(\ln sq_{n-1} + s). \tag{9.54}$$

In the other case, decompose J in maximal intervals T_κ such that for $j, j+1 \in T_\kappa$ we have $\hat{\theta}_{j+1} = \hat{\theta}_j + \frac{q_{n-1}}{2}\alpha$. Notice that the boundary points of an interval T_κ are either boundary points of J or satisfy $\|2\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_{n-1} \geq \frac{\Delta_{n-2}}{2}$. Assuming $T_\kappa \neq J$, there exists $j \in T_\kappa$ such that $\|2\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_{n-1} \geq \frac{\Delta_{n-2}}{2}$. An estimate similar to (9.54) gives

$$\sum_{j\in T_\kappa} \ln|\sin 2\pi\hat{\theta}_j| > -|T_\kappa|\ln q_{n-1} - C(\ln sq_{n-1} + |T_\kappa|). \tag{9.55}$$

If T_κ does not contain a boundary point of J (in particular $|T_\kappa| \leq |J|-2 = s-2$ and $[\frac{q_n}{q_{n-1}}] \geq 2s \geq 6$), then T_κ does not contain any j with $\|2\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_{n-2}}{10} < \frac{\Delta_{n-2}}{2} - \Delta_{n-1}$ (otherwise $|T_\kappa| - 1 \geq \frac{\Delta_{n-2}}{\Delta_{n-1}} - 2 \geq \frac{q_n}{2q_{n-1}} - 2 \geq s-2$, which is impossible) and hence

$$\sum_{j\in T_\kappa} \ln|\sin 2\pi\hat{\theta}_j| > -|T_\kappa|(\ln q_{n-1} + C). \tag{9.56}$$

Putting together all T_κ, using (9.55) for the ones that intersect the boundary of J and (9.56) for the others, we get

$$\sum_{j\in J}\ln|\sin 2\pi\hat\theta_j| > s\ln\frac{s}{q_n} - C(\ln sq_{n-1}+s) \tag{9.57}$$

in all cases.

Putting together $J=J_1$ and $J=J_2$ we have

$$\sum_{j\in J_1\cup J_2}\ln|\sin 2\pi\hat\theta_j| > 2s\ln\frac{s}{q_n} - C(\ln sq_{n-1}+s) \tag{9.58}$$

Combining it with (9.50) we obtain

$$\Sigma_+ > -2sq_{n-1}\ln 2 + 2s(\ln\frac{s}{q_n} - C\ln q_{n-1}) \tag{9.59}$$

Putting together (9.59),(9.49), and (9.38) gives

$$\sum_{\substack{j\in I_1\cup I_2\\ j\neq i}}\ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j| > 4s(\ln\frac{sq_{n-1}}{q_n} - C\ln q_{n-1}) - 2sq_{n-1}\ln 2 \tag{9.60}$$

This together with (9.40) yields

$$\max_{j\in I_1\cup I_2}\prod_{\substack{\ell\in I_1\cup I_2\\ \ell\neq j}}\frac{|z-\cos 2\pi\theta_\ell|}{|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell|} < e^{-4s\ln\frac{sq_{n-1}}{q_n}+Cs\ln q_{n-1}}$$

as desired. □

By Lemmas 9.3 and 9.9 at least one of $\theta_j,\ j\in I_1\cup I_2$, is not in $A_{2sq_{n-1}-1,L+\frac{2\ln\frac{sq_{n-1}}{q_n}}{q_{n-1}}-\epsilon}$ where ϵ can be made arbitrarily small for large n. By Lemma 9.2 and singularity of 0,[7] we have that for all $j\in I_1,\ \theta_j\in A_{2sq_{n-1}-1,L+\frac{2\ln\frac{sq_{n-1}}{q_n}}{q_{n-1}}-\epsilon}$ (using that $(s+1)q_{n-1} > q_n^{8/9}$ and the bound $\frac{\ln q_n}{q_{n-1}} < L$). Let $j_0\in I_2$ be such that $\theta_{j_0}\notin A_{2sq_{n-1}-1,L+\frac{2\ln\frac{sq_{n-1}}{q_n}}{q_{n-1}}-\epsilon}$. Set $I=[j_0-sq_{n-1}+1, j_0+sq_{n-1}-1]=[x_1,x_2]$. Then by (9.6),(9.7),

$$\begin{aligned}|G_I(k,x_i)| &< e^{(L+\epsilon_1)(2sq_{n-1}-2-|k-x_i|)-2sq_{n-1}(L+\frac{2\ln\frac{sq_{n-1}}{q_n}}{q_{n-1}}-\epsilon)} \\ &< e^{-(L+\epsilon_1)|k-x_i|-4sq_{n-1}\frac{\ln\frac{sq_{n-1}}{q_n}}{q_{n-1}}+(\epsilon_1+\epsilon)sq_{n-1}}.\end{aligned} \tag{9.61}$$

Since

$$|k-x_i| \geq [\frac{sq_{n-1}}{2}]-1, \tag{9.62}$$

we obtain that

$$|G_I(k,x_i)| < e^{-(L+9\frac{\ln\frac{sq_{n-1}}{q_n}}{q_{n-1}}-\epsilon)|k-x_i|} \tag{9.63}$$

which in view of $(s+1)q_{n-1} > q_n^{8/9}$ gives the statement of the first part of Lemma 9.4.

We now assume $s=0$. In this case α is "Diophantine" on the scale q_{n-1} however some caution is needed as it may not be so on the scale q_n. Let $k=mq_n\pm k_0,\ \max\{\frac{1}{20}q_{n-1}, q_n^{8/9}\} < k_0 < q_{n-1}$. We will assume that $m=q_n+k_0$, the other case being analogous.

We distinguish three cases.

[7]To get what we need here one can take in Lemma 9.2, besides $y=0$, also $\epsilon=\frac{99}{100}L$ and $\delta=\frac{99}{400}$.

(1) If $\frac{1}{20}q_{n-1} < k_0 \le \frac{4}{5}q_{n-1}$, set $I_1 = [-[\frac{19}{40}q_{n-1}]+1, [\frac{19}{40}q_{n-1}]]$ and $I_2 = [mq_n + [\frac{19}{40}q_{n-1}] + 1, mq_n + 2[\frac{q_{n-1}}{2}] - [\frac{19}{40}q_{n-1}]]$.
(2) If $\frac{4}{5}q_{n-1} < k_0 < q_{n-1}$ and $q_n \le 2q_{n-1}$, define $I_1 = [-[\frac{q_n}{4}]+1, [\frac{q_n}{4}]]$ and $I_2 = [mq_n + [\frac{q_n}{4}] + 1, mq_n + 2[\frac{q_n}{2}] - [\frac{q_n}{4}]]$.
(3) If $\frac{4}{5}q_{n-1} < k_0 < q_{n-1}$ and $q_n > 2q_{n-1}$, set $I_1 = [-[\frac{q_{n-1}}{2}]+1, q_{n-1} - [\frac{q_{n-1}}{2}]]$ and $I_2 = [mq_n + q_{n-1} - [\frac{q_{n-1}}{2}] + 1, mq_n + 2q_{n-1} - [\frac{q_{n-1}}{2}]]$.

Set $\theta_j = \theta + j\alpha, j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j\in I_1\cup I_2}$ consists of $2[\frac{q_{n-1}}{2}]$ elements in the first case, $2[\frac{q_n}{2}]$ elements in the second case, and of $2q_{n-1}$ elements in the third case.

Lemma 9.10. *For any $\epsilon > 0$ and sufficiently large n, the set $\{\theta_j\}_{j\in I_1\cup I_2}$ is ϵ-uniform.*

Proof. Consider first the case $k_0 \le \frac{4}{5}q_{n-1}$. We will assume q_{n-1} is even, the other case needing obvious adjustments. As in the proof of Lemma 9.4 we will first estimate the numerator in (9.8). We have

$$\begin{aligned}(9.64)\qquad &\sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \\ &= \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin 2\pi\frac{a+\theta_j}{2}| + \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin 2\pi\frac{a-\theta_j}{2}| + (q_{n-1}-1)\ln 2 \\ &= \Sigma_+ + \Sigma_- + (q_{n-1}-1)\ln 2.\end{aligned}$$

Both Σ_+ and Σ_- are of the form (9.34) with $\ell_k \in \{0, m\}$[8] and $r = n$ plus a minimum term minus $\ln|\sin 2\pi\frac{a\pm\theta_i}{2}|$, so that the last two cancel each other for the purpose of the upper bound. Therefore, by (9.34)

$$\begin{aligned}(9.65)\qquad \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| &\le (1-q_{n-1})\ln 2 + 2\ln q_{n-1} + C(\Delta_{n-1} + m\Delta_n)q_{n-1}\ln q_{n-1} \\ &\le -q_{n-1}\ln 2 + Cq_{n-1}^{8/9}\ln q_{n-1}.\end{aligned}$$

To estimate the denominator of (9.8) we write it in the form (9.64) with $a = \theta_i$. Then

$$(9.66)\qquad \Sigma_- = \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin\pi(i-j)\alpha|$$

is exactly of the form (9.34). Therefore, by (9.34),

$$(9.67)\qquad \Sigma_- > (1-q_{n-1})\ln 2 - \ln q_{n-1} - C(\frac{1}{q_n} + \frac{1}{q_{n+1}^{1/9}})q_{n-1}\ln q_{n-1} > -q_{n-1}\ln 2 - Cq_{n-1}^{8/9}\ln q_{n-1}.$$

Similarly, for Σ_+ we have

$$\begin{aligned}(9.68)\qquad \Sigma_+ &> (1-q_{n-1})\ln 2 + \ln \min_{\ell\in I_1\cup I_2} |\sin 2\pi(\theta + \frac{(i+\ell)\alpha}{2})| - Cq_{n-1}^{8/9}\ln q_{n-1} \\ &> -q_{n-1}\ln 2 - Cq_{n-1}^{8/9}\ln q_{n-1}.\end{aligned}$$

[8]Recall that m is chosen so that $k = mq_n + k_0$, where $k \le \max\{50q_{n+1}^{8/9}, q_n/20\}$. We have the bound $m \le 50q_{n+1}^{8/9}/q_n$, so (9.34) really applies.

Here, we use the estimate

$$\ln \min_{\ell \in I_1 \cup I_2} |\sin 2\pi(\theta + \frac{(i+\ell)\alpha}{2})| > -C \ln q_{n-1} \tag{9.69}$$

which is obtained by considering separately two cases $q_{n+1} > q_{n-1}^C$ and $q_{n+1} < q_{n-1}^C$, and arguing in the same way as in (9.52),(9.53). Combining (9.65),(9.64),(9.67) and (9.68), we arrive at

$$\max_{j \in I_1 \cup I_2} \prod_{\substack{\ell \in I_1 \cup I_2 \\ \ell \neq j}} \frac{|z - \cos 2\pi\theta_\ell)|}{|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)|} < e^{C q_{n-1}^{8/9} \ln q_{n-1}} < e^{\epsilon q_{n-1}} \tag{9.70}$$

for any $\epsilon > 0$ and sufficiently large n, as stated.

For the other cases, $k > \frac{4}{5} q_{n-1}$, the proof is very similar. If $q_n \leq 2q_{n-1}$, the argument is the same (replacing q_{n-1} by q_n). We will concentrate on the case $q_n > 2q_{n-1}$ where the changes are slightly more substantial. Arguing as above we obtain by (9.34)

$$\sum_{\substack{j \in I_1 \cup I_2 \\ j \neq i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leq -2q_{n-1} \ln 2 + 4 \ln q_{n-1} + C(\Delta_{n-1} + m\Delta_n) q_{n-1} \ln q_{n-1} \tag{9.71}$$

$$< -2q_{n-1} \ln 2 + C q_{n-1}^{8/9} \ln q_{n-1}$$

The denominator in (9.8) can be again split as $\Sigma_+ + \Sigma_- + (2q_{n-1} - 1)\ln 2$. Both Σ_+ and Σ_- are, up to a constant, the sums of two terms of the form (9.34) plus minimum terms (two for Σ_+ and one for Σ_-). For the minimum terms of Σ_+ the estimate (9.69) holds so that we obtain

$$\Sigma_+ > -2q_{n-1} \ln 2 - C q_{n-1}^{8/9} \ln q_{n-1}. \tag{9.72}$$

For the minimum term of Σ_-, that is, $\ln \min |\sin \pi(i-j)\alpha|$ (where the minimum is taken over all j which belong to the interval I_1 or I_2 that does not contain i) we observe that it is achieved at j_0 such that $\|(i - j_0)\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \Delta_{n-1} - m\Delta_n$ (since the possible values of $|i-j|$ are contained in $[mq_n + 1, mq_n + 2q_{n-1} - 1]$ and $q_n > 2q_{n-1}$ by hypothesis). Thus, recalling that in the present situation we have $q_n^{8/9} < q_{n-1}$,

$$\ln \min |\sin \pi(i-j)\alpha| > \ln(\Delta_{n-1} - m\Delta_n) > \ln(\frac{1}{2q_n} - \frac{50}{q_{n+1}^{1/9} q_n}) > -C \ln q_n > -C \ln q_{n-1}. \tag{9.73}$$

Therefore, by (9.34),

$$\Sigma_- > -2q_{n-1} \ln 2 - C q_{n-1}^{8/9} \ln q_{n-1}. \tag{9.74}$$

Combining (9.71),(9.64),(9.74) and (9.72), gives (9.70), as desired. □

By Lemmas 9.3 and 9.10 at least one of θ_j, $j \in I_1 \cup I_2$, is not in $A_{2[\frac{q_{n-1}}{2}]-1, L-\epsilon}$ if $k_0 \leq \frac{4}{5} q_{n-1}$, not in $A_{2[\frac{q_n}{2}]-1, L-\epsilon}$ if $\frac{4}{5} q_{n-1} < k_0 < q_{n-1}$ and $q_n \leq 2q_{n-1}$, and not in $A_{2q_{n-1}-1, L-\epsilon}$ if $k_0 > \frac{4}{5} q_{n-1}$ and $q_n > 2q_{n-1}$, where ϵ can be made arbitrarily small for large n. By Lemma 9.2 and singularity of 0, we have that, in all three cases, for all $j \in I_1$, θ_j belongs to the corresponding $A_{\cdot, L-\epsilon}$. Let $j_0 \in I_2$ be such that $\theta_{j_0} \notin A_{L-\epsilon}$.

For $k_0 \leq \frac{4}{5} q_{n-1}$ set $I = [j_0 - [\frac{q_{n-1}}{2}] + 1, j_0 + [\frac{q_{n-1}}{2}]] = [x_1, x_2]$. We then have

$$|k - x_i| > \frac{q_{n-1}}{40}. \tag{9.75}$$

Then by (9.6),(9.7),

$$|G_I(k, x_i)| < e^{(L+\epsilon_1)(q_{n-1} - 2 - |k - x_i|) - q_{n-1}(L-\epsilon)} < e^{-(L+\epsilon_1 - 40(\epsilon_1 + \epsilon))|k - x_i|}, \tag{9.76}$$

as desired.

For $k_0 > \frac{4}{5}q_{n-1}$ and $q_n \leq 2q_{n-1}$, set $I = [j_0 - [\frac{q_n}{2}] + 1, j_0 + [\frac{q_n}{2}]] = [x_1, x_2]$. Then

$$|k - x_i| > \frac{q_n}{10}, \tag{9.77}$$

since $k - x_1 > \frac{4}{5}q_{n-1} - \frac{q_n}{4} \geq \frac{3}{10}q_{n-1}$ and $x_2 - k > \frac{3q_n}{4} - q_{n-1} = \frac{3q_{n-2} - q_{n-1}}{4} > \frac{q_{n-1}}{5}$ (using that $\frac{4}{5}q_{n-1} < k_0 \leq \frac{q_n}{2} = \frac{q_{n-1} + q_{n-2}}{2}$). Thus for any $\epsilon > 0$ and sufficiently large n, by (9.6),(9.7), and estimating as in (9.76)

$$|G_I(k, x_i)| < e^{-(L-\epsilon)|k - x_i|}. \tag{9.78}$$

For $k_0 > \frac{4}{5}q_{n-1}$ and $q_n > 2q_{n-1}$ set $I = [j_0 - q_{n-1} + 1, j_0 + q_{n-1} - 1] = [x_1, x_2]$. Then

$$|k - x_i| > \frac{3q_{n-1}}{10}. \tag{9.79}$$

This implies as before that (9.78) holds for any $\epsilon > 0$ and sufficiently large n. This concludes the proof of Lemma 9.4 in all cases. □

The estimates in the proof of Lemma 9.4 have the following corollary which will be necessary later (when dealing with the resonant case).

Lemma 9.11. *Fix $\epsilon > 0$. Assume $b_n < k \leq \max\{\frac{q_n}{20}, 50q_{n+1}^{8/9}\}$. Let $d = \operatorname{dist}(k, \{\ell q_n\}_{\ell \geq 0}) > \frac{1}{10}q_n$. Let $\phi = \phi_E$ be a generalised eigenfunction. Assume that either*

(1) $q_n \geq q_{n-1}^{10/9}$, *or*

(2) $q_n < q_{n-1}^{10/9}$ *and $k < q_n^C$ for some $C < \infty$.*

Then, for sufficiently large n ($n > n_0(\epsilon, c, E)$ in the first case, $n > n_0(\epsilon, c, E, C)$ in the second case),

$$|\phi(k)| < e^{-(L-\epsilon)\frac{d}{2}}.$$

Proof. Recall that the previous estimates in this subsection were obtained, under the non-resonance hypothesis $\operatorname{dist}(k, \{\ell q_n\}_{\ell \geq 0}) > b_n$, for $b_n < k \leq \max\{\frac{q_n}{20}, 50q_{n+1}^{8/9}\}$.

If $q_n \geq q_{n-1}^{10/9}$, we have $s \geq [\frac{q_n^{1/10}}{10}]$, and the statement follows immediately from (9.4), (9.2) and (9.63), (9.62).

In case $q_n < q_{n-1}^{10/9}$, (9.63), (9.62), (9.76), (9.75), (9.78), (9.77), (9.79) only lead to $|\phi(k)| < e^{-(L-\epsilon)cd}$ with certain $c < 1/2$. In order to prove the Lemma as stated we will need an additional "patching" argument, which is very similar to the one used in subsection 9.1.

We will show that in this case

$$|\phi(k)| < e^{-(L-\epsilon)(d - \frac{q_{n-1}}{20})} \tag{9.80}$$

from which the statement of the lemma follows. Assume $\ell q_n < k < (\ell+1)q_n$. Using (9.63), (9.62), (9.76),(9.75) and (9.78), (9.77), (9.79) we obtain that for every $y \in [\ell q_n, (\ell+1)q_n]$ with $\operatorname{dist}(y, \{\ell q_n, (\ell+1)q_n\}) > \frac{1}{20}q_{n-1}$, there exists an interval $y \in I(y) = [x_1, x_2] \subset [(\ell-1)q_n, (\ell+2)q_n]$ such that

$$\operatorname{dist}(y, \partial I(y)) > \frac{q_{n-1}}{40}, \tag{9.81}$$

$$G_{I(y)}(y, x_i) < e^{-(L-\epsilon)|y - x_i|},\ i = 1, 2 \tag{9.82}$$

(notice that under the condition $q_n < q_{n-1}^{10/9}$ we have $b_n = \frac{q_{n-1}}{20}$).

We here denote the boundary of the interval $I(y)$, the set $\{x_1, x_2\}$, by $\partial I(y)$. For $z \in \partial I(y)$ we let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 < (\ell+1)q_n - \frac{1}{20}q_{n-1}$, we expand $\phi(x_2+1)$ in (9.4) iterating (9.4) with $I = I(x_2+1)$, and if $x_1 - 1 > \ell q_n + \frac{1}{20}q_{n-1}$, we expand $\phi(x_1 - 1)$ in (9.4) iterating (9.4) with $I = I(x_1 - 1)$. We continue to expand each term of the form $\phi(z)$ in the same fashion until we arrive to z such that either $z+1 \geq (\ell+1)q_n - \frac{1}{20}q_{n-1}$, $z-1 \leq \ell q_n + \frac{1}{20}q_{n-1}$, or the number of G_I terms in the product becomes $[\frac{40d}{q_{n-1}}]$, whichever comes first. We then obtain an expression of the form

$$\phi(k) = \sum_{s;z_{i+1}\in\partial I(z'_i)} G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\phi(z'_{s+1}). \tag{9.83}$$

where in each term of the summation we have $\ell q_n + \frac{1}{20}q_{n-1}+1 < z_i < (\ell+1)q_n - \frac{1}{20}q_{n-1}-1$, $i=1,\ldots,s$, and either $z_{s+1} \notin [\ell q_n + \frac{1}{20}q_{n-1}+1, (\ell+1)q_n - \frac{1}{20}q_{n-1}-1]$, $s+1 < [\frac{40d}{q_{n-1}}]$, or $s+1 = [\frac{40d}{q_{n-1}}]$.

By construction, for each z'_i, $i \leq s$, we have that $I(z'_i)$ is well-defined and satisfies (9.81) and (9.82). We now consider the two cases, $z_{s+1} \notin [\ell q_n + \frac{1}{20}q_{n-1}+1, (\ell+1)q_n - \frac{1}{20}q_{n-1}-1]$, $s+1 < [\frac{40d}{q_{n-1}}]$, and $s+1 = [\frac{40d}{q_{n-1}}]$ separately. If $z_{s+1} \notin [\ell q_n + \frac{1}{20}q_{n-1}+1, (\ell+1)q_n - \frac{1}{20}q_{n-1}-1]$, $s+1 < [\frac{40d}{q_{n-1}}]$, we have, by (9.82) and (9.2),

$$\begin{aligned} &|G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\phi(z'_{s+1})| \\ &\leq e^{-(L-\epsilon)(|k-z_1|+\sum_{i=1}^{s}|z'_i-z_{i+1}|)}(1+(\ell+2)q_n) \\ &\leq e^{-(L-\epsilon)(|k-z_{s+1}|-(s+1))}(1+(\ell+2)q_n) \leq e^{-(L-\epsilon)(d-\frac{q_{n-1}}{20}-\frac{40d}{q_{n-1}})}(1+q_{n-1}^C). \end{aligned} \tag{9.84}$$

If $s+1 = [\frac{40d}{q_{n-1}}]$, using again (9.2), (9.82), and also (9.81) we obtain

$$|G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\phi(z'_{s+1})| \leq e^{-(L-\epsilon)\frac{q_{n-1}}{40}\frac{40d}{q_{n-1}}}(1+q_{n-1}^C).$$

In either case,

$$|G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\phi(z'_{s+1})| \leq e^{-(L-2\epsilon)(d-\frac{q_{n-1}}{20})} \tag{9.85}$$

for n sufficiently large. Finally, we observe that the total number of terms in (9.83) is bounded above by $2^{[\frac{40d}{q_{n-1}}]}$. Combining it with (9.83), (9.85) we obtain

$$|\phi(k)| \leq 2^{[\frac{40d}{q_{n-1}}]}e^{-(L-2\epsilon)(d-\frac{q_{n-1}}{20})} < e^{-(L-3\epsilon)(d-\frac{q_{n-1}}{20})}$$

for large n. □

9.4. **Resonant case. Proof of Lemma 9.5.** Notice that, under the condition that k is resonant, we have $k \geq \frac{q_n}{2}$, which implies $q_{n+1}^{8/9} \geq \frac{q_n}{2}$. This is an implicit hypothesis in the next lemma.

Lemma 9.12. *For any $\epsilon > 0$, for sufficiently large n, and any $b \in [-\frac{13}{8}q_n, -\frac{3}{8}q_n] \cap \mathbb{Z}$, we have $\theta + (b+q_n-1)\alpha \in A_{2q_n-1,\frac{23L}{32}+\epsilon}$.*

Proof. Let $b_1 = b-1$, $b_2 = b+2q_n-1$.

Applying Lemma 9.11 we obtain that for $i=1,2$,

$$|\phi_E(b_i)| < \begin{cases} e^{-(L-\epsilon)(b/2+q_n)}, & -\frac{13}{8}q_n \leq b \leq -\frac{3}{2}q_n, \\ e^{-(L-\epsilon)|\frac{b+q_n}{2}|}, & -\frac{3}{2}q_n \leq b \leq -\frac{q_n}{2},\ |b+q_n| > \frac{q_n}{4}, \\ e^{(L-\epsilon)\frac{b}{2}}, & -\frac{q_n}{2} \leq b \leq -\frac{3}{8}q_n. \end{cases} \tag{9.86}$$

Using (9.4) with $I=[b,b+2q_n-2]$ we get

$$\max(|G_I(0,b)|,|G_I(0,b+2q_n-2)|) > \begin{cases} e^{(L-\epsilon)(b/2+q_n)}, & -\frac{13}{8}q_n \le b \le -\frac{3}{2}q_n, \\ e^{(L-\epsilon)|\frac{b+q_n}{2}|}, & -\frac{3}{2}q_n \le b \le -\frac{q_n}{2},\ |b+q_n| > \frac{q_n}{4}, \\ e^{-(L-\epsilon)\frac{b}{2}}, & -\frac{q_n}{2} \le b \le -\frac{3}{8}q_n, \\ e^{-\epsilon q_n}, & |b+q_n| < \frac{q_n}{4}. \end{cases} \tag{9.87}$$

By (9.6),(9.7),

$$\begin{aligned} &|Q_{2q_n-1}(\cos 2\pi(\theta+(b+q_n-1)\alpha))| \\ &\quad = |P_{2q_n-1}(\theta+b\alpha)| < \min\{|G_I(0,b)|^{-1}e^{(L+\epsilon_1)(b+2q_n-2)}, |G_I(0,b+2q_n-2)|^{-1}e^{-(L+\epsilon_1)b}\}. \end{aligned} \tag{9.88}$$

Therefore, using (9.87),(9.88) we obtain that $\theta+(b+q_n-1)\alpha$ belongs to

- $A_{2q_n-1,\frac{23L}{32}+\epsilon}$, if $-\frac{13}{8}q_n \le b \le -\frac{3}{2}q_n$ or $-\frac{q_n}{2} \le b \le -\frac{3}{8}q_n$,
- $A_{2q_n-1,\frac{5L}{8}+\epsilon}$, if $-\frac{3}{2}q_n \le b \le -\frac{q_n}{2}$.

for any $\epsilon>0$ and sufficiently large n. □

Fix $1 \le \ell \le q_{n+1}^{8/9}/q_n$. Set $I_1 = [-[\frac{5}{8}q_n],[\frac{5}{8}q_n]-1]$ and $I_2 = [(\ell-1)q_n+[\frac{5}{8}q_n],(\ell+1)q_n-[\frac{5}{8}q_n]-1]$. Set $\theta_j=\theta+j\alpha, j\in I_1\cup I_2$.

Lemma 9.13. *Assume $L > \frac{16}{9}\beta$. There exists an $\epsilon>0$ such that for sufficiently large n, set $\{\theta_j,\ j\in I_1\cup I_2\}$ is $(\frac{9L}{32}-\epsilon)$-uniform.*

We will now finish the proof of Lemma 9.5 and prove Lemma 9.13 at the end of the subsection.

Let k be resonant. Assume without loss of generality that $k=\ell q_n+r$, $0\le r\le \max\{q_n^{8/9},\frac{q_{n-1}}{20}\}$, $1\le \ell\le q_{n+1}^{8/9}/q_n$.

By Lemmas 9.3,9.12,9.13 there is $j_0\in I_2$ such that $\theta+j_0\alpha\notin A_{2q_n-1,\frac{23L}{32}+\epsilon}$. Set $I=[j_0-q_n+1,j_0+q_n-1]=[x_1,x_2]$. Then

$$|G_I(k,x_i)| < e^{(L+\epsilon_1)(2q_n-2-|k-x_i|)-2q_n(\frac{23L}{32}+\epsilon)} < e^{q_n(\frac{9L}{16}+\epsilon)-(L+\epsilon)|k-x_i|}.$$

Since, by a simple computation, $|k-x_i| > (5/8-\epsilon-\frac{1}{20})q_n$, we obtain that

$$|G_I(k,x_i)| < e^{(-\frac{L}{46}+\epsilon)|k-x_i|} \tag{9.89}$$

which gives the statement of Lemma 9.4. □

Proof of Lemma 9.13. As in the proof of Lemma 9.4 we will first estimate the numerator in (9.8). We have

$$\begin{aligned} &\sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\cos 2\pi a-\cos 2\pi\theta_j| \\ &\qquad = \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin 2\pi\frac{a+\theta_j}{2}| + \sum_{\substack{j\in I_1\cup I_2\\ j\ne i}} \ln|\sin 2\pi\frac{a-\theta_j}{2}| + (2q_n-1)\ln 2 \\ &\qquad = \Sigma_+ + \Sigma_- + (2q_n-1)\ln 2. \end{aligned} \tag{9.90}$$

Both Σ_+ and Σ_- consist of 2 terms of the form of (9.34) with $r = n$, plus two terms of the form $\ln \min_{k=1,\ldots,q_n} |\sin 2\pi(x + \frac{(k+\ell_k q_n)\alpha}{2})|$, where $\ell_k \in \{0, \pm(\ell-1), \pm\ell\}$, $k = 1, \ldots, q_n$, minus $\ln|\sin 2\pi \frac{a \pm \theta_i}{2}|$.

Therefore, by (9.34)

$$\sum_{\substack{j \in I_1 \cup I_2 \\ j \neq i}} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leq (2 - 2q_n)\ln 2 + 4\ln q_n + C\ell\Delta_n q_n \ln q_n. \tag{9.91}$$

To estimate the denominator of (9.8) we write it in the form (9.90) with $a = \theta_i$. Then $\Sigma_- = \sum_{\substack{j \in I_1 \cup I_2 \\ j \neq i}} \ln|\sin \pi(i-j)\alpha|$ can be split into two sums of the form (9.34) plus the minimum term. The corresponding minimum term is achieved at $|i - j_0|$ of the form q_n or ℓq_n. Therefore, for any $\epsilon_1 > 0$ and sufficiently large n

$$\begin{aligned} \Sigma_- &> -2q_n \ln 2 + \ln|\sin \pi q_n \alpha| - C\max(\ln q_n, \ell\Delta_n q_n \ln q_n) \\ &> -2q_n \ln 2 - \ln q_{n+1} - C\max(\ln q_n, \ell\Delta_n q_n \ln q_n). \end{aligned} \tag{9.92}$$

Since

$$\sin 2\pi(\theta + \frac{(k+i+\ell_k q_n)\alpha}{2}) = \sin 2\pi(\theta + \frac{(k+i)\alpha}{2})\cos \pi\ell_k\Delta_n \pm \cos 2\pi(\theta + \frac{(k+i)\alpha}{2})\sin \pi\ell_k\Delta_n$$

(the $\pm$ depending on the sign of $q_n\alpha - p_n$) we have, by (9.1) that if $q_{n+1} \geq q_n^{10}$ then

$$\min_{k,i \in [-q_n, q_n-1],\, \ell_k \in \{0, \pm(\ell-1), \pm\ell\}} |\sin 2\pi(\theta + \frac{(k+i+\ell_k q_n)\alpha}{2})| > \frac{1}{10} q_n^{-2}, \tag{9.93}$$

and if $q_{n+1} < q_n^{10}$ then we have the obvious

$$\min_{k,i \in [-q_n, q_n-1],\, \ell_k \in \{0, \pm(\ell-1), \pm\ell\}} |\sin 2\pi(\theta + \frac{(k+i+\ell_k q_n)\alpha}{2})| > \frac{1}{5} q_{n+1}^{-2} > \frac{1}{5} q_n^{-20}. \tag{9.94}$$

As before, Σ_+ can be split into two sums of the form (9.34) plus two minimum terms minus $\ln|\sin 2\pi(\theta + i\alpha)|$. Therefore,

$$\Sigma_+ > -2q_n \ln 2 - C\max(\ln q_n, \ell\Delta_n q_n \ln q_n) \tag{9.95}$$

Combining (9.90),(9.91),(9.92) and (9.95) we obtain

$$\max_{j \in I_1 \cup I_2} \prod_{\substack{\ell \in I_1 \cup I_2 \\ \ell \neq j}} \frac{|z - \cos 2\pi\theta_\ell)|}{|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)|} < q_n^C e^{(\beta+\epsilon_1)q_n} \tag{9.96}$$

For $\beta < \frac{9}{16}L$ this gives the desired bound. □

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