

COMBINATORIAL RIGIDITY FOR UNICRITICAL POLYNOMIALS

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ABSTRACT. We prove that any unicritical polynomial $f_c : z \mapsto z^d + c$ which is at most finitely renormalizable and has only repelling periodic points is combinatorially rigid. It implies that the connectedness locus (the “Multibrot set”) is locally connected at the corresponding parameter values. It generalizes Yoccoz’s Theorem for quadratics to the higher degree case.

1. INTRODUCTION

Let us consider the one-parameter family of *unicritical* polynomials

$$f_c : z \mapsto z^d + c, \quad c \in \mathbb{C},$$

of degree $d \geq 2$. Let $\mathcal{M} = \mathcal{M}_d = \{c \in \mathbb{C}, \text{ the Julia set of } f_c \text{ is connected}\}$ be the *connectedness locus* of this family. In the case of quadratic polynomials ($d = 2$), it is known as the Mandelbrot set, while in the higher degree case it is sometimes called the *Multibrot set* (see [Sc2]).

Rigidity is one of the most remarkable phenomena observed in holomorphic dynamics. In the unicritical case this phenomenon assumes (conjecturally) a particularly strong form of *combinatorial rigidity*: combinatorially equivalent non-hyperbolic maps are conformally equivalent. This Rigidity Conjecture is equivalent to the local connectivity of the Multibrot sets \mathcal{M}_d . In the quadratic case, we are dealing with the famous MLC conjecture asserting that the Mandelbrot set is locally connected.

About 15 years ago Yoccoz proved that the Mandelbrot set is locally connected at all non-hyperbolic parameter values which are at most finitely renormalizable, see [H]. In fact, this theorem consists of two independent parts dealing respectively with maps that have neutral periodic points or not. In the presence of neutral points, Yoccoz’s method extends readily to the higher degree case. However, the proof in the absence of neutral points was linked to the quadratic case in a very significant way.¹

Rigidity Theorem. *Let $f_c, c \in \mathcal{M}_d$, be an at most finitely renormalizable unicritical polynomial with all periodic points repelling. Then f_c is combinatorially rigid.*

Our work uses “complex bounds” recently proved in [KL2], which in turn are based on new analytic techniques developed in [KL1].

While combinatorial rigidity is a statement about polynomials with exactly the same combinatorics in all scales, our further analysis (geometric and measure-theoretical) of the parameter plane [ALS] (with applications to the real case) will

¹See also [ALM, K, L1, R] for other proofs of this result in the quadratic case.

depend on comparison of polynomials whose combinatorics coincide only up to a certain scale. For such maps one can consider *pseudo-conjugacies*, that is, homeomorphisms which are equivariant up to that scale. In the course of the proof of the Rigidity Theorem, we will show that these pseudo-conjugacies can be selected uniformly quasi-conformally, generalizing part of the analysis of [L1] in the quadratic case.

Let us point out that our argument for existence of pseudo-conjugacies is considerably simpler than the previous arguments, while needing much weaker geometric control of the dynamics. Also, though we restrict ourselves to the unicritical case in order not to overshadow the idea of the method, our argument can be extended to the multicritical case.

In conclusion, let us briefly outline the structure of the paper. In §2 we construct a “favorite nest” of puzzle pieces and transfer *a priori bounds* of [KL2] to this nest. The next section, §3, is central in the paper: here we prove, using the *a priori* bounds, that the respective favorite puzzle pieces of two maps with the same combinatorics stay a bounded Teichmüller distance apart. In the last section, §4, we derive from it, by means of the “pullback argument”, our Rigidity Theorem.

Note finally that for real polynomials of any degree, the real version of the Rigidity Theorem has been recently proved in [KSS].

Basic Notations and terminology.

$\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D} = \mathbb{D}_1$, $\mathbb{T} = \partial\mathbb{D}$;

$\text{Dom } R$ will stand for the domain of a map R ;

Connected components will be referred to as “components”;

Pullbacks of an *open* topological disk V under f are components of $f^{-1}(V)$;

Pullbacks of a *closed* disk V are the closures of the pullbacks of $\text{int } V$.

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2. THE COMPLEX BOUNDS

In this section we fix a map $f = f_c : z \mapsto z^d + c$. The constants below may depend implicitly on its degree d , but not on c .

Let V be an (open or closed) Jordan disk $V \subset \mathbb{C}$. We say that V is *nice* if $f^k(\partial V) \cap \text{int } V = \emptyset$ for all $k \geq 1$.

Let $R_V : \text{Dom } R_V \rightarrow V$ be the *first return map* for f to a nice *open* disk $V \ni 0$. This map has a nice structure: its restriction to each component U of $\text{Dom } R_V$ is a proper map onto V . The degree of this restriction is d or 1 depending on whether $0 \in U$ or otherwise. In the former case, U is called the *central component* of R_V .

If V is a *closed* nice disk with $0 \in \text{int } V$, then we can apply the previous discussion to $\text{int } V$. Somewhat abusing notations, we will denote $\text{Dom } R_V$ the closure of the $\text{Dom } R_{\text{int } V}$ (and we consider R_V only on $\text{Dom } R_V$). Then the *central piece* W of $\text{Dom } R_V$ is defined as the closure of the central component U of $\text{Dom } R_{\text{int } V}$. Notice that W is not necessarily a component of $\text{Dom } R_V$.

The *first landing map* L_V to a nice domain $V \ni 0$ has even nicer properties: it univalently maps each component of $\text{Dom } L_V$ onto V (one of these components is V itself, and $L_V = \text{id}$ on it). In case when V is closed, we will apply to the domain of the landing maps the same conventions as for the return map.

Let us consider two nice disks, V and V' , containing 0 in their interior. We say that V' is a *child* of V if there exists $t \geq 1$ such that $f^t : V' \rightarrow V$ is a branched

covering of degree d . (Note that $V' \subset V$.) We can alternatively say that V is the *parent* of V' (notice that any child has a single parent but not the other way around).

The children of V are naturally ordered by inclusion. Notice that the *first child* of V coincides with the central piece of R_V (whenever it exists). We say that V' is a *good child* if $f^t(0) \in \text{int } U$ where U is the first child of V . In this case, the first child U' of V' is contained in $(f^t|_{V'})^{-1}(U)$. In particular

$$\text{mod}(V' \setminus U') \geq \frac{1}{d} \text{mod}(V \setminus U).$$

A *puzzle* is a graded (by the *depth* $k \geq 0$) collection of nice closed Jordan disks called *puzzle pieces*, such that for each $k \geq 0$ the puzzle pieces of depth k have disjoint interiors, and the puzzle pieces of depth $k + 1$ are the pullbacks of the puzzle pieces of depth k under f .

It may happen that the first child U of V is good: then U is called *spoiled*. In this case $R_V(0) \in U$ and the first return to V is called *central*.

We say that a sequence of nested puzzle pieces $W^m \subset V^m \subset \dots \subset W^0 \subset V^0$ is a *modified principal nest* if

- (1) W^i is the first child of V^i ,
- (2) V^{i+1} is the oldest unspoiled child of W^i . (In other words, V^{i+1} is the first child of W^i if the return to W^i is non-central, and is the second child otherwise.)

See §2.2 of [KL2] for a detailed discussion of the combinatorics of this nest.

The following is the main technical result of [KL2]:

Theorem 2.1. *There exists $\delta > 0$ such that for every $\epsilon > 0$ there exists $n_0 > 0$ with the following property. Let $W^m \subset V^m \subset \dots \subset W^0 \subset V^0$ be a modified principal nest for f . If $\text{mod}(V^0 \setminus W^0) > \epsilon$ and $n_0 \leq n \leq m$ then $\text{mod}(V^n \setminus W^n) > \delta$.*

We will need a slight variation of this result.

Given a nice domain Q , let $\text{m}(Q) = \inf \text{mod}(Q \setminus D)$ where the infimum is taken over all components D of $\text{Dom } R_Q$.

Lemma 2.2. *Let U be the first child of V , and let V' be any child of V . Then*

$$\text{m}(V') \geq \frac{1}{d} \text{mod}(V \setminus U).$$

Proof. Let $k > 0$ be such that $f^k(V') = V$. Given a component D of $\text{Dom } R_{V'}$, let $\Omega = f^k(D)$. Notice that $f^j(V') \cap V' = \emptyset$ for $1 \leq j < k$, so that $\Omega \subset \text{Dom } L_{V'}$. Let us show that

$$(2.1) \quad \text{mod}(V \setminus \Omega) \geq \text{mod}(V \setminus U).$$

It is obvious if $\Omega \subset U$. Otherwise let us consider the following inclusions:

$$\Omega \subset V \cap \text{Dom } L_{V'} \subset V \cap \text{Dom } L_U \subset \text{Dom } R_V \subset V.$$

Let Ω' be the component of $\text{Dom } L_U$ containing Ω , and let Ω'' be the component of $\text{Dom } R_V$ containing Ω' . Then

$$\text{mod}(V \setminus \Omega) \geq \text{mod}(\Omega'' \setminus \Omega') \geq \text{mod}(V \setminus U),$$

and (2.1) follows.

Consequently,

$$\text{mod}(V' \setminus D) \geq \frac{1}{d} \text{mod}(V \setminus \Omega) \geq \frac{1}{d} \text{mod}(V \setminus U).$$

□

The *favorite child* Q' of Q is the oldest good unspoiled child of Q . It is constructed as follows. Let P be the first child of Q . Let $k > 0$ be the first moment when $R_Q^k(0) \in Q \setminus P$, and let $l > 0$ be the first moment when $R_Q^{k+l}(0) \in P$ (so, $k+l$ is the moment of the first return back to P after first escape from P under iterates of R_Q). Then Q' is the pullback of Q under R_Q^{k+l} that contains 0. (Compare with the construction of the domain \tilde{A} in Lemma 2.5 of [KL2].) Note that the first child is never favorite.

Lemma 2.3. *Let us consider a nest of four puzzle pieces, $P' \subset Q' \subset P \subset Q$, such that P is the first child of Q , P' is the first child of Q' , and Q' is the favorite child of Q . If V is a puzzle piece containing Q and whose first child U is contained in Q then*

$$\text{mod}(Q' \setminus P') \geq \frac{1}{d^2} m(V).$$

Proof. Let the moments k and l have the same meaning as in the above construction of the favorite child. Then $R_Q^{k+l}|_{P'}$ is a d -to-1 branched covering onto some domain $D \subset P \subset U$ which is a component of $\text{Dom } L_{Q'}$. Hence

$$\text{mod}(Q' \setminus P') = \frac{1}{d} \text{mod}(Q \setminus D).$$

Assume $Q' \neq D$. Then D is contained in a component of the domain of the first return map to U . Hence by Lemma 2.2,

$$\text{mod}(Q \setminus D) \geq \text{mod}(U \setminus D) \geq m(U) \geq \frac{1}{d} m(V),$$

and the conclusion follows.

Assume now that $Q' = D$, and let $\Omega = R_V(Q')$. Then $\Omega \neq Q$ since Q' is not the first child of Q . Hence Ω returns to $Q \subset V$ sometime, so that it is contained in a component of $\text{Dom } R_V$. Thus

$$\text{mod}(Q \setminus Q') \geq \text{mod}(U \setminus Q') = \frac{1}{d} \text{mod}(V \setminus \Omega) \geq \frac{1}{d} m(V),$$

and the result follows. □

Proposition 2.4. *There exists $\delta > 0$ such that for every $\epsilon > 0$ there exists $n_0 > 0$ with the following property. Let $P^m \subset Q^m \subset \dots \subset P^0 \subset Q^0$ be a nest of puzzle pieces such that P^i is the first child of Q^i and Q^{i+1} is the favorite child of Q^i . If $\text{mod}(Q^0 \setminus P^0) > \epsilon$ and $n_0 \leq n \leq m$ then $\text{mod}(Q^n \setminus P^n) > \delta$.*

Proof. Let us consider the modified principal nest $W^k \subset V^k \subset \dots \subset W^0 \subset V^0$ which begins with $V^0 = Q^0$ and ends at the maximal level k such that $V^k \supset Q^m$. For any $n = 0, 1, \dots, m$, let us define $k(n) \in [0, k]$ as the maximal level such that $V^{k(n)} \supset Q^n$. (In particular, $k = k(m)$ by definition.)

Let us show that if $n \leq m - 2$ then $k(n+2) > k(n)$. Indeed, since Q^{n+1} is a child of $Q^n \subset V^{k(n)}$, it is contained in the first child $W^{k(n)}$ of $V^{k(n)}$. Since Q^{n+2} is not younger than the second child of $Q^{n+1} \subset W^{k(n)}$, it is contained in the second child of $W^{k(n)}$, and the latter is contained in $V^{k(n)+1}$. Hence $k(n+2) \geq k(n) + 1$.

By Theorem 2.1 and Lemma 2.2, it is enough to show that for every natural $n \in [2, m-2]$ we have:

$$(*) \text{ Either } \text{mod}(Q^n \setminus P^n) \geq C^{-1}m(V^{k(n)}) \text{ or } \text{mod}(Q^{n+1} \setminus P^{n+1}) \geq C^{-1}m(V^{k(n)})$$

for some constant $C > 0$ which depends only on d .

If $W^{k(n)} \subset Q^n$, Lemma 2.3 yields the latter estimate with $C = d^2$. So, assume $W^{k(n)} \supset Q^n$.

Let $Z^0 = W^{k(n)}$, and let Z^{i+1} be the first child of Z^i .² If $Z^1 \subset Q^n \subset Z^0$, Lemmas 2.3 and 2.2 imply that

$$\text{mod}(Q^{n+1} \setminus P^{n+1}) \geq \frac{1}{d^2}m(Z_0) \geq \frac{1}{d^3}m(V^{k(n)}),$$

and we are done.

So, assume $Q^n \Subset Z^1$. Let us consider the first return map $R = R_{Z^0}$, and let us find the level $j > 0$ such that³ $R(0) \in Z^{j-1} \setminus Z^j$. If $j = 1$ then $Z^1 = V^{k(n)+1} \subset Q^n$, contradicting the assumption. So $j > 1$.

Let $D \subset Z^0 \setminus Z^1$ be the component of $\text{Dom } R$ containing $R^j(0)$. Then $V^{k(n)+1}$ is the component of $(R^j)^{-1}(D)$ containing 0. Let $D' \subset D$ be the component of $\text{Dom } L_{Z^j}$ containing $R^j(0)$. Then Z^{j+1} is the component of $(R^j)^{-1}(D')$ containing 0. So we have $Z^j \supset V^{k(n)+1} \supset Z^{j+1}$ and

$$\text{mod}(V^{k(n)+1} \setminus Z^{j+1}) = \frac{1}{d} \text{mod}(D \setminus D') \geq \frac{1}{d} \text{mod}(Z^0 \setminus Z^1).$$

If $Q^n \subset Z^j$ then $P^n \subset Z^{j+1}$, and we obtain the following nest:

$$Z^j \supset Q^n \supset V^{k(n)+1} \supset Z^{j+1} \supset P^n.$$

It follows that

$$\text{mod}(Q^n \setminus P^n) \geq \text{mod}(V^{k(n)+1} \setminus Z^{j+1}) \geq \frac{1}{d} \text{mod}(Z^0 \setminus Z^1) \geq \frac{1}{d^2}m(V^{k(n)})$$

and we are done.

If $Z^j \subset Q^n \Subset Z^1$ then Q^n is the first child of $R(Q^n)$. Since every child has a single parent, $R(Q^n) = Q^{n-1}$. This is a contradiction since by definition, Q^n is the favorite (and hence not the first) child of Q^{n-1} . \square

3. TEICHMÜLLER DISTANCE BETWEEN PUZZLE PIECES

A *good nest* is a sequence $Q_m \subset \dots \subset Q_0$ such that Q_i is a good child of Q_{i-1} , $1 \leq i \leq m$.

Theorem 3.1. *Let $c, \tilde{c} \in \mathbb{C}$, and let $f = f_c$, $\tilde{f} = f_{\tilde{c}}$. Let $Q^m \subset \dots \subset Q^0$, $\tilde{Q}^m \subset \dots \subset \tilde{Q}^0$, be good nests for f , \tilde{f} , such that there exists a homeomorphism $h : \mathbb{C} \rightarrow \mathbb{C}$ with $h(Q^i) = \tilde{Q}^i$, $0 \leq i \leq m$, and $h \circ f(x) = \tilde{f} \circ h(x)$, $x \notin Q^m$.⁴ Let P^i and \tilde{P}^i , $0 \leq i \leq m-1$, be the first kids of Q^i and \tilde{Q}^i respectively. Assume that*

- (1) $\text{mod}(Q^i \setminus P^i) > \delta$ and $\text{mod}(\tilde{Q}^i \setminus \tilde{P}^i) > \delta$, $0 \leq i \leq m-1$;
- (2) $h|_{\partial Q^0}$ extends to a K -qc map $(Q^0, 0) \rightarrow (\tilde{Q}^0, 0)$.

Then $h|_{\partial Q^m}$ extends to a K' -qc map $(Q^m, 0) \rightarrow (\tilde{Q}^m, 0)$ where $K' = K'(\delta, K)$.

The basic step of the proof of Theorem 3.1 is the following lemma on covering maps of the disk.

²So $Z^0 \supset Z^1 \supset \dots$ is the *principal nest* that begins with Z^0 , see [L1].

³Thus, $Z^0 \supset Z^1 \supset \dots \supset Z^j$ is a central cascade of puzzle pieces, compare §2.2 of [KL2].

⁴We refer to this property as *combinatorial equivalence* of the nests.

Lemma 3.2. *For every $0 < \rho < r < 1$ there exists $K_0 = K_0(\rho, r)$ with the following property. Let $g, \tilde{g} : (\mathbb{D}, 0) \rightarrow (\mathbb{D}, 0)$ be holomorphic proper maps of degree d . Let $h, h' : \mathbb{T} \rightarrow \mathbb{T}$, be such that $\tilde{g} \circ h' = h \circ g$. Assume that*

- (1) *The critical values of g, \tilde{g} are contained in $\overline{\mathbb{D}}_\rho$;*
- (2) *h admits a K -qc extension $H : \mathbb{D} \rightarrow \mathbb{D}$ which is the identity on \mathbb{D}_r .*

Then h' admits a K' -qc extension $H' : \mathbb{D} \rightarrow \mathbb{D}$ which is the identity on \mathbb{D}_r , where $K' = \max\{K, K_0\}$.

Proof. Let \mathcal{G}_ρ be the family of proper holomorphic maps $G : (\mathbb{D}, 0) \rightarrow (\mathbb{D}, 0)$ of degree d whose critical values are contained in $\overline{\mathbb{D}}_\rho$, endowed with the strong topology of their extensions to rational maps of degree d . This family is compact. One can see it, e.g., by checking normality of this family on the whole Riemann sphere. Normality is obvious on \mathbb{D} and $\mathbb{C} \setminus \overline{\mathbb{D}}$. To see normality near the unit circle \mathbb{T} , notice that the full preimages $G^{-1}(\mathbb{D}_r \cup (\overline{\mathbb{C}} \setminus \mathbb{D}_{1/r}))$, $G \in \mathcal{G}_\rho$, contain 0 and ∞ , and omit a definite symmetric annulus around \mathbb{T} (of modulus $d^{-1}2 \log r$).

For any $G \in \mathcal{G}_\rho$, the domain $U_G = G^{-1}(\mathbb{D}_r)$ is a Jordan disk with analytic anti-clockwise oriented boundary. By the Schwarz Lemma, $\mathbb{D}_r \Subset U_G$.

Let $G \in \mathcal{G}_\rho$, and let $\phi : \partial\mathbb{D}_{r^{1/d}} \rightarrow \partial U_G$ be an orientation preserving homeomorphism such that $G \circ \phi(z) = z^d$. Then ϕ is an analytic diffeomorphism, and there exists a L -qc map $H_\phi : \mathbb{D} \rightarrow \mathbb{D}$ such that $H_\phi|_{\mathbb{D}_r} = \text{id}$ and $H_\phi|_{\partial\mathbb{D}_{r^{1/d}}} = \phi$. Moreover $L = L(r, \rho)$ by compactness of \mathcal{G}_ρ .

Furthermore, the given map $H : \mathbb{D} \setminus \overline{\mathbb{D}}_r \rightarrow \mathbb{D} \setminus \overline{\mathbb{D}}_r$ lifts to a K -qc map

$$\hat{H} : \mathbb{D} \setminus \overline{U}_g \rightarrow \mathbb{D} \setminus \overline{U}_{\tilde{g}}$$

such that $\hat{H}|_{\partial\mathbb{D}} = h'$ and $\tilde{g} \circ \hat{H} = H \circ g$ on $\mathbb{D} \setminus \overline{U}_g$. By the previous discussion, \hat{H} extends to a qc map $H' : \mathbb{D} \rightarrow \mathbb{D}$ such that $H'|_{U_g}$ is the composition of two L -qc maps which are the identity on \mathbb{D}_r . The result follows with $K_0 = L^2$. \square

Proof of Theorem 3.1. Let us consider moments t_i , $1 \leq i \leq m$, such that $f^{t_i}(Q^i) = Q^{i-1}$. By combinatorial equivalence of our nests, $f^{t_i}(\tilde{Q}^i) = \tilde{Q}^{i-1}$. Then $f^{t_i} : Q^i \rightarrow Q^{i-1}$ are proper holomorphic maps of degree d , and similarly for the second nest.

Let $v_i = f^{t_{i+1} + \dots + t_m}(0)$, $0 \leq i \leq m$. Since $f^{t_m}(0) \in P^{m-1}$ and $f^{t_i}(P^i) \subset P^{i-1}$, we have: $v_i \in P^i$, $0 \leq i \leq m-1$.

Let $\psi_i : (Q^i, v_i) \rightarrow (\mathbb{D}, 0)$ be the uniformizations of the domains under consideration by the unit disk, and let $g_i = \psi_{i-1} \circ f^{t_i} \circ \psi_i^{-1}$. The maps $g_i : (\mathbb{D}, 0) \rightarrow (\mathbb{D}, 0)$ are unicritical proper holomorphic maps of degree d . Let $u_i = \psi_i(0)$ stand for the critical points of these maps.

The corresponding objects for the second nest will be marked with tilde.

Let us also consider homeomorphisms $h_i : \mathbb{T} \rightarrow \mathbb{T}$ given by $h_i = \tilde{\psi}_i \circ h \circ \psi_i^{-1}$. They are equivariant with respect to the g -actions, i.e., $h_{i-1} \circ g_i = \tilde{g}_i \circ h_i$.

Let $\psi_i(P^i) = \Omega^i$. Since $\text{mod}(\mathbb{D} \setminus \Omega^i) \geq \delta$ and $\Omega^i \ni \psi_i(v_i) = 0$, $0 \leq i \leq m$, these domains are contained in some disk \mathbb{D}_ρ with $\rho = \rho(\delta) < 1$. Since $f^{t_i}(0) \in P^{i-1}$, we conclude that $g_i(u_i) \in \Omega^{i-1} \subset \mathbb{D}_\rho$, $0 \leq i \leq m$. The same assertions hold for the second nest. So, all the maps g_i and \tilde{g}_i satisfy the assumptions of Lemma 3.2.

By Assumption (2) of the theorem we are proving, h_0 extends to a K -qc map $(\mathbb{D}, u_0) \rightarrow (\mathbb{D}, \tilde{u}_0)$. Fix some $r \in (\rho, 1)$. Since $u_0, \tilde{u}_0 \in \Omega^0 \subset \mathbb{D}_\rho$, we conclude that h_0 extends to an L -qc map $\mathbb{D} \rightarrow \mathbb{D}$ which is the identity on \mathbb{D}_r , where $L = L(K, \delta)$.

Let $K_0 = K_0(\rho, r)$ be as in Lemma 3.2, and let $K' = \max\{L, K_0\}$. Consecutive applications of Lemma 3.2 show that for $i = 1, \dots, m$, the maps h_i admit a K' -qc

extension $H_i : \mathbb{D} \rightarrow \mathbb{D}$ which are the identity on \mathbb{D}_r . The desired extension of $h|_{\partial Q^m}$ is now obtained by taking $\tilde{\psi}_m^{-1} \circ H_m \circ \psi_m$. \square

4. PULLBACK ARGUMENT

In this section we will derive the Rigidity Theorem from the bound on the Teichmüller distance between the central puzzle pieces by means of the “Pullback Argument” in the Yoccoz puzzle framework. This method is standard in holomorphic dynamics.

4.1. Combinatorics of a map. Let \mathcal{A} stand for the set of parameters c for which the map $f_c : z \mapsto z^d + c$ has an attracting fixed point. In the quadratic case, it is a domain bounded by the main cardioid of the Mandelbrot set. In the higher degree case, \mathcal{A} is a domain bounded by a simple closed curve with $d - 1$ cusps.

For the construction of the Yoccoz puzzle for a map f_c with $c \in \mathcal{M} \setminus \overline{\mathcal{A}}$, the reader can consult [KL2], §2.3. Keeping in mind future applications, here we will extend the construction (up to a certain depth) to some parameters outside \mathcal{M} .

The set $\mathcal{M} \setminus \overline{\mathcal{A}}$ is disconnected. Each connected component of $\mathcal{M} \setminus \overline{\mathcal{A}}$ is called a *limb*. The closure of a limb intersects $\overline{\mathcal{A}}$ at a single point called the *root* of the limb. There are two external rays landing at the root. Their union divides \mathbb{C} into two (open) connected components: the one containing the limb is called a *parabolic wake* (see [DH, M2, Sc1]).

For c inside a limb, the map f_c has a unique *dividing* repelling fixed point α : the rays landing at it, together with α itself, disconnect the plane into $q \geq 2$ domains. This repelling fixed point, and the q external rays landing at it, have an analytic continuation through the whole parabolic wake.

Let us truncate the parabolic wake by an equipotential of height h . For c in the truncated parabolic wake, the Yoccoz puzzle pieces of depth 0 are obtained by taking the closure of the connected components of $\mathbb{C} \setminus \{\text{external rays landing at } \alpha\}$ truncated by the equipotential of height h .

We say that f has well defined combinatorics up to depth n if

$$f^k(0) \in \bigcup_j \text{int } Y_j^0, \quad 0 \leq k \leq n.$$

In this case we define Yoccoz puzzle pieces of depth n as the pullbacks of the Yoccoz puzzle pieces of depth 0 under f^n . The puzzle pieces of depth n will be denoted by Y_j^n , where the label j stands for the angles of the external rays that bound Y_j^n . The puzzle piece of depth n whose interior contains 0 is called the *critical puzzle piece* of depth n and it is also denoted Y^n . The *combinatorics of f up to depth n* (provided it is well defined) is the set of labels of puzzle pieces of depth n . Note that the combinatorics up to depth $n + t$ determines the puzzle piece Y_j^n containing the critical value $f^t(0)$.

If f does not have well defined combinatorics of all depths, then either the Julia set of f is disconnected or the critical point is eventually mapped to the repelling fixed point α . Otherwise there are critical puzzle pieces of all depth. In this case, we say that f is *combinatorially recurrent* if the critical point returns to all critical puzzle pieces. Combinatorially recurrent maps can be either renormalizable or non-renormalizable, see [KL2], §2.3.

Two non-renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics up to an arbitrary depth. (See §4.3 for a definition of combinatorial equivalence in the renormalizable case.)

The following result treats the main special case of the Rigidity Theorem.

Theorem 4.1. *Let $f : z \mapsto z^d + c$ be a non-renormalizable combinatorially recurrent map. If $\tilde{f} : z \mapsto z^d + \tilde{c}$ is combinatorially equivalent to f , then f and \tilde{f} are quasiconformally conjugate.*

In the next section we will deduce Theorem 4.1 from a more general statement regarding pseudo-conjugacies.

4.2. Pseudo-conjugacies and rigidity. In this section f will stand for a map satisfying assumptions of Theorem 4.1. For such a map, the construction of the favorite child preceding Lemma 2.3 and the discussion of the modified principal nest (see [KL2], §2.2-2.3) yield:

- (1) Every critical puzzle piece Y^s has a favorite child.
- (2) Let $l > 0$ be the minimal moment for which $f^{lq}(0) \notin Y^1$. Then the first child of Y^{lq} is contained in $\text{int } Y^{lq}$.

This allows us to construct an infinite nest $Q^0 \supset P^0 \supset Q^1 \supset P^1 \supset \dots$ as follows. Take $Q^0 = Y^{lq}$, let Q^{i+1} be the favorite child of Q^i , and let P^i be the first child of Q^i .

If f and \tilde{f} have the same combinatorics up to depth n , a *weak pseudo-conjugacy* (up to depth n) between f and \tilde{f} is an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that $H(Y_j^0) = \tilde{Y}_j^0$ and $H \circ f = \tilde{f} \circ H$ outside the interior of the puzzle pieces of depth n . If the last equation is satisfied everywhere outside the central puzzle piece Y^n , then H is called a *pseudo-conjugacy* (up to depth n).

A (weak) pseudo-conjugacy is said to *match the Böttcher marking* if near ∞ it becomes the identity in the Böttcher coordinates for f and \tilde{f} . (Then by equivariance it is the identity in the Böttcher coordinates outside $\cup_j Y_j^n$ and $\cup_j \tilde{Y}_j^n$.) *In what follows all (weak) pseudo-conjugacies are assumed to match the Böttcher marking.*

The following lemma provides us with a weak pseudo-conjugacy (between f and \tilde{f}) with a weak dilatation control.

Lemma 4.2. *If f and \tilde{f} have the same combinatorics up to depth n then there exists a K_n -qc weak pseudo-conjugacy between f and \tilde{f} . (Here K_n depends on the maps f and \tilde{f} .)*

Proof. The case $n = 0$ can be dealt with by means of holomorphic motions. We will only sketch the construction; details can be found in [R] (in the case $d = 2$ which at this point does not differ from the higher degree case).

The property that f and \tilde{f} have the same combinatorics up to depth 0 just means that c and \tilde{c} belong to the same truncated parabolic wake. Inside the truncated parabolic wake, the q external rays landing at the α fixed point, and the equipotential of height h , move holomorphically in $\mathbb{C} \setminus \{0\}$. Namely, there exists a family of injective maps ϕ_b , parametrized by a parameter b in the truncated parabolic wake, which map the rays and equipotential in question for c to the corresponding curves for b (matching the Böttcher marking), and such that $b \mapsto \phi_b(z)$ is holomorphic, $\phi_c = \text{id}$.

Outside the equipotential of height h , this holomorphic motion extends to a motion holomorphic in both variables (b, z) and tangent to the identity at ∞ (it

comes from the Böttcher coordinate near ∞). By [BR], the map ϕ_b extends to a K_0 -qc map $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, where K_0 depends only on the hyperbolic distance between c and b inside the truncated parabolic wake. This is the desired qc weak pseudo-conjugacy H_0 for $n = 0$.

We will now treat the general case by induction. Assuming that it holds for some $n - 1 \geq 0$, let us modify the qc weak pseudo-conjugacy H_{n-1} up to depth $n - 1$ inside the puzzle piece of depth $n - 1$ containing the critical value c , so that it takes c to \tilde{c} . The resulting map H'_{n-1} is still a weak pseudo-conjugacy up to depth $n - 1$, and can be taken quasiconformal. We now define the desired qc weak pseudo-conjugacy H_n up to depth n as the lift of H'_{n-1} (i.e., $\tilde{f} \circ H_n = H'_{n-1} \circ f$) normalized so that $H_n = H_{n-1}$ near infinity. \square

The following lemma gives a two-fold refinement of the previous one: first, it improves equivariance properties of a weak pseudo-conjugacy H turning it into a pseudo-conjugacy H' ; more importantly, it provides us with a dilatation control of H' in terms of the Teichmüller distance between the deepest puzzle pieces.

Lemma 4.3. *Let H be a qc weak pseudo-conjugacy up to depth n between f and \tilde{f} . Assume that $H|_{\partial Y^n}$ admits a K -qc extension $(\text{int } Y^n, 0) \rightarrow (\text{int } \tilde{Y}^n, 0)$. Then there exists a K -qc pseudo-conjugacy (up to depth n) H' between f and \tilde{f} .*

Proof. We may assume that $H|_{\text{int } Y^n}$ is K -qc. Let $H^{(0)} = H$. Let us construct by induction a sequence of weak pseudo-conjugacies (up to depth n) $H^{(j)}$ as follows. Assume $H^{(j-1)}$ has been already constructed. Since the maps

$$f : \overline{\mathbb{C}} \setminus Y^n \rightarrow \overline{\mathbb{C}} \setminus f(Y^n) \quad \text{and} \quad \tilde{f} : \overline{\mathbb{C}} \setminus \tilde{Y}^n \rightarrow \overline{\mathbb{C}} \setminus \tilde{f}(\tilde{Y}^n)$$

are unbranched coverings of the same degree, the homeomorphism

$$H^{(j-1)} : \overline{\mathbb{C}} \setminus f(Y^n) \rightarrow \overline{\mathbb{C}} \setminus \tilde{f}(\tilde{Y}^n)$$

lifts to a homeomorphism $H^{(j)} : \overline{\mathbb{C}} \setminus Y^n \rightarrow \overline{\mathbb{C}} \setminus \tilde{Y}^n$ satisfying the equation $H^{(j-1)} \circ f = \tilde{f} \circ H^{(j)}$ and matching the Böttcher coordinate outside the union of puzzle pieces of depth n . In particular, it matches the Böttcher coordinate on ∂Y^n , so it can be extended to Y^n as H .

We obtain a sequence $\{H^{(j)}\}_{j \geq 0}$ of qc weak pseudo-conjugacies with non-increasing dilatation. Hence it is precompact in the uniform topology. Moreover, $H^{(j)} = H^{(j-1)}$ outside the union of puzzle pieces of depth $n + j - 1$. Thus, the sequence $\{H^{(j)}\}$ converges pointwise outside the filled Julia set $K(f)$. Since $K(f)$ has empty interior, we conclude that $H^{(j)}$ converges uniformly on the whole plane to some qc weak pseudo-conjugacy H' up to depth n .

By construction, H' coincides with H on $\text{int } Y^n$ and also outside the union of puzzle pieces of depth n (in particular it matches the Böttcher marking near ∞). Moreover, $H' \circ f = \tilde{f} \circ H'$ outside Y^n , so that H' is a qc pseudo-conjugacy. It follows that the dilatation of H' is bounded by K except possibly on the set $X = \{x \in J(f) : f^k(x) \notin \text{int } Y^n, \quad k \geq 0\}$ (here $J(f)$ stands for the Julia set of f). This set is uniformly expanding, and hence has zero Lebesgue measure. The result follows. \square

Remark. One can construct the above map H' more directly as follows. First define H' on the pieces of $\text{Dom } L_{Y^n}$ as the univalent pullbacks of H (this map is K -qc). Then define H' on $F(f) \setminus \text{Dom } L_{Y^n}$ (where $F(f) = \overline{\mathbb{C}} \setminus J(f)$ is the Fatou

set of f) to be the identity in the Böttcher coordinates (this map is conformal). These two maps match on the common boundary of the pieces since H respects the Böttcher marking on ∂Y^n . Since the residual set X is hyperbolic, one can show that this map admits a K -qc extension to the whole plane.

Let q_m (respectively, p_m) be the depth of the puzzle piece Q^m (respectively, P^m), i.e., $Q^m = Y^{q_m}$ (respectively, $P^m = Y^{p_m}$).

Theorem 4.4. *Assume that f is combinatorially recurrent and non-renormalizable. If \tilde{f} has the same combinatorics as f up to depth $q_m + p_{m-1} - q_{m-1}$, then there exists a K -qc pseudo-conjugacy between f and \tilde{f} up to depth q_m , where $K = K(f, \tilde{f})$.*

Proof. For $k = 0, \dots, m$, let h_k be the weak pseudo-conjugacies, up to depth q_k , constructed in Lemmas 4.2 and 4.3 (with the weak dilatation control at this moment).

Consider the sequence of puzzle pieces $\tilde{Q}^k = h_m(Q^k) = \tilde{Y}^{q_k}$ for \tilde{f} . Let us show that \tilde{Q}^k is the favorite child of \tilde{Q}^{k-1} for $1 \leq k \leq m$. Indeed, it is clear that \tilde{Q}^k is a child of \tilde{Q}^{k-1} , and that this child is not the first. Moreover, the combinatorics up to level $p_{k-1} + q_k - q_{k-1}$ determines the puzzle piece of depth p_{k-1} containing the critical value of the map $\tilde{f}^{q_k - q_{k-1}} : \tilde{Q}^k \rightarrow \tilde{Q}^{k-1}$. Hence $\tilde{f}^{q_k - q_{k-1}}(0) \in \text{int } \tilde{P}^{k-1}$, so \tilde{Q}^k is a good child of \tilde{Q}^{k-1} . To see that \tilde{Q}^k is a favorite child of \tilde{Q}^{k-1} , we reverse this reasoning to conclude that for $l \in (q_{k-1}, q_k)$, the piece \tilde{Y}^l cannot be a good non-spoiled child of \tilde{Q}^{k-1} , for otherwise Y^l would be a good non-spoiled child of Q^{k-1} .

Since $h_m|\partial Q^0 = h_0|\partial Q^0$, $h_m|\partial Q^0$ extends to a K_{q_0} -qc map $Q^0 \rightarrow Q^0$ with $K_{q_0} = K_{q_0}(f, \tilde{f})$. Moreover, by Proposition 2.4, *a priori bounds* (1) of Theorem 3.1 hold for the respective nests of f and \tilde{f} . Applying this theorem, we conclude that $h_m|\partial Q^m$ extends to a K' -qc map $Q^m \rightarrow \tilde{Q}^m$, where $K' = K'(f, \tilde{f})$. The result now follows from Lemma 4.3. \square

Remark 4.1. The proof shows that $K(f, \tilde{f})$ only depends on $K_{q_0}(f, \tilde{f})$, and on $\text{mod}(Q^0 \setminus P^0), \text{mod}(\tilde{Q}^0 \setminus \tilde{P}^0)$.

Proof of Theorem 4.1. Let h_n be the pseudo-conjugacy up to depth q_n between f and \tilde{f} given by Theorem 4.4. Since the h_n have uniformly bounded dilatations, we can take a limit map h . Then h is a qc map satisfying $h \circ f = \tilde{f} \circ h$ outside the filled Julia set $K(f)$. Since $K(f)$ has empty interior, $h \circ f = \tilde{f} \circ h$ holds everywhere by continuity. The result follows. \square

4.3. Final Remarks. The Rigidity Theorem stated in the Introduction is reduced to Theorem 4.1 by standard means:

- The non-combinatorially recurrent case is simple, and is treated in the same way as in the quadratic case (see [M1]).
- Rigidity follows from the qc equivalence of combinatorially equivalent maps by an open-closed argument. This argument can be summarized as follows (see e.g., §5 of [L2]). Combinatorial classes of maps with only repelling periodic orbits are closed subsets of the parameter plane, while qc classes are either singletons or open (in one-parameter families) by the Ahlfors-Bers Theorem. Thus, if some combinatorial class coincides with a qc class, it must be a singleton.
- The case of at most finitely renormalizable maps is reduced to the case of non-renormalizable maps by means of straightening. Namely, let us consider two

maps $f : z \mapsto z^d + c$ and $\tilde{f} : z \mapsto z^d + \tilde{c}$, which are exactly n times renormalizable. Then there is a nest of little Multibrot copies,

$$\mathcal{M} \supset \mathcal{M}^1 \supset \cdots \supset \mathcal{M}^n \ni \{c, \tilde{c}\},$$

such that under the straightening $\sigma : \mathcal{M}^n \rightarrow \mathcal{M}$ the parameters c and \tilde{c} become non-renormalizable. We say that f and \tilde{f} are combinatorially equivalent if the corresponding non-renormalizable maps $z \mapsto z^d + \sigma(c)$ and $z \mapsto z^d + \sigma(\tilde{c})$ are⁵ (see discussion in [Sc2]). If so then by the non-renormalizable case of the Rigidity Theorem, $\sigma(c) = \sigma(\tilde{c})$, and we are done.

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⁵Two infinitely renormalizable maps are called *combinatorially equivalent* if they belong to the same infinite nest of little Multibrot copies.

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