DENSITY OF POSITIVE LYAPUNOV EXPONENTS FOR QUASIPERIODIC $\mathrm{SL}(2,\mathbb{R})$ COCYCLES IN ARBITRARY DIMENSION

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Abstract. We show that given a fixed irrational rotation of the d-dimensional torus, any analytic $\mathrm{SL}(2,\mathbb{R})$ cocycle can be perturbed so that the Lyapunov exponent becomes positive. This result strengthens and generalizes previous results of Krikorian [K] and Fayad-Krikorian [FK]. The key technique is the analiticity of m-functions (under the hypothesis of stability of zero Lyapunov exponents), first observed and used in the solution of the Ten Martini Problem [AJ]. In the appendix, we discuss the smoothness of m-functions for a larger class of systems including the skew-shift.

1. Introduction

Let $d \geq 1$ be an integer. A (d-dimensional) *quasiperiodic* $\mathrm{SL}(2,\mathbb{R})$ *cocycle* is a pair $(\alpha, A) \in \mathbb{R}^d \times C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ understood as a linear skew-product:

$$(x, w) \mapsto (x + \alpha, A(x) \cdot w). \tag{1.1}$$

The Lyapunov exponent is defined by

$$L(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n(x)\| dx, \tag{1.2}$$

where $A_n \in C^0(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ is given by

$$A_n(x) = A(x + (n-1)\alpha) \cdots A(x), \tag{1.3}$$

so that

$$(\alpha, A)^n = (n\alpha, A_n). \tag{1.4}$$

Our main result is the following. Let us say that $\alpha \in \mathbb{R}^d$ is irrational if $x \mapsto x + \alpha$ is transitive on $\mathbb{R}^d/\mathbb{Z}^d$.

Theorem 1.1. *Let $\alpha \in \mathbb{R}^d$ be irrational. Then there is a dense set of $A \in C^\omega(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ (in the usual inductive limit topology) such that $L(\alpha, A) > 0$.*

Remark 1.1. The result is false if $\alpha \in \mathbb{Q}^d$, but remains true if $\alpha \in \mathbb{R}^d \setminus \mathbb{Q}^d$. We will carry the proof assuming irrationality of α to simplify the exposition.

In [FK], this result was proved for $d = 1$ and in the C^∞-topology (improving on [K] where an additional arithmetic condition on α, stronger than Diophantine, was imposed). Our result implies theirs, since the inclusion from C^ω to C^∞ is continuous with dense image. But the methods are quite different: [FK] uses renormalization techniques (as in [K]), which seem to be bound to the one-dimensional case, while we use a complex analytic approach (as in [AJ]). We notice that the arithmetic properties of α are almost irrelevant in our approach.

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It is reasonable to expect that positive Lyapunov exponent is dense in the much broader context of cocycles over smooth dynamical systems.[1] This is known to be the case for "fast systems" (expanding/hyperbolic), and more generally for systems which present periodic orbits of arbitrarily large period. The proof in those cases can be carried out using a result of Kotani (originally stated for Schrödinger cocycles, but which generalizes to the context of $\mathrm{SL}(2,\mathbb{R})$ cocycles): stability of zero Lyapunov exponent implies a continuous conjugation to a cocycle of rotations.

The case of quasiperiodic systems will be treated here via an improvement of Kotani result: in this setting Kotani's continuous conjugacy to rotations turns out to be analytic (this was first remarked in [AJ] in the particular case of "almost Mathieu" cocycles as a step in the solution of the "Ten Martini Problem"). This fact easily leads to the density of positive Lyapunov exponent result, since analytic cocycles over rotations are easy to perturb.

Though we are unable to treat at this point the problem of density of positive Lyapunov exponent for general dynamical systems, we would like to point out that one can still improve Kotani's Theorem for a large class of "slow systems" (including the skew-shift $(x, y) \mapsto (x + \alpha, y + x)$). We will give this argument in the appendix.

If α is irrational and $L(\alpha, A) > 0$ then Oseledets theorem implies that there exists a measurable decomposition of $\mathbb{R}^2$ as $E^s(x) \oplus E^u(x)$, $x \in \mathbb{R}^d/\mathbb{Z}^d$ such that $\lim \frac{1}{n} \ln \|A_n(x) \cdot w\| = -L(\alpha, A)$ for $w \in E^s(x) \setminus \{0\}$ and $\lim \frac{1}{n} \ln \|A_{-n}(x) \cdot w\| = -L(\alpha, A)$ for $w \in E^u(x)$. If $E^s(x)$ and $E^u(x)$ are actually continuous functions of x then (α, A) is said to be *uniformly hyperbolic*. Otherwise it is said to be *non-uniformly hyperbolic*. Thus our result states that hyperbolicity is dense in this context, though it does not specify whether one gets uniform or non-uniform hyperbolicity. Notice that uniform hyperbolicity is not dense in general, since there are topological obstructions (since uniform hyperbolicity implies that the cocycle is homotopic to the identity). It seems likely that uniform hyperbolicity is not dense in $d \geq 2$ frequencies even in the case of cocycles homotopic to the identity (this is indicated by the work of Chulaevsky-Sinai [CS]). It is an interesting open problem to show that uniform hyperbolicity is dense in the one-frequency, homotopic to the identity context.

1.1. **Outline.** We consider perturbations of a fixed cocycle obtained by postcomposition with a rotation. If the Lyapunov exponent is zero for all such (small) perturbation, Kotani Theorem implies that the cocycle is continuously conjugate to a cocycle of rotations. (This is also the starting point of [K] and [FK].)

An analytic extension argument (first used in this context in [AJ]) improves this result: the cocycle is actually analytically conjugate to a cocycle of rotations. After a small perturbation (needed to treat non-Diophantine frequencies), such a cocycle can be put into a normal form. It is easy to perturb directly the normal form to get a positive Lyapunov exponent.

2. Analiticity of m-functions

Let

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{2.1}$$

Our aim is to prove the following result.

Theorem 2.1. *Let $\alpha \in \mathbb{R}^d$ and let $A \in C^\omega(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$. If there exists an open interval $0 \in I \subset \mathbb{R}$ such that $L(\alpha, R_{-\theta}A) = 0$ for $\theta \in I$ then there exists $B \in C^\omega(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ such that*

$$B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R}). \tag{2.2}$$

[1]For our purposes, a smooth dynamical systems is assumed to be supplied with a probability measure with full support.

Consider the action of $\mathrm{SL}(2,\mathbb{C})$ on $\overline{\mathbb{C}}$ by Moebius transformations: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. Let $\mathbb{H} = \{z \in \mathbb{C}, \Im z > 0\}$. If $m : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{H}$ satisfies

$$A(x) \cdot m(x) = m(x+\alpha), \tag{2.3}$$

then

$$B(x) = \begin{pmatrix} \frac{\Re m(x)}{|m(x)|(\Im m(x))^{1/2}} & -\frac{|m(x)|}{(\Im m(x))^{1/2}} \\ \frac{(\Im m(x))^{1/2}}{|m(x)|} & 0 \end{pmatrix} \tag{2.4}$$

satisfies (2.2). Thus it is enough to find $m : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{H}$ which is analytic and satisfies (2.3).

Lemma 2.2. *Let $\alpha \in \mathbb{R}^d$ and let $A \in C^\omega(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$. Then there exists a unique analytic map $m : \mathbb{H} \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{H}$ satisfying*

$$R_{-\theta}A(x) \cdot m(\theta, x) = m(\theta, x+\alpha). \tag{2.5}$$

Proof. Notice that $R_{-\theta}A(x) \cdot \overline{\mathbb{H}} \subset K_{\Im\theta}$ where $K_t \subset \mathbb{H}$ is the closed disk of radius $\frac{1}{\sinh 4\pi t}$ and center $i\frac{\cosh 4\pi t}{\sinh 4\pi t}$. It follows that there exists $\Omega \subset \mathbb{H} \times \mathbb{C}^d/\mathbb{Z}^d$ such that $R_{-\theta}A(x) \cdot \overline{\mathbb{H}} \subset K_{\frac{\Im\theta}{2}}$. We may assume that $(\theta, x) \in \Omega$ implies that $(\theta, x+y) \in \Omega$, $y \in \mathbb{R}^d$. The Schwarz Lemma then implies that

$$R_{-\theta}A(x-\alpha) \cdots R_{-\theta}A(x-n\alpha) \cdot \overline{\mathbb{H}} \tag{2.6}$$

shrinks (exponentially fast) to a point $m(\theta, x)$, for every $(\theta, x) \in \Omega$. Thus $m(\theta, x)$ is the only function satisfying $R_{-\theta}A(x) \cdot m(x) = m(x+\alpha)$. Moreover, m is the limit of the sequence of holomorphic functions

$$m_n(x) = R_{-\theta}A(x-\alpha) \cdots R_{-\theta}A(x-n\alpha) \cdot i, \tag{2.7}$$

and hence is holomorphic. □

The following lemma gives a well-known generalization of a theorem of Kotani for Schrödinger cocycles (see [Sim]), adapted to the general context of analytic cocycles. We point the reader to [AK] for the full proof (of an interesting generalization).

Lemma 2.3. *Let α, A, m be as in the previous lemma. If there exists an open interval $0 \in I \subset \mathbb{R}$ such that $L(\alpha, R_\theta A) = 0$ for $\theta \in I$ then there exists a continuous extension $m : (\mathbb{C} \setminus (\mathbb{R} \setminus I)) \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{H}$ such that $\theta \mapsto m(\theta, x)$ is analytic for $x \in \mathbb{R}^d/\mathbb{Z}^d$.*

Proof of Theorem 2.1. It is enough to show that $x \mapsto m(0, x)$ is analytic where $m : (\mathbb{C} \setminus (\mathbb{R} \setminus I)) \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{H}$ is the function given in the previous lemma. Let

$$m(\theta, x) = \sum_{k \in \mathbb{Z}^d} \hat{m}_\theta(k) e^{2\pi i \langle x, k\rangle}, \tag{2.8}$$

and let

$$a(\theta) = \limsup_{|k| \to \infty} \frac{1}{|k|} \ln |\hat{m}_\theta(k)|. \tag{2.9}$$

Obviously $a(\theta) \leq 0$ for all θ, and $a(\theta) < 0$ exactly when $x \mapsto m(\theta, x)$ is analytic in x (in particular if $\theta \in \mathbb{H}$). We must show that $a(0) < 0$. Let $r > 0$ be small so that $\mathbb{D}_r$ is compactly contained in $\mathbb{C} \setminus (\mathbb{R} \setminus I))$. Then $|m(\theta, x)| < C$ for $\theta \in \mathbb{D}_r$, $x \in \mathbb{R}^d/\mathbb{Z}^d$. Thus $a|\mathbb{D}_r$ is the $\limsup$ of a sequence of non-positive subharmonic functions

$$\frac{1}{|k|} \ln \frac{|\hat{m}_\theta(k)|}{C}. \tag{2.10}$$

It follows that

$$a(0) \leq \frac{1}{\pi r^2} \int_{\mathbb{D}_r} a(\theta) d\theta. \tag{2.11}$$

Since $a(\theta) \leq 0$, $\theta \in \mathbb{D}_r$ and $a(\theta) < 0$, $\theta \in \mathbb{D}_r \cap \mathbb{H}$, we conclude that $a(0) < 0$. □

3. Proof of Theorem 1.1

We may assume that there exists an open interval $0 \in I \subset \mathbb{R}$ such that $L(\alpha, R_{-\theta} A) = 0$, $\theta \in I$. Then there exists $B \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ such that $B(x+\alpha)A(x)B(x)^{-1} = R_{\langle x,s\rangle + \phi(x)}$ for some $s \in \mathbb{Z}^d$, $\phi : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$. Perturb ϕ to a trigonometrical polynomial $\tilde{\phi}$ with

$$\theta = \int \tilde{\phi}(x) dx = \langle \alpha, k \rangle + l, \tag{3.1}$$

for some $k \in \mathbb{Z}^d$, $l \in \mathbb{Z}$. We may solve the equation

$$\psi(x+\alpha) - \psi(x) = \tilde{\phi}(x) - \theta \tag{3.2}$$

and $\psi : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is analytic. Then $C(x) = R_{-\langle x,k\rangle - \psi(x)}$ satisfies

$$C(x+\alpha) R_{\langle x,s\rangle + \tilde{\phi}(x)} C(x)^{-1} = R_{\langle s,x\rangle}. \tag{3.3}$$

Thus it is enough to find $X \in \mathrm{SL}(2, \mathbb{R})$ arbitrarily close to id such that

$$L(\alpha, R_{\langle s,x\rangle} X) > 0 \tag{3.4}$$

for in this case $E \in C^\omega(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2, \mathbb{R})$ given by

$$E(x) = B(x+\alpha)^{-1} C(x+\alpha)^{-1} R_{\langle s,x\rangle} X C(x) B(x) \tag{3.5}$$

is close to A and satisfies $L(\alpha, E) > 0$.

There are two possibilities. If $s = 0$, just take X a hyperbolic matrix close to id. If $s \neq 0$, let $X_0 \in \mathrm{SL}(2, \mathbb{R}) \setminus \mathrm{SO}(2, \mathbb{R})$ be close to id. By [AB],

$$\int_0^1 L(\alpha, R_\gamma R_{\langle x,s\rangle} X_0) d\gamma = \ln \frac{\|X_0\| + \|X_0\|^{-1}}{2} > 0. \tag{3.6}$$

Take γ with $L(\alpha, R_\gamma R_{\langle x,s\rangle} X_0) > 0$ and let $X = R_\gamma X_0 R_{-\gamma}$. Then X is close to id and we have

$$\left(\frac{\gamma s}{|s|^2}, R_\gamma \right) (\alpha, R_\gamma R_{\langle x,s\rangle} X_0) \left(\frac{\gamma s}{|s|^2}, R_\gamma \right)^{-1} = (\alpha, R_{\langle x,s\rangle} X), \tag{3.7}$$

so that

$$L(\alpha, R_{\langle s,x\rangle} X) = L(\alpha, R_\gamma R_{\langle s,x\rangle} X_0) > 0. \tag{3.8}$$

Appendix A. Smoothness of m-functions for subexponential dynamics

The proof of analyticity of m-functions (under the hypothesis of stability of zero Lyapunov exponents) carried in [AJ] and slightly generalized here uses in a quite fundamental way the fact that the underlying dynamics is a translation. If one tries to apply the same argument to, say, the skew-shift

$$(x, y) \mapsto (x+\alpha, y+x), \quad x, y \in \mathbb{R}^d/\mathbb{Z}^d, \alpha \in \mathbb{R}^d, \tag{A.1}$$

one runs into problems because when complexifying the system ($x, y \in \mathbb{C}^d/\mathbb{Z}^d$) we get orbits escaping to infinity, and thus we can not define m-functions. (Complexification just in the second variable does work, and leads to analyticity in y, but does not elucidate the smoothness in x.)

We can however obtain smoothness of m-functions for a class of dynamical systems including the skew-shift. Call a C^∞ dynamical system $T : X \mapsto X$ (X a compact smooth manifold), preserving a measure μ with $\operatorname{supp}\mu = X$, *subexponential* if

$$\lim_{n\to\pm\infty} \sup_{x\in X} \frac{1}{n} |\ln \|DT^n(x)\|| = 0. \tag{A.2}$$

Given a continuous $A : X \to \mathrm{SL}(2,\mathbb{R})$ we can consider the cocycle (T, A),

$$(x, w) \mapsto (T(x), A(x) \cdot w). \tag{A.3}$$

and we define the Lyapunov exponent

$$L(T, A) = \lim_{n\to\infty} \frac{1}{n} \int \ln \|A(T^{n-1}(x)) \cdots A(x)\| d\mu(x). \tag{A.4}$$

Theorem A.1. *Let (T, A) be a C^∞ cocycle with T subexponential. If $0 \in I \subset \mathbb{R}$ is an open interval such that $L(T, R_\theta A) = 0$ for $\theta \in I$ then there exists $B : X \to \mathrm{SL}(2,\mathbb{R})$ which is C^∞ and satisfies*

$$B(T(x))A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R}). \tag{A.5}$$

Proof. As before, Kotani Theory implies that there exists a continuous function $m : (\mathbb{C}\backslash(\mathbb{R}\backslash I))\times X \to \mathbb{H}$ such that $\theta \mapsto m(\theta, x)$ is analytic and we have

$$R_\theta A(x) \cdot m(\theta, x) = m(\theta, T(x)). \tag{A.6}$$

As before, it is enough for us to establish that $x \mapsto m(0, x)$ is C^∞.

It is easy to see that $x \mapsto m(\theta, x)$ is C^∞ for $\theta \in \mathbb{H}$, either by direct computation or by applying general partial hyperbolic theory [HPS]. Indeed one gets the stronger fact that (in charts) $(\theta, x) \mapsto \partial_x^k m(\theta, x)$ is a bounded function for all k and for (θ, x) in compact subsets of $\mathbb{H} \times X$. Fixing $\theta_0 \in \mathbb{H}$, we have

$$\partial_x^s m(\theta, x) = \sum \partial^s a_k(x)(\theta - \theta_0)^k, \quad |\theta - \theta_0| < \Im\theta_0, \tag{A.7}$$

where $x \mapsto a_k(x)$ is C^∞ and satisfies

$$\|\partial^s a_k(x)\| \le C(s, \delta)\delta^{-k} \quad 0 < \delta < \Im\theta_0, k \ge 0, s \ge 0. \tag{A.8}$$

Since $m : (\mathbb{C} \setminus (\mathbb{R} \setminus I)) \times X \to \mathbb{H}$ is continuous, this estimate can be improved to

$$|a_k(x)| \le C(\rho)\rho^{-k} \quad 0 < \rho < \operatorname{dist}(\theta_0, \mathbb{R} \setminus I), k \ge 0. \tag{A.9}$$

By convexity estimates

$$\|\partial^r f(x)\| \le C(r, s)\|f(x)\|^{\frac{s-r}{s}} \|\partial^s f(x)\|^{\frac{r}{s}}, \quad 0 \le r \le s, \tag{A.10}$$

we get

$$\|\partial^r a_k(x)\| \le C(r, s)C(\rho)^{\frac{s-r}{s}} \rho^{-k\frac{s-r}{s}} C(s, \delta)^{\frac{r}{s}} \delta^{-k\frac{r}{s}} \le C(r, s)C(\rho)^{\frac{s-r}{s}} C(s, \delta)^{\frac{r}{s}} \left(\delta^{\frac{r}{s}} \rho^{\frac{s-r}{s}}\right)^{-k}, \tag{A.11}$$

where we must choose

$$0 < \delta < \Im\theta_0, \quad 0 < \rho < \operatorname{dist}(\theta_0, \mathbb{R} \setminus I), \quad k \ge 0, \quad 0 \le r \le s. \tag{A.12}$$

Taking $\theta_0 = 2\epsilon i$ for some small ϵ, we can thus take for any $r \ge 0$

$$\delta = \epsilon, \quad \rho = 4\epsilon, \quad s = 4r, \tag{A.13}$$

which implies

$$\|\partial^r a_k(x)\| \le C(r, 4r)C(\rho)^{\frac{3}{4}} C(4r, \delta)^{\frac{1}{4}} (4^{\frac{3}{4}} \epsilon)^{-k}, \quad k \ge 0, r \ge 0, \tag{A.14}$$

which implies that $x \mapsto m(0, x)$ is C^∞ (since $|\theta_0| = 2\epsilon < 4^{\frac{3}{4}}\epsilon$). □

Remark A.1. By the previous theorem, the question of (C^∞) density of positive Lyapunov exponents (for subexponential dynamics) is thus reduced to whether one can approach a C^∞ cocycle of rotations by a $\mathrm{SL}(2,\mathbb{R})$ cocycle with positive Lyapunov exponent. Though this is very easy to do in the case of rotations, the case of the skew-shift is already (as far as we know) open.

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