WEAK MIXING DIRECTIONS IN NON-ARITHMETIC VEECH SURFACES

ARTUR AVILA AND VINCENT DELECROIX

Abstract. We show that the billiard in a regular polygon is weak mixing in almost every invariant surface, except in the trivial cases which give rise to lattices in the plane (triangle, square and hexagon). More generally, we study the problem of prevalence of weak mixing for the directional flow in an arbitrary non-arithmetic Veech surface, and show that the Hausdorff dimension of the set of non-weak mixing directions is not full. We also provide a necessary condition, verified for instance by the Veech surface corresponding to the billiard in the pentagon, for the set of non-weak mixing directions to have positive Hausdorff dimension.

1. Introduction

1.1. **Weak-mixing directions for billiards in regular polygons.** Let $n \geq 3$ be an integer and consider the billiard in an n-sided regular polygon P_n. It is readily seen that the 3-dimensional phase space (the unit tangent bundle $T^1 P_n$) decomposes into a one-parameter family of invariant surfaces, as there is a clear integral of motion. In such a setting, it is thus natural to try to understand the dynamics restricted to each of, or at least most of, the invariant surfaces.

The cases $n = 3, 4, 6$ are simple to analyze, essentially because they correspond to a lattice tiling of the plane: the dynamics is given by a linear flow on a torus, so for a countable set of surfaces all trajectories[1] are periodic, and for all others the flow is quasiperiodic and all trajectories are equidistributed with respect to Lebesgue measure on the surface.

For $n \neq 3, 4, 6$, the invariant surfaces have higher genus, and quasiperiodicity can not take place. However, W. Veech [Ve89] showed that a dichotomy still holds: for a countable set of surfaces all infinite trajectories are periodic, and for all others all trajectories are equidistributed with respect to Lebesgue measure.

The most basic question that follows Veech work is whether weak mixing takes place (absence of mixing is a general property in the more general class of translation flows, which is known from earlier work of A. Katok [Ka80]). Recall that weak mixing means that there is no remaining of periodicity or quasiperiodicity from the measurable point of view (i.e., there is no factor which is periodic or quasiperiodic), and can thus be interpreted as the complete breakdown of the nice lattice behavior seen for $n = 3, 4, 6$.

While results about the prevalence of weak mixing were obtained in the more general context of translation flows ([AF07], [AF]), the case of regular polygons

[1] Here we restrict consideration to orbits that do not end in a singularity (i.e., a corner of the billiard table) in finite time.

proved to be much more resistent. The basic reason is that the most succesful approaches so far were dependent on the presence of a suitable number of parameters which can be used in a probabilistic exclusion argument, and as a consequence they were not adapted to study the rigid situation of a specific billiard table. In this paper we address directly the problem of weak mixing for exceptionally symmetric translation flows, which include the ones arising from regular polygonal billiards.

Theorem 1. *If $n \neq 3, 4, 6$ then the restriction of the billiard flow in P_n to almost every invariant surface is weak mixing.*

Of course, in view of Veech's remarkably precise answer to the problem of equidistribution, one could wonder whether weak mixing is not only a prevalent property, but one that might hold outside a countable set of exceptions. It turns out that this is not the case in general, and in fact we will show that the set of exceptions can be relatively large and have positive Hausdorff dimension. However, we will prove that it can never have full dimension.

1.2. **Non-arithmetic Veech surfaces.** We now turn to the general framework in which the previous result fits. A *translation surface* is a compact surface S which is equipped with an atlas defined on the complement of a finite and non-empty set of "marked points" Σ, such that the coordinate changes are translations in $\mathbb{R}^2$ and each marked point p has a punctured neighborhood isomorphic to a finite cover a punctured disk in $\mathbb{R}^2$. The geodesic flow in any translation surface has an obvious integral of motion, given by the angle of the direction, which decomposes it into separate dynamical systems, the *directional flows*.

An *affine diffeomorphism* of a translation surface (S, Σ) is a homeomorphism of S which fixes Σ pointwise and which is affine and orientation preserving in the charts. The linear part of such diffeomorphism is well defined in $\mathrm{SL}(2, \mathbb{R})$, and allows one to define a homomorphism from the group of affine diffeomorphisms to $\mathrm{SL}(2, \mathbb{R})$: its image is a discrete subgroup called the *Veech group* of the translation surface.

A *Veech surface* is an "exceptionally symmetric" translation surface, in the sense that the Veech group is a (finite co-volume) lattice in $\mathrm{SL}(2, \mathbb{R})$ (it is easily seen that the Veech group is never co-compact). Simple examples of Veech surfaces are *square-tiled surfaces*, obtained by gluing finitely many copies of the unit square $[0, 1]^2$ along their sides: in this case the Veech group is commensurable with $\mathrm{SL}(2, \mathbb{Z})$. Veech surfaces that can be derived from square-tiled ones by an affine diffeomorphism are called *arithmetic*. Arithmetic Veech surfaces are branched covers of flat tori, so their directional flows are never topologically weak mixing (they admit a *continuous* almost periodic factor).

The first examples of non-arithmetic Veech surfaces were described by Veech, and correspond precisely to billiard flows on regular polygons. It is easy to see that the billiard flow in P_n corresponds, up to finite cover, to the geodesic flow on a translation surface obtained by gluing the opposite sides of P_n (when n is even), or the corresponding sides of P_n and $-P_n$ (when n is odd). This construction yields indeed a Veech surface S_n which is non-arithmetic precisely when $n \neq 3, 4, 6$. The genus g of S_n is related to n by $g = \frac{n-1}{2}$ (n odd) and $g = [\frac{n}{4}]$ (n even).

We can now state the main result of this paper:

Theorem 2. *The geodesic flow in a non-arithmetic Veech surface is weakly mixing in almost every direction. Indeed the Hausdorff dimension of the set of exceptional directions is less than one.*

An important algebraic object associated to the Veech group Γ of a Veech surface is the trace field k, the extension of $\mathbb{Q}$ by the traces of the elements of Γ. Its degree $r = [k:\mathbb{Q}]$ satisfies $1 \leq r \leq g$, where g is the genus of S, and we have $r = 1$ if and only if S is arithmetic. As an example, the trace field of S_n is $\mathbb{Q}[\cos \frac{\pi}{n}]$ if n is odd or $\mathbb{Q}[\cos \frac{2\pi}{n}]$ if n is even.

Theorem 3. *Let S be a Veech surface with a quadratic trace field (i.e., $r = 2$). Then the set of directions for which the directional flow is not even topologically weak mixing has positive Hausdorff dimension.*

Notice that this covers the case of certain polygonal billiards ($\mathbb{Q}[\cos \frac{\pi}{n}]$ is quadratic if and only if $n \in \{4, 5, 6\}$, hence the above result holds for S_n with $n \in \{5, 8, 10, 12\}$), and of all non-arithmetic Veech surfaces in genus 2. We point out that Theorem 3 is a particular case of a more general result, Theorem 31, which does cover some non-arithmetic Veech surfaces with non-quadratic trace fields (indeed it applies to all non-arithmetic S_n with $n \leq 16$, the degrees of their trace fields ranging from 2 to 6) and could possibly apply to all non-arithmetic Veech surfaces. Let us also note that the non-weak mixing directions obtained in Theorem 31 have multiple rationally independent eigenvalues.

One crucial aspect of our analysis is a better description of the possible eigenvalues (in any minimal direction) in a Veech surface. Using the algebraic nature of Veech surfaces, we are able to conclude several restriction on the group of eigenvalues. For non-arithmetic Veech surfaces, the ratio of two eigenvalues always belong to k and the number of rationally independent eigenvalues is always at most $[k:\mathbb{Q}]$. Moreover, the group of eigenvalues is finitely generated (so the *Kronecker factor* is always a minimal translation of a finite dimensional torus). Whereas for the case of arithmetic Veech surfaces, we obtain that all eigenvalues come from a ramified cover over a torus.

We expect that, for a non-arithmetic Veech surface and along any minimal direction that is not weak mixing, there are always exactly $[k:\mathbb{Q}]$ independent eigenvalues, and that they are either all continuous or all discontinuous. This is the case along directions for which the corresponding forward Teichmüller geodesic is bounded in moduli space, see Remark 7.1.

1.3. **Further comments.** The strategy to prove weak mixing for a directional flow on a translation surface S is to show that the associated unitary flow has no non-trivial eigenvalues. It is convenient to first rotate the surface so that the directional flow goes along the vertical direction. The small scale behavior of eigenfunctions associated to a possible eigenvalue can then be studied using renormalization. Technically, one parametrizes all possible eigenvalues by the line in $H^1(S;\mathbb{R})$ through the imaginary part of the tautological one form (the Abelian differential corresponding to the translation structure) and then apply the so-called Kontsevich-Zorich cocycle over the Teichmüller flow in moduli space. According to the Veech criterion any actual eigenvalue is parametrized by an element of the "weak-stable lamination" associated to an acceleration of the Kontsevich-Zorich cocycle acting modulo $H^1(S;\mathbb{Z})$. Intuitively, eigenfunctions parametrized by an eigenvalue outside the weak stable lamination would exhibit so much oscillation in small scales that measurability must be violated. The core of [AF07] is a probabilistic method to exclude non-trivial intersections of an arbitrary fixed line in $H^1(S;\mathbb{R})$ with the weak stable lamination, which uses basic information on the non-degeneracy of the cocycle.

The problem of weak mixing in the case of S_5 was asked during a talk by the first author on [AF07] by C. McMullen in 2004. It was realized during discussions with P. Hubert that the probabilistic method behind [AF07] fails for Veech surfaces, due essentially to a degeneracy (non-twisting) of the Kontsevich-Zorich cocycle. Attempts to improve the probabilistic argument using Diophantine properties of invariant subspaces turned out to lead to too weak estimates.

In this paper we prove that the locus of possible eigenvalues is much more constrained in the case of Veech surfaces: eigenvalues must be parametrized by an element in the "strong stable lamination", consisting of all the integer translates of the stable space, which is a much simpler object than the weak stable lamination (considered modulo the strong stable space, the former is countable, while the latter is typically uncountable). Direct geometric estimates on the locus of intersection can be then obtained using much less information on the non-degeneracy of the cocycle.

In order to obtain this stronger constraint on the locus of possible eigenvalues we will first carry out an analysis of the associated eigenfunctions at scales corresponding to renormalizations belonging to a large compact part of the moduli space (this refines Veech's criterion, which handles compact sets that are small enough to be represented in spaces of interval exchange transformations). This is followed by a detailed analysis of the excursion to the non-compact part of the moduli space, which is used to forbid the occurrence of an integer "jump" in cohomology along such an excursion. In doing so, we use fundamentally the particularly simple nature of the renormalization dynamics in the non-compact part of the moduli space $\mathrm{SL}(2,\mathbb{R})/\Gamma$ of a Veech surface (a finite union of cusps).

We should point out that it is reasonable to expect that, in the general case of translation flows, one can construct examples of eigenvalues which do not come from the strong stable lamination. Indeed M. Guenais and F. Parreau construct suspension flows over ergodic interval exchange transformations and with piecewise constant roof function admitting infinitely many independent eigenvalues (see Theorem 3 of [GP]), and this provides many eigenvalues that do not come from the strong stable lamination (which can only be responsible for a subgroup of eigenvalues of finite rank). See also [BDM2], section 6, for a different example in a related context.

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2. Preliminaries

2.1. Translation surfaces, moduli space, $\mathrm{SL}(2,\mathbb{R})$-action.

A *translation surface* can be also defined as a triple (S,Σ,ω) of a closed surface S, a non-empty finite set $\Sigma \subset S$, and an Abelian differential ω on S whose zeros are contained in Σ (ω is holomorphic for a unique complex structure on S). Writing in local coordinates $\omega = dz$, we get canonical charts to $\mathbb{C}$ such that transition maps are translations.

Such map exists at $x \in S$ if and only if ω is non zero at x. The zeros of ω are the *singularities* of the translation surface. Reciprocally, a translation surface S defined as in the introduction (in terms of a suitable atlas on $S \setminus \Sigma$) allows one to recover the Abelian differential ω by declaring that $\omega = dz$ and extending it uniquely to the marked points. We write (S, ω) for the translation surface for which Σ is the set of zeros of ω.

Let (S, Σ, ω) be a translation surface. The form $|\omega|$ defines a flat metric except at the singularities. For each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we define the *directional flow* in the direction θ as the flow $\phi_T^{S,\theta} : S \to S$ obtained by integration of the unique vector field X_θ such that $\omega(X_\theta) = e^{i\theta}$. In local charts z such that $\omega = dz$ we have $\phi_t^{S,\theta}(z) = z + te^{i\theta}$ for small t, so the directional flows are also called $t$ranslation flows. The *(vertical) flow* of (S, ω) is the flow $\phi^{S,\pi/2}$ in the vertical direction. The flow is not defined at the zeros of ω and hence the flow is not defined for all positive times on backward orbits of the singularities. The flows $\phi_t^{S,\theta}$ preserve the volume form $\frac{1}{2i}\omega \wedge \overline{\omega}$ and the ergodic properties of translation flows we will discuss below refer to this measure.

Translation surfaces were introduced to study rational billiards as the example of the regular polygons P_n in the introduction. Each rational billiard may be seen as a translation surface by a well known construction called unfolding or Zemliakov-Katok construction (see [MT02]).

Several results are known to holds for an arbitrary translation surface: the directional flows is minimal except for a countable set of directions [Ke75], the translation flow is uniquely ergodic except for a set of directions of Hausdorff dimension at most 1/2 [KMS86], [Ma92], and the translation flow is not mixing in any direction [Ka80].

The weak mixing property is more subtle. Indeed, translation flows in a genus one translation surface are never weakly mixing. The same property holds for the branched coverings of genus one translation surfaces, which form a *dense* subset of translation surfaces. However, for *almost every* translation surface of genus at least two, the translation flow is weakly mixing in almost every direction [AF07]. The implicit topological and measure-theoretical notions above refer to a natural structure on the space of translation structure that we introduce now.

Let $g, s \geq 1$ and let S be a closed surface of genus g, let $\Sigma \subset S$ be a subset with $\#S = s$ and let $\kappa = (\kappa_x)_{x \in \Sigma}$ be a family of non-negative integers such that $\sum \kappa_i = 2g - 2$. The set of translation structures on S with prescribed conical angle $(\kappa_x + 1)2\pi$ at x, modulo isotopy relative to Σ forms a manifold $\mathcal{T}_{S,\Sigma}(\kappa)$. The manifold structure on $\mathcal{T}_{S,\Sigma}(\kappa)$ is described by the so-called *period map* $\Theta : \mathcal{T}_{S,\Sigma}(\kappa) \to H^1(S, \Sigma; \mathbb{C})$ which associates to ω its cohomology class in $H^1(S, \Sigma; \mathbb{C})$. The period map is locally bijective and provides natural charts to $\mathcal{T}_{S,\Sigma}(\kappa)$ as well as an affine structure and a canonical Lebesgue measure. We denote by $\mathcal{T}_{S,\Sigma}(\kappa)^{(1)}$ the hypersurface of area 1 translation surfaces.

The group $\mathrm{SL}(2, \mathbb{R})$ acts (on left) on $\mathcal{T}_{S,\Sigma}(\kappa)$ by postcomposition on the charts and preserves the hypersurface $\mathcal{T}_{S,\Sigma}(\kappa)^{(1)}$. The subgroup of rotations $r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ acts by multiplication by $e^{i\theta}$ on ω. The action of the diagonal subgroup $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is called the *Teichmüller flow* and transforms $\omega = \mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega)$ into

$g_t \cdot \omega = e^t \operatorname{Re}(\omega) + ie^{-t}\operatorname{Im}(\omega)$. The *stable* and *unstable horocycle flows* are the action of the matrices $h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and $h_s^+ = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

The *modular group* $\mathrm{MCG}(S,\Sigma)$ of (S,Σ) is the group of diffeomorphisms of S fixing Σ pointwise modulo isotopy relative to Σ. It acts discretely discontinuously (on right) on the spaces $\mathcal{T}_{S,\Sigma}(\kappa)$ and $\mathcal{T}^{(1)}_{S,\Sigma}(\kappa)$ via $(S,\omega) \mapsto (S, \omega \circ d\phi)$. Their quotient, denoted $\mathcal{M}_{S,\Sigma}(\kappa)$ and $\mathcal{M}^{(1)}_{S,\Sigma}(\kappa)$ is called a *stratum of the moduli space of translation surfaces of genus g and s marked points* or shortly a *stratum*. The space $\mathcal{M}_{S,\Sigma}(\kappa)$ inherits from $\mathcal{T}_{S,\Sigma}(\kappa)$ a complex affine orbifold structure. The $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{MCG}(S,\Sigma)$ actions on $\mathcal{T}_{S,\Sigma}(\kappa)$ commutes, hence the $\mathrm{SL}(2,\mathbb{R})$ action is well defined on the quotient $\mathcal{M}_{S,\Sigma}(\kappa)$. The Lebesgue measure projects on $\mathcal{M}_{S,\Sigma}(\kappa)$ (respectively $\mathcal{M}^{(1)}_{S,\Sigma}(\kappa)$) into a measure ν (resp. $\nu^{(1)}$) called the Masur-Veech measure. Masur ([Ma82]) and Veech ([Ve82]) proved independently that the measure $\nu^{(1)}$ has finite total mass, the action of $\mathrm{SL}(2,\mathbb{R})$ on each $\mathcal{M}_{S,\Sigma}(\kappa)$ preserves it and moreover that the Teichmüller flow g_t is ergodic on each connected component of $\mathcal{M}^{(1)}_{S,\Sigma}(\kappa)$ with respect to that measure.

We will also sometimes use the notation $\mathcal{M}_g(\kappa)$ to denote $\mathcal{M}_{S,\Sigma}(\kappa)$, where the $(\kappa_j) 1 \leq j \leq s$ is obtained by reordering the $(\kappa_x)_{x \in S}$ in non-increasing order. As an example, for n even, the surface S_n built from a regular n-gon belongs to the stratum $\mathcal{M}_{\lfloor n/4 \rfloor}((n-4)/2)$ if $n \equiv 0 \mod 4$ or $\mathcal{M}_{\lfloor n/4 \rfloor}((n-6)/4, (n-6)/4)$ if $n \equiv 2 \mod 4$.

Over the Teichmüller space, let us consider the trivial cocycle $g_t \times id$ on $\mathcal{T}_{S,\Sigma}(\kappa) \times H^1(S;\mathbb{R})$. The modular group $\mathrm{MCG}(S,\Sigma)$ acts on $\mathcal{T}_{S,\Sigma}(\kappa) \times H^1(S;\mathbb{R})$ and the quotient bundle is a flat orbifold vector bundle over $\mathcal{M}_{S,\Sigma}(\kappa)$ called the *Hodge bundle*. The *Kontsevich-Zorich cocycle* is the projection of $g_t \times id$ to the Hodge bundle. We will also need a slightly different form of the Kontsevich-Zorich cocycle, namely the projection of $g_t \times id$ on $\mathcal{T}_{S,\Sigma}(\kappa) \times H^1(S \backslash \Sigma;\mathbb{R})$ that we call the *extended Kontsevich-Zorich cocycle* (on the *extended Hodge bundle*). The moduli space, the Teichmüller flow and the Kontsevich-Zorich cocycle are of main importance in the results we mentioned above about the ergodic properties of translation flows.

Let μ be a g_t invariant ergodic probability measure on $\mathcal{M}_{S,\Sigma}(\kappa)$. Because the modular group acts by symplectic (with respect to the intersection form) transformations on $H^1(S;\mathbb{R})$, the $2g$ Lyapunov exponents $\lambda_1^\mu \geq \lambda_2^\mu \geq \ldots \geq \lambda_{2g}^\mu$ of the Kontsevich-Zorich cocycle satisfy

$$\forall 1 \leq k \leq g, \quad \lambda_k^\mu = -\lambda_{2g-k-1} \geq 0.$$

Because of the natural injection $H^1(S;\mathbb{R}) \to H^1(S \backslash \Sigma;\mathbb{R})$, the Lyapunov spectrum of the extended Kontsevich-Zorich cocycle contains the one of the Kontsevich-Zorich cocycle. It may be proved that the remaining exponents are $s-1$ zeros where s is the cardinality of Σ.

The *tautological bundle* is the subbundle of the Hodge bundle whose fiber over (S,Σ,ω) is $V = \mathbb{R}\operatorname{Re}\omega \oplus \mathbb{R}\operatorname{Im}\omega$. The Kontsevich-Zorich cocycle preserves the tautological bundle and one sees directly from the definition that the Lyapunov exponents on the tautological bundle are 1 and -1. For the remaining exponents we have the following inequality.

Theorem 4 (Forni [Fo02]). *Let μ be an ergodic invariant probability measure of the Teichmüller flow on some stratum $\mathcal{M}_{S,\Sigma}(\kappa)$. Then the second Lyapunov exponent λ_2^μ of the Kontsevich-Zorich cocycle satisfies $1 > \lambda_2^\mu$.*

Remark 2.1. *Forni's proof indeed shows that there exists a natural Hodge norm on the Hodge bundle such that for any $x \in \mathcal{M}_{S,\Sigma}(\kappa)$, the Kontsevich-Zorich cocycle starting from x has norm strictly less than e^t at any time $t > 0$, when restricted to the symplectic orthogonal to the tautological space.*

2.2. **Veech surfaces.** Our goal in this article is to study the weak-mixing property for the directional flows in a translation surfaces with closed $\mathrm{SL}(2,\mathbb{R})$-orbits in $\mathcal{M}_{S,\Sigma}(\kappa)$, which are called Veech surfaces.

Let us recall that an *affine homeomorphism* of a translation surface (S, Σ, ω) is a homeomorphism of S which preserves Σ pointwise and is affine in the charts of S compatible with the translation structure. An affine homeomorphism ϕ has a well defined linear part, denoted by, $d\phi \in \mathrm{SL}(2,\mathbb{R})$, which is the derivative of the action of ϕ in charts. The set of linear parts of affine diffeomorphisms forms a discrete subgroup $\Gamma(S, \Sigma, \omega)$ of $\mathrm{SL}(2,\mathbb{R})$ called the *Veech group* of (S, Σ, ω). A translation surface is called a *Veech surface* if its Veech group is a lattice. The $\mathrm{SL}(2,\mathbb{R})$ orbit of a Veech surface is closed in $\mathcal{M}_{S,\Sigma}(\kappa)$ and naturally identifies with the quotient $\mathcal{C} = \mathrm{SL}(2,\mathbb{R})/\Gamma(S, \Sigma, \omega)$. The $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{C}$ preserves the natural Haar measure and the Teichmüller flow g_t is the geodesic flow on the unit tangent bundle of the hyperbolic surface $\mathbb{H}/\Gamma(S, \Sigma, \omega)$.

Veech proved that the Veech group of a Veech surface is never co-compact. Moreover, the cusp excursion may be measured in terms of systoles as in the well known case of lattices with $\mathrm{SL}(2,\mathbb{Z})$. A *saddle connection* in a translation surface (S, Σ, ω) is a geodesic segment for the metric $|\omega|$ that start and ends in Σ and whose interior is disjoint from Σ. For the square torus, $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}i)$ with the 1-form dz the set of saddle connections identifies with primitive vectors (ie vector of the form $pi + q$ with p and q relatively prime integers). For a translation surface (S, ω) the *systole* $\mathrm{sys}(S, \omega)$ of (S, ω) is the length of the shortest saddle connection in (S, ω). Assume that (S, ω) is a Veech surface and denote $\mathcal{C}$ its $\mathrm{SL}(2,\mathbb{R})$-orbit in $\mathcal{M}_{S,\Sigma}(\kappa)$. Then the set $\mathcal{C}_\varepsilon = \{(S, \omega) \in \mathcal{C}; \mathrm{sys}(S, \omega) \geq \varepsilon\}$ forms an exhaustion of $\mathcal{C}$ by compact sets.

Beyond arithmetic surfaces (cover of the torus ramified over one point) the first examples of Veech surfaces are the translations surfaces S_n associated to the billiard in the regular polygon with n sides $P_n \subset \mathbb{R}^2$ which is built from P_n (n even) or from the disjoint union of P_n and $-P_n$ (n odd) [Ve89]. In either case, S_n is defined by identifying every side of P_n with the unique other side (of either P_n or $-P_n$ according to the parity of n) parallel to it, via translations. For them, the Veech group as well as the trace field was computed by Veech.

Theorem 5 ([Ve89]). *Let S_n be the Veech surface associated to the billiard in the regular polygon with n sides:*

(1) *if n is odd, the Veech group of S_n is the triangle group $\Delta(2, n, \infty)$ with trace field $\mathbb{Q}[\cos(\pi/n)]$,*
(2) *if n is even, the Veech group of S_n is the triangle group $\Delta(n/2, \infty, \infty)$ with trace field $\mathbb{Q}[\cos(2\pi/n))]$.*

Notice that for n even, $\Delta(n/2, \infty, \infty)$ is a subgroup of index 2 of $\Delta(2, n, \infty)$.

Calta [Ca04] and McMullen [McM03] made a classification of Veech surfaces in genus 2. In the stratum $\mathcal{M}_2(1,1)$ there is only one non arithmetic Veech surface which may be built from the decagon (the surface S_{10}). In the stratum $\mathcal{M}_2(2)$, there is a countable family of Veech surfaces that have quadratic trace fields (the family includes S_5 and S_8). They are all obtained from billiards in a rectangular L-shape table. But for those examples, the structure of the Veech group is rather mysterious. Other surfaces with quadratic trace field were discovered by McMullen [McM06] in the strata $\mathcal{M}_3(4)$ and $\mathcal{M}_4(6)$ and further studied by Lanneau and Nguyen [LN]. More recently, Bouw and Möller [BM10], generalizing Veech examples, introduce a family of Veech surfaces $S_{m,n}$ for which the Veech group is the triangle group $\Delta(m,n,\infty)$. They prove that some of them may be obtained as unfolding billiards. An explicit construction of those surfaces using polygons is given in [Ho12] (see also [Wr]).

2.3. **Translation flow of Veech surfaces.** Let (S,Σ,ω) be a translation surface and assume that there exists an affine diffeomorphism ϕ whose image under the Veech group g is parabolic. The direction determined by the eigenvector of g in $\mathbb{R}^2$ is a *completely periodic direction* in S: the surface (S,Σ,ω) decomposes into a finite union of cylinders whose waist curves are parallel to it. Morever, ϕ preserves each cylinder and acts as a power of a Dehn-twist in each of them. We may assume that the eigendirection is vertical, and hence $g = h_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ for some real number s. Let h_1, h_2, …, h_k and w_1, w_2, …, w_k be the widths and the heights of the cylinders C_1, …, C_k in the vertical direction. For each cylinder C_i, let ϕ_i be the Dehn twist in C_i. Then its linear part is $g_i = \begin{pmatrix} 1 & 0 \\ \mu(C_i)^{-1} & 1 \end{pmatrix}$ where $\mu(C_i) = w_i/h_i$ is the *modulus* of C_i. The real number s is such $s/\mu(C_i)$ are integers. Reciprocally, a completely periodic direction admits a non trivial stabilizer in $\mathrm{SL}(2,\mathbb{R})$ if and only if the moduli $\mu(C_i)$ of the cylinders are commensurable (their ratio are rational numbers). Such a direction is called *parabolic*.

Keane and Kerckhoff-Masur-Smillie theorems about minimality and unique ergodicity of translation flows (see Section 2.1) have the following refinement.

Theorem 6 (Veech alternative, [Ve89])**.** *Let (S,Σ,ω) be a Veech surface. Then*

(1) *either there exists a vertical saddle connection and (S,Σ,ω) is parabolic,*
(2) *or the vertical flow is uniquely ergodic.*

The $\mathrm{SL}(2,\mathbb{R})$ orbit of a Veech surface is never compact (any fixed saddle connection can be shrinked arbitrarily by means of the $\mathrm{SL}(2,\mathbb{R})$ action, thus escaping any compact subset of the moduli space). Nevertheless, the geometry of flat surfaces in the cusps is well understood and will be crucial in the study of eigenvalues (see Section 4). If ζ and ζ' are two saddle connections in a flat surface (S,ω) we denote by $\zeta \wedge_\omega \zeta'$ the number in $\mathbb{C}$ that corresponds to the (signed) area of the parallelogram determined by $\omega(\zeta)$ and $\omega(\zeta')$.

Theorem 7 (No small triangles)**.** *Let (S,ω) be a Veech surface. Then there exists $\kappa > 0$ such that for any pair of saddle connections ζ and ζ'*

- *either $|\zeta \wedge_\omega \zeta'| > \kappa$,*
- *or ζ and ζ' are parallel (i.e. $\zeta \wedge_\omega \zeta' = 0$).*

The above theorem is actually the easy part of a characterization of Veech surfaces proved in [SW10]. Note that the quantity $\zeta \wedge_\omega \zeta'$ is invariant under the Teichmüller flow (i.e. $\zeta \wedge_{g\cdot\omega} \zeta' = \zeta \wedge \zeta'$ for any $g \in \mathrm{SL}(2,\mathbb{R})$) and corresponds to twice the area of a (virtual) triangle delimited by ζ and ζ'. We deduce in particular, that if there exists a small saddle connection in a Veech surface, then any other short saddle connection would be parallel to it and that the smallness is uniform for the whole $\mathrm{SL}(2,\mathbb{R})$-orbit. More precisely,

Corollary 8. *Let $\mathcal{C}$ be a closed $\mathrm{SL}(2,\mathbb{R})$-orbit in some stratum. Then there exists $\varepsilon > 0$ such that for any $\omega \notin \mathcal{C}_\varepsilon$ the saddle connections in (S,Σ,ω) shorter than 1 are parallel to the direction of the shortest saddle connection in (S,Σ,ω).*

2.4. **Holonomy field and conjugates of Veech group.** Let (S,Σ,ω) be a translation surface and let $\Lambda = \omega(H_1(S;\mathbb{Z})) \subset \mathbb{C} \simeq \mathbb{R}^2$ be the free-module of periods. In what follows, periods will be sometimes called holonomies. Let e_1 and e_2 be two non-parallel elements in Λ. The *holonomy field* of (S,Σ,ω) is the smallest field k of $\mathbb{R}$ such that any element in Λ may be written as k-linear combination of e_1 and e_2. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a group, the *trace field* of Γ is the group generated by the traces of the element of Γ.

Theorem 9 (Gutkin-Judge [GJ00], Kenyon-Smillie [KS00])**.** *Let (S,Σ,ω) be a Veech surface. Then its holonomy field k coincides with the trace field of its Veech group. The degree of k over $\mathbb{Q}$ is at most the genus of S and the rank of $\Lambda = \omega(H_1(S;\mathbb{Z}))$ is twice the degree of k over $\mathbb{Q}$.*

We will need two important facts about the holonomy field of a Veech surface.

Theorem 10 (Gutkin-Judge [GJ00])**.** *A Veech surface (S,Σ,ω) is arithmetic (ie a ramified cover of a torus over one point) if and only if its holonomy field is $\mathbb{Q}$.*

Theorem 11 ([LH06])**.** *The holonomy field of a Veech surface is totally real.*[2]

The later result uses the fact that the Veech group of a Veech surfaces contains parabolic elements.

Now, we define the Galois conjugate of the Veech group. Let (S,Σ,ω) be a Veech surface, let Γ be its Veech group and let k be its holonomy field. Let e_1 and e_2 be two non-parallel elements in the set of holonomies $\Lambda = \omega(H_1(S;\mathbb{Z}))$. For each element $v \in H_1(S;\mathbb{Z})$ there exist unique elements α and β of k such that $\omega(v) = \alpha e_1 + \beta e_2$. The maps α and β are linear with values in k, in other words they belong to $H^1(S;k)$, and moreover, the tautological space $V = \mathbb{R}\,\mathrm{Re}(\omega) \oplus \mathbb{R}\,\mathrm{Im}(\omega)$ can be rewritten as $V = \mathbb{R}\alpha \oplus \mathbb{R}\beta$ in $H^1(S;\mathbb{R})$. Note that an alternative definition of the trace field would be the field of definition of the plane $\mathbb{R}\,\mathrm{Re}(\omega) + \mathbb{R}\,\mathrm{Im}(\omega)$ in $H^1(S;\mathbb{R})$. For any embedding $\sigma : k \to \mathbb{R}$, we may define new linear forms $\sigma \circ \alpha$ and $\sigma \circ \beta$. Those linear forms generate a 2 dimensional subspace in $H^1(S;\mathbb{R})$. The subspace does not depend on the choice of α and β and we call it the *conjugate by σ* of the tautological subspace V. This subspace (which is indeed defined in $H^1(S;k)$) will be denoted by V^σ. Because the action of the affine group on homology is defined over $\mathbb{Z}$ and preserves V, it preserves as well the conjugates V^σ.

Theorem 12. *Let (S,Σ,ω) be a Veech surface, k its holonomy field and $V = \mathbb{R}\,\mathrm{Re}\,\omega \oplus \mathbb{R}\,\mathrm{Im}\,\omega$ be the tautological subspace. Then for any embedding $\sigma : k \to \mathbb{R}$ the*

[2]Recall that a subfield $k \subset \mathbb{R}$ is called totally real if its image under any embedding $k \to \mathbb{C}$ is contained in $\mathbb{R}$.

subspace V^σ *is invariant under the action of the affine group of* (S,Σ,ω)*. Moreover, the space generated by the* $[k:\mathbb{Q}]$ *subspaces* $V^\sigma \subset H^1(S;\mathbb{R})$ *is the smallest rational subspace of* $H^1(S;\mathbb{R})$ *containing* V *and is the direct sum of the* V^σ*.*

The fact that the sum is direct follows from the presence of hyperbolic elements in the Veech group (see Theorem 28 of [KS00]). The subspaces V^σ are well defined over the whole Teichmüller curve: they form subbundles of the Hodge bundle invariant for the Kontsevich-Zorich cocycle. In other words, we may restrict the Kontsevich-Zorich cocycle to any of the V^σ and consider the associated pair $(\lambda^\sigma, -\lambda^\sigma)$ of Lyapunov exponents.

The Veech group Γ is canonically identified to the action of the affine group on the tautological subspace $V = \mathbb{R}\,\mathrm{Re}\,\omega \oplus \mathbb{R}\,\mathrm{Im}\,\omega$. The choice of two elements of $H_1(S;\mathbb{Z})$ with non parallel holonomy provides an identification of Γ as a subgroup of $\mathrm{SL}(2,k)$. Given an embedding of k in $\mathbb{R}$ we may conjugate the coefficients of the matrices in $\mathrm{SL}(2,k)$ and get a new embedding of Γ into $\mathrm{SL}(2,\mathbb{R})$. This embedding is canonically identified to the action of the affine group on the conjugate of the tautological bundle V^σ. We denote by Γ^σ this group and call it the *conjugate of the Veech group* by σ.

3. Markov model

3.1. **Locally constant cocycles.** Let Δ be a measurable space, and let μ be a finite probability (reference) measure on Δ. Let $\Delta^{(l)}$, $l \in \mathbb{Z}$ be a partition μ-mod 0 of Δ into sets of positive μ-measure. Let $T : \Delta \to \Delta$ be a measurable map such that $T|\Delta^{(l)} : \Delta^{(l)} \to \Delta$ is a bimeasurable map.

Let Ω be the space of all finite sequences of integers. The length of $\underline{l} \in \Omega$ will be denoted by $|\underline{l}|$.

For $\underline{l} = (l_1, ..., l_n) \in \Omega$, we let $\Delta^{\underline{l}}$ be the set of all $x \in \Delta$ such that $T^{j-1}(x) \in \Delta^{l_j}$ for $1 \le j \le n$. We say that T has *bounded distortion* if there exists $C_0 > 0$ such that every $\Delta^{\underline{l}}$ has positive μ-measure and $\mu^{\underline{l}} = \frac{1}{\mu(\Delta^{\underline{l}})} T_*^{|\underline{l}|}(\mu|\Delta^{\underline{l}})$ satisfies $\frac{1}{C_0}\mu \le \mu^{\underline{l}} \le C_0\mu$. In particular, for every $n \ge 1$, $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} T_*^k\mu$ satisfies $\frac{1}{C_0}\mu \le \mu_n \le C_0\mu$. Taking a weak limit, one sees that there exists an invariant measure ν satisfying $\frac{1}{C_0}\mu \le \nu \le C_0\mu$. It is easy to see that this invariant measure is ergodic provided the σ-algebra of μ-measurable sets is generated (mod 0) by the $\Delta^{\underline{l}}$.

Let H be a finite dimensional (real or complex) vector space, and let $\mathrm{SL}(H)$ denote the space of linear automorphisms of H with determinant 1. Given such a T, we can define a locally constant $\mathrm{SL}(H)$-cocycle over T by specifying a sequence $A^{(l)} \in \mathrm{SL}(H)$, $l \in \mathbb{Z}$: Take $A(x) = A^{(l)}$ for $x \in \Delta^{(l)}$ and $(T, A) : (x, w) \mapsto (T(x), A(x))$. Then the cocycle iterates are given by $(T, A)^n = (T^n, A_n)$ where $A_n(x) = A(T^{n-1}(x)) \cdots A(x)$. Notice that if $\underline{l} = (l_1, ..., l_n)$ then $A_n(x) = A^{\underline{l}}$ for $x \in \Delta^{\underline{l}}$ with $A^{\underline{l}} = A^{(l_n)} \cdots A^{(l_1)}$.

For a matrix A we note, $\|A\|_+ = \max(\|A\|, \|A^{-1}\|)$. We say that T is *fast decaying* if there exists $C_1 > 0$, $\alpha_1 > 0$ such that

$$\sum_{\mu(\Delta^{(l)}) \le \varepsilon} \mu(\Delta^{(l)}) \le C_1 \varepsilon^{\alpha_1}, \qquad \text{for } 0 < \varepsilon < 1,$$

and we say that A is *fast decaying* if there exists $C_2 > 0$, $\alpha_2 > 0$ such that

$$\sum_{\|A^{(l)}\|_+ \ge n} \mu(\Delta^{(l)}) \le C_2 n^{-\alpha_2}.$$

Note that fast decay implies that (T, A) is an integrable cocycle with respect to the invariant measure ν, i.e., $\int \ln \|A\|_+ d\nu < \infty$.

In our applications, Δ will be a simplex in $\mathbb{PR}^p$, i.e., the image of $\mathbb{PR}^p_+$ by a projective transformation, μ is the Lebesgue measure, and $T|\Delta^{(l)}$ is a projective transformation for every $l \in \mathbb{Z}$. In fact, we will be mostly interested in the case $p = 2$, and we will use freely the identification of $\mathbb{PR}^2$ with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

We note that if there exists a simplex $\Delta \Subset \Delta'$ such that the projective extension of $(T|\Delta^{(l)})^{-1}$ maps Δ' into itself for every $l \in \mathbb{Z}$, then the bounded distortion property holds, see section 2 of [AF07]. In this case we will say that T is a *projective expanding map*. For such maps, the invariant measure ν is ergodic.

3.2. **The Markov model for the geodesic flow on** $\mathrm{SL}(2, \mathbb{R})/\Gamma$**.** We will need the following nice Markov model for the geodesic flow on $\mathcal{C} = \mathrm{SL}(2, \mathbb{R})/\Gamma$ where Γ has finite covolume. Let us fix any point $x \in \mathcal{C}$. Then we can find a small smooth rectangle Q through x, which is transverse to the geodesic flow and provides us with a nice Poincaré section, in the sense that the first return map to Q under the geodesic flow has a particularly simple structure.

More precisely, let $p : \mathbb{R}^2 \to \mathcal{C}$ be given by $p(u, s) = h^-_s(h^+_u(x))$. Then for any $\varepsilon > 0$, we can find $u_- < 0 < u+$ and $s_- < 0 < s_+$ with $u_+ - u_- < \varepsilon$ and $s_+ - s_- < \varepsilon$ such that, letting $\Delta = (u_-, u_+) \subset \mathbb{R}$ and $\widehat{\Delta} = \{(u, s) \in \Delta \times \mathbb{R};\ s_- < \frac{s}{1+su} < s_+\}$, then we can take $Q = p(\widehat{\Delta})$. It is clear that Q is transverse to the geodesic flow. Let F denote the first return map to Q. Then

(1) There exist countably many disjoint open intervals $\Delta^{(l)} \subset \Delta$, such that the domain of F is the union of the $p(\widehat{\Delta}^{(l)})$, where $\widehat{\Delta}^{(l)} = \widehat{\Delta} \cap (\Delta^{(l)} \times \mathbb{R})$.
(2) There exists a function $r : \bigcup \Delta^{(l)} \to \mathbb{R}_+$ such that if (u, s) belongs to some $\widehat{\Delta}^{(l)}$ then the return time of $p(u, s)$ to Q is $r(u)$. Moreover, r is globally bounded away from zero, and its restriction to each $\Delta^{(l)}$ is given by the logarithm of the restriction of a projective map $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$.
(3) There exist functions $T : \bigcup \Delta^{(l)} \to \Delta$ and $S : \bigcup \Delta^{(l)} \to \mathbb{R}$ such that if (u, s) belongs to some $\widehat{\Delta}^{(l)}$ then $F(p(u, s)) = p(T(u), S(u) - e^{-2r(u)}s)$. Moreover, the restriction of T to each $\Delta^{(l)}$ coincides with the restriction of a projective map $T_l : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, and the restriction of S to each $\Delta^{(l)}$ coincides with the restriction of an affine map $S_l : \mathbb{R} \to \mathbb{R}$.
(4) There exists a bounded open interval Δ' containing $\overline{\Delta}$ such that $T_l^{-1}(\Delta') \subset \Delta'$ for every $l \in \mathbb{Z}$.

The basic idea of the construction is to guarantee that the forward orbit of the "unstable" frame $\delta_u(Q) = p\{(u, s) \in \partial\widehat{\Delta};\ u = u_\pm\}$ and the backward orbit of the "stable" frame $\delta_s(Q) = p\{(u, s) \in \partial\widehat{\Delta};\ \frac{s}{1+su} = s_\pm\}$ never come back to Q.[3] This easily yields the Markovian structure and the remaining properties follow from direct computation (or, for the last property, by shrinking ε).

Remark 3.1. *Given $u_0 \in \Delta^{(l)}$, knowledge of $T(u_0)$, $S(u_0)$ and $r(u_0)$ allows one to easily compute T, S, and r restricted to $\Delta^{(l)}$. Indeed, for $u \in \Delta^{(l)}$, $g_{r(u_0)}(p(u, 0)) = h^+_{e^{2r(u_0)}(u-u_0)} F(p(u_0, 0))$. To move it to Q, we must apply g_{-t} where t is bounded*

[3]This is easy enough to do when x is not a periodic orbit of small period: In this case we can use the uniform hyperbolicity of g_t to select $u_\pm$ and $s_\pm$ so that $g_t h^+_{u_\pm}(x)$ and $g_{-t} h^-_{s_\pm}(x)$ remain away from the $C\varepsilon$-neighborhood of x for every $t \geq 1$ for some fixed large C. (If x is a periodic orbits of small period, one needs to be slightly more careful as one can not choose C large.)

(indeed at most of order ε). Using that $F(p(u_0,0)) = p(T(u_0), S(u_0))$ one gets $e^t = 1 + e^{2r(u_0)}(u-u_0)S(u_0)$ and then the formulas

$$e^{r(u)} = \frac{e^{r(u_0)}}{1+e^{2r(u_0)}(u-u_0)S(u_0)},$$

$$T(u) = T(u_0) + \frac{e^{2r(u_0)}(u-u_0)}{1+e^{2r(u_0)}(u-u_0)S(u_0)}$$

$$S(u) = S(u_0)(1+e^{2r(u_0)}(u-u_0)S(u_0)).$$

Note that

$$DT(u) = e^{2r(u)}, \tag{1}$$

and that for every $l \in \mathbb{Z}$, $r \circ T_l^{-1} : \Delta \to \mathbb{R}$ has uniformly bounded derivative.

Note that T is a projective expanding map with bounded distortion, so it admits an ergodic invariant measure ν equivalent to Lebesgue measure. In order to obtain an upper bound on the Hausdorff dimension of the set of non-weak mixing directions, we will also need to use that T is fast decaying. Using (1), we see that fast decay is implied by the following well known exponential tail estimate on return times: there exists $\delta > 0$ (depending on Q) such that

$$\int_{u_-}^{u_+} e^{-\delta r(u)} du < \infty.$$

Remark 3.2. *The exponential tail estimate is usually proved using a finite Markov model for the full geodesic flow (as opposed to the infinite Markov model for a Poincaré return map that we consider here). However, it can also be proved using some more general information about the geodesic flow. Namely, one can show that fixing a small $\varepsilon_0 > 0$, there exists $C_0, C_1, \delta_0 > 0$ such that:*

1. *If $x \in \mathcal{C}_{\varepsilon_0}$ then the Lebesgue measure of the set of all $u \in (0,1)$ such that $g_t(h_u^+(x)) \notin \mathcal{C}_{\varepsilon_0}$ for $1 \le t \le T$ is at most $C_0 e^{-\delta_0 T}$,*
2. *If $x \in \mathcal{C}_{\varepsilon_0}$ then $g_t(h_u^+(x)) \in Q$ for some $0 < u < 1$ and $0 < t < C_1$.*

One can use those two estimates to show that if $x \in \mathcal{C}_{\varepsilon_0}$ then the Lebesgue measure of the set of all $u \in (0,1)$ such that $g_t(h_u^+(x)) \notin Q$ for $1 \le t \le T$ is at most $C_2 e^{-\delta_1 T}$ for appropriate constants $C_2, \delta_1 > 0$.

Details of this approach are carried out in [AD], *where it is used to obtain exponential tails for the return time for a Markov model of an arbitrary affine* $\mathrm{SL}(2,\mathbb{R})$*-invariant measure in moduli space.*

Remark 3.3. *As remarked before, $r \circ T_l^{-1} : \Delta \to \mathbb{R}$ has uniformly bounded derivative. Since T is expanding, this estimate can be iterated as follows. Denote by $r_n(u) = \sum_{k=0}^{n-1} r(T^k(u))$ be the n-th return time of $p(u,0)$ to Q, and write $T_{\underline{l}} = T_{l_n} \circ \cdots \circ T_{l_1} : \Delta^{\underline{l}} \to \Delta$ for any $\underline{l} = (l_1, ..., l_n)$. Then $r_n \circ T_{\underline{l}}^{-1} : \Delta \to \mathbb{R}$ also has uniformly bounded derivative (independent of n).*

Let now $\mathcal{C} = \mathrm{SL}(2,\mathbb{R})/\Gamma$ be the $\mathrm{SL}(2,\mathbb{R})$-orbit in of some Veech surface. Then the Kontsevich-Zorich cocycle over $\mathcal{C}$ gives rise to a locally constant cocycle over T as follows.

For $y \in \mathcal{C}$, let us denote by H_y the fiber of the Hodge bundle over y and let $V_y \subset H_y$ be the tautological bundle. Since Q is simply connected, there is a unique continuous identification between H_x and H_y for $y \in Q$ which preserves the integer lattice $H^1(S;\mathbb{Z})$ and the tautological bundle $V = \mathbb{R}\operatorname{Re}\omega \oplus \mathbb{R}\operatorname{Im}\omega$. If $y \in p(\widehat{\Delta}^{(l)})$ for some j (i.e., y belongs to the domain of the first return map F to Q), then the Kontsevich-Zorich cocycle provides a symplectic linear map $H_y \to H_{F(y)}$ preserving the integer lattices and the tautological bundle. Using the identification, we get an element $A(y)$ of the discrete group G_x of sympletic automorphisms of H_x preserving the integer lattice such that $A(y) \cdot V_x = V_x$. Note that $A(y)$ depends continuously on $y \in p(\widehat{\Delta}_j)$, so it must be in fact a constant, denoted by $A^{(l)}$.

If $\underline{l} = (l_1, ..., l_n)$ with $n \geq 1$ then F^n has a unique fixed point $(S, \Sigma, \omega) = p(u_{\underline{l}}, s_{\underline{l}})$ with $(u_{\underline{l}}, s_{\underline{l}}) \in \widehat{\Delta}^{\underline{l}} = (\Delta^{\underline{l}} \times \mathbb{R}) \cap \widehat{\Delta}$, and $A^{\underline{l}}|V_x$ is hyperbolic with unstable direction $\operatorname{Im}\omega$, stable direction $\operatorname{Re}\omega$, and Lyapunov exponent $r_n(u_{\underline{l}})$. In particular, $\|A^{\underline{l}}|V\|$ is of order $e^{r_n(u_{\underline{l}})}$ (up to uniformly bounded multiplicative constants), since the angle between $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$ is uniformly bounded over Q.

Note that that $\|A^{\underline{l}}\|$ is also of order $e^{r_n(u_{\underline{l}})}$ (this follows for instance from Remark 2.1). In particular ($n = 1$), the exponential tail estimate implies that A is fast decaying.

Since the geodesic flow on $\mathcal{C}$ is ergodic, the Lyapunov exponents of the Kontsevich-Zorich cocycle on $\mathcal{C}$ (with respect to the Haar measure) are the same as the Lyapunov exponents of the locally constant cocycle (T, A), with respect to the invariant measure ν, up to the normalization factor $\overline{r} = \int r(u) d\nu(u)$.

3.3. **Some simple applications.** The following is due to [BM10].

Lemma 13. *Let (S, Σ, ω) be a Veech surface, k its holonomy field and V be the tautological subbundle of the Hodge bundle of its* $\mathrm{SL}(2, \mathbb{R})$ *orbit. Then for any non identity embedding $\sigma : k \to \mathbb{R}$, the non-negative Lyapunov exponent λ^σ of the Kontsevich-Zorich cocycle restricted to V^σ satisfies $1 > \lambda^\sigma > 0$.*

Proof. The upper is a bound is Theorem 4.

For the lower bound, we use the Markov model (T, A) associated to an appropriate small Poincaré section Q. Let $V = V_x$ be the tautological space, let $G \subset \mathrm{SL}(V)$ denote the group generated by the restriction of the $A^{(l)}$ to V, and let G^σ be its Galois conjugate. If the Lyapunov exponents of $(T, A|V^\sigma)$ are non-zero then G^σ is "degenerate": it is either contained in a compact subgroup of $\mathrm{SL}(V^\sigma)$, or all elements leave invariant a direction in $\mathbb{P}V^\sigma$, or all elements leave invariant a pair of directions in $\mathbb{P}V^\sigma$ (though not necessarily leaving invariant each individual direction). This is a version of Furstenberg's criterion for positivity of the Lyapunov exponent for i.i.d. matrix products in $\mathrm{SL}(2, \mathbb{R})$, which can be obtained in our setting (where T has bounded distortion) by applying the criterion for the simplicity of the Lyapunov spectrum of [AV07] (which is stated in terms of "pinching" and "twisting" properties that are easily derived in case of non-degeneracy). Note that if G^σ is degenerate then it must be solvable, which implies that G is not solvable either.

To check that G is not solvable, we construct a copy of a free group on two generators contained in it. For any $l \in \mathbb{Z}$, the restriction of $A^{(l)}$ to the tautological bundle $V = V_x$ gives an element $g_l \in \mathrm{SL}(V)$. As remarked before g_l is always hyperbolic, with unstable direction given by $\operatorname{Re}\omega$ and stable direction given by $\operatorname{Im}\omega$, where (S, Σ, ω) is the unique fixed point of F in $p(\widehat{\Delta}^{(l)})$. It follows (using that Q is small) that for distinct $l_1, l_2 \in \mathbb{Z}$, the unstable and stable directions of g_{l_1}, g_{l_2} are all different. Thus for sufficiently large $p \in \mathbb{N}$, $g_{l_1}^p$ and $g_{l_2}^p$ generate a free subgroup of $\mathrm{SL}(V)$. □

We can also deduce non-discreteness of conjugates of the Veech group. We will need the following general result:

Lemma 14. *Let* Γ *be a lattice in* $\mathrm{SL}(2,\mathbb{R})$ *and* $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ *an injective homomorphism. Let* E *be the flat bundle over* $\mathrm{SL}(2,\mathbb{R})/\Gamma$ *associated to* ρ *and let* λ *be the non-negative exponent of the parallel transport in* E *along the geodesic flow on* $\mathrm{SL}(2,\mathbb{R})/\Gamma$*. Then* $\lambda < 1$ *implies that* ρ *is non discrete.*

Before proceeding to the proof, we recall the construction of the flat bundle associated to ρ. As in the construction of the moduli space, we consider the trivial bundle $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^2$ over $\mathrm{SL}(2,\mathbb{R})$. It has an action of Γ given by $g \cdot (z, v) := (g \cdot z, \rho(g)^{-1} \cdot v)$. The quotient $(\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^2)/\Gamma$ is by definition the flat bundle associated to ρ.

Proof. If Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ we denote

$$N_\Gamma(R) = \{g \in \Gamma; \log \|g\| \leq R\}.$$

We have

$$\limsup_{R\to\infty} \frac{\log N_\Gamma(R)}{R} \leq 2.$$

Our strategy is to notice that if Γ is of finite covolume, then the limit of the quotient exists and is 2. This statement will be contained in our proof and follows from the fact that we have a stable and unstable foliation with uniform contraction and dilatation properties. (As seen by G. Margulis in his thesis, a more precise asymptotic may be obtained using the mixing property of the geodesic flow, but this will play no role here.) By definition of the Lyapunov exponent, for most elements $g \in \Gamma$, the ratio $\log \|\rho(g)\|/\log \|g\|$ is nearby λ. This implies that if ρ was injective, the group Γ' would contain at least e^R elements of norm less than λR which contradicts the above asymptotic if Γ' was discrete.

For the formal argument, let us consider the n-th iterate of (T, A) for large n. Then with probability close to 1 we have $\frac{1}{n} \ln \|A_n|V\|$ close to $\overline{r} = \int r d\nu$ and $\frac{1}{n} \ln \|A_n|V^\sigma\|$ close to $\lambda\overline{r}$. Since the length of $\Delta^{\underline{l}}$ is comparable with $\|A^{\underline{l}}|V\|^{-2}$, we see that the number of distinct $\underline{l}$ with $|\underline{l}| = n$ and such that $\frac{1}{n} \ln \|A^{\underline{l}}|V^\sigma\|$ is close to $\lambda \int r d\nu$ is at least $e^{-2n(\overline{r}-\varepsilon)}$. Since the $A^{\underline{l}}|V$ are all distinct (as they are hyperbolic elements with distinct unstable and stable directions), the result follows. □

Corollary 15. *Let* (S, Σ, ω) *be a Veech surface,* k *its holonomy field and* Γ *its Veech group. Then, for any non identity embedding* $\sigma : k \to \mathbb{R}$ *the group* Γ^σ *is non discrete.*

3.4. **Reduction to the Markov model.** An eigenfunction $f : S \to \mathbb{C}$ with eigenvalue $\nu \in \mathbb{R}$ of a translation flow $\phi_t : S \to S$ is a measurable function such that $f \circ \phi_t = e^{2\pi i\nu t} f$. Note that if f is a measurable or continuous eigenfunction for the vertical flow on a translation surface $z = (S, \Sigma, \omega) \in \mathcal{M}_{S,\Sigma}(\kappa)$, then f is also an eigenfunction for $g \cdot z$ for any $g \in \mathrm{SL}(2,\mathbb{R})$ which fixes the vertical direction, and in particular for any g of the form $h_s^- g_t$, $s, t \in \mathbb{R}$.

Lemma 16. *Let* $\mathcal{C}$ *be a closed* $\mathrm{SL}(2,\mathbb{R})$ *orbit in some* $\mathcal{M}_{S,\Sigma}(\kappa)$*. Let* I *be a non-empty open subset of am* $\mathrm{SO}(2,\mathbb{R})$ *orbit and let* J *be a non-empty open subset of an unstable horocycle. Then:*

(1) *For every $z_0 \in I$, there exists a diffeomorphism $z \mapsto x$ from an open neighborhood $I' \subset I$ to a subinterval of J, such that the stable horocycle through z intersects the geodesic through x,*
(2) *For every $x_0 \in I$, there exists a diffeomorphism $x \mapsto z$ from an open neighborhood $J' \subset J$ to a subinterval of I, such that the stable horocycle through z intersects the geodesic through x.*

In particular, if Λ is any subset of $\mathcal{C}$ which is invariant by the stable horocycle and geodesic flows (such as the set of translation surfaces for which the vertical flow admits a continuous eigenfunction, or a measurable but discontinuous eigenfunction), $\mathrm{HD}(I \cap \Lambda) = \mathrm{HD}(J \cap \Lambda)$.

Proof. We prove the first statement. Fix $z \in I$ and some compact segment $J_0 \subset J$. Then we can choose t_0 large such that there exists $y \in J_0$ with $g_{t_0}(y)$ close to z (indeed as $t \to \infty$, $g_t \cdot I_0$ is becoming dense in $\mathcal{C}$). Thus for every $\theta \in \mathbb{R}$ close to 0 we can write $r_\theta z = h_s^- h_u^+ g_{t_0+t} y$ in a unique way with s, t, u small, and moreover $\theta \mapsto u$ is a diffeomorphism.

The second statement is analogous. □

4. Eigenfunctions in Veech surfaces

Let (S, Σ, ω) be a translation surface and $\phi_t : S \to S$ the vertical flow. We say that $\nu \in \mathbb{R}$ is an *eigenvalue* of ω if there exists a non-zero measurable function $f : S \to \mathbb{C}$ such that for almost every $x \in S$,we have $f(\phi_T(x)) = \exp(2\pi i \nu T) f(x)$ for all T. The eigenvalue is *continuous* if the map f may be chosen continuous. The flow is *weak-mixing* if it admits no eigenvalue except 0 with multiplicity one.

The Veech criterion that appeared in [Ve84] was of main importance in [AF07] to prove the genericity of weak-mixing among translation flows. This criterion depends on the consideration of appropriate compact transversal to the Teichmüller flow which is "small enough" to fit inside "zippered rectangles" charts and also satisfy some additional boundedness properties.

Theorem 17 (Veech criterion)**.** *Let $\mathcal{M}_g(\kappa)$ be a stratum of translation surfaces. For all (S, Σ, ω) in $\mathcal{M}_g(\kappa)$ there exists a small compact transversal of the Teichmüller flow containing (S, Σ, ω) such that for all (S, Σ, τ) in that transversal that is recurrent and admits an eigenvalue ν, the values of the Kontsevich-Zorich cocycle $A_n(\tau)$ for that transversal satisfy*

$$\lim_{n\to\infty} \mathrm{dist}(A_n \cdot (\nu \operatorname{Im} \tau), H^1(S \backslash \Sigma, \mathbb{Z})) = 0.$$

The above theorem holds in great generality as soon as the dynamical system is described by Rokhlin towers (see [BDM1]).

Veech's criterion tells us that we can prove that the vertical flow is weak mixing if we can show that the line $\mathbb{R} \operatorname{Im} \omega$ intersects the "weak stable lamination", the set of all $w \in H^1(S \backslash \Sigma; \mathbb{R})$ such that

$$\lim_{n\to\infty} \mathrm{dist}(A_n \cdot w, H^1(S \backslash \Sigma, \mathbb{Z})) = 0,$$

only at the origin. Unfortunately, the nature of the weak stable lamination is rather complicated. It is of course a union of translates of the stable space, the set of all $w \in H^1(S \backslash \Sigma; \mathbb{R})$ such that

$$\lim_{n\to\infty} \mathrm{dist}(A_n \cdot w, 0) = 0,$$

and it contains the "strong stable space" consisting of the integer translates of the stable space. However, in general it is much larger, being transversely uncountable.

The main objective of this section is to show that for Veech surfaces, any eigenvalue ν must be such that $\nu \operatorname{Im} \omega$ belongs must belong to the smaller (and much simpler) strong stable space.

Theorem 18. *Let (S, Σ, ω) be a Veech surface with no vertical saddle connection and whose linear flow admits an eigenvalue ν. Consider a compact transversal for the Teichmüller flow and denote by $A_n \in \mathrm{Sp}(H^1(S \backslash \Sigma, \mathbb{Z}))$ the associated Kontsevich-Zorich cocycle. Then there exists $v \in H^1(S \backslash \Sigma; \mathbb{Z})$ such that*

$$\lim_{n \to \infty} A_n \cdot (\nu \operatorname{Im}(\omega) - v) = 0.$$

In order to prove Theorem 18, we will need to control all the renormalizations of eigenfunctions, and not only those corresponding to returns to a small compact transversal. It will be crucial for our strategy that for a Veech surface, there exists a fixed compact set $\mathcal{C}_\varepsilon = \{\omega;\, \mathrm{sys}(S, \Sigma, \omega) \geq \varepsilon\}$ such that the set of "moments of compactness" $\{t > 0;\, g_t \cdot \omega \in \mathcal{C}_\varepsilon\}$ for the forward Teichmüller geodesic is unbounded if and only if there are no vertical saddle connections, and (most importantly) any orbit segment away from the moments of compactness (the cusp excursions) can be easily described geometrically.

4.1. **Tunneling curves and a dual Veech criterion.** In this section (which is not restricted to Veech surfaces) we show that the existence of eigenfunctions yields information about all times of the Teichmüller flow and not only return times to a (small or large) compact transversal. This is based on a refinement of the Veech criterion which is formulated in terms of homology cycles called *tunneling curves* (which are designed to follow closely the vertical flow in a suitable sense), which are shown to always see the expected property of approximation to integers. In a second time we prove that in any compact part of the moduli space, the tunneling curves generate $H_1(S \backslash \Sigma; \mathbb{Z})$. Those two results together allow us to remove the smallness condition on the transversal in the formulation of the Veech criterion, hence allowing us to consider the large compact set $\mathcal{C}_\varepsilon$ when analysing Veech surfaces later.

Before defining tunneling curves we need the notion of cycle of rectangles. A *rectangle* for (S, Σ, ω) is an isometric immersion of an euclidean rectangle with horizontal and vertical sides. In other words, a rectangle is a map $R : [0, w] \times [0, h] \to S \setminus \Sigma$ such that $R^*(\mathrm{Re}(\omega)) = \pm dx$ and $R^*(\mathrm{Im}(\omega)) = \pm dy$. The number w is called the *width* of the rectangle and the number h its *height*. Note that with our convention we may not identify a rectangle with its image in S, we care about the direction: a rectangle is determined by its image in S and an element of $\{+1, -1\} \times \{+1, -1\}$.

Definition 19. *A (k, δ, h)-*cycle of rectangles *for ω is a set of $2k$ rectangles denoted H_j and V_j for $j \in \mathbb{Z}/k\mathbb{Z}$ such that*

- *the height of H_j is δ and its width is $w_j \geq \delta$,*
- *the width of V_j is δ and its height is $\delta \leq h_j \leq h$,*
- *$H_j^*(\mathrm{Re}(\omega)) = \pm dx$ and $H_j^*(\mathrm{Im}(\omega)) = dy$*
- *$V_j^*(\mathrm{Re}(\omega)) = dx$ and $V_j^*(\mathrm{Im}(\omega)) = \pm dy$*
- *each rectangle H_j is embedded in the surface,*
- *for each j, $H_j([0, \delta] \times [0, \delta]) = V_{j-1}([0, \delta] \times [h_j - \delta, h_j])$ and $H_j([w_j - \delta, w_j] \times [0, \delta]) = V_j([0, \delta] \times [0, \delta])$.*

In other words, a (k,δ,h)-cycle of rectangles is a thin tube of width δ in (S,Σ,ω) made of k horizontal and k vertical pieces and that forms a cycle in the surface. We will sometimes drop the condition on the heights and write (k,δ) for (k,δ,∞).

A tunneling curve is a curve which belongs in a cycle of rectangles. More precisely, let $R = (H_j, V_j)_{j\in\mathbb{Z}/k\mathbb{Z}}$ be a (k,δ,h)-cycle of rectangles for ω. We may build a curve ζ as follows: for $j \in \mathbb{Z}/k\mathbb{Z}$, we define vertical segments $\zeta_j^v : [0, h_j - \delta] \to S\backslash\Sigma$ by $\zeta_v^j(t) = V_j(\frac{\delta}{2}, t + \frac{\delta}{2})$ and horizontal segments $\zeta_j^h : [0, w_j - \delta] \to S\backslash\Sigma$ by $\zeta_j^h(t) = H_j(t + \frac{\delta}{2}, \frac{\delta}{2})$. The curve ζ is the concatenation of ζ_1^h, ζ_1^v,..., ζ_k^h, ζ_k^v and forms a loop in the surface $S\backslash\Sigma$. The homology class of ζ in $H_1(S\backslash\Sigma;\mathbb{Z})$ is the *homology class* of the cycle of rectangles R.

A homology class $\zeta \in H_1(S\backslash\Sigma;\mathbb{Z})$ is said to be (k,δ,h)-*tunneling* if there exists a set of (k_i,δ,h)-cycles of rectangles for $i = 1,\ldots,n$ such that $k_1 + \ldots + k_n \le k$ and whose homology classes ζ_i satisfy $\sum \zeta_i = \zeta$.

Note that if ζ is a tunneling curve in a (k,δ,h)-cycle of rectangles, then

$$|\operatorname{Re}(\omega)(\zeta)| \le \int_\zeta |\operatorname{Re}(\omega)| \le \frac{k}{\delta}\operatorname{Area}(\omega) \quad \text{and} \quad |\operatorname{Im}(\omega)(\zeta)| \le \int_\zeta |\operatorname{Im}(\omega)| \le kh.$$

In particular, in a fixed translation surface (S,Σ,ω) there is only a finite number of (k,δ,h)-tunneling curves. The set of (k,δ,h)-tunneling homology classes in $H_1(S\backslash\Sigma;\mathbb{Z})$ for ω is noted $TC_{k,\delta,h}(\omega)$. The set $TC_{k,\delta}(\omega) = TC_{k,\delta,\infty}(\omega)$ denotes the set of (k,δ)-tunneling homology classes. Note that, if $k' \le k$, $\delta' \le \delta$ and $h' \ge h$ then a (k',δ',h')-tunneling curve is also (k,δ,h)-tunneling.

We will now adapt Veech's original proof of his criterion in Veech [Ve84] to obtain a dual version with respect to the tunneling curves in $TC_{k,\delta}$.

Theorem 20 (dual Veech criterion)**.** *Let (S,Σ,ω) be a translation surface without vertical saddle connections and that admits an eigenvalue ν. Then, for any positive integer k and positive real number δ we have*

$$\lim_{t\to\infty} \sup_{\zeta\in TC_{k,\delta}(g_t\cdot\omega)} \operatorname{dist}(\nu \operatorname{Im}(\omega)(\zeta), \mathbb{Z}) = 0.$$

Proof. Fix k and δ. We fix a small number α and we prove that for t big enough, all (k,δ)-tunneling curves for ω_t are such that $\operatorname{dist}(\nu\operatorname{Im}(\omega)(\zeta),\mathbb{Z}) < k\alpha$. It is enough to prove the theorem for a curve that belongs to a cycle of rectangles (recall that a tunneling curve may be a sum of curves associated to cycle of rectangles).

Let $(H_j,V_j)_{j\in\mathbb{Z}/k\mathbb{Z}}$ be a (k,δ)-cycle of rectangles for ω_t and let ζ be its homology class. We define the *signed height* of the vertical rectangle V_j by $\widetilde{h_j} = h_j - \delta$ if $(V_j)^*(\operatorname{Im}(\omega)) = dy$ and $\widetilde{h_j} = \delta - h_j$ otherwise (it is precisely the value of the integral of $\operatorname{Im}(\omega)$ along the component ζ_j^v of a curve ζ). In particular, the integral of $\operatorname{Im}\omega$ over ζ is $\widetilde{h_1} + \ldots + \widetilde{h_k}$. For each j, we define $I_j = H_j([0,w_j] \times \{\delta/2\})$ the middle interval of the rectangle H_j. The segment I_j is an horizontal interval of length $e^{-t}w_j$ for ω.

We assume that the surface S admits a non trivial eigenvalue $\nu \notin \mathbb{Z}$ and note f an associated eigenfunction $f : S \to \mathbb{C}$ with $|f| = 1$. Up to modifying f on a zero measure set, we may assume that there exists a measurable subset $\Omega \subset S$ of full area, consisting of points for which the vertical flow is defined for all times, such that $f(\phi_t(x)) = e^{2\pi i\nu} f(x)$ for every $x \in \Omega$. Note that Ω intersects any horizontal segment in a subset of full linear measure. Define measurable functions $f_j : [0,w_j] \to \mathbb{C}$ by

$f_j(t) = f(H_j(t, \delta/2))$. In particular, for each j and almost every $x \in [0, \delta]$ we have

$$\begin{array}{ll} f_j(w_j - \delta + x) = e^{-2\pi i \nu \widetilde{h_j}} f_{j+1}(x) & \text{if } H_j^*(\mathrm{Re}(\omega)) = dx \text{ and } H_{j+1}^*(\mathrm{Re}(\omega)) = dx \\ f_j(w_j - x) = e^{-2\pi i \nu \widetilde{h_j}} f_{j+1}(x) & \text{if } H_j^*(\mathrm{Re}(\omega)) = -dx \text{ and } H_{j+1}^*(\mathrm{Re}(\omega)) = dx \\ f_j(w - \delta + x) = e^{-2\pi i \nu \widetilde{h_j}} f_{j+1}(\delta - x) & \text{if } H_j^*(\mathrm{Re}(\omega)) = dx \text{ and } H_{j+1}^*(\mathrm{Re}(\omega)) = -dx \\ f_j(w_j - x) = e^{-2\pi i \nu \widetilde{h_j}} f_{j+1}(\delta - x) & \text{if } H_j^*(\mathrm{Re}(\omega)) = -dx \text{ and } H_{j+1}^*(\mathrm{Re}(\omega)) = -dx \end{array}$$

Those formulas can be rewritten as follows. Let $s_j : [0, \delta] \to [0, \delta]$ be given by $s_j(x) = \delta - x$ if $H_j^*(\mathrm{Re}\,\omega)) = dx$ and by $s_j(x) = x$ if $H_j^*(\mathrm{Re}\,\omega) = -dx$. Then $f_j(w_j - s_j(x)) = e^{-2\pi i \nu \widetilde{h_j}} f_{j+1}(\delta - s_{j+1}(x))$ in all cases.

The strategy of the proof, consists in proving that if t is big enough, independently of the choice of the (k, δ)-cycle of rectangles, we may find points $x_j \in [0, \delta]$, $j \in \mathbb{Z}/k\mathbb{Z}$ such that for each j we have $|f_j(\delta - s_j(x_{j-1})) - f_j(w_j - s_j(x_j))| < \alpha$. In particular, we can write $\frac{f_j(\delta - s_j(x_{j-1}))}{f_j(w_j - s_j(x_j))} = e^{2\pi i \lambda_j}$ where $\lambda_j \in (-\alpha, \alpha)$.

Assuming that such points do exist, we prove how to derive our theorem. Using the points x_j we may write

$$\begin{aligned} 1 &= \prod_{j \in \mathbb{Z}/k\mathbb{Z}} \frac{f_j(w_j - s_j(x_j))}{f_{j-1}(\widetilde{w_{j-1}} - s_{j-1}(x_{j-1}))} = \prod_{j \in \mathbb{Z}/k\mathbb{Z}} e^{2\pi i \nu \widetilde{h_{j-1}}} \frac{f_j(w_j - s_j(x_j))}{f_j(\delta - s_j(x_{j-1}))} \\ &= \prod_{j \in \mathbb{Z}/k\mathbb{Z}} e^{2\pi i \nu h_{j-1}} e^{-2\pi i \lambda_j} \end{aligned}$$

so that $\mathrm{Im}(\omega)(\zeta) = \sum \lambda_j \mod \mathbb{Z}$, implying the result.

Now, we show how to find points x_j using a measure theoretic argument. More precisely, we prove that the measure of the set of points $(x, y) \in [0, \delta] \times [w_j - \delta, w_j]$ such that $|f_j(x) - f_j(y)| < \alpha$ becomes arbitrarily close to δ^2 as t goes to infinity independently of the choice of the cycle of rectangles. Since $\delta \leq w_j \leq \delta^{-1}$, it is enough to show that there exists a compact subset $K_j \subset I_j$ with probability close to 1 such that $|f(x) - f(y)| < \alpha$ for every $x, y \in K_j$.

Fix some small constant $\chi > 0$. By Lusin's Theorem, there exists a compact subset $K \subset S$ of measure $1 - \chi$ such that $f|K$ is continuous. In particular, there exists $\varepsilon > 0$ such that if $x, y \in K$ are ε-close then $|f(x) - f(y)| < \alpha$.

Notice that the rectangle H_j has width $e^{-t} w_j$ and height $e^t \delta$ in (S, Σ, ω). Recall that $\delta \leq w_j \leq \frac{1}{\delta}$, and in particular the area of H_j is at least δ^2, so K must intersect it in a subset of probability at least $1 - \delta^{-2}\kappa$. It follows that some (full) horizontal segment $I'_j = H_j([0, w_j] \times \{T\})$ in this rectangle intersects K into a subset K'_j of probability at least $1 - \delta^{-2}\chi$ as well. Take t so large that $e^{-t}\delta^{-1} < \varepsilon$. Then $|f(x) - f(y)| < \alpha$ for every $x, y \in K'_j$. Note that $I'_j = \phi_{\pm e^t (T - \delta/2)}(I_j)$ (the same sign as when writing $H_j^*(\mathrm{Im}\,\omega) = \pm dy$) so by the functional equation we have $|f(x) - f(y)| < \alpha$ for every $x, y \in K_j = \phi_{\mp e^t (T - \delta/2)}(K'_j)$, as desired. □

We now prove that any translation surface admits tunneling basis and the constant may be taken uniform in compact sets.

Lemma 21. *Let $\mathcal{M}^{(1)}_{S,\Sigma}(\kappa)$ be a stratum in moduli space and let $\varepsilon > 0$. Let $K_\varepsilon \subset \mathcal{M}_{S,\Sigma}(\kappa)$ be the set of translation surfaces whose systole is at least ε. Then there exists (k, δ, h) such that for any translation surface $\omega \in K_\varepsilon$, the (k, δ, h)-tunneling curves for ω generate $H_1(S \backslash \Sigma; \mathbb{Z})$.*

Proof. The set $A_{k,\delta,h}$ of surfaces in $\mathcal{M}_{S,\Sigma}(\kappa)$ that admits a (k, δ', h')-tunneling basis with $\delta' > \delta$ and $h' < h$ is an open set. From compactness of K_ε it is hence enough to prove that for any translation surface in $\mathcal{M}_{S,\Sigma}(\kappa)$, every closed curve in $S \setminus \Sigma$

is homotopic to a (k, δ, h)-tunneling curve, for some k, δ and h. Indeed, up to homotopy we may assume that a closed curve is built by concatenating small (and hence embedded) horizontal and vertical segments (in alternation). Those segments can then be slightly thickened to build the desired cycle of rectangles. □

4.2. **Excursions in cusps.** In this section we prove Theorem 18.

We first give another formulation of Theorem 18 in terms of tunneling basis (in order to use Theorem 20). Let $\mathcal{C} \subset \mathcal{M}_{S,\Sigma}(\kappa)$ be the $\mathrm{SL}(2,\mathbb{R})$ orbit of a Veech surface, let $\varepsilon > 0$ be small, and let $\mathcal{C}_\varepsilon$ be the set of surfaces in $\mathcal{C}$ whose systole is at least ε. From Lemma 21, we get k, δ and h such that each translation structure $\omega \in \mathcal{C}_\varepsilon$ has a (k, δ, h)-tunneling basis in $H_1(S \backslash \Sigma; \mathbb{Z})$. Using compactness and the finiteness of (k, δ, h)-tunneling curves, there exists a constant $M > 1$ such that for any translation structure ω in $\mathcal{C}_\varepsilon$ any (k, δ, h)-tunneling basis $\{\zeta_j\}$ and any (k, δ, h)-tunneling curve ζ for ω, the coefficients of $\zeta = \sum c_j \zeta_j$ with respect to the basis satisfy $\sum |c_j| < M$.

Let $\omega \in \mathcal{C}$ be a translation surface that admits an eigenvalue ν. From Theorem 20, there exists t_0 such that for any t larger than t_0, to each (k, δ, h)-tunneling curve ζ for ω_t, we may associate a unique $n_t(\zeta) \in \mathbb{Z}$ such that $|\nu \operatorname{Im}_\omega(\zeta) - n_t(\zeta)| < 1/(2M)$. Let $t \geq t_0$ be such that $\omega_t \in \mathcal{C}_\varepsilon$. If $\{\zeta_j\}$ is a (k, δ, h)-tunneling basis and $\zeta = \sum c_j \zeta_j$ is a (k, δ, h)-tunneling curve, then

$$|\nu \operatorname{Im}_\omega(\zeta) - \sum c_j n_t(\zeta_j)| = |\sum c_j (\nu \operatorname{Im}_\omega(\zeta_j) - n_t(\zeta_j))| < \frac{1}{2M} \sum |c_j| < \frac{1}{2},$$

so that $n_t(\zeta) = \sum c_j n_t(\zeta_j)$. Thus the mapping $n_t : TC_{k,\delta,h}(\omega_t) \to \mathbb{Z}$ extends in a unique way to a linear map $n_t : H_1(S \backslash \Sigma; \mathbb{Z}) \to \mathbb{Z}$. The convergence to an integer element in Theorem 18 is then equivalent to the following statement.

Lemma 22. *Let ω be a Veech surface in $\mathcal{C}$ without vertical saddle connections for which the translation flow admits an eigenvalue ν and let $\omega_t = g_t \cdot \omega$. Let $\varepsilon > 0$ and let t_0 and $n_t \in H^1(S \backslash \Sigma; \mathbb{Z})$ be as above. Then the family $(n_t)_{t \geq t_0, \omega_t \in \mathcal{C}_\varepsilon}$ is eventually constant.*

The proof of Lemma 22 follows by analyzing parts of Teichmüller geodesics that go off $\mathcal{C}_\varepsilon$ because, by construction, n_t is locally constant.

Until the end of this section, fix $\varepsilon > 0$ such that the cusps of $\mathcal{C}$ are isolated in the complement of $\mathcal{C}_\varepsilon$ and that the conclusion of Corollary 8 holds (the former is actually a consequence of the latter). A *cusp excursion* of length $t > 0$ is a segment of a Teichmüller orbit $\omega_s = g_s \cdot \omega_0$, $s \in [0, t]$, such that $\omega_s \in \mathcal{C}_\varepsilon$ only for $s = 0, t$. Note that the shortest saddle connection at the beginning of a cusp excursion is never horizontal or vertical, and indeed we have $|\operatorname{Im}_{\omega_0}(\gamma)| > |\operatorname{Re}_{\omega_0}(\gamma)| > 0$, with the length of the cusp excursion given by $t = \log \frac{|\operatorname{Im}_{\omega_0}(\gamma)|}{|\operatorname{Re}_{\omega_0}(\gamma)|}$.

If ω belong to $\partial \mathcal{C}_\varepsilon = \{(S, \Sigma, \omega) \in \mathcal{C};\ \mathrm{sys}(S, \Sigma, \omega) = \varepsilon\}$, then S admits a canonical decomposition as a finite union of maximal cylinders C_i, $1 \leq i \leq c$, with waist curve γ_i parallel to the shortes saddle connection γ.

Lemma 23. *For any (k, δ, h) with $\delta > 0$ small enough and $h > 0$ big enough, there exists an integer $k' \geq k$ with the following property. Let us consider a cusp excursion ω_s of length t. Let ε be the sign of $\frac{\operatorname{Im}_{\omega_0}(\gamma)}{\operatorname{Re}_{\omega_0}(\gamma)}$ and let $m_i = \left\lfloor \dfrac{e^t}{\mu(C_i)} \right\rfloor$, where $\mu(C_i)$ is the modulus of the cylinder C_i in the canonical decomposition of*

(S, Σ, ω_0). *For each* (k, δ, h)*-tunneling curve* ζ *for* ω_0 *the class* $\zeta - \varepsilon \sum_{i=1}^{c} m_i \langle \zeta, \gamma_i \rangle \gamma_i$ *is* (k', δ, h)*-tunneling for* ω_t *and for any integers* ℓ_i *such that* $0 \leq \ell_i \leq m_i$ *the classes* $\zeta - \varepsilon \sum_{i=1}^{c} \ell_i \langle \zeta, \gamma_i \rangle \gamma_i$ *are* (k', δ)*-tunneling for* ω_0.

Proof. We first reduce our study to (non-necessarily closed) paths inside a single cylinder. Let $X = \{x_1, x_2, \ldots, x_p\}$ be the middle points of saddle connections in the direction of the shortest saddle connection for ω_0. We call *transversal*, a flat geodesic segment γ' that joins two points of X and disjoint from saddle connections parallel to the shortest one. Any curve in (S, Σ, ω_0) is freely homotopic in $S \backslash \Sigma$ to a concatenation of transversals such that $|\operatorname{Im}_{\omega_0}(\gamma')| < 2M$ and $|\operatorname{Re}_{\omega_0}(\gamma')| < 2M$ where the constant M may be choosen independently of (S, Σ, ω) in $\partial\mathcal{C}_\varepsilon$. Moreover, if ζ is a (k, δ, h)-tunneling curve for ω_0, the minimal number of pieces is uniformly bounded in terms of ε, k, δ and h.

Let us fix x and y on the boundary of some cylinder C_i and denote by $T(x, y)$ the set of transversals that join x to y. Considering a curve in $T(x, y)$ up to homotopy in $S \setminus \Sigma$ fixing the boundary points, one obtains an element of $H_1(S \setminus \Sigma, \{x, y\}; \mathbb{Z})$.

We build rectangles around the curves in $T(x, y)$ in order to be able to reconstruct a cycle of rectangles. A (k', δ, h)-*path of rectangles* for x and y is a set of rectangles H_1, V_1, $\ldots$, $H_{k'}$, $V_{k'}$, $H_{k'+1}$ such that

- the height of H_j is δ and its width is $w_j \geq \delta$,
- the width of V_j is δ and its height h_j satisfies $\delta \leq h_j \leq h$,
- $H_j^*(\operatorname{Re}(\omega)) = \pm dx$ and $H_j^*(\operatorname{Im}(\omega)) = dy$
- $V_j^*(\operatorname{Re}(\omega)) = dx$ and $V_j^*(\operatorname{Im}(\omega)) = \pm dy$
- each rectangle H_j is embedded in the surface,
- for each j, $H_j([0, \delta] \times [0, \delta]) = V_{j-1}([0, \delta] \times [h_j - \delta, h_j])$ and $H_j([w_j - \delta, w_j] \times [0, \delta]) = V_j([0, \delta] \times [0, \delta])$.
- $H_1(\delta/2, \delta/2) = x$ and $H_{k'+1}(w_{k'+1} - \delta/2, \delta/2) = y$.

As we did for cycles of rectangles, to a (k', δ, h)-path of rectangles for x and y we may associate its homology class in $H_1(S \setminus \Sigma, \{x, y\}; \mathbb{Z})$.

Let us fix a transversal γ' joining x and y inside some cylinder C_i and such that $|\operatorname{Im}_{\omega_0}(\gamma')| < 2M$ and $|\operatorname{Re}_{\omega_0}(\gamma')| < 2M$. Since any (k, δ, h)-tunneling curve can be decomposed into a uniformly bounded number of such transversals, it will be enough for us to prove that there exists $k' > 0$ (only depending on $\varepsilon, \delta, h, M$) with the following properties:

(1) The transversal $\gamma' \in T(x, y)$ in the class of $\gamma' - \varepsilon m_i \langle \gamma', \gamma_i \rangle \gamma_i$ is a (k', δ, h)-tunneling path for ω_t,
(2) For every $0 \leq \ell \leq m_i$, the class of $\ell \gamma_i$ is (k', δ)-tunneling for ω_0.

Note that the width $w(C_i)$, the height $h(C_i)$, and the modulus, $\mu(C_i)$ are all bounded away from zero and infinity, independently of i, through $\partial\mathcal{C}_\varepsilon$. In particular, we may assume that $w(C_i)$ and $h(C_i)$ are bigger than 10δ. Note also that $\langle \gamma', \gamma_i \rangle = \pm 1$.

Write $\frac{\operatorname{Im}_{\omega_0}(\gamma)}{\operatorname{Re}_{\omega_0}(\gamma)}$ as $\varepsilon \tan\theta$ with $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. We may assume that the cusp excursion of the surface (S, Σ, ω_0) in $\mathcal{C}_\varepsilon$ is long enough, so that in particular $m_i \geq 10$ and $\tan\theta > \frac{2M}{h(C_i)}$.

Let us first show the second property. It is enough to show that for $0 \leq \ell \leq \frac{3m_i}{4}$, $\ell\gamma_i$ can be represented by a $(2,\delta)$-tunneling curve. Let $\hat{C}_i$ denote the "core of C_i", obtained by removing a $\frac{h(C_i)}{8}$-neighborhood of its boundary. If $l \leq \frac{3m_i}{4}$, then we can represent $\ell\gamma_i$ by a concatenation γ of a vertical path of length $\ell w(C_i)\sin\theta$ and a horizontal path of length $\ell w(C_i)\cos\theta$ inside $\hat{C}_i$.[4] It easily follows that the δ-enlargement of the horizontal part of γ is embedded in C_i, while the δ-enlargement of the vertical part of γ is contained in C_i, so that γ is $(2,\delta)$-tunneling.

Let us now show the first property. By compactness considerations, it will be enough to show that the transversal γ' has bounded length with respect to ω_t. Up to changing the orientation of γ', we may assume that $\langle\gamma',\gamma_i\rangle = 1$. Then, for the imaginary part we have

$$|\operatorname{Im}_{\omega_t}(\gamma'-\varepsilon m_i\gamma_i)| = e^{-t}|\operatorname{Im}_{\omega_0}(\gamma'-\varepsilon m_i\gamma_i)| < e^{-t}2M + e^{-t}\left\lfloor\frac{e^t}{\mu(C_i)}\right\rfloor \leq 2M + \frac{1}{\mu(C_i)}.$$

For the real part, note first that

$$\begin{aligned}\operatorname{Re}_{\omega_t}(\gamma'-\varepsilon m_i\gamma_i) &= e^t \operatorname{Re}_{\omega_0}(\gamma'-\varepsilon m_i\gamma_i)\\ &= \frac{|\operatorname{Im}_{\omega_0}(\gamma_i)|}{|\operatorname{Re}_{\omega_0}(\gamma_i)|}\left(\operatorname{Re}_{\omega_0}(\gamma') - \varepsilon m_i \operatorname{Re}_{\omega_0}(\gamma_i)\right)\\ &= \pm|\operatorname{Im}_{\omega_0}(\gamma_i)|\left(\frac{\operatorname{Re}_{\omega_0}(\gamma')}{\operatorname{Re}_{\omega_0}(\gamma_i)} - \varepsilon m_i\right).\end{aligned}$$

Recall that $m_i = \lfloor\frac{h(C_i)}{u(C_i)}\tan\theta\rfloor$, while $\operatorname{Im}_{\omega_0}(\gamma_i)$ is uniformly bounded. In order to conclude, let us show that $\frac{\operatorname{Re}_{\omega_0}(\gamma')}{\operatorname{Re}_{\omega_0}(\gamma_i)}$ is at a uniformly bounded distance from $\varepsilon\frac{h(C_i)}{w(C_i)}\tan\theta$. Using that $\langle\gamma',\gamma_i\rangle = 1$ and that $\tan\theta > \frac{2M}{h(C_i)}$, we see that $\operatorname{Re}_{\omega_0}(\gamma')$ and $\operatorname{Im}_{\omega_0}(\gamma_i)$ have the same sign,[5] which implies that $\frac{\operatorname{Re}_{\omega_0}(\gamma')}{\operatorname{Re}_{\omega_0}(\gamma_i)}$ and $\varepsilon\frac{h(C_i)}{w(C_i)}\tan\theta$ have the same sign as well. We have $|\operatorname{Re}_{\omega_0}(\gamma_i)| = w(C_i)\cos\theta$ and

$$|\operatorname{Re}_{\omega_0}(\gamma')| \pm |\operatorname{Im}_{\omega_0}(\gamma')|\cot\theta = \frac{h(C_i)}{\sin\theta},$$

so that

$$\frac{|\operatorname{Re}_{\omega_0}(\gamma')|}{|\operatorname{Re}_{\omega_0}(\gamma_i)|} = \frac{h(C_i)}{w(C_i)}\frac{1}{\sin\theta\cos\theta} \mp \frac{|\operatorname{Im}_{\omega_0}(\gamma')|}{w(C_i)\sin\theta}.$$

It follows that

$$\frac{|\operatorname{Re}_{\omega_0}(\gamma')|}{|\operatorname{Re}_{\omega_0}(\gamma_i)|} - \frac{h(C_i)}{w(C_i)}\tan\theta = \frac{h(C_i)}{w(C_i)}\cot\theta \mp \frac{|\operatorname{Im}_{\omega_0}(\gamma')|}{w(C_i)\sin\theta}.$$

Since $h(C_i)$ is uniformly bounded away from infinity, $w(C_i)$ is uniformly bounded away from 0, $\cot\theta < 1$, $\sin\theta > 2^{-1/2}$ and $|\operatorname{Im}_{\omega_0}(\gamma')| < 2M$, the result follows. □

We now prove how Lemma 23 may be used to conclude the proof of Lemma 22.

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[4]To see that such a concatenation lies inside $\hat{C}_i$, note that the maximal length of a vertical path in $\hat{C}_i$ is exactly $\frac{3}{4}\frac{h(C_i)}{\cos\theta}$ and $\frac{3}{4}m_i w(C_i)\sin\theta \leq \frac{3}{4}h(C_i)\frac{\sin^2\theta}{\cos\theta} \leq \frac{3}{4}\frac{h(C_i)}{\cos\theta}$.

[5]Indeed, let us consider an horizontal path γ'' in C_i joining the boundaries of C_i, which is homotopic to γ' relative to ∂C_i. Since $\operatorname{Im}_{\omega_0}(\gamma') < 2M$, the condition $\tan\theta > \frac{2M}{h(C_i)}$ implies that the sign of $\operatorname{Re}_{\omega_0}(\gamma')$ is the same as the sign of $\operatorname{Re}_{\omega_0}(\gamma'')$, and since $\langle\gamma'',\gamma_i\rangle = \langle\gamma',\gamma_i\rangle = 1$, this must have the same sign as $\operatorname{Im}_{\omega_0}(\gamma_i)$.

Proof of Lemma 22. Let (k,δ,h) with δ small enough and h large enough that every surface in $\mathcal{C}_\varepsilon$ admits a (k,δ,h)-tunneling basis, and such that for every surface in $\partial\mathcal{C}_\varepsilon$, the waist curves γ_i of the canonical cylinder decomposition are (k,δ,h)-tunneling. Let $k' \geq k$ be such that the conclusion of Lemma 23 holds. Let M' be an upper bound for $\sum |c_\alpha|$ over all expressions $\sum c_\alpha \zeta_\alpha$ of a (k',δ,h)-tunneling curve in a (k',δ,h)-tunneling basis. Let (S,Σ,ω) be a surface that admits an eigenvalue ν. We know from Theorem 20 that there exists a time t_0 such that for every $t \geq t_0$ and every (k',δ)-tunneling curve α for $\omega_t = g_t \cdot \omega$ we have $\mathrm{dist}(\nu\,\mathrm{Im}_\omega(\alpha),\mathbb{Z}) < 1/(4M')$.

Recall that for $t \geq t_0$ such that $\omega_t \in \mathcal{C}_\varepsilon$, we may define $n_t \in H^1(S\backslash\Sigma;\mathbb{Z})$ from the nearest integer vectors of elements of (k',δ,h)-tunneling basis. By construction, n_t remains constant in interval of times for which $\omega_t \in \mathcal{C}_\varepsilon$. Let $t \geq t_0$ be such that ω_t is the beginning of a cusp excursion of length τ. We will prove that $n_{t+\tau} = n_t$.

From Lemma 23, we know that there exist basis $\{\zeta_j^0\}$ and $\{\zeta_j^1\}$ of $H_1(S\backslash\Sigma;\mathbb{Z})$ such that the ζ_j^0 are (k,δ,h)-tunnneling for ω_t, the ζ_j^1 are (k',δ,h)-tunneling for $\omega_{t+\tau}$ and for each j, $\zeta_j^1 - \zeta_j^0 = -\varepsilon \displaystyle\sum_{i=1}^{c} m_i \langle \zeta_j^0,\gamma_i\rangle \gamma_i$ is a sum of multiples of waist curves of cylinders in the canonical decomposition of (S,Σ,ω_t). Moreover, each partial sum $\zeta_j^\ell = \zeta_j^0 - \varepsilon \displaystyle\sum_{i=1}^{c} \ell_i \langle \zeta_j^0,\gamma_i\rangle\gamma_i$, with $0 \leq \ell_i \leq m_i$, is (k',δ)-tunneling for ω_t which implies by choice of $t \geq t_0$ that $\nu\,\mathrm{Im}_\omega(\zeta_j^\ell)$ is at distance less than $1/(4M')$ from an integer. Because we may pass from ζ_j^0 to ζ_j^1 through a sequence of ζ_j^ℓ with two consecutive ones differing by a single curve γ_i which is (k',δ,h)-tunneling for ω_t (and hence that $\nu\,\mathrm{Im}_\omega(\gamma_i)$ is $1/(4M')$-close to $n_t(\gamma_i)$), we deduce that $n_{t+\tau}(\zeta_j^1) = n_t(\zeta_j^0) - \varepsilon \sum_{i=1}^c m_i \langle \zeta_j^0,\gamma_i\rangle n_t(\gamma_i)$. It follows that n_t and $n_{t+\tau}$ coincide as elements of $H^1(S\backslash\Sigma;\mathbb{Z})$. $\square$

4.3. **On the group of eigenvalues.** Using Theorem 18, we will show that the Kronecker factor (the maximal measurable almost periodic factor) of the translation flow of a Veech surface is always small. For arithmetic Veech surfaces (square tiled surfaces), we will see that, in any minimal direction, this factor actually identifies with a maximal torus quotient of that surface. For non-aritmetic one, we obtain that the dimension of the Kronecker factor is at most the degree of the holonomy field.

Let (S,Σ,ω) be a Veech surface, $V = \mathrm{Re}\,\omega \oplus \mathrm{Im}\,\omega \subset H^1(S;\mathbb{R})$ the tautological bundle and k its trace field. For each embedding $\sigma : k \to \mathbb{R}$ we note V^σ the Galois conjugate of V. The subspace $W = \bigoplus V^\sigma \subset H^1(S;\mathbb{R})$ is defined over $\mathbb{Q}$ and has dimension $2[k:\mathbb{Q}]$.

The field k acts by multiplication on $H^1(S;\mathbb{R})$ preserving $H^1(S;\mathbb{Q})$ as follows: for $\lambda \in k$ consider the endomorphism of $H^1(S;\mathbb{R})$ that acts by multiplication by λ^σ in V^σ. In particular, the set of elements $\lambda \in k$ that preserves $H^1(S;\mathbb{Z})$ forms an order (a $\mathbb{Z}$-module of rank $[k:\mathbb{Q}]$, stable under multiplication). This phenomenon is actually much deeper as the action of k preserves the complex structure on $H^1(S;\mathbb{C})$ and the Hodge decomposition $H^1(S;\mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$ into holomorphic and anti-holomorphic one forms: Veech surfaces belong to so-called *real multiplication loci*, see [McM03] and [BM10].

Now we turn to the case of arithmetic surfaces and describe their maximal tori. Let (S,Σ,ω) be a square tiled surface. Let x be a point in S and Λ the subgroup of

$\mathbb{C}$ generated by the integration of ω along closed loops. Then we have a well defined map $f : S \to \mathbb{C}/\Lambda$ defined by $f(y) = \int_\gamma \omega \mod \Lambda$ where γ is any path that joins x to y. The intersection $H^1(S;\mathbb{Z}) \cap (\operatorname{Re}\omega \oplus \operatorname{Im}\omega)$ naturally identifies to $H^1(\mathbb{C}/\Lambda;\mathbb{Z})$ through f^* and we call $\mathbb{C}/\Lambda$ together with the projection f the *maximal torus* of S. Note that f is not necessarily ramified over only one point.

Theorem 24. *Let (S,Σ,ω) be a Veech surface, k its trace field. Then in each minimal direction the group of eigenvalues is finitely generated. Moreover,*

- *if (S,Σ,ω) is arithmetic ($k=\mathbb{Q}$) then, in each minimal direction, all eigenfunctions of the flow of S are lifts from the maximal torus of S. In particular, there are exactly 2 rationally independent continuous eigenvalues.*
- *if (S,Σ,ω) is non-arithemtic ($k \neq \mathbb{Q}$) then, the ratio of any two eigenvalues for the flow of S belongs to k. In particular, in each minimal direction, there are at most $[k:\mathbb{Q}]$ rationally independent eigenvalues.*

Proof. Let us assume that $\nu \in \mathbb{R}$ is an eigenvalue of the flow of (S,Σ,ω).

Let $W = \bigoplus V^\sigma$ and $W_\mathbb{Z} = W \cap H^1(S;\mathbb{Z})$. Let $E^s \subset W$ be the stable space of the Kontsevich-Zorich cocycle restricted to W and denote $E^{s,\sigma} = E^s \cap V^\sigma$. Note that $E^{s,\sigma}$ has dimension at most 1. From Theorem 18, if ν is an eigenvalue of the flow, there exists $v \in W_\mathbb{Z}$ such that $\nu \operatorname{Im}\omega - v \in E^s$. The map $\nu \mapsto v$ provides a isomorphism between the group of eigenvalues and a subgroup of $W_\mathbb{Z}$, so the group of eigenvalues is finitely generated.

Decomposing $v = \sum v_\sigma$ with respect to the direct sum $W = \bigoplus V^\sigma$ we get

- $\nu \operatorname{Im}\omega - v_{id} \in E^{s,id}$,
- for any $\sigma \neq id$, $v_\sigma \in E^{s,\sigma}$.

In particular, if the dimension of E^s is not maximal then there is no eigenvalue.

The action of $\mathcal{O}_k$ preserves the set of lines in each V^σ and hence preserves (globally) the stable space E^s. In particular, if $\nu \operatorname{Im}\omega - v \in E^s$ then for any $\lambda \in \mathcal{O}_k$ we have $\lambda\nu \operatorname{Im}\omega - \sum \sigma(\lambda) v_\sigma \in E^s$. So the set of potential eigenvalues

$$\Theta = \{\mu \in \mathbb{R}; \exists v \in W_\mathbb{Z}, \quad \mu \operatorname{Im}\omega - v \in E^s\}$$

is stable under multiplication by $\mathcal{O}_k$.

If $k = \mathbb{Q}$, then we saw that $W_\mathbb{Z}$ naturally identifies to the cohomology of the maximal torus of S. As all eigenvalues are contained in Θ, they all come from the maximal torus.

If $k \neq \mathbb{Q}$, we know that the dimension of Θ is at most $[k:\mathbb{Q}]+1$ (there are two dimension for V and one dimension for each V^σ with $\sigma \neq id$). But, as Θ is stable under multiplication by $\mathcal{O}_k$ its rank is a multiple of $[k:\mathbb{Q}]$ and hence is 0 or $[k:\mathbb{Q}]$. As the ratio of any two eigenvalues is the ratio of two elements of Θ it belongs to k. □

5. Anomalous Lyapunov behavior, large deviations and Hausdorff dimension

We have seen so far that weak mixing can be established by ruling out nontrivial intersections of $\operatorname{Im}\omega$ with integer translates of the strong stable space. We will later see how this criterion can be rephrased in terms of certain fixed vectors (projections of integer points on Galois conjugates of the tautological bundle) lying in the strong stable space, which implies in particular that its iterate must see a non-positive rate of expansion, instead of the expected rate (given by one of the positive Lyapunov exponents).

In this section we introduce techniques to bound anomalous Oseledets behavior in the setting of locally constant cocycles with bounded distortion. The Oseledets Theorem states that for a typical orbit, any vector will expand precisely at the rate of some Lyapunov exponent. For a given vector, one can consider the minimum expansion rate which can be seen with positive probability. We will first show a (finite time) upper bound on the probability of seeing less than such minimum expansion. Then we will show that such an estimate can be converted into an upper bound on the Hausdorff dimension of orbits exhibiting exceptionally small expansion.

5.1. **Large deviations.** Let $T : \Delta \to \Delta$ be a transformation with bounded distortion with respect to the reference measure μ, let ν be the invariant measure, and let (T, A) be a locally constant integrable cocycle over T. The *expansion constant* of (T, A) is the maximal $c \in \mathbb{R}$ such that for all $v \in \mathbb{R}^d \setminus \{0\}$ and for μ-almost every $x \in \Delta$ we have

$$\lim_{n\to\infty} \frac{1}{n} \ln \|A_n(x) \cdot v\| \geq c.$$

(The limit exists by Oseledets Theorem applied to ν.)

Theorem 25. *Assume that A is fast decaying. for every $c' < c$, there exist $C_3 > 0$, $\alpha_3 > 0$ such that for every unit vector $v \in \mathbb{R}^d$,*

$$\mu\{\|A_n(x) \cdot v\| \leq e^{c'n}\} \leq C_3 e^{-\alpha_3 n}.$$

Proof. For $v \in \mathbb{R}^d \setminus \{0\}$, let $I(x, n, v) = \frac{1}{n} \ln \frac{\|A_n(x)\cdot v\|}{\|v\|}$ and $I(\underline{l}, v) = \frac{1}{|\underline{l}|} \frac{\|A^{\underline{l}}\cdot v\|}{\|v\|}$.

We claim that

$$\lim_{n\to\infty} \inf_{|\underline{l}|=n} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \int I(x, n, v) d\mu^{\underline{l}}(x) \geq c.$$

First notice that we have, for every $\delta > 0$,

$$\lim_{n\to\infty} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \mu\{x; I(x, n, v) < c - \delta\} = 0.$$

Indeed, the definition of c gives

$$\sup_{v\in\mathbb{R}^d\setminus\{0\}} \lim_{n\to\infty} \mu\{x; I(x, n, v) < c - \delta\} = 0,$$

and we can use compactness to exchange quantifiers since v may be assumed to be a unit vector. See Lemma 3.1 of [AF07] for an elaboration.

Since $\frac{1}{C}\mu \leq \mu^{\underline{l}} \leq C\mu$, the claim now follows since $I(x, n, v) \leq \frac{1}{n} \ln \|A_n(x)\|$ and the sequence $\frac{1}{n} \ln \|A_n(x)\|$ is uniformly integrable: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $n \geq 1$,

$$\int_X \frac{1}{n} \ln \|A_n(x)\| d\mu < \varepsilon,$$

over every set X satisfying $\mu(X) < \delta$.

Using the claim, we see that there exists $n_0 \geq 1$, $\kappa > 0$ satisfying

$$\sup_{v\in\mathbb{R}^d\setminus\{0\}} \sup_{\underline{l}} \int c' - I(x, n_0, v) d\mu_{\underline{l}} < -\kappa.$$

By fast decay, there exists $C' > 0$, $\delta' > 0$ such that for $1 \leq n \leq n_0$, $\mu_{\underline{l}}\{x; I(x,n,v) \geq |t|\} \leq e^{-C'\delta' t}$, for every $v \in \mathbb{R}^d \setminus \{0\}$ and every $\underline{l}$. This implies that there exists $C'' > 0$, $\delta'' > 0$ such that for $s \in \mathbb{C}$ with $|s| < \delta''$, $1 \leq n \leq n_0$, $v \in \mathbb{R}^d \setminus \{0\}$ and every $\underline{l}$

$$\int |e^{sn(c'-I(x,n,v))}| d\mu_{\underline{l}}(x) \leq C'',$$

so that

$$s \mapsto \int e^{sn(c'-I(x,n,v))} d\mu_{\underline{l}}(x)$$

are uniformly bounded holomorphic functions of $|s| < \delta''$. Note that they equal to 1 at $s = 0$ and their derivative at $s = 0$ is $n \int c' - I(x,n,v) d\mu_{\underline{l}}(x)$. Thus there exists $\delta > 0$ such that

$$\sup_{v \in \mathbb{R}^d \setminus \{0\}} \sup_{\underline{l}} \int e^{\delta n_0 (c'-I(x,n_0,v))} d\mu_{\underline{l}} < e^{-\kappa n_0 \delta},$$

while, for every $1 \leq n \leq n_0 - 1$,

$$\sup_{v \in \mathbb{R}^d \setminus \{0\}} \sup_{\underline{l}} \int e^{\delta n (c'-I(x,n,v))} d\mu_{\underline{l}} < 2e^{-\kappa n \delta}.$$

Note that

$$\int e^{\delta(n+m)(c'-I(x,n+m,v))} d\mu(x) \leq \sum_{|\underline{l}|=n} \mu(\Delta^{\underline{l}}) e^{\delta n (c'-I(\underline{l},v))} \int e^{\delta m (c'-I(x,A^{\underline{l}} \cdot v))} d\mu^{\underline{l}}(x)$$

$$\leq \int e^{\delta n (c'-I(x,n,v))} d\mu(x) \sup_{|\underline{l}|=m} \int e^{\delta m (c'-I(x,m,A^{\underline{l}} \cdot v))} d\mu_{\underline{l}}.$$

It follows that for every $n \geq 1$,

$$\int e^{\delta n (c'-I(x,n,v))} d\mu(x) \leq 2e^{-\kappa n \delta},$$

so that

$$\mu\{x; I(x,n,v) \leq c'\} \leq 2e^{-\kappa n \delta}$$

giving the result. □

Remark 5.1. *The previous theorem can be somewhat refined: If A is fast decaying and for some vector $v \in \mathbb{R}^d \setminus \{0\}$ we have $\lim \frac{1}{n} \ln \|A_n(x) \cdot v\| > c'$ for a positive μ-measure set of $x \in \Delta$, then the μ-measure of the set of x such that $\frac{1}{n} \ln \|A_n(x) \cdot v\| \leq c'$ is exponentially small in n. This can be proved by reduction to the setting above after taking the quotient by an appropriate invariant subspace.*

5.2. **Hausdorff dimension.** The next result shows how to convert Theorem 25 into an estimate on Hausdorff dimension. We will assume that $T : \Delta \to \Delta$ is a transformation with bounded distortion, Δ is a simplex in $\mathbb{PR}^p$ for some $p \geq 2$ and $T|\Delta^{(l)}$ is a projective transformation for every $l \in \mathbb{Z}$.

Theorem 26. *Assume that T is fast decaying. For $n \geq 1$, let $X_n \subset \Delta$ be a union of $\Delta^{\underline{l}}$ with $|\underline{l}| = n$, and let $X = \liminf X_n$. Let*

$$\delta = \limsup_{n \to \infty} \frac{1}{n} \ln \mu(X_n),$$

then $\mathrm{HD}(X) \leq p - 1 - \min(\delta, \alpha_1)$ where α_1 is the fast decay constant of T.

We will need a preliminary result:

Lemma 27. *Assume that T has bounded distortion and is fast decaying. Then for $0 < \alpha_4 < \alpha_1$, there exists $C_4 > 0$ such that for every $n \geq 1$, we have*

$$\sum_{|\underline{l}|=n} \mu(\Delta^{\underline{l}})^{1-\alpha_4} \leq C_4^n.$$

Proof. Notice that for $0 < \varepsilon < \alpha_1$,

$$\sum_l \mu(\Delta^{(l)})^{1-\varepsilon} \leq \sum_{k \geq 0} 2^{-(1-\varepsilon)k} \frac{C_1 2^{-\alpha_1 k}}{2^{-k-1}} \leq 2C_1 \sum_{k\geq 0} 2^{(\varepsilon-\alpha_1)k}.$$

It follows that for every $\underline{l}$,

$$\sum_l \mu^{\underline{l}}(\Delta^{(l)})^{1-\varepsilon} \leq C_4.$$

On the other hand, it is clear that

$$\sum_{|\underline{l}'|=n+1} \mu(\Delta^{\underline{l}'})^{1-\varepsilon} = \sum_{|\underline{l}|=n} \mu(\Delta^{\underline{l}})^{1-\varepsilon} \sum_l \mu^{\underline{l}}(\Delta^{(l)})^{1-\varepsilon}.$$

The result follows by induction. □

Proof of Theorem 26. Notice that there exists $C' > 0$ such that if $0 < \rho \leq \rho'$, then any simplex with Lebesgue measure ρ' is contained in the union of $C' \frac{\rho'}{\rho^{p-1}}$ balls of diameter ρ.

Let $0 < \delta' < \alpha_1$ be such that $\mu(X_n) < e^{-\delta' C n}$ for infinitely many n, and fix an arbitrary such n. Fix $\delta' < \alpha_4 < \alpha_1$, let $C_4 > 0$ be as in the previous lemma, and let $C > 0$ be such that $C_4 e^{-C(\alpha_4-\delta')} < 1$. We are going to find a cover $\{B_i\}$ of X_n by balls of diameter at most e^{-Cn} satisfying

$$\sum_i \operatorname{diam}(B_i)^{p-1-\delta'} \leq 2C',$$

showing that $\operatorname{HD}(\liminf X_n) \leq p-1-\delta'$.

Let $X_n = Y_n \cup Z_n$, where Y_n is the union of those $\Delta^{\underline{l}}$ with $|\underline{l}| = n$ such that $\mu(\Delta^{\underline{l}}) > e^{-Cn}$ and Z_n is the complement. It follows that Y_n can be covered with at most $C' \mu(Y_n) e^{pCn}$ balls of diameter e^{-Cn}. This cover $\{B_i^Y\}$ satisfies

$$\sum_i \operatorname{diam}(B_i^Y)^{p-1-\delta'} \leq C' \mu(X_n) e^{\delta' C n} \leq C'.$$

Let us cover each $\Delta^{\underline{l}} \subset Z_n$ by the smallest possible number of balls of diameter $\mu(\Delta^{\underline{l}})$. The resulting cover $\{B_i^Z\}$ of Z_n then satisfies

$$\begin{aligned}\sum_i \operatorname{diam}(B_i^Z)^{p-1-\delta'} &\leq \sum_{|\underline{l}|=n,\, \mu(\Delta^{\underline{l}}) \leq e^{-Cn}} C' \mu(\Delta^{\underline{l}})^{1-\delta'} \\ &\leq \sum_{|\underline{l}|=n} C' \mu(\Delta^{\underline{l}})^{1-\alpha_4} e^{-Cn(\alpha_4-\delta')} \leq C' C_4^n e^{-Cn(\alpha_4-\delta')} \leq C'.\end{aligned} \tag{2}$$

The result follows. □

The following simple result will allow us to control the set of escaping points as well.

Theorem 28. *Assume that T is fast decaying. Let $\Delta^n \subset \Delta$ be the domain of T^n and let $\Delta^\infty = \bigcap_{n\in\mathbb{N}} \Delta^n$. Then $\mathrm{HD}(\Delta \setminus \Delta^\infty) \leq p - 1 - \frac{\alpha_1}{1+\alpha_1}$, where α_1 is the fast decay constant of T.*

Proof. Note that $\Delta^n \setminus \Delta^{n+1} = T^{-n}(\Delta \setminus \Delta^1)$, so $\mathrm{HD}(\Delta \setminus \Delta^\infty) = \mathrm{HD}(\Delta \setminus \Delta^1)$.

For simplicity, let us map Δ to the interior of the cube $W = [0,1]^{p-1}$ by a bi-Lipschitz map P. For $M \in \mathbb{N}$, let us partition W into $2^{M(p-1)}$ cubes of side $\delta = 2^{-M}$ in the natural way. and let us estimate the number N of cubes that are not contained in $P(\Delta^1)$. In order to do this, we estimate the total volume L of those cubes.

For fixed $\varepsilon > 0$, L is at most the sum L_0 of the volumes of all $P(\Delta^{(l)})$ with volume at most ε, plus the sum L_1 of the volumes of the $\sqrt{p-1}\delta$-neighborhood of the boundary of each $P(\Delta^{(l)})$ with volume at least ε.

By the fast decay of T, we obviously have $L_0 \leq C\varepsilon^{\alpha_1}$. On the other hand, the volume of the $\sqrt{p-1}\delta$-neighborhood of the boundary of each $P(\Delta^{(l)})$ is at most $C\delta$. Thus $L \leq C(\delta\varepsilon^{-1} + \varepsilon^{\alpha_1})$. Taking $\varepsilon = \delta^{\frac{1}{1+\alpha_1}}$, we get $L \leq 2C\delta^{\frac{\alpha_1}{1+\alpha_1}}$ and hence $N \leq 2C\delta^{-M+\frac{\alpha_1}{1+\alpha_1}}$. The result follows. □

6. Proof of Theorem 2

Let $x = (S, \Sigma, \omega)$ be a non arithmetic Veech surface, and let $\mathcal{C} \subset \mathcal{M}_{S,\Sigma}(\kappa)$ be its $\mathrm{SL}(2,\mathbb{R})$ orbit. Let us consider the Markov model (T, A) for the Kontsevich-Zorich cocycle associated to an appropriate Poincaré section Q through x. This Poincaré section contains the unstable horocycle segment $\{p(u, 0);\, u \in \Delta\}$. We will show that in this horocycle segment, the set of surfaces for which the the vertical flow is not weak mixing has Hausdorff dimension $d < 1$. By Lemma 16, for any surface in $\mathcal{C}$, the set of directions for which the directional flow is not weak mixing has Hausdorff dimension d as well.

Let k be the holonomy field of (S, Σ, ω). We recall from Section 2.4 that the Hodge bundle $H^1(S;\mathbb{R})$ admits $[k:\mathbb{Q}]$ invariant planes V^σ associated to the embeddings $\sigma : k \to \mathbb{R}$. We define $W = \bigoplus_\sigma V^\sigma$, and $\pi_\sigma : W \to V^\sigma$ the natural projection. The real vector space W is actually defined over $\mathbb{Q}$ and we note $W_\mathbb{Z} = W \cap H^1(S;\mathbb{Z})$ the integer lattice in W. By definition, π_σ restricted to $W_\mathbb{Z}$ is injective (because W is the smallest subspace defined over $\mathbb{Q}$ that contains $V = V^{\mathrm{id}}$).

Let Δ^∞ be the set of all $u \in \Delta$ which are in the domain of T^n for all $n \in \mathbb{N}$. For any $u \in \Delta^\infty$, we may define the stable space $E^s(u)$ of the Kontsevich-Zorich restricted to W^0 as the set of all $v \in W^0$ such that $A_n(u) \cdot v \to 0$ as $n \to \infty$ as $n \to \infty$.

By Theorem 18), if $u \in \Delta^\infty$ is such that $p(u, 0) = \omega$ admits an eigenvalue ν, then there exists a vector $v \in H^1(S \setminus \Sigma;\mathbb{Z})$ such that $\lim A_n(u) \cdot (v - \nu \operatorname{Im}\omega) = 0$. Notice that in this case, we necessarily have $v \in W$: Indeed, $\operatorname{Im}\omega \in W$, so the projection of $v - \nu \operatorname{Im}\omega$ on the quotient $H^1(S \setminus \Sigma;\mathbb{R})/W$ is contained in $H^1(S \setminus \Sigma;\mathbb{Z})/W_\mathbb{Z}$, if the projection of $v - \nu \operatorname{Im}\omega$ would be non-zero then the projection of $A_n(u) \cdot v - \nu \operatorname{Im}\omega$ would be a non-zero integer vector as well, and hence far away from zero.

Notice that if $v = 0$ then $\nu = 0$ (since $\operatorname{Im}\omega$ generates the strongest unstable subspace for the Kontsevich-Zorich cocycle), and any measurable eigenfunction with eigenvalue 0 must be constant by ergodicity (which follows from the recurrence hypothesis $u \in \Delta^\infty$).

Thus the set of $u \in \Delta^\infty$ such that $p(u,0)$ is not weak mixing is contained in

$$\mathcal{E} = \bigcup_{v \in W_{\mathbb{Z}} \setminus \{0\}} \bigcap_{\sigma \neq \mathrm{id}} \mathcal{E}(\pi_\sigma(v)),$$

where

$$\mathcal{E}(w) = \{u \in \Delta^\infty;\ w \in E^s(u)\}$$

By Theorem 28, $\mathrm{HD}(\Delta \setminus \Delta^\infty) < 1$, so the result will follow once we show that $\mathrm{HD}(\mathcal{E}) < 1$. In order to do this, we will show that for each $\sigma \neq \mathrm{id}$, there exists a constant $d_\sigma < 1$ such that for each $w \in V^\sigma \setminus \{0\}$, $\mathrm{HD}(\mathcal{E}(w)) \leq d_\sigma$.

Let us show that the expansion constant of $(T, A|V^\sigma)$ is the largest Lyapunov exponent of $(T, A|V^\sigma)$, which we recall is given by $\overline{r}\lambda^\sigma$ where $\lambda^\sigma > 0$ by Lemma 13. By the Oseledets Theorem, for any $w \in V^\sigma \setminus \{0\}$, and for μ-almost every $u \in \Delta$, we have $\lim \frac{1}{n} \ln \|A_n(u) \cdot w\| = \overline{r}\lambda^\sigma$, unless w belongs to the one-dimensional Oseledets subspace $E^\sigma_- \subset V^\sigma$ associated to the exponent $-\overline{r}\lambda^\sigma$. Assume that there exists a subset of Δ of positive μ-measure such that $w \in E^\sigma_-$. By a density point argument, E^σ_- is constant,[6] so that it must be invariant by all the $A^{(l)}$. This contradicts the fact that the group generated by all the $A^{(l)}$ is not solvable, see the proof of Lemma 13.

If $w \in E^s(u) \cap (V^\sigma \setminus \{0\})$, then of course $\limsup \frac{1}{n} \ln \|A_n(u) \cdot w\| \leq 0$. Since the expansion constant of $(T, A|V^\sigma)$ is strictly positive, we can apply Theorems 25 and 26 to conclude.

7. Construction of directions with non-trivial eigenfunctions

In this section, we provide a general construction of directional flows with non-trivial eigenfunctions in a Veech surface. This construction makes use of very particular elements in the Veech group called *Salem*. The presence of a single element will allow us to apply a somewhat more general geometric criterion for positivity of Hausdorff dimension of certain exceptional Oseledets behavior, which we will now describe in the setting of locally constant cocycles.

7.1. **Lower bound on Hausdorff dimension.** Let H be a finite dimensional (real or complex) vector space. We consider locally constant $\mathrm{SL}(H)$-cocycles (T, A) where $T : \bigcup_{l \in \mathbb{Z}} \Delta^{(l)} \to \Delta$ restricts to projective maps $\Delta^{(l)} \to \Delta$ between simplices in $\mathbb{PR}^p$, $p \geq 2$. We will assume that there exists some $\underline{l} \in \Omega$ such that $\Delta^{\underline{l}}$ is compactly contained in Δ, but we will not need to assume that T has bounded distortion or even that $\bigcup_{l \in \mathbb{Z}} \Delta^{(l)}$ has full measure in Δ.

Theorem 29. *Let (T, A) be a cocycle as above. Assume that for every $v \in H \setminus \{0\}$, there exists $\underline{l} \in \Omega$ such that $\|A^{\underline{l}} \cdot v\| < \|v\|$. Then there exists a finite subset $J \subset \mathbb{Z}$ such that for every $v \in H \setminus \{0\}$, there exists a compact set $K_v \subset \Delta$ with positive Hausdorff dimension such that for every $x \in K_v$ we have $T^n(x) \in \bigcup_{j \in J} \Delta^{(j)}$, $n \geq 0$, and $\limsup \frac{1}{n} \ln \|A_n(x) \cdot v\| < 0$.*

Proof. Fix two words $\underline{l}', \underline{l}'' \in \Omega$ such that $\Delta^{\underline{l}'}$ and $\Delta^{\underline{l}''}$ have disjoint closures contained in Δ.

[6] Indeed one can find $\underline{l}$ such that the probability that $u \in \Delta^{\underline{l}}$ is such that $w \in E^\sigma_-(u)$ is arbitrarily close to 1. Mapping $\Delta^{\underline{l}}$ to Δ by an iterate of T and using bounded distortion, we see that E^σ_- is constant on a subset of Δ of probability arbitrarily close to 1.

By compactness, there exists $\varepsilon > 0$ and a finite subset $F \subset \Omega$ such that for every $v \in H \setminus \{0\}$, there exists $\underline{l}(v) \in F$ such that $\|A^{\underline{l}(v)} \cdot v\| < e^{-\varepsilon}\|v\|$. Let $J \subset \mathbb{Z}$ be a finite subset containing all entries of words in F, as well as all entries of $\underline{l}'$ and $\underline{l}''$.

Let $F^n \subset \Omega$ be the subset consisting of the concatenation of n words (not necessarily distinct) in F. By induction, we see that for every $v \in H \setminus \{0\}$, there exists $\underline{l}^n(v) \in F^n$ such that $\|A^{\underline{l}^n(v)} \cdot v\| < e^{-n\varepsilon} v$ (just take $\underline{l}^1(v) = \underline{l}(v)$ and for $n \geq 2$ take $\underline{l}^n(v)$ as the concatenation of $\underline{l}(v)$ and $\underline{l}^{n-1}(A^{\underline{l}(v)} \cdot v)$).

Choose n such that $e^{-n\varepsilon} < \frac{1}{2} \max\{\|A^{\underline{l}'}\|, \|A^{\underline{l}''}\|\}$.

For $k \geq 1$ and a sequence $(t_0, ..., t_{k-1}) \in \{0,1\}^k$, let us define a word $\underline{l}(v,t)$ as follows. For $k = 1$, we let $\underline{l}(v,t) = \underline{l}^n(v)\underline{l}'$ if $t = (0)$ and $\underline{l}(v,t) = \underline{l}^n(v)\underline{l}''$ if $t = (1)$. For $k \geq 2$ and $t = (t_0, ..., t_{k-1})$, denoting $\sigma(t) = (t_1, ..., t_{k-1})$, we let $\underline{l}(v,t) = \underline{l}(v,t_0)\underline{l}(A^{\underline{l}(v,t_0)} \cdot v, \sigma(t))$.

Note that the diameter of $\Delta^{\underline{l}(v,t)}$ in the Hilbert metric of Δ is exponentially small in k: indeed, the diameter of $\Delta^{\underline{l}(v,t)}$ in $\Delta^{\underline{l}(v,t_0)}$ is equal to the diameter of $\Delta^{\underline{l}(v,\sigma(t))}$ in Δ, and the Hilbert metric of $\Delta^{\underline{l}(v,(t_0))}$ is strictly stronger than the Hilbert metric of Δ. Thus given an infinite sequence $t \in \{0,1\}^{\mathbb{N}}$, the sequence $\Delta^{\underline{l}(v,(t_0,...,t_{k-1}))}$ decreases to a point denoted by $\gamma_v(t) \in \Delta$. The map γ_v then provides a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and a Cantor set $K_v \subset \Delta$.

By definition, if $t \in \{0,1\}^k$ then $\|A^{\underline{l}(v,t)} \cdot v\| < 2^{-k}\|v\|$. It thus follows that for $x \in K_v$ we have $\limsup \frac{1}{n} \ln \|A_n(x) \cdot v\| \leq -\frac{\ln 2}{M}$, where M is the maximal length of all possible words $\underline{l}(v,t)$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in \{0,1\}$.

Let us endow $\{0,1\}^{\mathbb{N}}$ with the usual 2-adic metric d_2, where for $t \neq t'$ we let $d_2(t,t') = 2^{-k}$ where k is maximal such that $t_j = t'_j$ for $j < k$. With respect to this metric, $\{0,1\}^{\mathbb{N}}$ has Hausdorff dimension 1. To conclude, it is enough to show that $\gamma_v^{-1} : K \to \{0,1\}$ is α-Hölder for some $\alpha > 0$, as this will imply that the Hausdorff dimension of K is at least α.

Let d be the spherical metric on $\mathbb{P}H$. Let $\varepsilon_0 > 0$ be such that for every $x \in \partial\Delta$ and $y \in \bigcup_{v \in H\setminus\{0\}} \bigcup_{t \in \{0,1\}} \Delta^{\underline{l}(v,t)}$ we have $d(x,y) > \varepsilon_0$. Let $\Lambda > 1$ be an upper bound on the derivative of the projective actions of any $A^{\underline{l}(v,t)}$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in \{0,1\}$. For $k \in \mathbb{N}$, and $t \in \{0,1\}^{\mathbb{N}}$, $\gamma_v(t)$ is contained in $\Delta^{\underline{l}(v,(t_0,...,t_k))}$ and hence at distance at least $\varepsilon\Lambda^{-k}$ from $\partial\Delta^{\underline{l}(v,(t_0,...,t_{k-1}))}$. It follows that if $d_2(t,t') \geq 2^{1-k}$ then $d(\gamma_v(t), \gamma_v(t')) \geq \varepsilon_0\Lambda^{-k}$. The result the follows with $\alpha = \frac{\ln 2}{\ln \Lambda}$. □

The previous result would have been enough to construct continuous eigenfunctions. In order to construct discontinuous eigenfunctions as well, we will need the following more precise result.

Theorem 30. *Let (T, A) be a cocycle as above. Assume that for every $v \in H \setminus \{0\}$, there exist $\underline{l}, \underline{\tilde{l}} \in \Omega$ such that $\|A^{\underline{l}} \cdot v\| < \|v\| < \|A^{\underline{\tilde{l}}} \cdot v\|$. Then there exists a finite subset $J \subset \mathbb{Z}$ such that for every $v \in H \setminus \{0\}$, and for every sequence $a_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, such that $\sup_k |\ln a_k - \ln a_{k+1}| < \infty$, there exists a compact set $K_v \subset \Delta$ with positive Hausdorff dimension such that for every $x \in K_v$ we have $T^n(x) \in \bigcup_{l \in J} \Delta^{(l)}$, $n \geq 0$, and there exists a strictly increasing subsequence m_k, $k \in \mathbb{N}$, such that $\sup_k m_{k+1} - m_k < \infty$ and $\sup_k |\ln \|A_{m_k}(x) \cdot v\| - \ln a_k| < \infty$.*

Proof. Fix two words $\underline{l}', \underline{l}'' \in \Omega$ such that $\Delta^{\underline{l}'}$ and $\Delta^{\underline{l}''}$ have disjoint closures contained in Δ.

Let C_0 be an upper bound for $|\ln a_j - \ln a_{j+1}|$.

As in the proof of the previous theorem, define a finite set $F \subset \Omega$ such that for every $v \in \mathbb{R}^d \setminus \{0\}$, there exist $\underline{l}^c(v), \underline{l}^e(v) \in F$, such that

$$\max_{\underline{l} \in \{\underline{l}', \underline{l}''} \|A^{\underline{l}^c(v)\underline{l}} \cdot v\| < e^{-C_0}\|v\|,$$

$$\min_{\underline{l} \in \{\underline{l}', \underline{l}''} \|A^{\underline{l}^e(v)\underline{l}} \cdot v\| > e^{C_0}\|v\|.$$

Given $k \geq 1$ and a sequence $t = (t_0, ..., t_{k-1}) \in \{0,1\}^k$, define $\underline{l}(v,t)$ by induction as follows. If $k = 1$, then we let $\underline{l}(v,t) = \underline{l}^a \underline{l}^b$ where $\underline{l}^a = \underline{l}^c(v)$ if $\|v\| > a_0$, $\underline{l}^a = \underline{l}^e(v)$ if $\|v\| \leq a_0$, $\underline{l}^b = \underline{l}'$ if $t = 0$ and $\underline{l}^b = \underline{l}''$ if $t = 1$. If $k \geq 2$, we let $\underline{l}(v,t) = \underline{l}(v,t_0)\underline{l}(A^{\underline{l}(v,t_0)} \cdot v, \sigma(t))$, where $\sigma(t_0, ..., t_{k-1}) = (t_1, ..., t_{k-1})$.

Notice that the set $G \subset \Omega$ of possible words $\underline{l}(v,t)$ with $v \in \mathbb{R}^d \setminus \{0\}$ and $t \in \{0,1\}$ is finite.

By induction, we get $|\ln\|A^{\underline{l}(v,t)} \cdot v\| - \ln a_k| \leq |\ln\|v\| - \ln a_0| + C_1$, where

$$C_1 = \max_{\underline{l} \in G} \{\ln\|A^{\underline{l}}\|, \ln\|(A^{\underline{l}})^{-1}\|\}.$$

As in the proof of the previous theorem, we define $\gamma_v : \{0,1\}^{\mathbb{N}} \to \Delta$ so that $\gamma_v(t)$ is the intersection of the $\Delta^{\underline{l}(v,(t_0,...,t_{k-1}))}$, and conclude that $K_v = \gamma_v(\{0,1\}^{\mathbb{N}})$ is a Cantor set of positive Hausdorff dimension.

By construction, if $x = \gamma_v(t)$, then for every $n \in \mathbb{N}$ we have $A_n(x) \in \Delta^{(j)}$ for some entry j of some word in G. Moreover, $|\ln\|A_{m_k}(x) \cdot v\| - \ln a_k| \leq |\ln\|v\| - \ln a_0| + C_1$ where m_k is the length of $\underline{l}(v,(t_0,...,t_{k-1}))$. In particular, m_k is strictly increasing and $m_{k+1} - m_k$ is bounded by the maximal length of the words in G. □

7.2. **Salem elements and eigenfunctions.** A real number λ is a *Salem* number if it is an algebraic integer greater than 1, all its conjugates have absolute values no greater than 1 and at least one has absolute value 1. The last condition implies that the minimal polynomial of a Salem number is reciprocal and that all conjugates have modulus one except λ and $1/\lambda$. For $M \in \mathrm{SL}(2,\mathbb{R})$, we say that M is a *Salem* matrix if its dominant eigenvalue is a Salem number.

Let (S,Σ,ω) be a Veech surface, Γ its Veech group and k the trace field of Γ. We recall that the action of the Veech group on the tautological subspace $V = \mathbb{R}\,\mathrm{Re}(\omega) \oplus \mathbb{R}\,\mathrm{Im}(\omega)$ is naturally identified with the Veech group (see Section 2.2). For each $\sigma \in \mathrm{Gal}(k/\mathbb{Q})$ there is a well defined conjugate V^σ of V which is preserved by the affine group of (S,Σ,ω). These actions identifies to conjugates of the Veech group (see Section 2.4). Salem elements in Veech group have an alternative definition: an element of a Veech group is Salem if and only if it is direct hyperbolic and its Galois conjugates are elliptic.

Theorem 31. *Let (S,Σ,ω) be a non arithmetic Veech surface and assume that its Veech group contains a Salem element. Then*

(1) *the set of angles whose directional flow has a continuous eigenfunction has positive Hausdorff dimension,*
(2) *the set of angles whose directional flow has a measurable discontinuous eigenfunction has positive Hausdorff dimension.*

To build directions with eigenvalues, we use a criterion proved in [BDM1] which is a partial reciprocal of the Veech criterion (see Section 4). An earlier version of this criterion appears in the paper of Veech [Ve84]. The criterion of [BDM1] only concerns *linearly recurrent systems*: the translation flow of (S,ω) is linearly recurrent if there exists a constant K such that for any horizontal interval J embedded in (S,ω) the maximum return time to J is bounded by $K/|J|$. Equivalently, a translation surface is linearly recurrent if and only if the associated Teichmüller geodesic is bounded in the moduli space of translation surfaces.

Theorem 32 ([BDM1])**.** *Let U be a relatively compact open subset in the moduli space $\mathcal{M}_g(\kappa)$ in which the Hodge bundle admits a trivialization and let A_n be the associated Kontsevich Zorich cocycle. Let $(S,\Sigma,\omega) \in U$ be such that the return times to U have bounded gaps, then*

(1) *ν is a continuous eigenvalue of (S,ω) if and only if there exists an integer vector $v \in H^1(S;\mathbb{Z})\backslash\{0\}$ such that*
$$\sum_{n\geq 0} \|A_n(\omega)\cdot(\nu\,\mathrm{Im}(\omega)-v)\| < \infty.$$
(2) *ν is an L^2 eigenvalue of (S,Σ,ω) if and only if there exists an integer vector $v \in H^1(S;\mathbb{Z})\backslash\{0\}$ such that*
$$\sum_{n\geq 0} \|A_n(\omega)\cdot(\nu\,\mathrm{Im}(\omega)-v)\|^2 < \infty.$$

Actually, the criterion applies to the Cantor space obtained from the translation surface where each point that belongs to a singular leaf is doubled. The continuity in that space is weaker than the continuity on the surface. But from the cohomological equation, a continuous eigenfunction on the modified surface is well defined and continuous on the surface.

Remark 7.1. *Theorem 32 allows to strengthen the conclusion of Theorem 24 when the flow is linearly recurrent: if (S,Σ,ω) is a non-arithmetic Veech surface with trace field k and whose flow is linearly recurrent then it admits either 0 or $[k:\mathbb{Q}]$ rationally independent eigenvalues. Moreover, they are simultaneously contiuous or discontinuous. Indeed (following the proof of Theorem 24), any two non-zero "potential eigenvalues" $\mu,\mu' \in \mathbb{R}$ such that there exists $v,v' \in W_{\mathbb{Z}}$ with $\mu\,\mathrm{Im}(\omega)-v$ and $\mu'\,\mathrm{Im}(\omega')-v'$ belong to E^s are such that $\frac{\|A_n(\omega)\cdot(\mu\,\mathrm{Im}(\omega)-v)\|}{\|A_n(\omega)\cdot(\mu\,\mathrm{Im}(\omega)-v)\|}$ is uniformly bounded away from zero or infinity (independent of n). By Theorem 32, μ is a continuous eigenvalue if and only if μ' is, and μ is an L^2 eigenvalue if and only if μ' is. Since the set Θ of potential eigenvalues is either $\{0\}$ or has dimension $[k:\mathbb{Q}]$ over $\mathbb{Q}$, the result follows.*

Before going into details of the proof, we provide various examples of Veech surfaces which contain Salem elements. In particular the next result shows that Theorem 3 follows from Theorem 31.

Proposition 33 ([BBH], Proposition 1.7)**.** *A Veech surface with quadratic trace field has a Salem element in its Veech group.*

Proof. We follow [BBH]. The Veech group has only one conjugate and this conjugate is non discrete (see Proposition 15). On the other hand we know from a result of Beardon ([Be83] Theorem 8.4.1) that any non discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ contains an elliptic element with irrational angle. If g is an element of the Veech group whose conjugate is an irrational rotation then g can not be elliptic as it is of infinite order and the Veech group is discrete and g can not be parabolic because a conjugate of a parabolic element is again parabolic. Hence g is hyperbolic and g^2 is a Salem element of the Veech group. □

For more general Veech surfaces, we obtain examples through computational experiments (see appendix A for explicit matrices).

Proposition 34. *For respectively odd $q \leq 15$ and any $q \leq 15$ the triangle groups $\Delta(2,q,\infty)$ and $\Delta(q,\infty,\infty)$ contain Salem elements.*

In particular, Theorem 31 holds for many billiards in regular polygons P_n defined in the introduction.

Now, we proceed to the proof of Theorem 31.

Lemma 35. *Let λ be a Salem number and $\{\lambda, 1/\lambda, e^{i\alpha_1}, e^{-i\alpha_1}, \ldots, e^{i\alpha_k}, e^{-i\alpha_k}\}$ its Galois conjugates, then $\alpha_1, \ldots, \alpha_k, \pi$ are rationally independant.*

Proof. Let $n_1, \ldots n_k, m$ be integer such that

$$n_1\alpha_1 + \ldots + n_k\alpha_k = 2m\pi. \tag{3}$$

Then

$$\left(e^{i\alpha_1}\right)^{n_1} \ldots \left(e^{i\alpha_k}\right)^{n_k} = 1 \tag{4}$$

Each element of the Galois group is a field homomorphism and hence it preserves the partition $\{\lambda, 1/\lambda\}$, $\{e^{i\alpha_1}, e^{-i\alpha_1}\}$, ..., $\{e^{i\alpha_k}, e^{-i\alpha_k}\}$. By definition, the Galois group acts transitively on $\{\lambda, 1/\lambda, e^{i\alpha_1}, e^{-i\alpha_1}, \ldots, e^{i\alpha_k}, e^{-i\alpha_k}\}$ and for each $i = 1, \ldots, k$ there exists a field homomorphism that maps $e^{i\alpha_j}$ to λ and all other $e^{i\alpha_{j'}}$ to some $e^{\pm\alpha_{j''}}$. By applying this field homomorphism to the equality (4) and taking absolute value we get that $n_i = 0$ because $|\lambda| > 1$. Hence the relation (3) is trivial. □

We now show that the presence of Salem elements allows us to verify the hypothesis of Theorem 30.

Lemma 36. *Let (S, Σ, ω) be a Veech surface, k its holonomy field, $V = \mathbb{R}\operatorname{Re}\omega \oplus \operatorname{Im}\omega$ the tautological subspace and $W^0 = \bigoplus_{\sigma \in \mathrm{Gal}(k/\mathbb{Q})\setminus\{\mathrm{id}\}} V^\sigma$. Let γ be a Salem element of the Veech group and γ_j, $j \geq 1$, be such that for each $\sigma \neq \mathrm{id}$, the norm of the conjugates γ_j^σ grows to infinity. Denote by g and g_k their actions on W^0. Then for any $v \in W^0 \setminus \{0\}$ there exist positive integers n_-, n_+ and k_-, k_+ such that elements $g_- = g_{k_-} g^{n_-}$ and $g_+ = g_{k_+} g^{n_+}$ satisfy $\|g_- v\| < \|v\| < \|g_+ v\|$.*

Proof. Let $v \in W^0$ be a unit vector, and for $\sigma \neq \mathrm{id}$, let $\pi_\sigma : W^0 \to V^\sigma$ be the projection on the V^σ-coordinate. Let $D \subset \mathrm{Gal}(k/\mathbb{Q}) \setminus \{\mathrm{id}\}$ be the set of all σ such that $\pi_\sigma(v) \neq 0$. We show that there exists positive integers n and k such that for all $\sigma \in D$, we have $\|g_k g^n \pi_\sigma(v)\| < \frac{\|v\|}{\#D}$, which implies the first inequality with $n = n_-$ and $k = k_-$. The other inequality may be obtained by the very same argument.

Let θ_k^σ be the norm of γ_k^σ. By hypothesis, $\theta_k^\sigma > 1$ for every σ and every k sufficiently large. Let $F_-^\sigma \in \mathbb{P}V^\sigma$ be the most contracted direction of γ_k^σ in V^σ.

Consider the action of g on the torus $\prod_{\sigma \in D} \mathbb{P}V^\sigma$. By Lemma 35, this action is minimal. In particular there exist a sequence of non negative integer n_j such that $g^{n_j}\pi_\sigma(v)$ converges to a vector w_σ in $F_-^\sigma \setminus \{0\}$ for every $\sigma \in D$. In particular, for fixed $k \in \mathbb{N}$ we have $\lim_{j\to\infty} \|g_k g^{n_j} \pi_\sigma(v)\| = \frac{\|w_\sigma\|}{\theta_k^\sigma}$ for every $\sigma \in D$. The result follows by taking k such that $\frac{\|w_\sigma\|}{\theta_k^\sigma} < \frac{\|v\|}{\#D}$ (recall that $\theta_k^\sigma \to \infty$ as $k \to \infty$). □

Proof of Theorem 31. Let $V = \mathbb{R}\operatorname{Re}(\omega) \oplus \mathbb{R}\operatorname{Im}(\omega)$ be the tautological subspace of the cohomology $H^1(S;\mathbb{R})$ and $W = \oplus_{\sigma:k\to\mathbb{R}} V^\sigma$ and $W^0 = \oplus_{\sigma\neq\mathrm{id}} V^\sigma$. We are going to construct a Markov model $(T : \Delta \to \Delta, A)$ for the Kontsevich-Zorich over the $\mathrm{SL}(2,\mathbb{R})$ orbit $\mathcal{C}$ of (S,Σ,ω), a point $x \in \Delta$ and a positive integer n, and a sequence of points $y_k \in \Delta$ and positive integers n_k, such that $A_n(x)|V$ is a Salem element and $\lim_{k\to\infty} \inf_{\sigma\neq\mathrm{id}} \|A_{n_k}|V^\sigma\| = \infty$. First, let us show how this construction implies the result.

By Lemma 36, this implies that the hypothesis of Theorem 30 are satisfied for the cocycle $(T, A|W^0)$, so for every $w \in W^0 \setminus \{0\}$, there exist subsets $Z_c, Z_m \subset \Lambda$ of positive Hausdorff dimension such that for $u \in Z_c$ we have $\sum \|A_n(u)\cdot w\| < \infty$ and for $u \in Z_m$ we have $\sum \|A_n(u)\cdot w\|^2 < \infty$ and $\sum \|A_n(u)\cdot w\| = \infty$. Moreover, since Theorem 30 provides also that $T^n(u)$ visits only finitely many distinct $\Delta^{(l)}$, the return times $r(T^n(u))$ remain bounded so that the forward Teichmüller geodesic starting at u is bounded in moduli space.

Let us take w as the projection on W^0, along V, of a non-zero vector $v \in W \cap H^1(S;\mathbb{Z})$. Fix $u \in Z_c \cup Z_m$. Let $\omega = p(u,0)$ and write $v = w + \nu \operatorname{Im}\omega + \eta \operatorname{Re}\omega$. Then $A_n(u)\cdot(\nu \operatorname{Im}\omega - v) = -A_n(u)\cdot w - \eta A_n(u)\cdot \operatorname{Re}\omega$. Note that $A_n(u)\cdot\operatorname{Re}\omega$ decays exponentially fast, since $\operatorname{Re}\omega$ is in the direction of the strongest contracting subbundle of the Kontsevich-Zorich cocycle. Notice that $\nu \neq 0$, otherwise the integer non-zero vectors $A_n(u)\cdot v$ would converge to 0. By Theorem 32, if $u \in Z_c$, then the vertical flow for ω admits a continuous eigenfunction with eigenvalue ν and if $u \in Z_m$ then the vertical flow for ω admits a measurable eigenfunction, but no continuous eigenfunction, with eigenvalue ν.

We have thus obtained positive Hausdorff dimension subsets $p(Z_c \times \{0\})$ and $p(Z_m \times \{0\})$ of an unstable horocycle for which the vertical flow has continuous and measurable but discontinuous eigenfunctions. Using Lemma 16, we transfer the result to the directional flow in any surface in $\mathcal{C}$, giving the desired conclusion.

We now proceed with the construction of the Markov model. Recall that for each hyperbolic element γ in the Veech group, there exists a periodic orbit O_σ of the Teichmüller flow in the $\mathrm{SL}(2,\mathbb{R})$ orbit and a positive integer n_γ, such that the restriction to V of the n_γ-th iterate of the monodromy of the Kontsevich-Zorich cocycle along this periodic orbit is conjugate to γ.

Let γ be a Salem element in the Veech group, and let us consider the Markov model (T, A) for the Kontsevich-Zorich cocycle obtained by taking a small Poincaré section Q through some $x \in O_\gamma$. Then clearly $A_{n_\gamma}(x)|V$ is a Salem element.

On the other hand, by Lemma 13, $(T, A|V^\sigma)$ has a positive Lyapunov exponent for every σ. Thus for large n and for a set of y of probability close to 1, the norm of $\|A_n(y)|V^\sigma\|$ is large. In particular, for each $k \in \mathbb{N}$ there exists y_k and a positive integer n_k such that $\|A_{n_k}|V^\sigma\| > k$ for every $\sigma \neq \mathrm{id}$. The result follows. □

Appendix A. Salem elements in triangle groups

In that appendix we provide explicit Salem matrices in triangle groups $\Delta(2,q,\infty)$ and $\Delta(q,\infty,\infty)$ for trace field of degree greater than two and small values of q. The matrices are given in terms of the standard generators s, t of the triangle group $\Delta(p,q,r)$ that satisfy

$$s^p = t^q = (st)^r = \pm id.$$

Instead of writing down the minimal polynomial of the eigenvalue, we write it for half the trace. The roots of modulus less than one are cosine of the angles of the corresponding elliptic matrices.

The array stops at the values $q = 17$ for $\Delta(2, q, \infty)$ and $q = 16$ for $\Delta(q, \infty, \infty)$ for which we were unable to find Salem elements. All these examples were obtained using the mathematical software Sage [Sa].

A.1. **Salem elements in $\Delta(2, q, \infty)$.**

q	degree	matrix m
	minimal polynomial of trace(m)/2	
	approximate conjugates of trace(m)/2	
7	3	$t^3.s$
	$x^3 - 2x^2 - x + 1$	
	$2.247,\ 0.5550,\ -0.8019$	
9	3	$t^4.s$
	$x^3 - 3x^2 + 1$	
	$2.879,\ 0.6527,\ -0.5321$	
11	5	$t^5.s.t^4.s$
	$x^5 - \frac{39}{2}x^4 - 47x^3 - \frac{243}{8}x^2 - \frac{17}{16}x + \frac{89}{32}$	
	$21.73,\ 0.2425,\ -0.6156,\ -0.8781,\ -0.9764$	
13	6	$t^7.s.t^7.s.t^4.s$
	$x^6 - 227x^5 - 11x^4 + 318x^3 + 41x^2 - 110x - 25$	
	$227.0,\ 0.9072,\ 0.8412,\ -0.2464,\ -0.6697,\ -0.8746$	
15	4	$t^7.s$
	$x^4 - 4x^3 - 4x^2 + x + 1$	
	$4.783,\ 0.5112,\ -0.5473,\ -0.7472$	

A.2. Salem elements in $\Delta(q, \infty, \infty)$.

q	degree	matrix m
	minimal polynomial of trace(m)/2 approximate conjugates of trace(m)/2	
7	3	$t.s^3$
	$x^3 - 3x^2 - 4x - 1$ $4.049, \ -0.3569, \ -0.6920$	
8	4	$t.s^2.t.s^3$
	$x^4 - 24x^3 + 15x^2 + 4x + \frac{1}{8}$ $23.35, \ 0.8571, \ -0.03655, \ -0.1709$	
9	3	$t.s^2$
	$x^3 - 3x^2 + 1$ $2.879, \ 0.6527, \ -0.5321$	
10	4	$t.s^3.t.s^7$
	$x^4 - 49x^3 - \frac{441}{4}x^2 - \frac{291}{4}x - \frac{199}{16}$ $51.18, \ -0.2644, \ -0.9504, \ -0.9672$	
11	5	$t.s^4.t.s^7$
	$x^5 - \frac{155}{2}x^4 - 122x^3 - \frac{459}{8}x^2 - \frac{173}{16}x - \frac{23}{32}$ $79.05, \ -0.1907, \ -0.2214, \ -0.2388, \ -0.9015$	
12	4	$t.s^2.t.s^3$
	$x^4 - 24x^3 - 61x^2 - 48x - \frac{191}{16}$ $26.38, \ -0.5254, \ -0.9096, \ -0.9468$	
13	6	$t.s^4.t.s^5.t^{-1}.s^4.t^{-1}.s^5$
	$x^6 - \frac{43107}{2}x^5 - \frac{188297}{4}x^4 - 26514x^3 + \frac{53979}{8}x^2 + \frac{304515}{32}x + \frac{124175}{64}$ $21560., \ 0.5373, \ -0.3375, \ -0.7022, \ -0.8374, \ -0.8440$	
14	6	$t.s^5.t.s^9$
	$x^6 - 125x^5 - \frac{955}{4}x^4 - \frac{45}{4}x^3 + \frac{1653}{8}x^2 + \frac{967}{8}x + \frac{1009}{64}$ $126.9, \ 0.9692, \ -0.1912, \ -0.6930, \ -0.9794, \ -0.9879$	
15	4	$t.s^3$
	$x^4 - 4x^3 - 4x^2 + x + 1$ $4.783, \ 0.5112, \ -0.5473, \ -0.7472$	

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CNRS UMR 7586, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

E-mail address: artur@math.jussieu.fr

E-mail address: delecroix@math.jussieu.fr