GLOBAL THEORY OF ONE-FREQUENCY SCHRÖDINGER OPERATORS II: ACRITICALITY AND FINITENESS OF PHASE TRANSITIONS FOR TYPICAL POTENTIALS

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Abstract. We consider Schrödinger operators with a one-frequency analytic potential. Energies in the spectrum can be classified as subcritical, critical or supercritical, by analogy with the almost Mathieu operator. Here we show that the critical set is empty for an arbitrary frequency and almost every potential. Such acritical potentials also form an open set, and have several interesting properties: only finitely many "phase transitions" may happen, however never at any specific point *in the spectrum*, and the Lyapunov exponent is minorated in the region of the spectrum where it is positive. In the appendix, we give examples of potentials displaying (arbitrarily) many phase transitions.

1. Introduction

This work continues the global analysis of one-dimensional Schrödinger operators with an analytic one-frequency potential started in [A1], to which we refer the reader for further motivation.

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, let $H = H_{\alpha,v}$ be the Schrödinger operator

(1) $$(Hu)_n = u_{n+1} + u_{n-1} + v(n\alpha)u_n$$

on $\ell^2(\mathbb{Z})$ and let $\Sigma = \Sigma_{\alpha,v} \subset \mathbb{R}$ be its spectrum.

For any energy $E \in \mathbb{R}$, let

(2) $$A(x) = A^{(E-v)}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},$$

(3) $$A_n(x) = A(x + (n-1)\alpha) \cdots A(x),$$

which are analytic functions with values in $\mathrm{SL}(2, \mathbb{R})$. They are relevant to the analysis of H because a formal solution of $Hu = Eu$ satisfies $\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A_n(0) \cdot \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}$. The *Lyapunov exponent* $L(E)$ is given by

(4) $$\lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n(x)\| dx.$$

Energies $E \in \Sigma$ can be:

1. *supercritical*, if $L(E) > 0$,

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2. *subcritical*, if there is a uniform subexponential bound on the growth of $\|A_n(z)\|$ through some band $|\Im z| < \epsilon$,
3. *critical* otherwise.

Supercritical energies are usually called *nonuniformly hyperbolic*. The nonuniformly hyperbolic regime is stable by [BJ1]: if we perturb α in $\mathbb{R} \setminus \mathbb{Q}$, v in $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and E in $\mathbb{R}$ (but still belonging to the perturbed spectrum), we stay in the same regime. In [A1] it is shown that the subcritical regime is also stable. As we will see (Theorem 9), the critical regime in fact equals the boundary of the nonuniformly hyperbolic regime: if E is critical, we can perturb v so that E still belongs to the perturbed spectrum (with the same α) and becomes nonuniformly hyperbolic. Thus subcritical energies are also said to be *away from nonuniform hyperbolicity.*

In the most studied case of the almost Mathieu operator, $v(x) = 2\lambda \cos 2\pi(x+\theta)$, all energies are subcritical when $|\lambda| < 1$, supercritical when $|\lambda| > 1$ and critical when $|\lambda| = 1$. In general, the subcritical and supercritical regime can coexist in the spectrum of the same operator [Bj]. However, to go from one regime to the other it may not be necessary to pass through the critical regime, since one usually expects the spectrum to be a Cantor set. In this paper we show that this is the prevalent behavior. Let us say that H is acritical if no energy $E \in \Sigma$ is critical.

Main Theorem. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then for a (measure theoretically) typical $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, the operator $H_{\alpha,v}$ is acritical.*

The Main Theorem yields a precise description of the basic structure of the spectrum of typical operators with respect to the behavior of the Lyapunov exponent. Indeed the stability of the non-critical regimes [A1] yields immediately:

1. Acriticality is stable with respect to perturbations of both α and v, and the supercritical and subcritical parts of the spectrum define compact sets that depend continuously (in the Hausdorff topology) on the perturbation.
2. As a consequence, acritical operators have the nicest behavior from the point of view of bifurcations: There is at most a finite number of "alternances of regime", as one moves through the spectrum Σ in the following sense: there is $k \geq 1$ and points $a_1 < b_1 < ... < a_k < b_k$ in the spectrum such that $\Sigma \subset \bigcup_{i=1}^k [a_i, b_i]$ and energies alternate between supercritical and subcritical along the sequence $\{\Sigma \cap [a_i, b_i]\}_{i=1}^k$.
3. Another consequence is *spectral uniformity* through both subcritical and supercritical regimes: There exists $\epsilon > 0$ such that whenever E is supercritical we have $L(E) \geq \epsilon$ (by continuity of the Lyapunov exponent [BJ1]), and when E is subcritical we have uniform subexponential growth of $\|A_n(z)\|$ through the band $|\Im z| < \epsilon$ (again by continuity of the Lyapunov exponent, together with quantization of the acceleration [A1]).

As we will show in the appendix, the number of phase transitions can be arbitrarily large.

1.1. **The Spectral Dichotomy program.** The Main Theorem reduces the spectral theory of a typical one-frequency Schrödinger operator H to the separate "local theories" of (uniform) supercriticality and subcriticality. It is thus a key step in our program to establish the *Spectral Dichotomy*, the decomposition of a typical operator as a direct sum of operators with the spectral type of "large-like" and "small-like" operators. Below we comment briefly at the current state of the local theories.

The supercritical theory has been intensively developed in [BG], [GS1], [GS2], [GS3]. As far as the spectral type is concerned, perhaps the key result is that, up to a typical perturbation of the frequency, Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) holds through the supercritical regime. It is important to emphasize that these developments superseded several early results depending on suitable largeness conditions on the potentials, and that the change of focus towards the Lyapunov exponent can be in large part attributed to [J].

The concept of subcriticality has evolved more recently, and the development of the corresponding local theory originally centered on the concept of *almost reducibility*, which *by definition* generalizes the scope of applicability of the theory of small potentials (which is well understood by KAM and localization-duality methods). In particular, it was shown ([AJ], [A2], [A3]) that almost reducibility implies absolute continuity of spectral measures. In [AJ] the vanishing in a band of the Lyapunov exponent was suggested to be the sought after mirror condition to positivity of the Lyapunov exponent: more specifically, it was conjectured to be equivalent to almost reducibility (in the spectrum). Proving this *Almost Reducibility Conjecture* would at once provide an almost complete understanding of subcriticality, and partial results were obtained in [A2] and [A3].

We have recently proved the Almost Reduciblity Conjecture (for all frequencies), and we refer the reader to [A4] for a detailed account of spectral consequences.

1.2. **Prevalence.** Let us explain in more detail the notion of typical we use in this paper. Since in infinite-dimensional settings one lacks a translation invariant measure, it is common to replace the notion of "almost every" by "prevalence": one fixes some probability measure μ of compact support (a set of admissible perturbations w), and declare a property to be typical if it is satisfied for *almost every perturbation* $v+w$ of *every* starting condition v. In finite dimensional vector spaces, prevalence implies full Lebesgue measure.

In our case, we have quite a bit of flexibility for the choice of μ. For instance, though we do want to be able to perturb of all Fourier coefficients, we may impose arbitrarily strong restrictions on high Fourier mode perturbations. For definiteness, we will set $\Delta = \mathbb{D}^{\mathbb{N}}$ endowed with the probability measure μ given by the product of normalized Lebesgue measure. Given an arbitrary function $\varepsilon : \mathbb{N} \to \mathbb{R}_+$ which decays exponentially fast (the particular choice is quite irrelevant for us), we associate a probability measure μ_ε with compact support on $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ by push forward of μ under the map $(t_m)_{m\in\mathbb{N}} \mapsto \sum_{m\geq 1} \varepsilon(m) 2\Re[t_m e^{2\pi i m x}]$.

Our goal will be to show that for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, for μ_ε-almost every w, $H_{\alpha,v+w}$ is acritical.

Remark 1.1.
1. The notion of prevalence is usually formulated for separable Banach spaces (see [HSY]). Our result does imply prevalence of acriticality in any Banach space of analytic potentials which is continuously and densely embedded in C^ω.
2. The notion of prevalence (or rather, the corresponding smallness notion called shyness in [HSY]) was first introduced in [C], i.e., the complement of a prevalent set in a Banach space is what is called a *Haar-null set*. There is a stronger notion of smallness (and thus a corresponding stronger notion of typical) which is induced by the family of non-degenerate Gauss measures

in a Banach space: Gauss-null sets.[1] In a Banach space, a Borel set which has zero probability with respect to any affine embedding of the Hilbert cube (endowed with the natural product measure) which is non-degenerate (i.e., not contained in a proper closed affine subspace) is Gauss-null, see [BL], Section 6.2. While we have considered in the description above a particular family of embeddings of $\mathbb{D}^{\mathbb{N}}$, it is transparent from the proof that an arbitrary non-degenerate embedding of the Hilbert cube would work equally well, so acritical potentials are also typical in this stronger sense.

2. Cocycles

In what follows, Banach spaces of analytic functions on $\mathbb{R}/\mathbb{Z}$ with bounded holomorphic extensions to $|\Im z| < \delta$, continuous up to the boundary, will be denoted $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, *)$, $* = \mathbb{R}, \mathrm{SL}(2,\mathbb{R}), ...$, with norms denoted $\|\cdot\|_\delta$. Banach spaces of continuous functions will be denoted $C^0(\mathbb{R}/\mathbb{Z}, *)$ with norms denoted $\|\cdot\|_0$.

Our analysis of the operator $H_{\alpha,v}$ will be based on the dynamics of the associated family of Schrödinger cocycles.

Let us first introduce slightly more general $\mathrm{SL}(2,\mathbb{C})$ cocycle dynamics, and some key results of [A1], to which we refer for a thorough discussion. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. For $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$, define A_n by (3). We interpret the pair (α, A) as a skew-product dynamical system $(x, y) \mapsto (x + \alpha, A(x) \cdot y)$, and the n-th iterate of (α, A) is given by $(n\alpha, A_n)$. The Lyapunov exponent $L(\alpha, A)$ is given by (4). Since A is analytic, we can define an analytic family of deformations of A, denoted by A_ϵ, $\epsilon \in \mathbb{R}$ small, given by $A_\epsilon(x) = A(x + \epsilon i)$. The function $\epsilon \mapsto L(\alpha, A_\epsilon)$ is easily seen to be convex. In [A1], we have shown that there exists an integer $\omega(\alpha, A)$, called the *acceleration* of (α, A) such that

$$L(\alpha, A_\epsilon) - L(\alpha, A) = 2\pi\epsilon\omega(\alpha, A) \tag{5}$$

for every $\epsilon > 0$ small. If A takes values in $\mathrm{SL}(2,\mathbb{R})$, the acceleration is non-negative by convexity. The Lyapunov exponent is continuous on (α, A) throughout $(\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ ([BJ1], [JKS]) while the acceleration is upper semicontinuous. We say that (α, A) is *regular* if (5) holds for all ϵ small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near (α, A).

Given $H_{\alpha,v}$, we define the acceleration ω at energy $E \in \mathbb{R}$ by $\omega(E) = \omega(\alpha, A)$, where $A = A^{(E-v)}$ is given by (2). Then E is critical if and only if $L(E) = 0$ and $\omega(E) > 0$. If E is critical with acceleration k, we will call it also a *critical point of degree k*.

Our basic plan is to show that critical points of maximal degree $k \geq 1$ can be destroyed by a small typical perturbation by trigonometric polynomials of some large degree. This may give rise to many critical points of degree $\leq k - 1$, but by iterating this process we will eventually get rid of all of them.

More formally, let $\mathcal{A}_k \subset (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $k \geq 0$, be the set of all (α, v) such that $H_{\alpha,v}$ has only critical points of degree at most k. Hence $\mathcal{A}_k$ forms an increasing sequence of open sets with $\bigcup_{k\geq 0} \mathcal{A}_k = (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $\mathcal{A}_0$ is the set of all (α, v) such that $H_{\alpha,v}$ is acritical. Let $\mathcal{P}^n \subset C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be the space of trigonometric polynomials with degree at most n, let $\mathcal{P}^n_0 \subset \mathcal{P}^n$ be the subspace of zero average functions, and for $\epsilon > 0$ let $\mathcal{P}^n(\epsilon) \subset \mathcal{P}^n$ and $\mathcal{P}^n_0(\epsilon) \subset \mathcal{P}^n_0$ be the corresponding ϵ-balls with respect to the C^0 norm.

[1]The author learned this notion from Assaf Naor.

Our main estimate is the following:

Theorem 1. *For every* $(\alpha, v) \subset \mathcal{A}_k$ *there exists* $\epsilon > 0$ *and* $n \geq 1$ *such that for almost every* $w \in \mathcal{P}_0^n(\epsilon)$ *we have* $(\alpha, v + w) \in \mathcal{A}_{\max\{0,k-1\}}$.[2]

The Main Theorem follows immediately from this estimate.

3. Parameter exclusion argument

From now on, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is fixed.

In [A1], we proved that critical cocycles have "codimension one" among all cocycles. Earlier, in [AK1] and [AK2], we had shown that almost every cocycle in certain one-parameter families has either a positive Lyapunov exponent or admits a sequence of renormalizations converging to a good normal form. The techniques in those works are quite distinct, and our aim is to combine them to show that critical cocycles in fact have zero measure inside a codimension one subspace. The key difficulty we will face is in establishing an indefiniteness result for the derivative of the Lyapunov exponent, which will enable us to construct appropriate one-parameter families inside the locus where criticality might appear.

In this strategy, Theorem 1 is obtained as a consequence of the following. Let

$$A^{(v)}(x) = \begin{pmatrix} v(x) & -1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

and for every $k \geq 1$ let $\mathcal{C}^k$ be the set of all $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that $L(\alpha, A^{(v)}) = 0$ and $\omega(\alpha, A^{(v)}) = k$.

Theorem 2. *For every* $v_0 \subset \mathcal{C}^k$*, there exists* $\epsilon > 0$ *and* $n \geq 1$ *such that* $\{w \in \mathcal{P}^n(\epsilon),\ v_0 + w \in \mathcal{C}^k\}$ *has* $2n-1$*-dimensional Hausdorff measure zero.*

Theorem 2 implies Theorem 1 by projecting in the direction of the energy, using that the set of critical points of maximal degree for $H_{\alpha,v}$ is compact.

Through the remaining of the paper, $v_0 \in \mathcal{C}^k$ is also fixed. Fix $\xi' > \xi > 0$ such that $v_0 \in C^\omega_{\xi'}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Recall the usual identification $\mathbb{PC}^2$ with $\overline{\mathbb{C}}$, corresponding the direction through $\begin{pmatrix} z \\ w \end{pmatrix}$ with $\frac{z}{w}$. With this identification, $\mathrm{SL}(2, \mathbb{C})$ acts on $\overline{\mathbb{C}}$ through Möbius transformations in the usual way: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$.

If $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$, we say that (α, A) is *uniformly hyperbolic* if $L(\alpha, A) > 0$ and there exists a pair of distinct analytic *invariant sections*, called the unstable and stable directions, $u, s : \mathbb{R}/\mathbb{Z} \to \overline{\mathbb{C}}$, such that $A_n(x)$ contracts exponentially along the $s(x)$ (respectively $u(x)$) direction as $n \to \infty$ (respectively $n \to -\infty$). It is easy to see that $u(x) \neq s(x)$ for every $x \in \mathbb{R}/\mathbb{Z}$ and $A(x) \cdot u(x) = u(x + \alpha)$, $A(x) \cdot s(x) = s(x + \alpha)$. Let $\mathcal{UH} \subset C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ be the set of A such that (α, A) is uniformly hyperbolic. Then $\mathcal{UH}$ is open and $A \mapsto L(\alpha, A)$ is analytic over $\mathcal{UH}$.

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[2] A slightly stronger statement follows from our proof: if (α, v) also belongs to $(\mathbb{R} \setminus \mathbb{Q}) \times C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ then $n = n(\alpha, v)$ and $\epsilon = \epsilon(\alpha, v, \delta)$ may be chosen so that for every $(\alpha', v') \in (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that $|\alpha - \alpha'| < \epsilon$ and $\|v - v'\|_\delta < \epsilon$ and for almost every $w \in \mathcal{P}_0^n(\epsilon)$ we have $(\alpha', v' + w) \in \mathcal{A}_{\max\{0,k-1\}}$.

We have shown in [A1], Corollary 7, that if either $L(\alpha, A^{(v)}) > 0$ or $\omega(\alpha, A^{(v)}) > 0$ then for every $\epsilon > 0$ small, $\omega(\alpha, A^{(v)}_\epsilon) = \omega(\alpha, A^{(v)})$ and $A^{(v)}_\epsilon \in \mathcal{UH}$. Thus there exist an open neighborhood $\mathcal{V}$ of v_0 in $C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, and $0 < \xi_0 < \xi$ such that $A^{(v)}_{\xi_0} \in \mathcal{UH}$ and $\omega(\alpha, A^{(v)}_{\xi_0}) = k$ for every $v \in \mathcal{V}$. We fix such ξ_0 and let $L_{\xi,k} : \mathcal{V} \to \mathbb{R}$ be given by $L_{\xi,k}(v) = L(\alpha, A^{(v)}_{\xi_0}) - 2\pi k \xi_0$, which is analytic. Then for every $v \in \mathcal{V}$ such that $\omega(\alpha, A^{(v)}) = k$, $L_{\xi,k}(v) = L(\alpha, A^{(v)})$. Thus $\mathcal{C}^k \cap \mathcal{V} \subset L^{-1}_{\xi,k}(0)$.

Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 and let $v_\lambda \in \mathcal{V}$, $\lambda \in U$, be an analytic deformation of v_0. For any $\lambda_0 \in U$, let $D_{\lambda_0} v_\lambda : \mathbb{R}^n \to C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be the derivative $D_{\lambda_0} v_\lambda \cdot w = \frac{d}{dt} v_{\lambda_0 + tw}\big|_{t=0}$. The reader should keep in mind the family $v_\lambda = v_0 + P_n \lambda$, $\lambda \in \mathbb{R}^{2n+1}$, where $P_n : \mathbb{R}^{2n+1} \to \mathcal{P}^n$ is some fixed isomorphism (in this case $\mathrm{Im} D_0 v_\lambda = \mathcal{P}^n$).

We say that $a \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2,\mathbb{R}))$ is signed if $\det a(x) > 0$ for every $x \in \mathbb{R}/\mathbb{Z}$. Given $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, we say that $a \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is A-signed if there exists $b \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2,\mathbb{R}))$ such that

$$x \mapsto A(x)^{-1} b(x+\alpha) A(x) - b(x) + a(x) \tag{7}$$

is signed.

Given $v_0, w \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, we say that w is v_0-signed if $\begin{pmatrix} 0 & 0 \\ -w & 0 \end{pmatrix}$ is $A^{(v_0)}$-signed.

Remark 3.1. It is easy to see that if $\pm w(x) > 0$ for every $x \in \mathbb{R}/\mathbb{Z}$ then w is v-signed (indepently of v), just choose $b = \begin{pmatrix} 0 & 0 \\ \mp\epsilon & 0 \end{pmatrix}$ with sufficiently small $\epsilon > 0$ in (7).

Remark 3.2 (Interpretation of signedness). Recall that an analytic one-parameter family $A^\lambda \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, is said to be monotonic (in the sense of [AK2]) if for each $x \in \mathbb{R}$ and each unit vector m in $\mathbb{R}^2$, the derivative, with respect to λ, of the argument of $A^\lambda(x) \cdot m$ is non-zero. It is easy to see that this condition is equivalent to positivity of $\det a^\lambda$, where $a^\lambda = (A^\lambda)^{-1} \frac{d}{d\lambda} A^\lambda$. Monotonicity is a powerful concept that allows one to efficiently use complexification techniques in the analysis of the parameter space (generalizing Kotani Theory).

It turns out that monotonicity is not invariant under coordinate changes. Indeed, let us consider a one-parameter analytic family of coordinate changes $B^\lambda \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, giving rise to the family $\tilde{A}^\lambda(x) = B^\lambda(x+\alpha) A^\lambda(x) B^\lambda(x)^{-1}$, and define b^λ and $\tilde{a}^\lambda$ analogously to a^λ. Then

$$\tilde{a}^\lambda(x) = B^\lambda(x)(A^\lambda(x)^{-1} b^\lambda(x+\alpha) A^\lambda(x) - b^\lambda(x) + a^\lambda(x)) B^\lambda(x)^{-1}, \tag{8}$$

so that the determinant of $\tilde{a}^\lambda(x)$ is the same as that of $A^\lambda(x)^{-1} b^\lambda(x+\alpha) A^\lambda(x) - b^\lambda(x) + a^\lambda(x)$.

Thus a family A^λ can be made monotonic by coordinate change near some parameter λ_0 if and only if a^λ_0 is A^λ_0-signed.

Theorem 3. *Let v_λ, $\lambda \in U$ be an analytic family as above such that there exists a v_0-signed vector w in the image of $D_0 v_\lambda$ with $DL_{\xi,k}(v_0) \cdot w = 0$, but $DL_{\xi,k}(v_{\lambda_0})$ does not vanish over $D_0 v_\lambda$. Then there exists $\epsilon > 0$ such that the set of all λ which are ϵ-close to 0 and such that $v_\lambda \in \mathcal{C}^k$ has $n-1$-dimensional Hausdorff measure zero.*

In [A1], Theorem 8, it is shown that the linear functional $DL_{\xi,k}(v_0)$ has rank 1 (because $v_0 \in \mathcal{C}^k$), so Theorem 3 reduces the proof of Theorem 2 (and hence the Main Theorem as well) to the following indefiniteness estimate for the derivative of the Lyapunov exponent.

Theorem 4 (Indefiniteness of the derivative)**.** *There exists a v_0-signed trigonometrical polynomial w such that $DL_{\xi,k}(v_0) \cdot w = 0$.*

This is our main estimate and will be proved in section 5. For the moment, let us give the proof of Theorem 3.

Proof of Theorem 3. Let us say that (α, A) is L^2-conjugate to rotations if there exists $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ measurable such that $B(x+\alpha)A(x)B(x)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ for almost every x and $\int \|B(x)\|^2 dx < \infty$. It is clear that if (α, A) is L^2-conjugate to rotations then $L(\alpha, A) = 0$.

The following is a convenient restatement of a result of [AK2].

Theorem 5. *Let $v_\lambda \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be an analytic family defined for $\lambda \in \mathbb{R}$ near 0 such that $w = \frac{d}{d\lambda} v_\lambda\big|_{\lambda=0}$ is v_0-signed. Then for almost every λ near 0, $(\alpha, A^{(v_\lambda)})$ is L^2-conjugate to rotations.*

Proof. Let b be such that (7) is signed, and let $A^\lambda(x) = e^{\lambda b(x+\alpha)} A^{(v_\lambda)}(x) e^{-\lambda b(x)}$. Then $\lambda \mapsto A^\lambda$ is a monotonic family (in the sense of [AK2]), for λ near 0. By the generalized Kotani Theory of [AK2] (see Theorem 1.7 therein), for almost every λ near 0, (α, A^λ), and hence $(\alpha, A^{(v_\lambda)})$, is L^2-conjugate to rotations. □

Corollary 6. *If 0 is v_0-signed then $A^{(v_0)}$ is L^2-conjugate to rotations.*

Proof. Apply the previous theorem to the constant family $v_\lambda = v_0$. □

Let p_n/q_n be the sequence of continued fraction approximations of α. Let $\beta_n = (-1)^n(q_n\alpha - p_n) > 0$ and $\alpha_n = \beta_n/\beta_{n-1}$. If $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, we say that (α', A') is a n-th renormalization of (α, A) if $\alpha' = \alpha_n$, $A' \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, and there exist $x_0 \in \mathbb{R}/\mathbb{Z}$ and $N : \mathbb{R} \to \mathrm{SL}(2,\mathbb{R})$ analytic such that

$$N(x+1)A_{(-1)^{n-1}q_{n-1}}(x_0 + \beta_{n-1}x)N(x)^{-1} = id, \tag{9}$$

$$N(x+\alpha_n)A_{(-1)^n q_n}(x_0 + \beta_{n-1}x)N(x)^{-1} = A'(x). \tag{10}$$

Here $A_{-k}(x) = A_k(x - k\alpha)^{-1}$ for $k \geq 1$.[3]

Theorem 7 ([AK2], Theorem 4.3)**.** *Let $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be homotopic to a constant. If (α, A) is L^2-conjugate to rotations then for every $\epsilon > 0$ there exists n and $\theta \in \mathbb{R}$ such that (α, A) has an n-th renormalization (α', A') with $\|A' - R_\theta\|_{\epsilon^{-1}} < \epsilon$.*

Corollary 8. *If (α, A) is homotopic to a constant and L^2-conjugate to rotations then $\omega(\alpha, A) = 0$.*

[3] Heuristically, the n-th renormalization is obtained by inducing the cocycle dynamics to $[x, x+\beta_{n-1}]$ and then rescaling the interval to unit length. However, this does not output a one-frequency cocycle, since an appropriate gluing must be made (9). This gluing is not canonical, so the n-th renormalization (10) is only defined up to conjugation. See [AK1], [AK2].

Proof. Recall that $\beta_{n-1} = \frac{1}{q_n+\alpha_n q_{n-1}}$.

Let (α', A') be an n-th renormalization of (α, A), and let $N : \mathbb{R} \to \mathrm{SL}(2,\mathbb{R})$ be analytic satisfying (9) and (10). It follows that

$$A_{k(-1)^n q_n + l(-1)^{n-1} q_{n-1}}(x_0 + \beta_{n-1}x) = N(x + k\alpha' + l)^{-1} A'_k(x) N(x) \tag{11}$$

for $k, l \in \mathbb{Z}$ (naturally we define $A'_k(x) = A'(x+(k-1)\alpha') \cdots A'(x)$ using translations by α' and not by α).

Let $\epsilon_0 > 0$ be such that $A' \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ and N admits an analytic extension to an open neighborhood of $\mathbb{R}$ containing $Q = [0,2] \times [-\epsilon_0, \epsilon_0]$. Let $C_0 = \sup_{z\in Q} \|N(z)\|^2$. If k is an arbitrary integer, $l = l(k)$ is the unique integer such that $0 \leq k\alpha' + l < 1$ and $t = t(k) = k(-1)^n q_n + l(-1)^{n-1} q_{n-1}$ then we have

$$C_0^{-1} \leq \frac{\|A_t(y + \beta_{n-1}\epsilon i)\|}{\|A'_k(x + \epsilon i)\|} \leq C_0, \tag{12}$$

where $x, y \in \mathbb{C}/\mathbb{Z}$ are related by $y = x_0 + \beta_{n-1}x$ and we assume that $|\Im(x+\epsilon i)| < \epsilon_0$. It follows that

$$\begin{aligned} &\left| \int_{\mathbb{R}/\mathbb{Z}} \ln \|A'_k(x+\epsilon i)\| dx - \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_{(-1)^n t}(x + \beta_{n-1}\epsilon i)\| dx \right| \\ &= \left| \int_{\mathbb{R}/\mathbb{Z}} \ln \|A'_k(x+\epsilon i)\| dx - \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_t(x + \beta_{n-1}\epsilon i)\| dx \right| \leq \ln C_0. \end{aligned} \tag{13}$$

Notice that when k is large, t satisfies $(-1)^n \frac{t}{k} = q_n - \frac{l}{k} q_{n-1} = q_n + \alpha_n q_{n-1} + o(1) = \frac{1}{\beta_{n-1}} + o(1)$. It follows that for large k,

$$\frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A'_k(x+\epsilon i)\| dx = \frac{1+o(1)}{\beta_{n-1}} \frac{1}{(-1)^n t} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_{(-1)^n t}(x + \beta_{n-1}\epsilon i)\| dx, \tag{14}$$

and taking the limit we get

$$L(\alpha', A'_\epsilon) = \frac{1}{\beta_{n-1}} L(\alpha, A_{\beta_{n-1}\epsilon}), \tag{15}$$

from which follows $\omega(\alpha, A) = \omega(\alpha', A')$.

If (α, A) is L^2-conjugate to rotations, then by the previous theorem we can take $\|A' - R_\theta\|_1 < 1$. This easily implies that $L(\alpha', A'_\epsilon) < \ln 2$ for $0 < \epsilon < 1$, so $\omega(\alpha', A') \leq \frac{\ln 2}{2\pi} < 1$ by convexity, hence $\omega(\alpha', A') = 0$ by quantization. □

Now, since $DL_{\xi,k} \cdot D_0 v_\lambda$ is non-trivial, the implicit function theorem allows us to shrink U and change coordinates near 0 so that $L_{\xi,k}$ becomes a linear function $\tilde{L}(\lambda_1, ..., \lambda_n) = \lambda_n$.

The hypothesis implies that there exists $t_0 \in \mathbb{R}^n$ and such that $w = D_0 v_\lambda \cdot t_0$ is v_0-signed and $DL_{\xi,k}(v_0) \cdot w = 0$. By Corollaries 6 and 8, $t_0 \neq 0$, so we may assume that $t_0 = (1, 0, ..., 0)$.

Shrinking further U, we may assume that $D_{\lambda_0} v_\lambda \cdot t_0$ is v_{λ_0}-signed at every λ_0 near 0. By Theorem 5, for every $(\lambda_2, ..., \lambda_{n-1})$ and for almost every λ_1, if $(\alpha, A^{(v_{(\lambda_1,...,\lambda_{n-1},0)})})$ has zero Lyapunov exponent then it is L^2-conjugate to rotations, hence by Corollary 8, its acceleration is zero, and thus $v_{(\lambda_1,...,\lambda_{n-1},0)} \notin \mathcal{C}^k$. This concludes the proof of Theorem 3. □

4. The critical regime as the boundary of nonuniform hyperbolicity

The indefiniteness estimate (Theorem 4) also has the following consequence.

Theorem 9. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. If $E \in \Sigma_{\alpha,v}$ is critical, then there exists a trigonometric polynomial w, and arbitrarily small $t > 0$, such that E (belongs to the spectrum and) is supercritical for $H_{\alpha,v+tw}$.*

Proof. Let $\omega(\alpha, A^{(E-v)}) = k > 0$. Fix $\xi > 0$ such that $v \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, and define a function $L_{\xi,k} : \mathcal{V} \to \mathbb{R}$ on a neighborhood $v \in \mathcal{V} \subset C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ as before. Choose w a $E - v$-signed trigonometric polynomial such that the derivative of $v' \mapsto L_{\xi,k}(E - v')$ at $v' = v$ and in the direction of w is zero. Let v_λ be an analytic family of trigonometric polynomials with $v_0 = v$, tangent to w at 0 and satisfying $L_{\xi,k}(E - v_\lambda) = 0$. Let $N_{\alpha,v'} : \mathbb{R} \to \mathbb{R}$ denote the integrated density of states of $H_{\alpha,v'}$.

By the usual monotonicity argument (see, e.g., [AK2] Lemma 2.4), since w is $E - v$-signed, $\lambda \mapsto N_{\alpha,v_\lambda}(E)$ is either non-increasing or non-decreasing on λ small. Moreover, since $(\alpha, A^{(E-v)})$ is not uniformly hyperbolic, it can not be constant near 0.[4]

It follows that there exists a sequence $\lambda_n \to 0$ such that $N_{\alpha,v_\lambda}(E) \notin \mathbb{Z} \oplus \alpha\mathbb{Z}$. By the Gap Labelling Theorem, this implies that $E \in \Sigma_{\alpha,v_{\lambda_n}}$ and it is accumulated from both sides by points in $\Sigma_{\alpha,v_{\lambda_n}}$.

Let w' be a trigonometric polynomial such that the derivative of $v' \mapsto L_{\xi,k}(E-v')$ at $v' = v$ and in the direction of w' is positive. For every n, there exists a sequence $0 < \lambda'_{j,n} < 1/j$ such that $N_{\alpha,v_{j,n}}(E) \notin \mathbb{Z} \oplus \alpha\mathbb{Z}$, where $v_{j,n} = v_{\lambda_n} + \lambda'_{j,n} w'$. Taking n and j large then E is supercritical for $H_{\alpha,v_{j,n}}$: on one hand, E belongs to the spectrum (by the Gap Labelling Theorem), and on the other, $L_{\xi,k}(E - v_{j,n}) > 0$ by the choice of w', so by convexity we have $L(\alpha, A^{(E-v_{j,n})}) \geq L_{\xi,k}(E - v_{j,n})$.

Note that in the "generic case" $N_{\alpha,v}(E) \notin \mathbb{Z} \oplus \alpha\mathbb{Z}$, the result can be obtained in a much simpler way from [A1], since one can find directly a sequence $0 < \lambda'_j < 1/j$ such that $N_{\alpha,v+\lambda'_j}(E) \notin \mathbb{Z} \oplus \alpha\mathbb{Z}$. □

5. Indefiniteness

Recall the setting of Theorem 3. We will need the expression for the derivative of $L_{\xi,k} : \mathcal{V} \to \mathbb{R}$ at $v \in \mathcal{V}$ that was derived in [A1]. For each such v, there exists a maximal interval $\xi_0 \in (\xi_-(v), \xi_+(v)) \subset (0, \xi)$ such that the cocycle $(\alpha, A^{(v)})$ is uniformly hyperbolic through $\{\xi_-(v) < \Im z < \xi_+(v)\}$ (note in particular that $\xi_-(v_0) = 0$). Then the unstable and stable directions provide holomorphic functions u and s with values in $\overline{\mathbb{C}}$, such that $A^{(v)}(z) \cdot u(z) = u(z + \alpha)$ and $A^{(v)}(z) \cdot s(z) = s(z + \alpha)$, moreover $u(z) \neq s(z)$ for every z. If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ satisfies

[4] Indeed, there exists $\varepsilon \in \{-1, 1\}$ and $C > 1$ such that for all small λ $N_{\alpha,v_\lambda}(E)$ belongs to the closed interval bounded by $N_{\alpha,v}(E+\varepsilon C^{-1}\lambda)$ and $N_{\alpha,v}(E+\varepsilon C\lambda)$. This, or rather the correspondig estimate for the *fibered rotation number* $\rho = \frac{1-N}{2}$, comes from a comparison of the ρ-dependence in two monotonic families of cocycles (constructed by suitable coordinate change, see Remark 3.2). (A related argument appears in the proof of Lemma 3.6 of [AK2].) So $\lambda \mapsto N_{\alpha,v_\lambda}(E)$ is non-constant (near $\lambda = 0$) if and only if $\lambda \mapsto N_{\alpha,v}(E + \lambda)$ is, which happens if and only if E is in the spectrum of $H_{\alpha,v}$, that is, $(\alpha, A^{(E-v)})$ is not uniformly hyperbolic.

$u = \frac{a}{c}$ and $s = \frac{b}{d}$ and depends, say, continuously on z, then

$$B(z+\alpha)^{-1} A^{(v)}(z) B(z) = \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda(z)^{-1} \end{pmatrix}, \tag{16}$$

so $L(\alpha, A^{(v)}_\epsilon) = \int \ln |\lambda(x+\epsilon i)| dx$ for $\xi_-(v) < \epsilon < \xi_+(v)$.[5] Though a, b, c, d are not well defined, $ab, cd, ad + bc$ are and depend holomorphically on z. We let $q(z) = a(z)b(z)$. Notice that $q(z-\alpha) = c(z)d(z)$.

The expression for $DL_{\xi,k}(v)$ in a direction $w \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ is

$$DL_{\xi,k}(v) \cdot w = \Re \int q(x+\epsilon i) w(x+\epsilon i) dx, \quad \xi_-(v) < \epsilon < \xi_+(v). \tag{17}$$

We say that v is directed if $DL_{\xi,k}(v) \cdot w \neq 0$ for every real-symmetric trigonometric polynomial w with $w(x) > 0$ for every $x \in \mathbb{R}/\mathbb{Z}$.

The main step in the proof of Theorem 3 is the following:

Theorem 10. *Assume that v_0 is directed. Then*

1. *The non-tangential limits of u and s exist almost everywhere,*
2. *$\Im u(x)$ and $\Im s(x)$ are non-zero and have the same constant sign almost everywhere,*
3. *$\Re[u(x) - s(x)] > 0$ almost everywhere,*
4. *Let $D(x)$ be the open real-symmetric disk with $u(x), s(x) \in \partial \mathbb{D}$. Then $0 \notin D(x) \cap \mathbb{R}$, but for every $\epsilon > 0$, there exists a positive measure set of x with $D(x) \cap (-\epsilon, \epsilon) > 0$.*

We delay the proof to the next section. We will also need the following result, proved in section 7.

Theorem 11. *Let $v \in C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be non-identically zero. Then there exist a neighborhood $\mathcal{U}$ of $A^{(v)}$ in $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and analytic functions $\Phi : \mathcal{U} \to C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $\Psi : \mathcal{U} \to C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that*

1. *$\Psi(\tilde{A})(x+\alpha) \tilde{A}(x) \Psi(\tilde{A})(x)^{-1} = A^{(\Phi(\tilde{A}))}(x)$,*
2. *If $\tilde{A} = A^{(\tilde{v})}$ for some $\tilde{v}$ then $\Phi(\tilde{A}) = \tilde{v}$ and $\Psi(\tilde{A}) = id$.*

Proof of Theorem 4.

We start with the following simple consequence of Theorem 11.

Lemma 5.1. *There exist analytic families $v_t \in \mathcal{V}$ and $B_t \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, t near 0, such that $B_0 = id$, and for every $x \in \mathbb{R}/\mathbb{Z}$, $B_t(x+\alpha) A^{(v)}(x) B_t(x)^{-1} = A^{(v_t)}(x)$ and $\frac{d}{dt}[B_t(x) \cdot 0] > 0$ at $t = 0$ for every $x \in \mathbb{R}/\mathbb{Z}$.*

Proof. Recall that $v_0 \in C^\omega_{\xi'}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ for some $\xi' > \xi$. Let $b \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be a positive analytic funtion such that bv_0 is a trigonometric polynomial. Then there exists a unique trigonometric polynomial a such that $a(x) + a(x+\alpha) = -b(x)v_0(x)$. Set $c(x) = -b(x-\alpha)$, and let $\eta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Then for small s,

$$e^{s\eta}(x+\alpha) A^{(v_0)}(x) e^{-s\eta}(x) = A^{(v_0 + s\gamma)}(x) + O(s^2), \tag{18}$$

[5]From (16) it follows only that $L(\alpha, A^{(v)}_\epsilon) = |\int \ln |\lambda(x+\epsilon i)| dx|$, but since u is taken as the unstable direction we must have $\int \ln |\lambda(x+\epsilon i)| dx > 0$.

with $\gamma(x) = (a(x+\alpha) - a(x))v_0(x) + b(x+\alpha) - b(x-\alpha)$. By Theorem 11 (notice that v_0 is not identically zero since $\omega(\alpha, A^{(v_0)}) \neq 0$), there exists η_s and γ_s with $\|\eta_s\|_\xi = O(s^2)$ and $\|\gamma_s\|_\xi = O(s^2)$ such that

$$e^{\eta_s}(x+\alpha)e^{s\eta}(x+\alpha)A^{(v_0)}(x)e^{-s\eta}(x)e^{-\eta_s}(x) = A^{(v_0+s\gamma+\gamma_s)}(x), \tag{19}$$

Set $B_t = e^{\eta_t}e^{t\eta}$. Then

$$\frac{d}{dt}[B_t(x) \cdot 0] = b(x) \tag{20}$$

at $t = 0$. ☐

Lemma 5.2. *Let v_t be as in Lemma 5.1. There exists arbitrarily small $t \in \mathbb{R}$ such that v_t is not directed.*

Proof. We may assume that v_0 is directed, so there are disks $D(x)$ defined for almost every $x \in \mathbb{R}/\mathbb{Z}$ as in Theorem 10. Let B_t be as in Lemma 5.1, and let u_t, v_t be the unstable and stable directions for v_t. Notice that $B_t(z)u(z) = u_t(z)$ and $B_t(z) \cdot s(z) = u_s(z)$. By Theorem 10, if v_t is directed for every $|t| < \epsilon$, then for every measurable continuity point x_0 of $x \mapsto D(x)$, $B_t(x_0) \cdot D(x_0)$ must be a disk not containing 0. In particular, we must have $D(x_0) \cap M(x_0) = \varnothing$, where $M(x)$ is the set of all $B_t(x)^{-1} \cdot 0$, $|t| < \epsilon$.

Since there exists $\delta > 0$ such that $(-\delta, \delta) \subset M(x)$ for every x, this contradicts Theorem 10. ☐

Let v_t be as in the conclusion of Lemma 5.2. Since v_t is not directed, there exists a trigonometric polynomial w with $DL_{\xi,k}(v_t) \cdot w = 0$ and $\inf_{x \in \mathbb{R}/\mathbb{Z}} w(x) > 0$. Define an analytic family $A^\lambda \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $\lambda \in \mathbb{R}$, by $A^\lambda(x) = B_t(x+\alpha)^{-1}A^{(v_t+\lambda w)}(x)B_t(x)$, so that $A^0 = A^{(v_0)}$. Let Φ, Ψ be as in Theorem 11 with $v = v_0$ and $\delta = \xi$, and let $\tilde{v}_\lambda = \Phi(A^\lambda)$ and $\tilde{B}_\lambda = \Psi(A^\lambda)B_t^{-1}$ for $\lambda \in \mathbb{R}$ small, so that $\tilde{v}_0 = v_0$. Let $\tilde{w} = \frac{d}{d\lambda}\tilde{v}_\lambda\big|_{\lambda=0}$. and$d = \frac{d}{d\lambda}\tilde{B}_\lambda\Big|_{\lambda=0}\tilde{B}_0^{-1}$. By construction, we have

$$A^{(\tilde{v}_\lambda)}(x) = \tilde{B}_\lambda(x+\alpha)A^{(v_t+\lambda w)}(x)\tilde{B}_\lambda(x)^{-1}, \quad |\Im x| < \xi. \tag{21}$$

Differentiating (21) and then multiplying on the left by $(A^{(v_0)}(x))^{-1}$, we get

$$\begin{pmatrix} 0 & 0 \\ -\tilde{w}(x) & 0 \end{pmatrix} = A^{(v_0)}(x)^{-1}d(x+\alpha)A^{(v_0)}(x) - d(x) + B_t^{-1}(x)\begin{pmatrix} 0 & 0 \\ -w & 0 \end{pmatrix}B_t(x), \tag{22}$$

where $d = \frac{d}{d\lambda}\tilde{B}_\lambda\Big|_{\lambda=0}\tilde{B}_0^{-1}$.

From the definition of $L_{\xi,k}$, (21) implies $L_{\xi,k}(\tilde{v}_\lambda) = L(v_t+\lambda w)$, so that $DL_{\xi,k}(v_0) \cdot \tilde{w} = 0$.

Let us now show that $\begin{pmatrix} 0 & 0 \\ -\tilde{w} & 0 \end{pmatrix}$ is v_0-signed. Since $\inf_{x\in\mathbb{R}/\mathbb{Z}} w(x) > 0$, $\begin{pmatrix} 0 & 0 \\ -w & 0 \end{pmatrix}$ is v_t-signed (see Remark 3.1), so there exists $b \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2,\mathbb{R}))$ such that

$$a(x) = A^{(v_t)}(x)^{-1}b(x+\alpha)A^{(v_t)}(x) - b(x) + \begin{pmatrix} 0 & 0 \\ -w & 0 \end{pmatrix} \tag{23}$$

is signed, i.e., $\det a(x) > 0$. Notice that (23) gives

$$B_t(x)^{-1}a(x)B_t(x) = A^{(v_0)}(x)^{-1}B_t(x+\alpha)^{-1}b(x+\alpha)B_t(x+\alpha)A^{(v_0)}(x) - B_t(x)^{-1}b(x)B_t(x) + B_t(x)^{-1}\begin{pmatrix} 0 & 0 \\ -w(x) & 0 \end{pmatrix}B_t(x). \tag{24}$$

Let

$$\tilde{b} = B_t^{-1}bB_t - d \tag{25}$$

and let

$$\tilde{a}(x) = A^{(v_0)}(x)^{-1}\tilde{b}(x+\alpha)A^{(v_0)}(x) - \tilde{b}(x) + \begin{pmatrix} 0 & 0 \\ -\tilde{w} & 0 \end{pmatrix}. \tag{26}$$

Putting together (22), (24), (25) and (26), we get that $\tilde{a} = B_t^{-1}aB_t$, so that $\det \tilde{a} = \det a$. In particular, $\tilde{a}$ is signed, so by (26), $\tilde{w}$ is v_0-signed, as desired.

While $\tilde{w}$ is not necessarily a trigonometric polynomial, it can be approximated by a trigonometric polynomial in the kernel of $DL_{\xi,k}(v_0)$, which will be v_0-signed as well (since v_0-signedness is obviously an open condition). □

6. When the derivative of the Lyapunov exponent is a measure

In this section we prove Theorem 10. We let $v = v_0$ and $A = A^{(v)}$ for simplicity of notation.

The starting observation is that if v is directed then $DL_{\xi,k}(v)$ extends to a functional on $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ with norm $|DL_{\xi,k}(v) \cdot 1|$, which is either non-negative or non-positive on positive functions. By the Riesz representation Theorem, it is given by a measure with finite mass μ on $\mathbb{R}/\mathbb{Z}$. By (17), this means that for any $w \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ we have

$$\Re \int q(x+\epsilon i)w(x+\epsilon i)dx = \int w(x)d\mu(x), \quad 0 < \epsilon < \xi_0. \tag{27}$$

We will assume from now on that μ is non-negative, the other case being analogous.

Our plan is to show that the non-negativity of μ leads to good estimates for q which imply one of two conclusions:

(C1) Either u or s extend analytically through $\mathbb{R}/\mathbb{Z}$.
(C2) The conclusion of Theorem 10 holds.

Let us first show that (C1) implies $\omega(\alpha, A) = 0$, which contradicts the standing hypothesis that $v \in \mathcal{C}^k$.

Assume for simplicity that u extends analytically, then either $u(x) \in \overline{\mathbb{R}}$ for every $x \in \mathbb{R}/\mathbb{Z}$ or $u(x) \notin \overline{\mathbb{R}}$ for every $x \in \mathbb{R}/\mathbb{Z}$ (since the $\mathrm{SL}(2,\mathbb{R})$ action preserves $\overline{\mathbb{R}}$ and $x \mapsto x + \alpha$ is minimal).

If $u(x) \notin \overline{\mathbb{R}}$ for $x \in \mathbb{R}/\mathbb{Z}$, this holds still for $\Im z > 0$ small. In this case we can select $a = u$, $c = 1$ when defining $B(z)$, and it follows that $\lambda = u$, so $L(\alpha, A_\epsilon) = \int \ln |u(x+\epsilon i)|dx$ is independent of ϵ small (argument of u being always different from $k\pi$, $k \in \mathbb{Z}$), thus $\omega(\alpha, A) = 0$.

If $u(x) \in \overline{\mathbb{R}}$, we can use u to define analytic functions $A' : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ and $B' : \mathbb{R}/\mathbb{Z} \to \mathrm{PSL}(2,\mathbb{R})$ such that $A'(x)$ is upper triangular and $B'(x+\alpha)^{-1}A(x)B'(x) = A'(x)$: take the first column of B' parallel to $\begin{pmatrix} u \\ 1 \end{pmatrix}$. We have $\omega(\alpha, A') = \omega(\alpha, A)$, and we just need to show that the $\omega(\alpha, A') = 0$. But if

$A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$ then $L(\alpha, A'_\epsilon) = \int \ln |a'(x + \epsilon i)| dx$ for $\epsilon > 0$ small. This is independent of ϵ since for z near $\mathbb{R}/\mathbb{Z}$ the argument of $a'(z)$ is always different from $\frac{2k+1}{2}\pi$, $k \in \mathbb{Z}$. We conclude that $\omega(\alpha, A') = 0$.

The remaining of this section is dedicated to showing that one of (C1) or (C2) always holds.

6.1. **Non-tangential limits and analytic continuation.** Recall that for any bounded holomorphic function $f : \mathbb{D} \to \mathbb{C}$, the non-tangential limits $f(z) = \lim_{r \to 1-} f(rz)$ exist for almost every $z \in \partial\mathbb{D}$ (see, e.g., [G]), and the Poisson formula holds: $f(0) = \int_0^1 f(e^{2\pi i\theta}) d\theta$. Applying appropriate conformal maps, we see that if $U \subset \mathbb{C}$ is any real-symmetric domain, and $f : U \cap \mathbb{H} \to \mathbb{C}$ is a holomorphic function which is either bounded, or takes values on $\mathbb{H}$, or takes values on $\mathbb{C} \setminus (-\infty, 0]$, the non-tangential limits $f(x) = \lim_{\epsilon \to 0+} f(x + \epsilon i)$ also exist for almost every $x \in U \cap \mathbb{R}$.

We will use the following simple version of the Schwarz Reflection Principle.

Proposition 6.1. *Let U be a real-symmetric domain and let $f : U \cap \mathbb{H} \to \mathbb{C}$ be holomorphic. Then*

1. *If f takes values in $\mathbb{H}$ and the non-tangential limits at $U \cap \mathbb{R}$ are almost surely imaginary then f extends analytically to a function on U, and $f(\overline{z}) = -\overline{f(z)}$,*
2. *If $f : U \cap \mathbb{H} \to \mathbb{C} \setminus (-\infty, 0]$ is a holomorphic function whose non-tangential limits at $U \cap \mathbb{R}$ are almost surely real then f extends analytically to a function on U, and $f(\overline{z}) = \overline{f(z)}$.*

Proof. Assume that $\Re f(x) = 0$ (respectively, $\Im f(x) = 0$ and $\Re f(x) > 0$) for almost every $x \in U \cap \mathbb{R}$. Let $\phi : \mathbb{H} \to \mathbb{D}$ (respectively, $\phi : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{D}$) be a conformal map commuting with the symmetry about the imaginary axis (respectively, real axis). Then $\phi \circ f$ is bounded and its non-tangential limits are imaginary (real). Thus the usual Schwarz Reflection Principle applies.[6] Since $\phi \circ f$ extends, the same holds for f. □

6.2. **Initial estimates on q.** Let us write $q(z) = if(z) + g(z)$ with f analytic and real-symmetric for $x \in \mathbb{R}/\mathbb{Z}$ and a holomorphic function g with $\hat{g}_k = 0$ for $k \leq -1$. Thus g is defined on $\Im z > 0$ and is bounded at ∞.

Lemma 6.2. *We have $\Re g(z) \geq 0$ for every z such that $\Im z > 0$.*

Proof. Let $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a positive C^∞ function with $\hat{\phi}_0 = 1$. Let $g^\phi(z) = \int g(z + x)\phi(x) dx$. It suffices to show that $\Re g^\phi(z) \geq 0$ for every such ϕ. Let $h^\phi(x) = \int \phi(y - x) d\mu(y)$, which is a non-negative C^∞ function. For any real

[6] The Schwarz Reflection Principle is usually stated assuming continuity at the boundary, the version for bounded holomorphic functions following immediately (as we can consider convolution approximations satisfying the continuity requirement). See also [G], Exercise 12, page 95.

symmetric trigonometric polynomial w and any $\epsilon > 0$, we have

$$\begin{aligned}(28)\qquad \int \Re[g^\phi(x+\epsilon i)w(x+\epsilon i)]dx &= \int\int \Re[g(x+y+\epsilon i)\phi(y)w(x+\epsilon i)]dydx \\ &= \int \phi(y)\int \Re[g(x+y+\epsilon i)w(x+\epsilon i)]dxdy \\ &= \int \phi(y)\int \Re[g(x'+\epsilon i)w(x'-y+\epsilon i)]dx'dy \\ &= \int \phi(y)\int w(x'-y)d\mu(x')dy \\ &= \int\int \phi(y)w(x'-y)dyd\mu(x') \\ &= \int\int \phi(x'-y)w(y)dyd\mu(x') \\ &= \int\int \phi(x'-y)d\mu(x')w(y)dy \\ &= \int h^\phi(y)w(y)dy,\end{aligned}$$

where the first identity uses the definition of g^ϕ, the fourth is (27), and the last is the definition of h^ϕ. There exists a bounded holomorphic function H^ϕ on $\Im z > 0$ which extends smoothly to $\Im z \geq 0$ and satisfies $\Re H^\phi(x) = h^\phi(x)$ (constructed with the help of the Hilbert Transform). Obviously $H^\phi(z) > 0$ on $\Im z > 0$ by the Poisson formula. If w is a real symmetric trigonometric polynomial, we have

$$(29)\qquad \int \Re[H^\phi(x+\epsilon i)w(x+\epsilon i)]dx = \int h^\phi(x)w(x)dx$$

for every $\epsilon > 0$. Since both $\hat{H}^\phi_k = \hat{g}^\phi_k = 0$ for every $k \leq -1$, this implies that $\hat{H}^\phi_k = \hat{g}^\phi_k$ for every $k \geq 1$ and $\Re\hat{H}^\phi_0 = \Re\hat{g}^\phi_0$. Thus $g^\phi - H^\phi$ is a purely imaginary constant and $\Re g^\phi(z) = \Re H^\phi(z) > 0$ for $\Im z > 0$. $\square$

Since g takes values in a half plane, it admits non-tangential limits. This allows us to make conclusions for q as well, so that for almost every $x \in \mathbb{R}/\mathbb{Z}$ the non-tangential limits $q(x) = \lim_{\epsilon\to 0} q(x+\epsilon i)$ exist and satisfy $\Re q(x) \geq 0$. Notice that $\Re g(x) \in L^1$ (since $\Re g(z) > 0$),[7] and hence $\Re q(x) \in L^1$.

6.3. **The general case.** By quick computation, we conclude that limits also exist, almost everywhere, for the unstable and the stable directions. Indeed, from $q(x) = a(x)b(x)$, $q(x-\alpha) = a(x-\alpha)b(x-\alpha) = c(x)d(x)$, we get

$$(30)\qquad q(x) = \frac{u(x)s(x)}{u(x)-s(x)} \text{ and } q(x-\alpha) = \frac{1}{u(x)-s(x)},$$

from which we conclude that

$$(31)\qquad 1 + 4q(x)q(x-\alpha) = \left(\frac{u(x)+s(x)}{u(x)-s(x)}\right)^2.$$

Assume the non-tangential limits of q exist at x and $x-\alpha$ and are finite. If $q(x-\alpha) \neq 0$ then $u(x) - s(x) = \frac{1}{q(x-\alpha)}$ and $u(x)s(x) = \frac{q(x)}{q(x-\alpha)}$ define u and s

[7]Indeed, by pointwise convergence, $\int |\Re g(x)|dx$ is at most $\lim_{\epsilon\to 0} |\Re g(x+\epsilon i)|dx$, and since $\Re g(z) > 0$ we have $\int |\Re g(x+\epsilon i)|dx$ constant equal to $\Re\hat{g}_0$.

uniquely up to a choice of sign for the $\sqrt{1+4q(x)q(x-\alpha)}$. So the set of non-tangential accumulation values for each of u and s has one or two points, and since it must be connected the non-tangential limit must be well defined. If $q(x-\alpha)=0$, then as z approaches x non-tangentially, either $u(z)$ is close to ∞ and $s(z)$ is close to $q(x)$, or $s(z)$ is close to ∞ and $u(z)$ is close to $-q(x)$. By the same argument as before, the non-tangential limits of u and s also exist in this case. In either case, we also conclude that the existence and finiteness of the nontangential limits of $q(x)$ at x and $x-\alpha$ imply that $s(x) \neq u(x)$. Moreover, u and s must be finite almost everywhere by the following:

Lemma 6.3. *Let $w : \mathbb{R}/\mathbb{Z} \to \overline{\mathbb{C}}$ be measurable and satisfy $A(x) \cdot w(x) = w(x+\alpha)$. Then $w(x) \neq \infty$ almost everywhere.*

Proof. Otherwise, there would exist $k, l > 0$ and a positive measure set $X \subset \mathbb{R}/\mathbb{Z}$ such that $w(x)$ and $w(x+k\alpha) = \infty$, $w(x+(kl+1)\alpha) = \infty$ for every $x \in X$. It follows from analyticity that $A_k(x) \cdot \infty = \infty$ and $A_{kl+1}(x) \cdot \infty = \infty$ for every x. Thus $A(x) \cdot \infty = \infty$ for every x, which is impossible since $A(x) \cdot \infty = v(x)$. □

Consider now the following possibilities:

1. $s(x) = \overline{u}(x) \notin \overline{\mathbb{R}}$ for almost every $x \in \mathbb{R}/\mathbb{Z}$,
2. $s(x), u(x) \in \overline{\mathbb{R}}$ for almost every $x \in \mathbb{R}/\mathbb{Z}$,
3. $\overline{s}(x) \neq u(x)$ and $m(x) \in \mathbb{H}$ for almost every $x \in \mathbb{R}/\mathbb{Z}$, for some choice of $m = u, s, \overline{u}, \overline{s}$ (independent of x).

Those possibilities exhaust all cases since $x \mapsto x+\alpha$ is ergodic and $A(x)$ preserves $\mathbb{H}$, $x \in \mathbb{R}/\mathbb{Z}$. We will deal now with the first two cases, and leave the third case for the next section.

In the first case, assuming, say, that $s(x) \in \mathbb{H}$, we have $\Re q(x) = 0$ and $\Im q(x) > 0$ for almost every $x \in \mathbb{R}/\mathbb{Z}$. Consider a decomposition $q = if + g$ with f real-symmetric and g holomorphic on $\mathbb{H}$ and bounded at ∞. We may also assume that $f(x) < 0$ for $x \in \mathbb{R}/\mathbb{Z}$. As we saw, $\Re g \geq 0$ on $\Im z > 0$ and now we also get that the non-tangential limits satisfy $\Re g(x) = 0$ and $\Im g(x) > 0$ for almost every x. By Proposition 6.1, $-ig$ admits an analytic continuation. This implies successively that q, u and s also admit analytic continuations, so we have reached conclusion (C1).

In the second case, $\Im q(x) = 0$ for $x \in \mathbb{R}/\mathbb{Z}$. Consider a decomposition $q = f + g$ with f analytic real symmetric, $f(x) < 0$ for $x \in \mathbb{R}/\mathbb{Z}$, and g holomorphic on $\Im z > 0$ and bounded at ∞. By comparison with the decomposition considered before, $\Re g > 0$ on $0 < \Im z < \epsilon$. Since $\Im q(x) = 0$, $\Im g(x) = 0$ for almost every $x \in \mathbb{R}/\mathbb{Z}$, and by Corollary 6.1 ig admits an analytic continuation. Hence q, u and s also admit analytic continuations, so we have reached conclusion (C1).

6.4. **Many sections.** We consider now the third case. We will assume that we can take $m = u$, the other possibilities being analogous. Notice that (α, A) admits at least three invariant sections $u, s, \overline{u}$.[8]

Lemma 6.4. $\Re q(x) > 0$ *for almost every x.*

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[8]This implies that there exists a measurable function $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ such that $B(x+\alpha)A(x)B(x)^{-1} = \pm id$.

Proof. Notice that $\Re q(x) = 0$ implies that either $s(x+\alpha) = \infty$ or $u(x+\alpha) - s(x+\alpha)$ is purely imaginary and hence $\Im u \neq |\Im s|$. Let us show that the sets $X_\pm$ of $x \in \mathbb{R}/\mathbb{Z}$ with $\Re q(x) = 0$ and $\pm \Im u(x) > \pm |\Im s(x)|$ have zero Lebesgue measure.

If $X_\pm$ has positive measure then there exist $k, l > 0$ and a positive measure set of $x \in \mathbb{R}/\mathbb{Z}$ such that $x, x+k\alpha, x+(kl+1)\alpha \in X_\pm$. It follows that $A_k(x+\alpha) \cdot \infty = \infty$ and $A_{kl+1}(x+\alpha) \cdot \infty = \infty$.[9] Since this happens for a positive measure set of x, this implies that $A_k(x) \cdot \infty = \infty$, $A_{kl+1}(x) \cdot \infty = \infty$, and hence $A(x) \cdot \infty = \infty$, hold for every $x \in \mathbb{R}/\mathbb{Z}$. But $A(x) \cdot \infty = v(x) \neq \infty$, contradiction. $\square$

For real x, consider the real-symmetric open disk $D(x)$ containing u and s at the boundary. If $0 \in D$, then $\Re[u(x) - s(x)] > 0$ implies $\Re[u(x+\alpha) - s(x+\alpha)] < 0$, contradiction. So $0 \notin D$ for almost every x.

In order to show that (C2) holds, it remains to check that for every $\epsilon > 0$, there exists a positive measure set of $x \in \mathbb{R}/\mathbb{Z}$ such that $D(x)$ intersects $(-\epsilon, \epsilon)$.

Assume that this is not the case. Then $D(x) \cap \mathbb{R} \subset [-C, C]$, where $C = \frac{1}{\epsilon} + \|v\|_0$. We claim that there exists $\epsilon' > 0$ such that $\Re q(x-\alpha) > \epsilon'$ for almost every $x \in \mathbb{R}/\mathbb{Z}$. There are three cases to consider:

1. $\Im s(x) = 0$. Then $\Re q(x-\alpha)$ is the inverse of the diameter of D, so $\Re q(x-\alpha) = 1/(2C)$,
2. $\Im s(x) > 0$. Then $\Re q(x-\alpha)$ is the inverse of the diameter of the real-symmetric disk through $u(x) - \Im s(x)$ and $s(x) - \Im s(x)$, which is bigger than the diameter of of D, so we get $\Re q(x-\alpha) > 1/(2C)$,
3. $\Im s(x) < 0$. Then $\Re q(x-\alpha) = \frac{1}{u(x) - \overline{s(x)}} \frac{|u(x)-\overline{s(x)}|^2}{|u(x)-s(x)|^2}$. We have $\frac{1}{\overline{u(x)-\overline{s(x)}}} > \frac{1}{2C}$ as in the previous case, so we just have to show that $\frac{|u(x)-\overline{s(x)}|}{|u(x)-s(x)|}$ is bounded from below. This is equivalent to showing that $\frac{|u(x)-\overline{s}(x)|}{2\Im u(x)}$ is bounded from below, which is equivalent to showing that the hyperbolic distance in $\mathbb{H}$ between $u(x)$ and $\overline{s(x)}$, $d(x) > 0$, is bounded from below. Since the hyperbolic metric is invariant by the $\mathrm{SL}(2,\mathbb{R})$ action, $d(x) = d(x+\alpha)$ for almost every $x \in \mathbb{R}/\mathbb{Z}$, so by ergodicity $d(x)$ is constant.

It follows that $\Re q(z)$ is bounded away from 0 for every z with $0 < \Im z < \delta$ (δ small). Thus we can define $t(z) = \sqrt{1 + 4q(z)q(z-\alpha)}$, $\Re t(z) > 0$ for every z with $0 < \Im z < \delta$.[10] Thus

$$u(x) = \frac{\pm t(x) + 1}{2q(x-\alpha)}, \tag{32}$$

$$s(x) = \frac{\pm t(x) - 1}{2q(x-\alpha)}. \tag{33}$$

We have $t(x) = \pm \frac{u(x)+s(x)}{u(x)-s(x)}$. Notice that $\Re t = \pm \frac{|u|^2 - |s|^2}{|u-s|^2}$, so $\Re t > 0$ (by the choice of t) implies $\pm |u| > \pm |s|$. Notice that $\Re[u(x) - s(x)] > 0$ implies, together with $\pm |u| > \pm |s|$, that $\pm \Re u(x), \pm \Re s(x) > 0$.[11]

Thus for almost every $x \in \mathbb{R}/\mathbb{Z}$, $\pm D(x)$ is contained in the right half plane.

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[9] If $z_1, z_2 \in \mathbb{C}$ and $B \in \mathrm{SL}(2,\mathbb{R})$ are such that $\Re z_1 = \Re z_2$, $\Re[B \cdot z_1] = \Re[B \cdot z_2]$, $\pm|\Im z_2| < \pm \Im z_1$ and $\pm |\Im B \cdot z_2| < \pm \Im B \cdot z_1$, then $B \cdot \infty = \infty$.

[10] Notice that the arguments of $q(z)$ and $q(z-\alpha)$ can be taken in $(-\pi/2, \pi/2)$, hence the argument of $q(z)q(z-\alpha)$ can be taken in $(-\pi, \pi)$, so that $1 + 4q(z)q(z-\alpha) \notin (-\infty, 1]$.

[11] If $\pm D(x) \subset \{\Re z > 0\}$, we have $s(x) \in \partial D(x) \cap \{\Re z < u(x)\} \subset \partial D(x) \cap \{\pm |z| < \pm u(x)\}$, so $\pm |u(x)| > \pm |s(x)|$.

Let us assume that $D(x)$ is contained in the right half plane, so that $D(x) \cap \mathbb{R} \subset (\epsilon', C)$.

Let $\epsilon' \leq z^-(x) < z^+(x) \leq C$ be the extremes of $D(x) \cap \mathbb{R}$. Notice that

$$\ln \|A_n(x) \cdot \begin{pmatrix} z^\pm(x) \\ 1 \end{pmatrix}\| \geq c \sum_{k=0}^{n-1} \ln z^\pm(x+k\alpha), \tag{34}$$

where $c > 0$ depends only on ϵ' and c. Since the Lyapunov exponent is 0, we must have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k-1} \ln z^\pm(x+n\alpha)| = 0, \tag{35}$$

so that

$$\int_{\mathbb{R}/\mathbb{Z}} \ln z^\pm(x) dx = 0 \tag{36}$$

which is impossible since $z^+(x) > z^-(x)$ almost everywhere.

The case where $-D(x)$ is contained in the right half plane is analogous.

The proof of Theorem 10 is complete. □

7. Conjugating SL(2, ℝ) perturbations to Schrödinger form

Let $d \geq 1$ be an integer and let $\delta \in \mathbb{R}^d_+$. We let $\Omega_\delta = \{z \in \mathbb{C}^d/\mathbb{Z}^d, |\Im z_k| < \delta_k\}$, and let $C^\omega_\delta(\mathbb{R}^d/\mathbb{Z}^d, *)$ stand for spaces of analytic functions on $\mathbb{R}^d/\mathbb{Z}^d$ with continuous extensions to $\overline{\Omega}_\delta$ which are holomorphic on Ω_δ.

We will prove the following generalization of Theorem 11 to arbitrary dimensions.

Theorem 12. *Let $v \in C^\omega_\delta(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{R})$ be non-identically zero. There exists $\epsilon > 0$ such that if $A' \in C^\omega_\delta(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$ satisfies $\|A' - A^{(v)}\|_{C^\omega_\delta} < \epsilon$, then there exists $v' \in C^\omega_\delta(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{R})$ and $B' \in C^\omega_\delta(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{SL}(2,\mathbb{R}))$, depending analytically on A' and such that $B'(x+\alpha)A'(x)B'(x)^{-1} = A^{(v')}(x)$. Moreover, if A' is already of the form $A^{(\tilde{v})}$, then $v' = \tilde{v}$ and $B = id$.*

A version of this result, for smooth cocycles over more general dynamical systems, was obtained in [ABD]. The proof of [ABD] makes use of partitions of unity to localize perturbations to some small region with disjoint first few iterates, one then tries to define functions in disjoint closed regions of space without worrying about interaction. The only additional care is to select the localizing region away from the critical locus $v(x+\alpha) = 0$, where the relevant equations develop singularities. Our approach is different: we take a disconnected finite cover of the dynamical system to realize the non-interacting condition, and concentrate on the linearized version of the problem, which can be broken up into several subproblems each of which involves a perturbation "dominated" by $v(x+\alpha)$ in such a way to compensate the singularity.

Proof. Let $A = A^{(v)}$. Writing $A' = Ae^{s'}$, $B = e^w$ and $v' = v + t'$, we see that the linearized form of the problem is: For $s' \in C^\omega_\delta(\mathbb{R}^d/\mathbb{Z}^d, \mathrm{sl}(2,\mathbb{R}))$, solve the equation

$$A(x)^{-1} w(x+\alpha) A(x) + s'(x) - w(x) = t'(x) L, \tag{37}$$

where L stands for $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. We will show below how to obtain a solution (w, t') of (37), linear in s', and satisfying $\|w\|_{C^\omega_\delta} \leq C \|s'\|_{C^\omega_\delta}$ for some $C = C(v) > 0$.

Moreover, $C(v')$ will be uniformly bounded in a neighborhood of v. This allows one to construct the solution of the nonlinear problem by, say, Newton's method.

Let $s' = \begin{pmatrix} s_1' & s_2' \\ s_3' & -s_1' \end{pmatrix}$, and $s = s' + s_3' L$, $t = t' + s_3'$. Then (37) is equivalent to

$$A(x)^{-1} w(x+\alpha) A(x) + s(x) - w(x) = t(x) L. \tag{38}$$

We will in fact construct a solution (w, t) to (38) which is linear in s and satisfies the required bounds. Notice that when A' is already of the form $A^{(\tilde{v})}$, then $s = 0$, so $w = 0$, and the iterative procedure yields $v' = \tilde{v}$, $B = id$.

Choose $N \geq 4$ such that $|\sum_{k=2}^{N-2} v(x+k\alpha)^2| > 1$ for every $x \in \Omega_\delta$.[12]

Notice that N is constant in a neighborhood of v. Write

$$s_k(x) = \frac{v(x+k\alpha)^2}{\sum_{j=2}^{N-2} v(x+j\alpha)^2} s(x), \quad 2 \leq k \leq N-2. \tag{39}$$

Let us show that there are functions $w_{k,l}$, $2 \leq k \leq N-2$, $l \in \mathbb{Z}_N$ and $t_{k,l}(x)$, $l = k-1, k, k+1$, such that

1. $w_{k,0} = 0$,
2. $A(x)^{-1} w_{k,1}(x+\alpha) A(x) + s_k(x) - w_{k,0}(x) = 0$,
3. $A(x)^{-1} w_{k,l+1}(x+\alpha) A(x) - w_{k,l}(x) = t_{k,l}(x) L$, $l = k-1, k, k+1$,
4. $A(x)^{-1} w_{k,l+1}(x+\alpha) A(x) - w_{k,l}(x) = 0$, $l \neq 0, k-1, k, k+1$.

If we then set $w(x) = \sum_{k,l} w_{k,l}(x)$ and $t(x) = \sum_{2 \leq k \leq N-2} \sum_{l=k-1}^{k+1} t_{k,l}(x)$, we will have $A(x)^{-1} w(x+\alpha) A(x) + s(x) - w(x) = t(x) L$.

Conditions (1,2,4) clearly define all $w_{k,l}$ except $w_{k,k}$ and $w_{k,k+1}$, in particular

$$w_{k,k-1}(x) = -A_{k-1}(x-(k-1)\alpha) s_k(x-(k-1)\alpha) A_{k-1}(x-(k-1)\alpha)^{-1}. \tag{40}$$

Using (39) we see that

$$\|w_{k,k-1}(x)\| \leq C |v(x+\alpha)|^2 \|s(x-(k-1)\alpha)\|. \tag{41}$$

The key equations are thus

$$A(x)^{-1} w_{k,k}(x+\alpha) A(x) - w_{k,k-1}(x) = t_{k,k-1}(x) L, \tag{42}$$

$$A(x)^{-1} w_{k,k+1}(x+\alpha) A(x) - w_{k,k}(x) = t_{k,k}(x) L, \tag{43}$$

$$-w_{k,k+1}(x) = t_{k,k+1}(x) L. \tag{44}$$

From this we get an equation only involving unknown t's,

$$\begin{aligned} -w_{k,k-1}(x) =& t_{k,k-1}(x) L + A(x)^{-1} t_{k,k}(x+\alpha) L A(x) \\ &+ A(x)^{-1} A(x+\alpha)^{-1} t_{k,k+1}(x) L A(x+\alpha) A(x). \end{aligned} \tag{45}$$

Once t's are known satisfying (45), getting the w's is immediate, so from now on we try to solve (45). Rewriting this equation we get

$$-w_{k,k-1}(x) = t_{k,k-1}(x) L + t_{k,k} L_1(x) + t_{k,k+1} L_2(x) \tag{46}$$

where

$$L_1(x) = \begin{pmatrix} -v(x) & 1 \\ -v(x)^2 & v(x) \end{pmatrix} \tag{47}$$

[12]By unique ergodicity of $x \mapsto x + \alpha$ on $\mathbb{R}^d/\mathbb{Z}^d$, the Birkhoff averages of $v(z)^2$ converge uniformly to $\int_{\mathbb{R}^d/\mathbb{Z}^d} v(z+x)^2 dx$, which equals $\int_{\mathbb{R}^d/\mathbb{Z}^d} v(x)^2 dx > 0$ by holomorphicity.

and

$$L_2(x) = \begin{pmatrix} v(x+\alpha) - v(x)v(x+\alpha)^2 & v(x+\alpha)^2 \\ -(1-v(x)v(x+\alpha))^2 & -v(x+\alpha)+v(x)v(x+\alpha)^2 \end{pmatrix}. \tag{48}$$

Thus

$$L_1(x) - v(x)^2 L = \begin{pmatrix} -v(x) & 1 \\ 0 & v(x) \end{pmatrix} \tag{49}$$

and

$$L_2(x) - v(x+\alpha)^2 L_1(x) + (2v(x)v(x+\alpha)-1)L = \begin{pmatrix} v(x+\alpha) & 0 \\ 0 & -v(x+\alpha) \end{pmatrix}. \tag{50}$$

We conclude that if $v(x+\alpha) \neq 0$ then L, $L_1(x)$ and $L_2(x)$ span $\mathrm{sl}(2,\mathbb{C})$, and there exists a unique solution $(t_{k,k-1}, t_{k,k}, t_{k,k+1})$ of (45), in fact bounded by $C\frac{\|w_{k,k-1}(x)\|}{|v(x+\alpha)|}$. As mentioned before, the singularity that seems to arise when $v(x+\alpha) = 0$ was well understood to be one source of difficulties in this problem, but here it emerges from (41) that whenever $v(x+\alpha) \neq 0$, the solutions are bounded by a constant times $|v(x+\alpha)|$. Hence they extend continuously as zero to $\{v(x+\alpha)=0\}$, and by holomorphic removability we conclude holomorphicity in Ω_δ. The result follows. □

Appendix A. Coexistence near critical coupling

Here we show that perturbations of the critical almost Mathieu operator (with potential $v(x) = 2\cos 2\pi x$) may exhibit coexistence of subcritical and supercritical energies, and in fact that one may have arbitrarily many alternances between subcritical and supercritical regimes. As far as we know, all previous examples of coexistence present only a small number of alternances.

Theorem 13. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $n \geq 1$, and let $\{E_j\}_{j=1}^n$ be n distinct points in $\Sigma_{\alpha,v}$. Then for any $\delta > 0$, there exists a trigonometric polynomial $w \in C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ such that for every $\kappa \neq 0$ sufficiently small, and for every $1 \leq j \leq n$, there exists $E_j^\kappa \in \Sigma_{\alpha,v+\kappa w} \cap (E_j - \delta, E_j + \delta)$ such that E_j^κ is subcritical if $(-1)^j \kappa > 0$ and E_j^κ is supercritical if $(-1)^j \kappa < 0$.*

Proof. For $H_{\alpha,v}$, all energies in the spectrum are critical, with zero Lyapunov exponent and acceleration 1 (see the appendix of [A1]). For $E \in \mathbb{C} \setminus \Sigma_{\alpha,v}$, $(\alpha, A^{(E-v)})$ is uniformly hyperbolic (this is general) with zero acceleration (this is obvious for real energies and can be analytically continued to complex energies).

As for $(\alpha, A_\epsilon^{(E-v)})$, it is uniformly hyperbolic with zero acceleration for $0 < \epsilon < \frac{L(E)}{2\pi}$ and uniformly hyperbolic with acceleration 1 for $\epsilon > \frac{L(E)}{2\pi}$ (here $L(E) = L(\alpha, A^{(E-v)})$). This follows from the asymptotic estimate $L(\alpha, A_\epsilon^{(E-v)}) = 2\pi\epsilon$ for $\epsilon >> 1$ (see the proof of Theorem 10 of [A1]). Particularly, for $E \in \mathbb{C} \setminus \Sigma_{\alpha,v}$, $(\alpha, A_\epsilon^{(E-v)})$ is not uniformly hyperbolic for $\epsilon = \frac{L(E)}{2\pi}$.

Let U be the set of all E such that $L(\alpha, A^{(E-v)}) < 1$. It is an open neighborhood of $\Sigma_{\alpha,v}$.

Following section 5, for $E \in U$, define a holomorphic function q^E, on $\Im x > \frac{1}{2\pi}$ by $q^E(x) = a^E(x) b^E(x)$ where $B^E = \begin{pmatrix} a^E & b^E \\ c^E & d^E \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ has columns parallel to the unstable and the stable directions of $(\alpha, A^{(E-v)})$. Notice that q^E is holomorphic with respect to (E, x), and for each $E \in U$, q^E admits holomorphic extensions up to $\Im x > \frac{L(E)}{2\pi}$, so when $E \in \Sigma_{\alpha,v}$, q^E is defined in the entire upper half plane $\mathbb{H}$.

Fix some $1 \leq \xi_0 < 2$ and let $\mathcal{V} \subset C^\omega_2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be an open neighborhood of all $E-v$, $E \in U$ such that for every $v' \in \mathcal{V}$ the cocycle $(\alpha, A^{(v')}_{\xi_0})$ is uniformly hyperbolic with acceleration 1. Define $L_{2,1} : \mathcal{V} \to \mathbb{R}$ by $L_{2,1}(v') = L(\alpha, A^{(v')}_{\xi_0}) - 2\pi\xi_0$.

For $E \in U$ and $w \in C^\omega_2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, the derivative of $t \mapsto L_{2,1}(E - v - tw)$ at $t = 0$ is given by

$$-\int_{\mathbb{R}/\mathbb{Z}} \Re q^E(x + \epsilon i)w(x + \epsilon i)dx, \tag{51}$$

where ϵ can be chosen arbitrary with $\frac{L(E)}{2\pi} < \epsilon < 2$ (see section 5). Denote by Φ^E the (bounded) linear functional on $C^\omega_2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ taking w to (51).

We claim that for every finite subset $\mathcal{E} \subset \Sigma_{\alpha,v}$, and any $E_* \in \Sigma_{\alpha,v}$, there exists $E' \in \Sigma_{\alpha,v}$ arbitrarily close to E_* such that $\Phi^{E'}$ is not a linear combination of the $\{\Phi^E\}_{E\in\mathcal{E}}$. Once this has been done, one can obtain inductively points $E'_j \in \Sigma_{\alpha,v} \cap (E_j - \frac{\delta}{2}, E_j + \frac{\delta}{2})$ and a trigonometric polynomial $w \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that $(-1)^j \Phi^{E'_j} \cdot w < 0$. Choose $0 < r < \frac{\delta}{2}$ such that $E' \in U$ for every $E' \in K_j = [E'_j - r, E'_j + r]$ and moreover $(-1)^j \Phi^{E'} \cdot w < 0$. Then for $\kappa \neq 0$ small, and every $E' \in K_j$ we have $\kappa(-1)^j L_{2,1}(E' - v - \kappa w) < 0$. Notice that if $\kappa \neq 0$ is small then for every $E' \in \Sigma_{\alpha,v+\kappa w}$ we have:

1. If $L_{2,1}(E' - v - \kappa w) > 0$ then E' is supercritical for $H_{\alpha,v+\kappa w}$,
2. If $L_{2,1}(E' - v - \kappa w) < 0$ then E' is subcritical for $H_{\alpha,v+\kappa w}$.

Indeed, in the first case, we just use that $L \geq L_{2,1}$ (by convexity), and in the second case we notice that we must have $L \geq 0 > L_{2,1}$, so $\omega(\alpha, A^{(E'-v-\kappa w)}) < 1$ hence by quantization $\omega(\alpha, A^{(E'-v-\kappa w)}) = 0$ and $(\alpha, A^{(E'-v-\kappa w)})$ is regular. The result then follows since for every κ small, $\Sigma_{\alpha,v+\kappa w}$ intersects each of the $\mathrm{int}K_j$ (as $\kappa \mapsto \Sigma_{\alpha,v+\kappa w}$ is continuous in the Hausdorff topology).

To conclude, let us prove the claim. Note that by Theorem 8 of [A1], $\Phi^{E_*} \neq 0$, which in particular implies the claim when $\mathcal{E}$ is empty. We may assume that the Φ^E, $E \in \mathcal{E}$, are linearly independent. If the claim does not hold, then for every $E' \in \Sigma_{\alpha,v}$ close to E_*, $\Phi^{E'}$ is a linear combination of Φ^E, $E \in \mathcal{E}$, so that we can write (in a unique way), $\Phi^{E'} = \sum_{E\in\mathcal{E}} c_E(E')\Phi^E$. Note that the coefficients $c_E(E')$, originally defined for E' near E_* in $\Sigma_{\alpha,v}$, coincide with restrictions of real analytic functions defined in a small open interval I_* around E_*, which we still denote by $c_E(E')$.[13] Let $D \subset U$ be a small disk around E_* with $D \cap \mathbb{R} \subset I_*$ and such that the $E' \mapsto c_E(E')$ extends holomorphically to D.

For $E' \in D$, define $\gamma^{E'} = q^{E'} - \sum_{E\in\mathcal{E}} c_E(E')q^E$. Then for $E' \in \Sigma_{\alpha,v} \cap D$, $\gamma^{E'}$ is a holomorphic function defined on the upper half plane $\mathbb{H}$ such that $\int \Re\gamma(x + \epsilon i)w(x + \epsilon i) = 0$ for every $w \in C^\omega_2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and any $0 < \epsilon < 2$. This implies that for $E' \in \Sigma_{\alpha,v} \cap \mathbb{D}$, $\gamma^{E'}$ extends to an entire function, which is purely imaginary on $\mathbb{R}$ (see section 4 of [A1]). But for every $E' \in D$, $\gamma^{E'}$ defines a holomorphic function on $\{\Im z > \frac{1}{2\pi}\}$ which depends holomorphically on E'. Since $\Sigma_{\alpha,v} \cap D$ has positive logarithmic capacity (see Theorem 7.2 in [S]), $\gamma^{E'}$ must define an

[13] Fix a $\#\mathcal{E}$-dimensional subspace $F \subset C^\omega_2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that the $\Phi^E|F$, $E \in \mathcal{E}$, are linearly independent, and define $c_E(E')$ in a neighborhood of E_* so to have $\Phi^{E'}|F = \sum_{E\in\mathcal{E}} c_E(E')\Phi^E|F$.

entire function for every $E' \in D$ (just use Hartogs Theorem). It follows that $q^{E'} = \gamma^{E'} + \sum_{E \in \mathcal{E}} c_E(E') q^E$ defines a holomorphic function on $\mathbb{H}$ for every $E' \in D$. By a similar argument to section 6.3, for every $E' \in D$ we can analytically continuate the unstable and stable directions of $(\alpha, A^{(E'-v)})$ defined on $\Im x > \frac{L(E')}{2\pi}$ to holomorphic functions $u^E, s^E : \mathbb{H} \to \mathbb{PC}^2$ which satisfy $A^{(E'-v)}(x) \cdot u^{E'}(x) = u^{E'}(x+\alpha)$, $A^{(E'-v)}(x) \cdot s^{E'}(x) = s^{E'}(x+\alpha)$, and $u^{E'}(x) \neq s^{E'}(x)$ for every $x \in \mathbb{H}$.

Since $L(\alpha, A_\epsilon^{(E'-v)}) > 0$ for every E' and every $\epsilon > 0$, this implies that $(\alpha, A_\epsilon^{(E'-v)})$ is uniformly hyperbolic for every $\epsilon > 0$ and for every $E' \in D$. But this can not happen when $E' \in D \smallsetminus \Sigma_{\alpha,v}$, since $(\alpha, A_\epsilon^{(E'-v)})$ is not uniformly hyperbolic when $2\pi\epsilon = L(E')$. This gives a contradiction and proves the claim. □

Remark A.1. For perturbations of the almost Mathieu operator, the acceleration is bounded by 1, which implies that the number of alternances between the subcritical, critical and supercritical regimes is always finite. Indeed, for any $v' \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, and near any critical energy $E_0 \in \Sigma_{\alpha,v'}$ with acceleration 1, we can define an analytic function $L_{\xi,1}$ as before which has the property that energies $E \in \Sigma_{\alpha,v'}$ near E_0 are supercritical, critical, or subcritical according to whether $L_{\xi,1} > 0$, $L_{\xi,1} = 0$, or $L_{\xi,1} < 0$.

Remark A.2. For $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, $w \in C^\omega_\xi(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and κ small, one may investigate the transition from subcriticality to supercriticality within the one-parameter family of operators $H_{\alpha,\lambda(v+\kappa w)}$, $\lambda > 0$. It is convenient to look simultaneously at all $\Sigma_{\alpha,\lambda(v+\kappa w)}$ in the (E,λ)-plane. The results of [A1] imply that there is a (possibly disconnected) nearly horizontal analytic curve $L_{\xi,1}^{-1}(0)$,[14] close to $\Sigma_{\alpha,v} \times \{1\}$, which separates the subcritical energies (below it) and the supercritical energies (above it). From the point of view of this paper, the study of this family is straightforward, since transversality can be checked by the direct computation of $L_{\xi,1}$ in the almost Mathieu case. In particular, since the "critical curve" is nearly horizontal it defines a premonotonic families of cocycles, so the arguments in section 3 show that the intersection of this curve with the spectra has zero linear measure.

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