COMBINATORICS OF FAITHFULLY BALANCED MODULES

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Abstract. We study and classify faithfully balanced modules for the algebra of triangular n by n matrices and more generally for Nakayama algebras. The theory extends known results about tilting modules, which are classified by binary trees, and counted with the Catalan numbers. The number of faithfully balanced modules is a 2-factorial number. Among them are $n!$ modules with n indecomposable summands, which can be classified by interleaved binary trees or by increasing binary trees.

1. Introduction

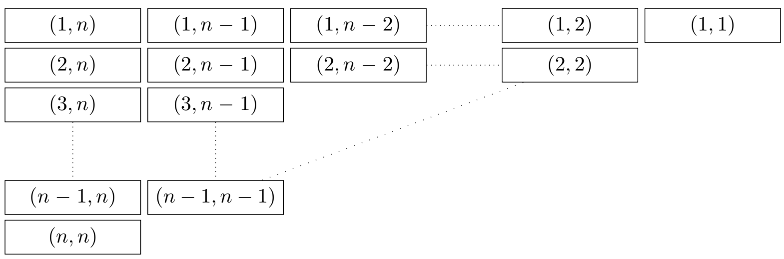
We consider the category Λ-mod of finitely generated left Λ-modules, where Λ is a finite-dimensional algebra over a field K, or more generally an artin algebra. Recall that a module M is said to be *balanced*, or to have the *double centralizer property* if the natural map $\Lambda \to \mathrm{End}_E(M)$ is surjective, where $E = \mathrm{End}_\Lambda(M)$, and it is said to be *faithfully balanced* if the natural map is bijective, or equivalently if M is faithful and balanced.

Balanced and faithfully balanced modules appear in various places in the literature on ring theory, such as Schur-Weyl duality (see for example [12]), and Thrall's notion of a QF-1 algebra [25]. The main known examples of faithfully balanced modules are faithful modules for a self-injective algebra, and more generally generators and cogenerators for any algebra, and tilting modules and cotilting modules. For more examples see [13].

In general the behaviour of faithfully balanced modules is rather mysterious. We shall illustrate this by studying these modules for the algebra Λ_n of $n \times n$ lower triangular matrices over K, or equivalently the path algebra of the linearly oriented A_n quiver

$$1 \to 2 \to \cdots \to n.$$

The indecomposable modules for Λ_n are indexed by the set $I_n = \{(i,j) : 1 \le i \le j \le n\}$, which we display as the blocks of a Young diagram of staircase shape



The element (i,j) corresponds to the module M_{ij} with top and socle the simple modules $S[i]$ and $S[j]$. The left hand column is the indecomposable projective modules, the top row is the indecomposable injective modules and the modules M_{ii} are the simple modules $S[i]$. The Auslander-Reiten quiver is the same picture, with irreducible maps going vertically and to the right, and the Auslander-Reiten translation $\tau = DTr$ takes each module M_{ij} with $j < n$ to $M_{i+1,j+1}$. By a *leaf* we mean an element of the set $L = \{(1,0),(2,1),\dots,(n+1,n)\}$. We define *cohooks* for $(i,j) \in I_n$ and *virtual cohooks* for $(i,j) \in L$ by the formula

$$\mathrm{cohook}(i,j) = \{M_{kj} : 1 \le k < i\} \cup \{M_{i\ell} : n \ge \ell > j\}.$$

In terms of the Young diagram, $\mathrm{cohook}(i,j)$ is the set consisting of the cells strictly to the left of or strictly above (i,j).

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In section 4 we prove the following theorem, along with its generalization to Nakayama algebras and a version for balanced modules.

Theorem 1.1. *A Λ_n-module M is faithfully balanced if and only if it satisfies the following conditions:*

(FB0) *M_{1n} is a summand of M;*
(FB1) *if M_{ij} is a summand of M, $(i,j) \neq (1,n)$, then* $\mathrm{cohook}(i,j)$ *contains a summand of M; and*
(FB2) *every virtual cohook contains a summand of M.*

For example the faithfully balanced modules for Λ_3 are given by taking copies of the indecomposable modules corresponding to the the black boxes ■ in one of the diagrams in Figure 1, together with an arbitrary subset of the shaded boxes ⊠. A module is *basic* if its indecomposable summands occur with multiplicity one. The diagrams show the $8+4+2+2+1+2+2=21$ basic faithfully balanced modules for Λ_3.

Given an algebra Λ and a module M, we write $\mathrm{add}(M)$ for the full subcategory of Λ-mod consisting of the direct summands of direct sums of copies of M, $\mathrm{gen}(M)$ for the category of modules generated by M, so quotients of a direct sum of copies of M, and $\mathrm{cogen}(M)$ for the category of modules cogenerated by M, so embeddable in a direct sum of copies of M. Recall from Pressland and Sauter [16, Definition 2.10], that if M is a Λ-module, then $\mathrm{gen}_1(M)$ is the category of modules X such that there is an exact sequence $M'' \to M' \to X \to 0$ with $M', M'' \in \mathrm{add}(M)$ and the sequence

$$\mathrm{Hom}(M, M'') \to \mathrm{Hom}(M, M') \to \mathrm{Hom}(M, X) \to 0$$

exact, and $\mathrm{cogen}^1(M)$ is the category of modules X such that there is an exact sequence $0 \to X \to M' \to M''$ with $M', M'' \in \mathrm{add}(M)$ and the sequence

$$\mathrm{Hom}(M'', M) \to \mathrm{Hom}(M', M) \to \mathrm{Hom}(X, M) \to 0$$

exact. These are full subcategories of Λ-mod, closed under direct sums and summands. It is known that M is faithfully balanced if and only if the projective modules are all in $\mathrm{cogen}^1(M)$ or equivalently the injective modules are all in $\mathrm{gen}_1(M)$ (see Lemma 2.4). It follows that the property of M being faithfully balanced only depends on $\mathrm{add}(M)$, and so one may assume that M is basic. By a (faithfully balanced) gen_1-*category* or cogen^1-*category* we mean a subcategory of Λ-mod of the form $\mathrm{gen}_1(M)$ or $\mathrm{cogen}^1(M)$ respectively, where M is some (faithfully balanced) module. We say that a Λ-module M is gen_1-*critical* if any proper summand N of M has $\mathrm{gen}_1(N) \neq \mathrm{gen}_1(M)$; similarly for cogen^1-critical. In section 5 we prove the following.

Theorem 1.2. *For the algebra Λ_n, or for any representation-directed algebra Λ, any gen_1-category $\mathcal{G}$ contains a gen_1-critical module M with $\mathrm{gen}_1(M) = \mathcal{G}$, which is unique up to isomorphism. For any module L, we have $\mathrm{gen}_1(L) = \mathcal{G}$ if and only if $\mathrm{add}(M) \subseteq \mathrm{add}(L) \subseteq \mathcal{G}$.*

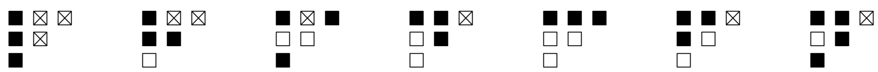

FIGURE 1. The seven faithfully balanced gen_1-categories for Λ_3. The black boxes ■ show the summands of the gen_1-critical module, and together with the shaded boxes ⊠ they show the category $\mathrm{gen}_1(M)$.

Figure 1 shows the faithfully balanced gen_1-categories for Λ_3. We say that a module is *minimal faithfully balanced* if it is faithfully balanced and any proper direct summand is not faithfully balanced. Clearly any minimal faithfully balanced module is gen_1- and cogen^1-critical. Any (generalized) tilting module T is faithfully balanced, see [27] and [26, Proposition 5]. In section 5 we prove the following.

Theorem 1.3. *If T is a basic classical tilting module for an artin algebra Λ, i.e. T has projective dimension ≤ 1, then T is gen_1-critical. If in addition Λ is hereditary, then T is minimal faithfully balanced.*

It follows that any τ-tilting module is gen_1-critical and balanced. In Figure 1, the first five gen_1-critical modules are tilting modules. These and the sixth module are minimal faithfully balanced. The last gen_1-critical module is not minimal faithfully balanced. Note that although all minimal faithfully balanced modules for Λ_3 have 3 indecomposable summands, the module

■ □ ■ □
□ □ ■
■ ■
□

is a minimal faithfully balanced module for Λ_4, but it has more than 4 indecomposable summands. To count faithfully balanced modules for Λ_n we prove the following in section 6.

Theorem 1.4. *In the expansion of the polynomial*

$$h_n(x_1, \dots, x_n) = \prod_{r=1}^{n} \left(\prod_{s=1}^{r} (1 + x_s) - 1 \right),$$

the coefficient of the monomial $x_1^{t_1} \dots x_n^{t_n}$ is the number of basic faithfully balanced Λ_n-modules M with t_i indecomposable summands having top $S[i]$ (or equivalently in row i of the Young diagram), for all i.

It follows that the number of basic faithfully balanced modules for Λ_n is the 2-factorial number

$$[n]_2! := \prod_{i=1}^{n} (2^i - 1).$$

For example there are $(2-1)(2^2-1)(2^3-1) = 21$ basic faithfully balanced Λ_3-modules. Also, any basic faithfully balanced module for Λ_n has at least n summands, and the number with exactly n summands is $n!$. For comparison, note that the number of basic tilting modules for Λ_n is the nth Catalan number, see [9, 11, 17]. In section 7 we use Theorem 1.4 to count faithfully balanced modules over certain quadratic Nakayama algebras. One should remark that the number of summands of a faithfully balanced module for a more general algebra may be less than the number of isomorphism classes of simple modules. For example the direct sum of the indecomposable projective-injective modules for an Auslander algebra is faithfully balanced. This can also happen for some non-linear orientations of the path algebra of A_n for $n \geq 5$.

In section 8, we investigate the combinatorics of faithfully balanced modules with exactly n indecomposable summands. Recall that an increasing binary tree with n vertices is a binary tree with a labelling of the vertices by the integers $1, \dots, n$, such that the label of any vertex is less that that of any of its children. See Definition 8.3 for the notion of an 'interleaved tree'.

Theorem 1.5. *Given n, there are explicit bijections between the following types of objects:*

(i) *faithfully balanced modules for Λ_n with exactly n indecomposable summands;*
(ii) *interleaved trees with n vertices;*
(iii) *increasing binary trees with n vertices;*
(iv) *functions $f : \{1, \dots, n\} \to \{1, \dots, n\}$ which are* self-bounded, *meaning that $f(i) \leq i$ for all i.*

These restrict to bijections between basic tilting modules; binary trees; well-ordered increasing binary trees and non-decreasing self-bounded functions[1].

Writing $fb(n)$ for the set of faithfully balanced Λ_n-modules with exactly n indecomposable summands, we also prove that there is a simple bijection between $fb(n)$ and the set of tree-like tableaux of size n in the sense of [6].

In section 9 we study the poset structure $\trianglelefteq$ on $fb(n)$ given by $N \trianglelefteq M$ if and only if $\mathrm{cogen}(N) \subseteq \mathrm{cogen}(M)$ and $\mathrm{gen}(N) \supseteq \mathrm{gen}(M)$. See Figure 2 for the cases $n = 3$ and 4. We prove the following.

Theorem 1.6. *Let $fb(n)$ be the set of basic faithfully balanced Λ_n-modules with exactly n indecomposable summands.*

(1) *The poset $(fb(n), \trianglelefteq)$ is a lattice.*
(2) *The Tamari lattice is a sub-lattice of $(fb(n), \trianglelefteq)$.*
(3) *The cover relations in $(fb(n), \trianglelefteq)$ are given by exchanging exactly one indecomposable summand.*

Experiments were carried out using the GAP-package QPA [22] and SageMath [23].

[1]These functions are in immediate bijection with the so-called Dyck Paths.

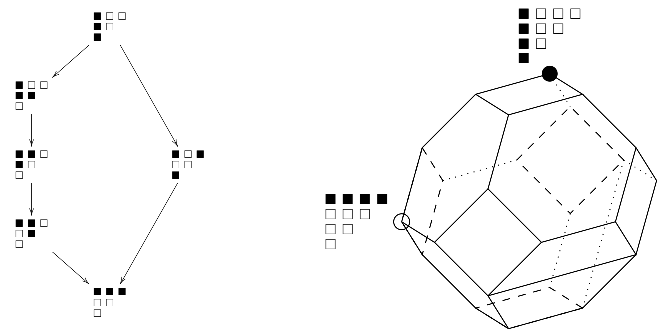

FIGURE 2. The Hasse diagram of $(fb(3), \trianglelefteq)$ and the graph of the Hasse diagram of $(fb(4), \trianglelefteq)$. For more details see Figures 8 and 9.

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2. Characterizations of (faithfully) balanced modules

In this section we consider finitely generated modules for an artin algebra Λ and write Hom for Hom_Λ. Recall that a morphism $f : X \to M'$ is called a *left* $\mathrm{add}(M)$*-approximation* of X if $M' \in \mathrm{add}(M)$ and any morphism $g : X \to M$ factors as $g = hf$ for some $h : M' \to M$. It is a *minimal left approximation* if in addition f is a *left minimal* morphism, which means that for any $\phi \in \mathrm{End}(M')$, if $\phi f = f$ then ϕ is an automorphism, or equivalently that $\mathrm{im}\,\phi$ is not contained in a proper direct summand of M', see [4, §1]. Dually there is the notion of a (minimal) right $\mathrm{add}(M)$-approximation. Minimal $\mathrm{add}(M)$-approximations exist, and are unique up to isomorphism, see [4, Propositions 3.9, 4.2]. The combination of (a) and (b) in the following lemma is the case $k = 1$ of [13, Lemma 2.2].

Lemma 2.1. *Let X and M be Λ-modules and let $E = \mathrm{End}(M)$.*
(a) The following are equivalent:

(i) *the natural map $X \to \mathrm{Hom}_E(\mathrm{Hom}(X, M), M)$ is injective;*
(ii) *$X \in \mathrm{cogen}(M)$;*
(iii) *the minimal left $\mathrm{add}(M)$-approximation $\theta : X \to M'$ is injective.*

(b) The following are equivalent:

(i) *the natural map $X \to \mathrm{Hom}_E(\mathrm{Hom}(X, M), M)$ is surjective;*
(ii) *there is a sequence $X \to M' \to M''$, exact in the middle and with $M', M'' \in \mathrm{add}(M)$, such that the sequence $\mathrm{Hom}(M'', M) \to \mathrm{Hom}(M', M) \to \mathrm{Hom}(X, M) \to 0$ is exact:*
(iii) *the minimal left $\mathrm{add}(M)$-approximation $\theta : X \to M'$ has cokernel in $\mathrm{cogen}(M)$.*

Proof. Part (a) is straightforward, see for example [3, Lemma VI.1.8]; we prove (b). For (i) implies (iii), we consider the sequence $X \xrightarrow{\theta} M' \xrightarrow{\phi} M''$ where ϕ is the composition of $M' \to \mathrm{coker}\,\theta$ and a left $\mathrm{add}(M)$-approximation of $\mathrm{coker}\,\theta$. Then the sequence

$$\mathrm{Hom}(M'', M) \to \mathrm{Hom}(M', M) \to \mathrm{Hom}(X, M) \to 0$$

is exact. This gives a commutative diagram

$$\begin{array}{ccccccc}
 & & X & \longrightarrow & M' & \longrightarrow & M'' \\
 & & {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow \\
0 & \longrightarrow & \mathrm{Hom}_E(\mathrm{Hom}(X, M), M) & \longrightarrow & \mathrm{Hom}_E(\mathrm{Hom}(M', M), M) & \longrightarrow & \mathrm{Hom}_E(\mathrm{Hom}(M'', M), M)
\end{array}$$

in which the bottom row is exact. Since $M' \in \operatorname{add}(M)$ it follows that g is an isomorphism. By (i) the map f is surjective. By diagram chasing the top row is exact, giving (iii).

(iii) implies (ii). One takes the sequence $X \xrightarrow{\theta} M' \xrightarrow{\phi} M''$ as in the previous part of the proof. Then this sequence is exact in the middle by (iii). The rest are straightforward.

(ii) implies (i). We have a commutative diagram as displayed above with exact rows. Since $M', M'' \in \operatorname{add}(M)$ the maps g, h are isomorphisms. It follows that f is a surjection. □

Considering the duals of X and M as Λ^{op}-modules (see [5, §II.3]) gives the following.

Lemma 2.2. *Let X and M be Λ-modules and let $E = \operatorname{End}(M)$.*
(a) The following are equivalent:
(i) *the natural map* $\operatorname{Hom}(M, X) \otimes_E M \to X$ *is surjective;*
(ii) $X \in \operatorname{gen}(M)$*;*
(iii) *the minimal right* $\operatorname{add}(M)$*-approximation* $\theta : M' \to X$ *is surjective.*

(b) The following are equivalent:
(i) *the natural map* $\operatorname{Hom}(M, X) \otimes_E M \to X$ *is injective;*
(ii) *there is a sequence* $M'' \to M' \to X$*, exact in the middle and with* $M', M'' \in \operatorname{add}(M)$*, such that the sequence* $\operatorname{Hom}(M, M'') \to \operatorname{Hom}(M, M') \to \operatorname{Hom}(M, X) \to 0$ *is exact;*
(iii) *the minimal right* $\operatorname{add}(M)$*-approximation* $\theta : M' \to X$ *has kernel in* $\operatorname{gen}(M)$*.*

Using the additivity property of minimal approximations, one gets the following.

Lemma 2.3. *For a module M the following are equivalent:*
(i) *M is balanced;*
(ii) *for every indecomposable projective module P, the minimal left* $\operatorname{add}(M)$*-approximation* $\theta : P \to M'$ *has cokernel in* $\operatorname{cogen}(M)$*;*
(iii) *for every indecomposable injective module I, the minimal right* $\operatorname{add}(M)$*-approximation* $\theta : M' \to I$ *has kernel in* $\operatorname{gen}(M)$*.*

As mentioned in the introduction, following Pressland and Sauter [16], we write $\operatorname{gen}_1(M)$ (respectively $\operatorname{cogen}^1(M)$) for the full subcatetegory of Λ-mod consisting of the modules X satisfying the conditions in parts (a) and (b) of Lemma 2.2 (respectively Lemma 2.1). These subcategories are closed under direct sums and summands. The following consequence is already in [16].

Lemma 2.4. *For a module M, the following conditions are equivalent.*
(i) *M is faithfully balanced;*
(ii) *all projective Λ-modules are in* $\operatorname{cogen}^1(M)$*;*
(iii) *all injective Λ-modules are in* $\operatorname{gen}_1(M)$*.*

3. Approximations for Nakayama algebras

In this section Λ is a Nakayama algebra, meaning that all indecomposable projective and injective modules are uniserial, see for example [5, §IV.2]. It follows that any indecomposable module X is uniserial, determined up to isomorphism by its length $\ell(X)$ and either its socle $\operatorname{soc}(X)$ or top $\operatorname{top}(X)$.

We fix an indecomposable module X.

Lemma 3.1. *Let $\phi : X \to U$ and $\phi' : X \to U'$ be non-zero homomorphisms with U, U' indecomposable.*
(i) *If $\theta \in \operatorname{End}(U)$ satisfies $\theta\phi = \phi$, then θ is invertible.*
(ii) *$\phi' = \theta\phi$ for some morphism $\theta : U \to U'$ $\Leftrightarrow$ $\ell(\ker\phi) \le \ell(\ker\phi')$ and $\ell(\operatorname{coker}\phi) \le \ell(\operatorname{coker}\phi')$.*
(iii) *$\phi' = \theta\phi$ for some isomorphism $\theta : U \to U'$ $\Leftrightarrow$ $\ell(\ker\phi) = \ell(\ker\phi')$ and $\ell(\operatorname{coker}\phi) = \ell(\operatorname{coker}\phi')$.*

Proof. (i) Since U is indecomposable and ϕ is non-zero, $\phi : X \to U$ is left minimal.

(ii) If there is θ, then trivially $\ell(\ker\phi) \le \ell(\ker\phi')$. Moreover since $\phi' \neq 0$, we have $\operatorname{im}\phi \not\subseteq \ker\theta$, so since U is uniserial, $\ker\theta \subseteq \operatorname{im}\phi$. Thus $\theta^{-1}(\operatorname{im}\phi') = \ker\theta + \operatorname{im}\phi = \operatorname{im}\phi$, so θ induces an injection from $\operatorname{coker}\phi$ to $\operatorname{coker}\phi'$, giving the other inequality.

Conversely if the inequalities hold, then $\operatorname{im}\phi$ and $\operatorname{im}\phi'$ are both quotients of the uniserial module X, so the inequality $\ell(\ker\phi) \le \ell(\ker\phi')$ ensures the existence of a surjective map $\alpha : \operatorname{im}\phi \to \operatorname{im}\phi'$ with $\phi' = \alpha\phi$. Taking the injective envelopes I and I' of $\operatorname{im}\phi$ and $\operatorname{im}\phi'$, the map α extends to a map

$\beta : I \to I'$. Now I and I' are indecomposable, hence uniserial, since the modules $\operatorname{im}\phi$ and $\operatorname{im}\phi'$ have simple socle. Moreover U embeds in I and U' in I'. Now $\ker\beta \cap \operatorname{im}\phi = \ker\alpha$, and $\operatorname{im}\phi \not\subseteq \ker\beta$, so since I is uniserial, $\ker\beta \subseteq \operatorname{im}\phi$, so $\ker\beta = \ker\alpha$. Then

$$\ell(\beta(U)) = \ell(U) - \ell(\ker\beta) = \ell(U) - \ell(\ker\alpha)$$
$$= \ell(U) - \ell(\operatorname{im}\phi) + \ell(\operatorname{im}\phi') = \ell(\operatorname{coker}\phi) + \ell(U') - \ell(\operatorname{coker}\phi') \leq \ell(U')$$

by the inequality. Thus $\beta(U) \subseteq U'$, and one can take θ to be the restriction of β to U.

(iii) Follows from (i) and (ii). □

In view of the lemma, when U is indecomposable, a non-zero morphism $\phi : X \to U$ is determined (up to an automorphism of U) by the pair of natural numbers $(s,t) = (\ell(\ker\phi), \ell(\operatorname{coker}\phi))$. We denote a representative of this morphism by $\phi_{st} : X \to X(s,t)$. Clearly if $s \leq s' < \ell(X)$, then there is a map $p^{s'}_{st} : X(s,t) \to X(s',t)$, necessarily an epimorphism, with $\phi_{s',t} = p^{s'}_{st}\phi_{st}$.

Now let M be an arbitrary module. We define M_X to be the set of pairs (s,t) such that $X(s,t)$ is a direct summand of M. The set M_X inherits the following partial ordering from $\mathbb{Z}^2$:

$$(s,t) \leq (s',t') \text{ in } \mathbb{Z}^2 \quad \Leftrightarrow \quad s \leq s' \text{ and } t \leq t'.$$

Lemma 3.2. *The map*

$$\phi = (\phi_i) : X \to \bigoplus_{i=1}^{k} X(s_i,t_i),$$

is a minimal left $\operatorname{add}(M)$*-approximation of* X*, where* $(s_1,t_1),\dots,(s_k,t_k)$ *are the minimal elements of* M_X*, ordered so that* $s_1 > \dots > s_k$ *and* $t_1 < \dots < t_k$*, and* $\phi_i = \phi_{s_i,t_i}$*. Assuming that* $k > 0$*, or equivalently that* $\operatorname{Hom}(X,M) \neq 0$*, we have*

$$\operatorname{coker}\phi \cong C_1 \oplus \dots \oplus C_k$$

where C_1 *is the quotient of* $X(s_1,t_1)$ *of length* t_1 *and* $C_i = X(s_{i-1},t_i)$ *for* $i > 1$*.*

Proof. The fact that ϕ is a left approximation follows immediately from part (ii) of Lemma 3.1. To show that ϕ is a minimal approximation, it suffices to show that if $\theta : X \to M'$ is a minimal $\operatorname{add}(M)$-approximation of X, then each $X(s_i,t_i)$ is a summand of M'. Now up to isomorphism we may write M' as a direct sum of modules $X(s,t)$ for various (s,t), with the components of θ being the maps ϕ_{st}. By assumption the map ϕ_i factors through θ. Consider a composition

$$X \xrightarrow{\phi_{st}} X(s,t) \xrightarrow{\alpha} X(s_i,t_i).$$

If $s > s_i$ then the first map has kernel of length $> s_i$, and hence so does the composition, so $\ell(\operatorname{im}\alpha\phi_{st}) < \ell(X) - s_i$. If $t > t_i$ then α has kernel of length at least $\ell(X(s,t)) - \ell(X(s_i,t_i)) = -s + t + s_i - t_i$, and so

$$\ell(\alpha(\operatorname{im}\phi_{st})) \leq \max\{\ell(X) + t_i - t - s_i, 0\} < \ell(X) - s_i.$$

Since the map ϕ_i factors through θ, and it has image of length $\ell(X) - s_i$, we deduce that some summand $X(s,t)$ has $(s,t) \leq (s_i,t_i)$. By minimality $(s,t) = (s_i,t_i)$, so $X(s_i,t_i)$ must occur as a summand of M'. Since the modules $X(s_i,t_i)$ have distinct lengths, we deduce that they are all summands of M', as required.

Now suppose $k > 0$. Let $\pi_1 : X(s_1,t_1) \to C_1$ be the projection. For $i > 1$, let $\pi_i = p^{s_{i-1}}_{s_i,t_i} : X(s_i,t_i) \to X(s_{i-1},t_i)$. Then the composition $\pi_i\phi_i$ is non-zero, it has kernel of length s_{i-1} and cokernel of length t_i, and $(s_{i-1},t_{i-1}) \leq (s_{i-1},t_i)$, so by Lemma 3.1 (ii) there is map $\sigma_i : X(s_{i-1},t_{i-1}) \to C_i$ with $\pi_i\phi_i = \sigma_i\phi_{i-1}$. This gives a sequence

$$X \xrightarrow{\phi} \bigoplus_{i=1}^{k} X(s_i,t_i) \xrightarrow{\psi} \bigoplus_{i=1}^{k} C_i \to 0$$

where

$$\psi = \begin{pmatrix} \pi_1 & 0 & & & \\ -\sigma_2 & \pi_2 & & & \\ & & \ddots & & \\ & & & \pi_{k-1} & 0 \\ 0 & & & -\sigma_k & \pi_k \end{pmatrix}.$$

Since the π_i are all epimorphisms, so is ψ. Also, it's easy to check (by computing lengths) that $\operatorname{im}\phi = \ker\psi$ and hence the sequence is exact. □

We may use Lemma 3.2 to compute $\operatorname{cogen}^1(M)$ for a module M, written as a direct sum of indecomposable modules, say $M = M_1 \oplus \cdots \oplus M_m$. Let X be an indecomposable module and let $\operatorname{coker}\phi = C_1 \oplus \cdots \oplus C_k$ as in Lemma 3.2.

Lemma 3.3. *We have the following for an indecomposable module X.*

(i) $X \in \operatorname{cogen}(M) \Leftrightarrow X$ *is isomorphic to a submodule of* M_j *for some* j.
(ii) $X \in \operatorname{cogen}^1(M) \Leftrightarrow X, C_1, \dots, C_k$ *are in* $\operatorname{cogen}(M)$.

4. (Faithfully) balanced modules for Nakayama algebras

In this section Λ is again a Nakayama algebra. Recall that a module X is a *subquotient* of Y if $X \cong Y''/Y'$ for some submodules $Y' \subseteq Y'' \subseteq Y$; it is a *proper* subquotient if $Y' \neq 0$ or $Y'' \neq Y$.

Theorem 4.1. *If Λ is Nakayama, then a module M is balanced if and only if it satisfies the following two conditions:*

(B1) *if X is an indecomposable summand of M and X is a proper subquotient of some indecomposable summand of M, then X is a proper submodule or proper quotient of some indecomposable summand of M, and*
(B2) *if S, T are simple modules with* $\operatorname{Ext}^1(T, S) \neq 0$ *and S or T is a composition factor of M, then* $\operatorname{Hom}(M, S) \neq 0$ *or* $\operatorname{Hom}(T, M) \neq 0$.

Proof. Assuming that M is balanced, we prove (B1). Let U be an indecomposable direct summand of M which is a proper subquotient of some other indecomposable summand, and suppose that U is not a proper submodule or quotient of any indecomposable summand of M. We derive a contradiction. Let $\theta : P \to U$ be the projective cover of U. Clearly P is indecomposable and we consider the minimal left $\operatorname{add}(M)$-approximation ϕ of P given by Lemma 3.2, involving modules of the form $P(s, t)$. Letting $s = \ell(\ker\theta)$, we can identify $U = P(s, 0)$ and $\theta = \phi_{s0}$. Then $(s, 0) \in M_P$, and it is minimal since if $s' < s$ then U is a proper quotient of $P(s', 0)$. Now U is a proper subquotient of some indecomposable summand Y of M. Say $U \cong Y''/Y'$ for $Y' \subseteq Y'' \subseteq Y$. Then θ lifts to a map $P \to Y''$, and by inclusion this gives a map $\psi : P \to Y$. We write it in the form $\phi_{s',t'} : P \to P(s', t')$. Now $(s, 0)$ is incomparable with (s', t'), so the minimal approximation of P has at least $k \geq 2$ terms, and the first term must be $(s_1, t_1) = (s, 0)$. The second term (s_2, t_2) gives rise to a summand $C_2 = P(s, t_2)$ of $\operatorname{coker}\phi$. Now $U = P(s, 0)$ embeds in C_2 properly since $t_2 > 0$. By assumption M is balanced, so $\operatorname{coker}\phi$ is cogenerated by M. Thus C_2 and hence also U embeds in an indecomposable summand of M, a contradiction. Thus there can be no such summand U, so (B1) holds.

Next, assuming still that M is balanced, we prove (B2). First assume that S is a composition factor of M. Let ϕ be the minimal $\operatorname{add}(M)$-approximation of the projective cover P of S given by Lemma 3.2. Since $\operatorname{Hom}(P, M) \neq 0$ we have $k > 0$ in the lemma. Now if $\operatorname{Hom}(M, S) = 0$, then $\operatorname{Hom}(P(s_1, t_1), S) = 0$, so S is not the top of $P(s_1, t_1)$, so ϕ_1 is not surjective. Thus $C_1 = \operatorname{coker}\phi_1$ is non-zero with socle T (since Λ is Nakayama). Since M is balanced, C_1 embeds in M, hence $\operatorname{Hom}(T, M) \neq 0$. On the other hand, if T is a composition factor of M, then since the dual module DM is a balanced Λ^{op}-module and $\operatorname{Ext}^1(DS, DT) \neq 0$, this argument shows that $\operatorname{Hom}(DM, DT) \neq 0$ or $\operatorname{Hom}(DS, DM) \neq 0$, so $\operatorname{Hom}(T, M) \neq 0$ or $\operatorname{Hom}(M, S) \neq 0$, as required.

For the converse, we now assume that (B1) and (B2) hold. Fix a simple S and consider the minimal left $\operatorname{add}(M)$-approximation ϕ of the projective cover P of S as in Lemma 3.2. We need to show that the summands C_i of $\operatorname{coker}\phi$ are in $\operatorname{cogen}(M)$. If $k = 0$ there is nothing to check, so suppose that $k > 0$. Thus S is a composition factor of M.

First we consider the term C_1. If $\operatorname{Hom}(M, S) \neq 0$, then M has a summand with top S. It follows that the first of the minimal elements of M_P is of the form $(s, 0)$. But then $C_1 = 0$, so there is nothing to check for this term. On the other hand, if $\operatorname{Hom}(M, S) = 0$, then the first of the minimal elements of M_P is of the form (s_1, t_1) with $t_1 \neq 0$. Then C_1 is non-zero, say with socle T. Clearly $\operatorname{Ext}^1(T, S) \neq 0$, so by condition $(B2)$ there is an indecomposable summand U of M with socle T, and, say, length h. Take h maximal with this property. If $h \geq \ell(C_1)$, then C_1 embeds in U, as required. Otherwise $h < \ell(C_1)$. Then U embeds in C_1, so it is a proper subquotient of $P(s_1, t_1)$. Thus by (B1), U is a proper quotient or submodule of a summand U' of M. Both are impossible. Indeed, if there is a proper surjection $\alpha : U' \to U$, then the top of $\ker \alpha$ is S, so $\ker \alpha$ is the image of a map $\psi : P \to U'$. But then $\ell(\operatorname{coker} \psi) = \ell(U) = h < \ell(C_1) = t_1 \le t_r$ for all r. This is impossible since $U' \cong P(s, h)$ for some s so $(s, h) \in M_P$, contradicting the fact that the (s_i, t_i) are the minimal elements. If U is a proper submodule of U' then h was not maximal.

Next we consider the term C_i for $1 < i \le k$. It is a quotient of $P(s_i, t_i)$ and it has a submodule isomorphic to $P(s_{i-1}, t_{i-1})$. Thus $P(s_{i-1}, t_{i-1})$ is a proper subquotient of an indecomposable summand of M. Since (s_{i-1}, t_{i-1}) is a minimal element of M_P, it follows that $P(s_{i-1}, t_{i-1})$ is not a proper quotient of any indecomposable summand of M. Thus by (B1) it is a proper submodule of an indecomposable summand U' of M. Take $\ell(U')$ to be maximal. If $\ell(U') \ge \ell(C_i)$, then C_i embeds in U', as required. Thus for a contradiction suppose that $\ell(U') < \ell(C_i)$. Then U' properly embeds in C_i. Thus U' is a subquotient of $P(s_i, t_i)$, so by condition (B1), U' is a proper submodule or quotient of an indecomposable summand W of M. If it is a proper submodule of W, then $\ell(U')$ is not maximal. Thus U' is a proper quotient of W. Now the composition f of ϕ_{i-1} with the inclusion $P(s_{i-1}, t_{i-1}) \to U'$ lifts to a map $g : P \to W$. Then $\ell(\operatorname{im} g) > \ell(\operatorname{im} f) = \ell(\operatorname{im} \phi_{i-1})$ and $\ell(\operatorname{coker} g) = \ell(\operatorname{coker} f) > \ell(\operatorname{coker} \phi_{i-1})$. By assumption $(\ell(\ker g), \ell(\operatorname{coker} g)) \ge (s_j, t_j)$ for some j. Since $\ell(\operatorname{im} g) > \ell(\operatorname{im} \phi_{i-1})$ we have $s_j \le \ell(\ker g) < \ell(\ker \phi_{i-1}) = s_{i-1}$, so $j \ge i$. On the other hand, $t_j \le \ell(\operatorname{coker} g) = \ell(\operatorname{coker} f) < \ell(\operatorname{coker} \alpha) = \ell(\operatorname{coker} \phi_i) = t_i$, where α is the composition of f with the inclusion $U' \to C_i$, so $j < i$, a contradiction. □

We recall that Nakayama algebras are QF-3, so a module is faithful if and only if it has every indecomposable projective-injective module as a summand, see [2, Theorem 32.2]. In addition, for a Nakayama algebra, every indecomposable is a subquotient of a projective-injective. For faithfully balanced modules, Theorem 4.1 takes the following form.

Corollary 4.2. *If Λ is Nakayama, then a module M is faithfully balanced if and only if it satisfies the following conditions:*

(FB0) *every indecomposable projective-injective module is a summand of M,*

(FB1) *if X is an indecomposable summand of M and X is not projective-injective, then X is a proper submodule or proper quotient of some indecomposable summand of M, and*

(FB2) *if S, T are simple modules with $\operatorname{Ext}^1(T, S) \neq 0$, then $\operatorname{Hom}(M, S) \neq 0$ or $\operatorname{Hom}(T, M) \neq 0$.*

Specializing to the algebra Λ_n, which is a Nakayama algebra, this gives Theorem 1.1.

5. Critical modules and minimal faithfully balanced modules

Let Λ be an artin algebra.

Lemma 5.1. *Given modules N, M, we have*

(i) $N \in \operatorname{gen}_1(M)$ *if and only if* $\operatorname{gen}_1(M \oplus N) = \operatorname{gen}_1(M)$.

(ii) $N \in \operatorname{cogen}^1(M)$ *if and only if* $\operatorname{cogen}^1(M \oplus N) = \operatorname{cogen}^1(M)$.

Proof. Part (ii) is due to Ma and Sauter [13, Lemma 3.3], and part (i) is dual. □

Recall that a (faithfully balanced) gen_1-*category* is a subcategory of Λ-mod of the form $\operatorname{gen}_1(M)$, where M is a (faithfully balanced) module.

Proof of Theorem 1.2. Clearly $\mathcal{G}$ contains at least one gen_1-critical module M with $\operatorname{gen}_1(M) = \mathcal{G}$. We shall show that M is uniquely determined.

By assumption Λ is representation-directed [3, §IX.3], so we can enumerate the indecomposable modules in $\mathcal{G}$ as $X_1, X_2, \dots, X_m$ with $\operatorname{Hom}(X_j, X_i) = 0$ for $j > i$ and each $\operatorname{End}(X_i)$ a division algebra.

We show by induction on i how to determine whether or not X_i is a summand of M. Let M_i be the direct sum of all X_j with $j < i$ which occur as summands of M. By the inductive hypothesis this is uniquely determined.

We show that X_i is a summand of M if and only if $X_i \notin \operatorname{gen}_1(M_i)$. Namely, if X_i is a summand of M, write $M = M' \oplus X_i$. By the ordering of the X_j, the minimal right $\operatorname{add}(M')$-approximation of X_i is the same as the minimal right $\operatorname{add}(M_i)$-approximation. Moreover the kernel of this approximation is in $\operatorname{gen}(M_i)$ if and only if it is in $\operatorname{gen}(M')$. It follows that if $X_i \in \operatorname{gen}_1(M_i)$, then $X_i \in \operatorname{gen}_1(M')$. But then $\operatorname{gen}_1(M') = \operatorname{gen}_1(M' \oplus X_i) = \operatorname{gen}_1(M)$ by Lemma 5.1, contradicting the criticality of M. Conversely, if X_i is not a summand of M, then the minimal right $\mathrm{add}(M)$-approximation of X_i is the same as the minimal right $\mathrm{add}(M_i)$-approximation. By directedness, the kernel of this approximation is in $\mathrm{gen}(M)$ if and only if it is in $\mathrm{gen}(M_i)$. Thus if $X_i \notin \mathrm{gen}_1(M_i)$ then $X_i \notin \mathrm{gen}_1(M) = \mathcal{G}$, which is nonsense.

The final part of the theorem follows from Lemma 5.1. □

In our example in the introduction, we have illustrated the faithfully balanced gen_1-categories for Λ_n with $n = 3$. The next proposition shows that in order to understand arbitrary gen_1-categories for Λ_n it is equivalent to understand faithfully balanced gen_1-categories for Λ_{n+1}. Let $\mathcal{C}$ be the category of Λ_{n+1}-modules vanishing at vertex 1, and $F : \mathcal{C} \to \Lambda_n$-mod the equivalence of categories which forgets vertex 1.

Proposition 5.2. *The assignment $\mathcal{G} \mapsto F(\mathcal{G} \cap \mathcal{C})$ gives a 1:1 correspondence between faithfully balanced* gen_1*-categories for* Λ_{n+1}-mod *and arbitrary* gen_1*-categories for* Λ_n*. The inverse sends $\mathcal{H}$ to the category of Λ_{n+1}-modules which are the direct sum of an injective module and a module in $\mathcal{C}$ whose image under F is in $\mathcal{H}$.*

Proof. Observe that any indecomposable Λ_{n+1}-module is either injective or in $\mathcal{C}$, and not both. Let $M \in \mathcal{C}$ and let I be injective. If $X \in \mathcal{C}$, then since $\mathrm{Hom}(I, X) = 0$, the minimal right $\mathrm{add}(M \oplus I)$-approximation of X is the same as the minimal right $\mathrm{add}(M)$-approximation. Moreover the kernel of this is in $\mathrm{gen}(M)$ if and only if it is in $\mathrm{gen}(M \oplus I)$, again using that injective modules have no non-zero maps to non-injective indecomposables. It follows that $X \in \mathrm{gen}_1(M \oplus I)$ if and only if $F(X) \in \mathrm{gen}_1(F(M))$.

Now if $\mathcal{G}$ is a faithfully balanced gen_1-category, then by Lemmas 2.4 and 5.1 it is of the form $\mathrm{gen}_1(M \oplus I)$ where I is the direct sum of all indecomposable injectives and $M \in \mathcal{C}$. The assignment sends this to $\mathrm{gen}_1(F(M))$, so it is a gen_1-category for Λ_n. The reverse mapping sends $\mathrm{gen}_1(F(M))$ to $\mathrm{gen}_1(I \oplus M)$, which is necessarily a faithfully balanced gen_1-category. □

According to our computer calculations, the number of faithfully balanced gen_1-categories in Λ_n-mod for $n = 1, \ldots, 6$ is $1, 2, 7, 39, 325, 3875$, and the number of minimal faithfully balanced Λ_n-modules is $1, 2, 6, 25, 134, 881$.

Lemma 5.3. *A basic module T with $\mathrm{pd}\, T \leq 1$ and which is rigid (i.e. $\mathrm{Ext}^1(T, T) = 0$), is* gen_1*-critical.*

Proof. Assume $T = M \oplus N$ and $\mathrm{gen}_1(M) = \mathrm{gen}_1(T)$. Then we have $N \in \mathrm{gen}_1(M)$ and so there is an exact sequence $M_1 \to M_0 \to N \to 0$ with $M_0, M_1 \in \mathrm{add}(M)$ and $\mathrm{Hom}(M, -)$ exact on it. Thus we obtain two short exact sequences

$$0 \to X_1 \to M_1 \to X_0 \to 0,$$
$$0 \to X_0 \to M_0 \to N \to 0.$$

Applying $\mathrm{Hom}(N, -)$ to the first exact sequence yields an exact sequence

$$0 = \mathrm{Ext}^1(N, M_1) \to \mathrm{Ext}^1(N, X_0) \to \mathrm{Ext}^2(N, X_1) = 0$$

since T is rigid and $\mathrm{pd}\, N \leq \mathrm{pd}\, T \leq 1$. This means the second short exact sequence is split and so $N \in \mathrm{add}(M)$. It follows that $\mathrm{add}(M) = \mathrm{add}(T)$ and therefore $M = T$ since T is basic. □

Proof of Theorem 1.3. The first part is a special case of Lemma 5.3. Now suppose that T is a basic tilting module and Λ is hereditary. Let M be a faithfully balanced summand of T. Recall that M being faithfully balanced is equivalent to $\Lambda \in \mathrm{cogen}^1(M)$ by Lemma 2.4. Thus we have two short exact sequences

$$0 \to \Lambda \to M_0 \to X \to 0,$$
$$0 \to X \to M_1 \to Y \to 0$$

with $M_i \in \mathrm{add}(M)$ such that $\mathrm{Hom}_\Lambda(-, M)$ is exact on both short exact sequences. It is straightforward to check that $T' = M \oplus X$ is a tilting module. By definition $T' \in \mathrm{gen}(T) \cap \mathrm{cogen}(T) = T^\perp \cap {}^\perp T$, so $T \oplus T'$ is rigid and since tilting modules are maximal rigid we conclude that $\mathrm{add}(T) = \mathrm{add}(T')$. By applying $\mathrm{Hom}(-, M)$ to the second short exact sequence we obtain $\mathrm{Ext}^1(Y, M) = 0$. By applying $\mathrm{Hom}(Y, -)$ to the first exact sequence we obtain $\mathrm{Ext}^1(Y, X) = 0$. Thus the second short exact sequence splits, so $X \in \mathrm{add}(M)$. This implies $\mathrm{add}(T) = \mathrm{add}(M \oplus X) = \mathrm{add}(M)$, and since T is basic, it follows that $M = T$. □

We refer to [1] for the notion of a support τ-tilting module.

Corollary 5.4. *Every basic support τ-tilting module is* gen_1*-critical and balanced.*

Proof. Let $I = \mathrm{ann}(\mathrm{M})$ be the annihilator ideal of a support τ-tilting module M. By [1, Proposition 2.2], ${}_{\Lambda/I}M$ is a classical tilting module, so it is faithfully balanced and gen_1-critical as a Λ/I-module by Theorem 1.3. This implies that ${}_{\Lambda}M$ is balanced. Assume $N \in \mathrm{add}({}_{\Lambda}M)$ such that $\mathrm{gen}_1(M) = \mathrm{gen}_1(N)$ and consider the fully faithful and exact functor $i\colon \Lambda/I\text{-mod} \to \Lambda\text{-mod}$ which has a left adjoint $q = \Lambda/I \otimes_{\Lambda} -$. Now we have $M \cong iq(M)$, therefore $N = i(N') \in \mathrm{add}(M) \subset \mathrm{im}\, i$. We have $i(\mathrm{gen}_1(N')) = \mathrm{gen}_1(N) \cap \mathrm{im}\, i = \mathrm{gen}_1(M) \cap \mathrm{im}\, i = i(\mathrm{gen}_1({}_{\Lambda/I}M))$ and since i is fully faithful, $\mathrm{gen}_1(N') = \mathrm{gen}_1({}_{\Lambda/I}M)$. This implies $\mathrm{add}(N') = \mathrm{add}({}_{\Lambda/I}M)$ since ${}_{\Lambda/I}M$ is gen_1-critical so apply i to conclude $\mathrm{add}(N) = \mathrm{add}(M)$. This proves M is gen_1-critical. □

The following result is due to Morita [14, Theorem 1.1].

Theorem 5.5. *If M is a faithfully balanced module for an algebra Λ and X is indecomposable, then $M \oplus X$ is faithfully balanced if and only if $X \in \mathrm{gen}(M)$ or $X \in \mathrm{cogen}(M)$.*

For convenience we give the proof of this in the next two lemmas. Observe that one direction, Lemma 5.6, holds with the weaker assumption that M is faithful.

Lemma 5.6. *Let M be faithful and X be indecomposable. If $M \oplus X$ is (faithfully) balanced, then we have either $X \in \mathrm{gen}(M)$ or $X \in \mathrm{cogen}(M)$.*

Proof. Let $E = \mathrm{End}_{\Lambda}(X)$, then E is a local ring and hence there is a unique simple E-module, say S. Define $X_1 = \sum_{f:M\to X} \mathrm{im}(f)$ and $X_0 = \bigcap_{g:X\to M} \ker(g)$. Then X_1 and X_0 are submodules of X. By definition, we have $X \in \mathrm{gen}(M)$ if and only if $X_1 = X$ and $X \in \mathrm{cogen}(M)$ if and only if $X_0 = 0$. Now assume $X_1 \neq X$ and $X_0 \neq 0$. The first assumption implies $X/X_1 \neq 0$ and hence has S as a quotient, the second implies S is a submodule of X_0. It follows that $\mathrm{Hom}_E(X/X_1, X_0) \neq 0$. Thus there exists a non-zero E-endomorphism $\theta : X \to X$ such that $X_1 \subseteq \ker(\theta)$ and $\mathrm{im}(\theta) \subseteq X_0$. Let $\Gamma = \mathrm{End}_{\Lambda}(M)$, then we have

$$\mathrm{End}_{\Lambda}(M \oplus X) = \begin{pmatrix} \Gamma & \mathrm{Hom}_{\Lambda}(X, M) \\ \mathrm{Hom}_{\Lambda}(M, X) & E \end{pmatrix},$$

and $M \oplus X$ is a left $\mathrm{End}_{\Lambda}(M \oplus X)$-module. We claim that $\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}$ is an $\mathrm{End}_{\Lambda}(M \oplus X)$-endomorphism of $M \oplus X$, that is, for any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}_{\Lambda}(M \oplus X)$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}.$$

To prove the claim we need to show $\theta c = 0$, $b\theta = 0$ and $\theta d = d\theta$. Now $\mathrm{im}(c) \subseteq X_1 \subseteq \ker(\theta)$ gives $\theta c = 0$, $\mathrm{im}(\theta) \subseteq X_0$ gives $b\theta = 0$ and the fact that θ is an E-endomorphism gives $\theta d = d\theta$. By assumption, $M \oplus X$ is balanced and this implies that the action of $\begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}$ is given by the multiplication of some element $\lambda \in \Lambda$. Now we must have $\lambda M = 0$ which forces $\lambda = 0$ since M is faithful as a Λ-module. Thus we have $\theta = 0$, a contradiction. □

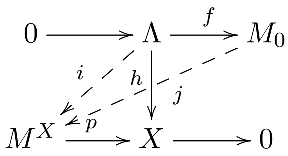
Lemma 5.7. *Let M be faithfully balanced. If either $X \in \mathrm{gen}(M)$ or $X \in \mathrm{cogen}(M)$, then $M \oplus X$ is also faithfully balanced.*

Proof. We will prove the case $X \in \mathrm{gen}(M)$; the case $X \in \mathrm{cogen}(M)$ is dual. Since M is faithfully balanced, there is an exact sequence

$$0 \to \Lambda \xrightarrow{f} M_0 \xrightarrow{g} M_1$$

such that f and $\mathrm{coker}(f) \to M_1$ are minimal left $\mathrm{add}(M)$-approximations. We claim that the map f is also a left $\mathrm{add}(M \oplus X)$-approximation. To this end, it is enough to show that any map $h : \Lambda \to X$

factors through f. Consider the following diagram



where p is the minimal right $\operatorname{add}(M)$-approximation of X. Since $X \in \operatorname{gen}(M)$, p is an epimorphism and so there is an $i : \Lambda \to M^X$ such that $h = pi$. Then i factors as $i = jf$ and we have $h = pi = (pj)f$. This proves the claim. Now since $\operatorname{coker}(f) \in \operatorname{cogen}(M) \subseteq \operatorname{cogen}(M \oplus X)$ we conclude that $\Lambda \in \operatorname{cogen}^1(M \oplus X)$. This proves $M \oplus X$ is faithfully balanced. □

For Nakayama algebras, the conditions (FB1) and (FB2) in Corollary 4.2 allow a different approach to minimal faithfully balanced modules. We begin with some constructions which work for a module M for an arbitrary algebra. Recall [4] that a module $X \in \operatorname{add}(M)$ is a *splitting projective* if every epimorphism $M' \to X$ with $M' \in \operatorname{add}(M)$ is a split epimorphism, and it is a *splitting injective* if every monomorphism $X \to M'$ is a split monomorphism. We write M^g for the direct sum of one copy of each of the splitting projective summands of M and M^c for the direct sum of one copy of each of the splitting injective summands of M. By [4, Theorem 2.3], $\operatorname{add}(M^g)$ is a minimal cover for $\operatorname{add}(M)$, so M^g is a minimal summand of M with $\operatorname{gen}(M^g) = \operatorname{gen}(M)$, and it is unique up to isomorphism with this property. Similarly for M^c with $\operatorname{cogen}(M^c) = \operatorname{cogen}(M)$.

For Nakayama algebras, Morita's Theorem 5.5 can be used to construct all faithfully balanced modules from minimal faithfully balanced modules. This follows from the following lemma:

Lemma 5.8. *Let Λ be a Nakayama algebra. If M is faithfully balanced but not minimal faithfully balanced, then there is a faithfully balanced summand N of M with $|M| = |N| + 1$.*

Proof. Let L be a faithfully balanced proper summand of M. Let $M = L \oplus U$. Pick an indecomposable summand $U' \in \operatorname{add}(U)$ of minimal length and let $U = U' \oplus V$. Then $N := V \oplus L$ still fulfills the conditions in Corollary 4.2 and therefore is a faithfully balanced module. Indeed, the condition (FB2) is satisfied by the summand L and the hypothesis on the length of U' implies that no other summands of U are generated or cogenerated by U' so condition (FB1) also holds. □

Remark 5.9. We don't know whether this result holds without the assumption that Λ is Nakayama.

Lemma 5.10. *If M is a minimal faithfully balanced module for a Nakayama algebra Λ, then any indecomposable summand X of M is a summand of M^g or M^c, and X is a summand of both if and only if X is projective-injective. Thus*

$$M \oplus P \cong M^g \oplus M^c,$$

where P is the direct sum of the indecomposable projective-injective Λ-modules.

Proof. Since M is faithfully balanced, by condition (FB1) in Corollary 4.2, every indecomposable summand X of M which is not projective-injective is a proper submodule or quotient of another summand of M. Thus X cannot be a summand of both M^g and M^c. On the other hand, if X is a summand of neither, then it is both a proper submodule and quotient of other summands of M. But then the complement of X still satisfies the conditions of Corollary 4.2, so is faithfully balanced, contradicting minimality. □

Theorem 5.11. *Let M be a minimal faithfully balanced module for a Nakayama algebra Λ. If N is a module with $\operatorname{gen}(N) \cap \operatorname{cogen}(N) = \operatorname{gen}(M) \cap \operatorname{cogen}(M)$, then N is faithfully balanced and M is a summand of N.*

Proof. Clearly $\operatorname{gen}(\operatorname{gen}(M) \cap \operatorname{cogen}(M)) = \operatorname{gen}(M)$ and $\operatorname{cogen}(\operatorname{gen}(M) \cap \operatorname{cogen}(M)) = \operatorname{cogen}(M)$, so we have $\operatorname{gen}(N) = \operatorname{gen}(M)$ and $\operatorname{cogen}(N) = \operatorname{cogen}(M)$. By the uniqueness of minimal covers and cocovers, $M^g \cong N^g$ and $N^c \cong M^c$. By Lemma 5.10, we conclude that M is a summand of N. Now N is faithfully balanced by Theorem 5.5. □

6. Counting faithfully balanced modules

In this section we prove Theorem 1.4. Given a module M for Λ_n, we write $t_r(M)$ for the number non-isomorphic indecomposable summands of M with top $S[r]$, or equivalently in row r in the Young diagram. We consider indeterminates $x_1, \dots, x_n$, and define

$$k_n(x_1, \dots, x_n) = \sum_M \prod_{r=1}^n x_r^{t_r(M)} \in \mathbb{Z}[x_1, \dots, x_n]$$

where the sum is over all basic faithfully balanced Λ_n-modules M. We define

$$p_n(x_1, \dots, x_n) = \sum_M \prod_{r=1}^n x_r^{t_r(M)} \in \mathbb{Z}[x_1, \dots, x_n]$$

where the sum is over all modules M satisfying (FB0) and (FB1) in the statement of Theorem 1.1.

Let $[2, n] := \{k \in \mathbb{Z} : 2 \leq k \leq n\}$. In condition (FB2) in Theorem 1.1, it follows from (FB0) that M contains summands in the virtual cohooks associated to the leaves $(1, 0)$ and $(n+1, n)$. Thus we may replace (FB2) by the conditions $(\mathrm{FB2})_k$ that M has a summand in $\mathrm{cohook}(k, k-1)$, for all $k \in [2, n]$. Given a subset $I \subseteq [2, n]$, we define $s_n^I(x_1, \dots, x_n)$ to be the sum of $\prod_{r=1}^n x_r^{t_r(M)}$ over all M which satisfy (FB0), (FB1) and $(\mathrm{FB2})_k$ for all $k \in I$, and $f_n^I(x_1, \dots, x_n)$ to be the sum over all M which satisfy (FB0), (FB1) and fail $(\mathrm{FB2})_k$ for all $k \in I$.

We write $\underline{x}_n = (x_1, \dots, x_n)$ and for a subset $J = \{j_1 < \cdots < j_m\}$ of $[2, n]$, we write

$$\underline{x}_n^J = (x_1, \dots, \hat{x}_{j_1}, \dots, \hat{x}_{j_m}, \dots, x_n)$$

where $\hat{x}_p$ means that the term x_p is omitted.

Lemma 6.1. *We have the following.*

(i) $f_n^I(\underline{x}_n) = p_{n-|I|}(\underline{x}_n^I)$.
(ii) $s_n^I(\underline{x}_n) = \sum_{J \subseteq I} (-1)^{|J|} p_{n-|J|}(\underline{x}_n^J)$.
(iii) $k_n(\underline{x}_n) = \sum_{J \subseteq [2,n]} (-1)^{|J|} p_{n-|J|}(\underline{x}_n^J)$.
(iv) $p_n(\underline{x}_n) = \sum_{J \subseteq [2,n]} k_{n-|J|}(\underline{x}_n^J)$.

Proof. (i) To fail the condition $(\mathrm{FB2})_k$ means that row k and column $k-1$ of the Young diagram must be empty. If so we can shrink the diagram to obtain a Young diagram for a smaller n.

(ii) Follows by (i) and the inclusion-exclusion principle.

(iii) This is a special case of (ii).

(iv) Using (iii), the right hand side becomes

$$\sum_{J \subseteq [2,n]} \sum_{I \subseteq [2,n] \setminus J} (-1)^{|I|} p_{n-|I \cup J|}(\underline{x}_n^{I \cup J}) = \sum_{L \subseteq [2,n]} n_L p_{n-|L|}(x_n^L)$$

where $L = I \cup J$ and

$$n_L = \sum_{I \subseteq L} (-1)^{|I|} = \begin{cases} 1 & (L = \emptyset) \\ 0 & (L \neq \emptyset) \end{cases}.$$

□

Lemma 6.2. *We have*

$$p_{n+1}(\underline{x}_{n+1}) = \left(\prod_{i=1}^{n+1} (1 + x_i) \right) \sum_{I \subseteq [2,n]} k_{n-|I|}(\underline{x}_n^I) \cdot \prod_{i \in I} \frac{1}{1 + x_i}.$$

Proof. Given a subset $I \subseteq [1, n+1]$ and a basic Λ_n-module M, we obtain a basic Λ_{n+1}-module via

$$M' = \left(\bigoplus_{i \in I} M_{ii} \right) \oplus \left(\bigoplus_{M_{ij} \in \mathrm{add}(M)} M_{i,j+1} \right),$$

and every basic Λ_{n+1}-module arises in this way for a unique I and M. Moreover M' satisfies (FB0) and (FB1) if and only if M satisfies (FB0), (FB1) and (FB2)$_k$ for $k \in [2,n] \cap I$. Thus

$$p_{n+1}(\underline{x}_{n+1}) = (1+x_1)(1+x_{n+1}) \cdot \sum_{I \subseteq [2,n]} \left(\prod_{i \in I} x_i \right) s_n^I(\underline{x}_n).$$

By Lemma 6.1(ii) this becomes

$$(1+x_1)(1+x_{n+1}) \cdot \sum_{I \subseteq [2,n]} \left(\prod_{i \in I} x_i \right) \left(\sum_{J \subseteq I} (-1)^{|J|} p_{n-|J|}(\underline{x}_n^J) \right).$$

Letting $L = I \setminus J$ we can rewrite this as

$$(1+x_1)(1+x_{n+1}) \cdot \sum_{J \subseteq [2,n]} \left(\prod_{j \in J} x_j \right) (-1)^{|J|} p_{n-|J|}(\underline{x}_n^J) \sum_{L \subseteq [2,n] \setminus J} \left(\prod_{\ell \in L} x_\ell \right).$$

$$= (1+x_1)(1+x_{n+1}) \cdot \sum_{J \subseteq [2,n]} \left(\prod_{j \in J} x_j \right) (-1)^{|J|} p_{n-|J|}(\underline{x}_n^J) \cdot \left(\prod_{i \in [2,n] \setminus J} (1+x_i) \right).$$

$$= \left(\prod_{i=1}^{n+1} (1+x_i) \right) \sum_{J \subseteq [2,n]} (-1)^{|J|} \left(\prod_{j \in J} \frac{x_j}{1+x_j} \right) p_{n-|J|}(\underline{x}_n^J).$$

By part (iv) of Lemma 6.1 this becomes

$$\left(\prod_{i=1}^{n+1} (1+x_i) \right) \sum_{J \subseteq [2,n]} (-1)^{|J|} \left(\prod_{j \in J} \frac{x_j}{1+x_j} \right) \sum_{K \subseteq [2,n] \setminus J} k_{n-|J|-|K|}(\underline{x}_n^{J \cup K}).$$

Letting $I = J \cup K$ this becomes

$$\left(\prod_{i=1}^{n+1} (1+x_i) \right) \sum_{I \subseteq [2,n]} k_{n-|I|}(\underline{x}_n^I) \sum_{J \subseteq I} (-1)^{|J|} \left(\prod_{j \in J} \frac{x_j}{1+x_j} \right)$$

$$= \left(\prod_{i=1}^{n+1} (1+x_i) \right) \sum_{I \subseteq [2,n]} k_{n-|I|}(\underline{x}_n^I) \left(\prod_{i \in I} \frac{1}{1+x_i} \right)$$

as claimed. □

Proof of Theorem 1.4. Define

$$h_n(\underline{x}_n) = \prod_{r=1}^{n} \left(\prod_{s=1}^{r} (1+x_s) - 1 \right)$$

as in the statement of the theorem. Suppose by induction that $k_m(\underline{x}_m) = h_m(\underline{x}_m)$ for all $m \leq n$. We show that $k_{n+1}(\underline{x}_{n+1}) = h_{n+1}(\underline{x}_{n+1})$. For a subset I of $[2,n]$ we have

$$\left(\prod_{i=1}^{n+1} (1+x_i) \right) h_{n-|I|}(\underline{x}_n^I) \prod_{i \in I} \frac{1}{1+x_i} = h_{n-|I|}(\underline{x}_n^I) \prod_{i \in [1,n+1] \setminus I} (1+x_i)$$

$$= h_{n-|I|}(\underline{x}_n^I) \left(\prod_{i \in [1,n+1] \setminus I} (1+x_i) - 1 \right) + h_{n-|I|}(\underline{x}_n^I)$$

$$= h_{n-|I|+1}(\underline{x}_{n+1}^I) + h_{n-|I|}(\underline{x}_{n+1}^{I \cup \{n+1\}}).$$

By Lemma 6.2 and the inductive hypothesis this gives

$$p_{n+1}(\underline{x}_{n+1}) = \sum_{I \subseteq [2,n]} \left(h_{n-|I|+1}(\underline{x}_{n+1}^I) + h_{n-|I|}(\underline{x}_{n+1}^{I \cup \{n+1\}}) \right) = \sum_{I \subseteq [2,n+1]} h_{n+1-|I|}(\underline{x}_{n+1}^I).$$

On the other hand, by Lemma 6.1(iv),

$$p_{n+1}(\underline{x}_{n+1}) = \sum_{I\subseteq[2,n+1]} k_{n+1-|I|}(\underline{x}^I_{n+1}).$$

By the inductive hypothesis, we can equate terms, giving $k_{n+1}(\underline{x}_{n+1}) = h_{n+1}(\underline{x}_{n+1})$, as required. □

Recall the notation $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \dots [n]_q$.

Corollary 6.3. *For Λ_n we have the following.*

(i) *If $k_{n,s}$ denotes the number of basic faithfully balanced modules with s summands, then*

$$\sum_s k_{n,s}x^s = \prod_{i=1}^{n}((1+x)^i - 1).$$

(ii) $k_{n,s} = \sum_{(j_1,j_2,\dots,j_n):\, 1\le j_r\le r, \sum_{r=1}^n j_r = s} \binom{1}{j_1}\binom{2}{j_2}\cdots\binom{n}{j_n}$.

(iii) *The number of basic faithfully balanced modules is $[n]_2! = \prod_{i=1}^n (2^i - 1)$. The number of faithfully balanced modules in which the indecomposable summands have multiplicity at most m is $\prod_{i=1}^n ((1+m)^i - 1)$.*

(iv) *Any basic faithfully balanced module for Λ_n has at least n indecomposable summands, and the number of basic faithfully balanced modules with n indecomposable summands is $n!$.*

(v) *The direct sum of all indecomposable modules is the only faithfully balanced module with $N = n(n+1)/2$ indecomposable summands; there are $N-1$ basic faithfully balanced modules with $N-1$ summands.*

7. Number of faithfully balanced modules for quadratic Nakayama algebras

In order to study quadratic Nakayama algebras, we begin with a lemma about faithfully balanced modules for Λ_n. For $n \ge 1$ and $0 \le k \le 2$, we define $N_k(n) \in \mathbb{N}$ by

$$N_0(n) = [n]_2!, \quad N_1(n) = 2^{n-1}[n-1]_2!, \quad N_2(n) = \begin{cases} 1 & (n \le 2) \\ 2^{n-3}(2^n-1)[n-2]_2! & (n \ge 3). \end{cases}$$

Recall that $S[n]$ is a simple projective module for Λ_n, and $S[1]$ is a simple injective.

Lemma 7.1. *Fix a subset of the set $\{S[1], S[n]\}$ of cardinality k. The number of basic faithfully balanced modules for Λ_n having all of the modules in this subset as direct summands is $N_k(n)$.*

Proof. The case $k = 0$ is Corollary 6.3(iii). The two possible subsets of size $k = 1$ give the same number of faithfully balanced modules by duality, so we may assume that the subset is $\{S[n]\}$. If the Young diagram for a faithfully balanced module has a non-empty second column (starting from the left), then the simple $S[n]$ is *irrelevant* for the faithfully balanced condition in Theorem 1.1. On the other hand, if the second column is empty then $S[n]$ must be a direct summand of M. Let M be a faithfully balanced module for Λ_n with empty second column. Removing the second column and the simple $S[n]$ and shrinking the diagram gives a faithfully balanced module for Λ_{n-1}. In other words, there is a bijection between faithfully balanced modules for Λ_{n-1} and faithfully balanced modules for Λ_n with an empty second column. If we denote by t_n the number of faithfully balanced modules for Λ_n having non-empty second column and $S[n]$ as a summand, we have $N_0(n) = N_0(n-1) + 2t_n$, and the number of faithfully balanced modules with $S[n]$ as a direct summand is $N_0(n-1) + t_n = N_1(n)$.

The case $k = 2$ is similar, but slightly more technical so we only sketch the arguments. The case $n \le 2$ is clear, so assume $n \ge 3$. If we denote by A the set of summands of M in the second column and by B the summands in the second row (starting from the top), we can split the set of faithfully balanced modules into 4 subsets accordingly to the emptiness or non-emptiness of A and B.

Let r be the number of faithfully balanced modules having $A \ne \emptyset$ and $B \ne \emptyset$ and having $S[1]$ and $S[n]$ as direct summands. In this case the modules $S[1]$ and $S[n]$ are both irrelevant for the condition of being faithfully balanced module.

Let s be the number of faithfully balanced modules having $A \ne \emptyset$ and $B = \emptyset$ and having $S[1]$ and $S[n]$ as direct summands. In this case the module $S[n]$ is irrelevant for the condition of being faithfully balanced module.

Note that duality induces a bijection between the case $A \neq \emptyset, B = \emptyset$ and the case $A = \emptyset, B \neq \emptyset$. Moreover, the shrinking argument used in the first part shows that there is a bijection between the set of faithfully balanced modules having $A = \emptyset, B = \emptyset$ and the set of faithfully balanced modules for Λ_{n-2}. In other words, we have

$$N_0(n) = 4r + 4s + N_0(n-2).$$

Looking at the modules having $B = \emptyset$ and using a shrinking argument, we have

$$N_0(n-1) = 2s + N_0(n-2).$$

Now the number of faithfully balanced modules with both $S[1]$ and $S[n]$ as summands is

$$r + 2s + N_0(n-2) = N_2(n),$$

as required. □

Now let Λ be a *quadratic Nakayama* algebra, say of the form KQ/I where Q is a linearly oriented quiver of type A_n or $\widetilde{A}_n$ (see [18, Theorem 10.3]), and I is an admissible ideal generated by paths of length 2. Let $P_1, \dots, P_t$ be the indecomposable projective-injective Λ-modules, say of lengths $n_1, \dots, n_t$, let $\mathcal{G} = \text{gen}(\text{D}\Lambda) \cap \text{cogen}(\Lambda)$ and let g be the number of simple Λ-modules in $\mathcal{G}$, equivalently the number of simples which occur as the socle of some P_i and the top of some P_j. Define k_i to be 2 if $\operatorname{soc} P_i$ and $\operatorname{top} P_i$ are both in $\mathcal{G}$, 1 if only one is in $\mathcal{G}$, and 0 otherwise.

Theorem 7.2. *If Λ is a quadratic Nakayama algebra as above, then the number of basic faithfully balanced Λ-modules is $2^g N_{k_1}(n_1) \dots N_{k_t}(n_t)$.*

Proof. It is easy to see that the Auslander-Reiten quiver of Λ is a concatenation of the Auslander-Reiten quivers of the algebras Λ_{n_i}. See Figure 3 for an example where Q is of type A_n. If Q is of type $\widetilde{A}_n$, the diagram is similar, but the bottom left and top right vertices must be identified, and the corresponding simple is in $\mathcal{G}$.

Corollary 4.2 implies that adding or deleting a simple in $\mathcal{G}$ as a summand of a module M does not affect whether or not M is faithfully balanced. Thus the number of basic faithfully balanced modules is 2^g times the number of those which have all simples in $\mathcal{G}$ as a summand. Clearly the basic modules M which have all simples in $\mathcal{G}$ as summands are in 1-1 correspondence with collections of basic modules $M_1, \dots, M_t$ for the algebras $\Lambda_{n_1}, \dots, \Lambda_{n_t}$, where M_i has a copy of $S[n_i]$ as a summand if $\operatorname{soc} P_i \in \mathcal{G}$ and a copy of $S[1]$ as a summand if $\operatorname{top} P_i \in \mathcal{G}$. Now Corollary 4.2 shows that M is faithfully balanced if and only if the M_i are faithfully balanced. The result thus follows from Lemma 7.1. □

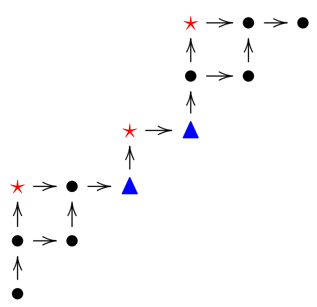


FIGURE 3. Auslander-Reiten quiver of the algebra given by a quiver of type A_6 modulo the ideal generated by the paths from 2 to 4 and from 3 to 5. The red stars are the indecomposable projective-injective modules and the blue triangles are the simples in $\mathcal{G}$. The number of basic faithfully balanced modules is $2^2 N_1(3) N_2(2) N_1(3) = 576$.

8. Tree-like combinatorics for faithfully balanced modules

The purpose of the whole section is to prove Theorem 1.5. As explained in Theorem 1.1, a basic faithfully balanced module can be identified with a collection of vertices in a staircase Young diagram. In order to be consistent with the literature on binary trees, we apply a rotation by an angle of $-\frac{\pi}{4}$ of the grid and we adopt the usual terminology of binary trees.

Let M be a faithfully balanced module for Λ_n. The black box corresponding to the projective module M_{1n} is at the top of the grid and is called the *root*. The black boxes in the Young diagram are called *vertices*. The first vertex in the cohook of a vertex v which is on its right side (resp. its left side) is called, if it exists, the *right parent* (resp. *left parent*) of v. Conversely we say that v is a *right child* (resp. *left child*) of its left parent (resp. right parent). The conditions (FB1) and (FB2) imply that every vertex (including the leaves) has at least one parent.

We turn the collection of vertices into a graph in the Young diagram by adding a straight edge between each vertex (including the leaves) and each one of its parents. If M is a faithfully balanced module, we denote by T_M the graph obtained as explained above and we call it the graph of M. From now on, we reserve the name *vertex of* T_M for the vertices that are not the leaves.

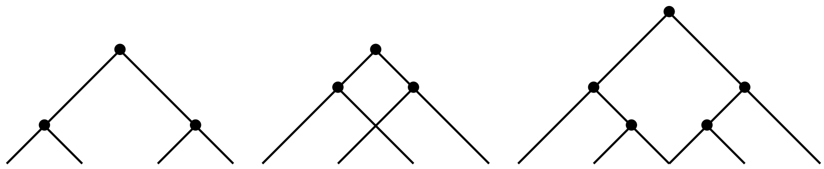


Figure 4. The left-most and middle trees correspond to faithfully balanced modules for Λ_3. The right-most example is a minimal faithfully balanced module with 5 summands for Λ_4.

Lemma 8.1. *Let $n \in \mathbb{N}$, let M be a minimal faithfully balanced module for Λ_n and let T_M be its graph. Then*

(1) *T_M is a connected graph.*
(2) *The number of vertices in T_M is at least n.*[2]
(3) *T_M is a rooted binary tree if and only if it has n vertices.*

Proof. By using (FB1) and (FB2) we see that each vertex and each leaf is connected to the root of T_M.

If a vertex v has a left and a right parent we can remove it without breaking the conditions (FB0), (FB1) and (FB2). As a consequence, in a minimal faithfully balanced module every (non-leaf) vertex has one left parent or one right parent but not both. This implies that there is a unique path in T_M between two vertices which does not go through the leaves. Moreover, all the vertices but the root are trivalent.

The second point is proved by induction on n. For $n = 0$ and $n = 1$ there is nothing to prove. Assume $n \geqslant 2$. The root of T_M has at least one child, say S. We consider the subgraph T_S of T_M that consists of all the vertices connected to S in $T_M - \{R\}$ where R is the root of T_M. Let T_R be the graph obtained by cutting between R and S and removing all the vertices of T_S. We add a leaf at the former position of S and we remove all the leaves which are no longer connected to R.

Since there is a unique path between two vertices, the number of vertices of T_M is equal to the sum of the numbers of vertices of T_S and T_R. However, a leaf may appear in T_S and in T_R (see Figure 4 for an example).

We denote by n_S and n_R the number of leaves of T_S and T_R. The graphs T_S and T_R satisfy the conditions (FB0), (FB1) and (FB2) so they can be identified with graphs of faithfully balanced modules for Λ_{n_S-1} and Λ_{n_R-1} respectively. By induction, we see that the number of direct summands in M is larger than or equal to $n_S + n_R - 2$. At least one leaf occurs in both T_S and T_R (the one that we add at the former position of S in T_R), so $n_S + n_R \geq n + 1 + 1$ and the result follows.

For the last point, we have already noticed that between any two vertices in T_M there is a unique path which does not go through the leaves. Hence T_M is a tree if and only if there is no leaf with more

[2]This is a combinatorial proof of Corollary 6.3 (iv).

than one parent. If we decompose T_M as T_S and T_R as above, we have that T_M is a tree if and only if T_S and T_R are two trees and they have no common leaf (other than the one at S). The result follows by induction.

□

Remark 8.2. The two trees on the left of Figure 4 give the same abstract graph but two different faithfully balanced modules. For us it is important to keep the 'shape' of the tree. This can be done by considering it in the Young diagram, or alternatively by fixing the positions of the root and the leaves of the tree.

Recall that binary trees can be defined inductively as follows. A binary tree is either the empty set or a tuple (r, L, R) where r is a singleton set and L and R are two binary trees. The empty set has no vertex but has one leaf. The set of leaves of $T = (r, L, R)$ is the disjoint union of the set of leaves of L and R. The *size* of the tree is its number of vertices (equivalently the number of leaves minus 1). As can be seen in Figure 4, we draw the trees with their root on the top and the leaves on the bottom. We will always implicitly label the leaves of a tree of size n from 1 to $n+1$ starting from the right-most leaf. Let us give an inductive definition for the graphs T_M associated to faithfully balanced modules for Λ_n.

Definition 8.3. An *interleaved tree* with 0 vertices is the empty set. An interleaved tree with $n > 0$ vertices is the data of

- A singleton set r called the root.
- Two interleaved trees T_R and T_L with, respectively, n_R and n_L vertices such that $n = n_R + n_L + 1$.
- A strictly increasing function $\mathrm{lea}_R : \{2, \ldots, n_R + 1\} \to \{2, \ldots, n\}$.

The function lea_R is called the *interleaving* function.

Remark 8.4. Let T be an interleaved tree. If the interleaving function satisfies $\mathrm{lea}_R(i) = i$ for all i we say that it is a *trivial* interleaving function and we say that T has *trivial interleaving*. The classical binary trees can be seen as interleaved trees which are inductively constructed from interleaved trees with trivial interleaving functions.

Lemma 8.5. *Let M be a faithfully balanced module for Λ_n with exactly n summands. The graph T_M can be naturally seen as an interleaved tree.*

Proof. This graph has a root and left and right subtrees denoted by T_L and T_R. The two subtrees correspond to faithfully balanced modules for Λ_{n_L} and Λ_{n_R} respectively. The function lea_R is defined by $\mathrm{lea}_R(i) = k$ if the ith leaf of T_R is the kth leaf of T for $i \in \{2, \ldots, n_R\}$. □

The interleaving function lea_R determines another strictly increasing function $\mathrm{lea}_L : \{1, \ldots, n_L\} \to \{2, \ldots, n\} \setminus \mathrm{Im}(\mathrm{lea}_R)$. Note that the function lea_R is not defined at 1. This is just for convenience: this function gives the positions of the leaves of the right subtree and the first leaf of the right subtree is always 1. Similarly, the function lea_L is not defined at $n_L + 1$ because the last leaf of the left subtree is always $n+1$.

Proposition 8.6. *Let $n \in \mathbb{N}$.*

(1) *The map sending a faithfully balanced module M for Λ_n to the interleaved tree T_M is a bijection between the set of isomorphism classes of basic faithfully balanced modules for Λ_n with exactly n summands and the set of interleaved trees with n (non-leaf) vertices.*
(2) *It restricts to a bijection between the set of isomorphism classes of basic tilting modules for Λ_n and the set of binary trees with n (non-leaf) vertices.*

Proof. By Lemma 8.5 the graph T_M is an interleaved tree. Conversely, let $T = (r, T_R, T_L, \mathrm{lea}_R)$ be a interleaved tree with n vertices. We aim to show that there is a bijection between interleaved trees and collections of n vertices in the Young diagram of triangular shape that satisfy the conditions of Theorem 1.1. Hence, we place T in the Young diagram of staircase shape as follows:

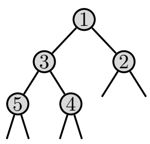
The root is placed in the box with coordinate $(1, n)$. The leaves of T_R are placed according to the function lea_R and the leaves of T_L are placed according to the function lea_L. The position of each vertex is determined by the positions of the leaves. Precisely, if v is the root of the subtree with right-most leaf i_r and left-most leaf i_l, then it is in the box with coordinates $(i_r, i_l - 1)$.

Since the root of T is placed at $(1, n)$, the condition (FB0) holds. The right subtree T_R and T share the same right-most leaf, hence the root of T_R is in the same row, on the left of the root of T. Similarly, the root of the left subtree T_L is in the same column, below the root the T. Hence the cohooks of these new vertices are non-empty. By induction, we see that (FB1) holds. Since each leaf has a parent, the virtual cohooks are non-empty and (FB2) holds.

For the second point, we remark that a faithfully balanced module with n summands is a tilting module if and only if it has no self extensions. It remains to see that there is an extension between two indecomposable modules if and only if there is a non-trivial interleaving in the corresponding tree. There is a non-trivial interleaving in T if and only if there are two indecomposable summands M_{ac} and M_{bd} with the property that $a < b < c + 1 \leq d$. By Lemma 8.1 of [11], $\mathrm{Ext}^1(M_{ac}, M_{bd}) \neq 0$ if and only if $a < b \leq c + 1 \leq d$. The case $b = c + 1$ cannot appear since T_M is a tree. We can also see that the bijection restricts to the one sketched in Section 9 of [11]. □

Using this inductive definition we can construct a simple bijection between the set of interleaved binary trees and the set of increasing binary trees introduced by Françon in Section 2 of [8]. The bijection uses two intermediate functions that we call *untangling* and *reordering*. At the level of abstract trees the functions do nothing, but they will change the positions of the leaves, and so the interleaving of the trees. These changes will be encoded in a labelling of the vertices of the tree.

We start by considering interleaved trees which are *labelled* by integers. Let T be an interleaved tree with n vertices. A label of T is a sequence of pairwise distinct integers $V = (v_1, v_2, \ldots, v_n)$. The integer v_i is the label of the i-th vertex in the pre-order traversal of T (recursively visit the root, the right subtree and the left subtree). Concretely, we first visit the root of T, then we move to the right subtree and recursively apply the algorithm in it. That is visit the new root and move to the new right subtree. When all the vertices of a right subtree have been visited, we recursively perform the algorithm in the left subtree. In the example below, the first vertex visited by the algorithm is labeled by 1, the second by 2 and so on.



Then v_1 is the label of the root of T and if T_R has n_R vertices, the sequence $(v_2, \ldots, v_{n_R+1})$ labels the vertices of the subtree T_R. The remaining are the labels of the left subtree. Note that the ordering of the elements of the sequence is important!

Definition 8.7. An increasing interleaved tree is an interleaved tree T together with a labelling of its vertices by pairwise distinct integers such that if v is a child of w, then the label of w is smaller than the label of v.

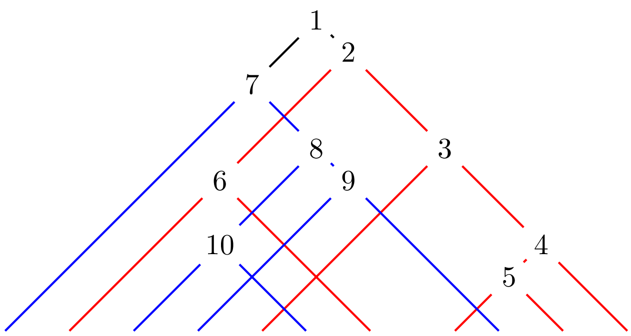


FIGURE 5. An example of an increasing interleaved tree.

If T is an interleaved tree of size n, we can always turn it into an increasing tree by associating to it the sequence of labels $(1, 2, \ldots, n)$. We say that the labelled interleaved tree (T, V) is *well-ordered* if the sequence V is strictly increasing for the usual ordering of the integers.

The first step of the bijection is given by the *untangling* function that takes a well-ordered increasing interleaved tree and gives an interleaved binary tree with trivial interleaving function.

Let $\big(T = (r, T_R, T_L, \mathrm{lea}_R), V\big)$ be a well-ordered increasing interleaved tree. Let $\mathrm{Unt}(T) = (r, T_R, T_L, \mathrm{triv})$ where triv is the trivial interleaving function. Let $\mathrm{Unt}(V)$ be the sequence consisting of the v_is where i runs first through the positions of the leaves of the right subtree and then through the positions of the leaves of the left subtree. In other words, $\mathrm{Unt}(V)$ is the sequence obtained by concatenation of the sequences (v_1), $(v_{\mathrm{lea}_R(i)})_{i\in\{2,\dots,n_R\}}$ and $(v_{\mathrm{lea}_L(i)})_{i\in\{1,\dots,n_L\}}$. The untangling function sends (T, V) to $(\mathrm{Unt}(T), \mathrm{Unt}(\mathrm{V}))$. See Figure 6 (left) for an illustration.

Conversely we define a *reordering* function that takes an increasing interleaved tree with trivial interleaving function and well-ordered subtrees and produces a well-ordered interleaved tree.

Let $\big(T' = (r, T'_R, T'_L, \mathrm{triv}), V'\big)$ be an increasing interleaved tree. Let $\mathrm{Reo}(V')$ be the sequence putting the elements of V' in a strictly increasing order. Let $\mathrm{Reo}(T') = (r, T'_L, T'_R, \mathrm{lea}'_R)$ be the interleaved binary tree where $\mathrm{lea}'_R(i)$ is defined as the position of v_i in $\mathrm{Reo}(V')$.

Lemma 8.8. *The function* Unt *and* Reo *are mutually inverse bijections between the set of well-ordered interleaved binary trees and the set of increasing interleaved binary trees with trivial interleaving function and well-ordered subtrees.*

Proof. Let (T, V) be a well-ordered increasing tree with n vertices and (T', V') be an increasing interleaved tree with trivial interleaving function and well-ordered left and right subtrees.

By construction $\mathrm{Unt}(T, V)$ is an interleaved tree with trivial function. Since the interleaving functions are strictly increasing we see that the subtrees T_R and T_L of $\mathrm{Unt}(T)$ are well-ordered.

Conversely, the left and right subtrees of T' are well-ordered, so the function lea'_R in $\mathrm{Reo}(T', V')$ is strictly increasing. So $\mathrm{Reo}(T')$ is an interleaved tree and it is by construction well-ordered.

For $i \in \{2, \dots, n_R + 1\}$, the i-th element of $\mathrm{Unt}(V)$ is $v_{\mathrm{lea}_R(i)}$. Since V is well-ordered, $v_{\mathrm{lea}_R(i)}$ is the $\mathrm{lea}_R(i)$-th largest element of V. It follows that Reo and Unt are two mutually inverse bijections. □

We can now describe a bijection between the set of interleaved binary trees and the set of increasing binary trees.

Starting with an interleaved tree with n vertices, we see it as an increasing interleaved tree with label $V = (1, 2, \dots, n)$. Applying the function Unt we obtain an increasing interleaved tree with a trivial interleaving function and well-ordered left and right subtrees. Then, we continue the process by inductively applying the untangling function to the left and right subtrees. Since at each step we go down in the tree, the process ends. Since we inductively remove the non trivial interleaving, the result is an increasing binary tree. We call this algorithm the *untangling procedure*.

Conversely, starting with an increasing binary tree we inductively apply the function Reo to the subtrees of increasing size. The result is an interleaved tree labelled by $(1, 2, \dots, n)$. We call this algorithm the *reordering procedure*.

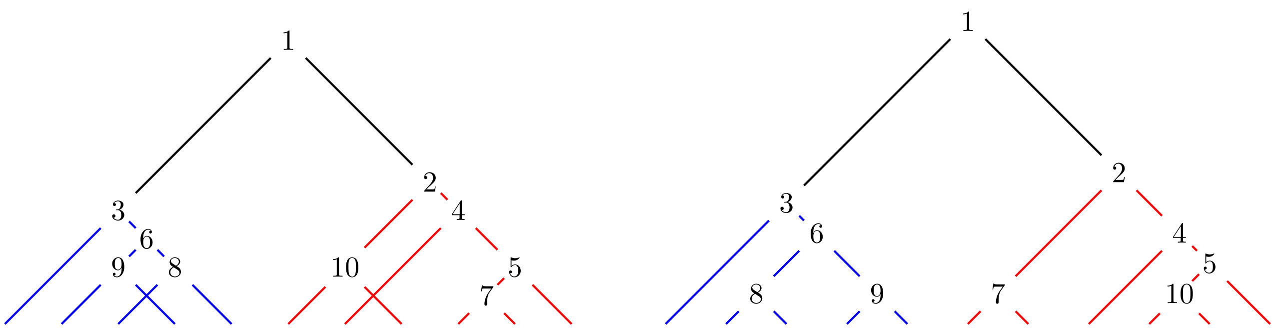


Figure 6. First and last steps of the untangling procedure applied to the example of Figure 5.

Proposition 8.9. *Let n be an integer.*

(1) *The untangling procedure induces a bijection between the set of interleaved trees with n vertices and the set of increasing binary trees with n-vertices with inverse bijection given by the reordering procedure.*

(2) *The map that sends an interleaved tree to the word obtained by reading in in-order (left subtree, root, right subtree) the label of the increasing binary tree given by the untangling procedure induces a bijection between the set of interleaved trees with n vertices and the set of permutations on $\{1, 2, \dots, n\}$.*

Proof. The first point follows from Lemma 8.8. It is classical that reading the labels of the vertices of an increasing tree in in-order induces a bijection between the set of increasing binary trees and the set of permutations (see Section 2 of [8] for more details). □

Remark 8.10. It is clear that the untangling procedure restricts to a bijection between binary trees and well-ordered increasing binary trees since all the untangling functions are the identity if we start with a binary tree.

The bijection between interleaved trees and increasing binary trees is natural, however the induced bijection with the set of permutations does not seem to reflect the interesting combinatorial properties that we observed in Corollary 6.3. For that, we consider another classical family counted by $n!$.

Definition 8.11. A function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is *self-bounded* if $f(i) \leq i$ for $i \in \{1, 2, \dots, n\}$. (These functions are called 'décroissantes' by Françon in [8].)

The untangling procedure is also a way of labelling the vertices of an interleaved tree: if T is an interleaved tree, the untangling procedure gives an increasing binary tree. Reading the binary tree using a traversal gives a sequence of labels that we use to label the vertices of T using the same traversal.

Due to the recursive nature of the interleaved trees, this labelling has recursive description. The description is based on the following algorithm with data a pair (T, V) where T is an interleaved tree with n vertices and V is a sequence of n integers. The outcome of the algorithm is the label of the root of T by the first element of V and two pairs (T_R, V_R) and (T_L, V_L) each consisting of an interleaved tree and a sequence of labels. The trees T_R and T_L are just the right and left subtrees of T. If $V = (v_1, v_2, \cdots, v_n)$, then $V_R = (v_{\mathrm{lea}_R(i)})_{i \in \{2, \cdots, n_R+1\}}$ where n_R is the number of vertices of T_R. Similarly, $V_L = (v_{\mathrm{lea}_L(i)})_{i \in \{1, \cdots, n_L\}}$ where n_L is the number of vertices of T_L.

To label the tree T, we start the algorithm with $(T, (1, 2, \cdots, n))$. It labels the root of T by 1 and produces two sequences of labels V_R and V_L. The sequence V_R is the sequence of the positions of the leaves (without the right-most leaf) of T_R in T. Then, we apply the algorithm to (T_R, V_R) and (T_L, V_L) and we continue so until reaching all the vertices of T.

We illustrate the procedure by doing a few steps with the tree of Figure 5: we start with $(T, (1, 2, \cdots, 10))$. The root of T is labelled by 1. Since the leaves of T_R have positions $(2, 4, 5, 7, 10)$ we have $(T_R, (2, 4, 5, 7, 10))$ and $(T_L, (3, 6, 8, 9))$ as output of the first algorithm. Then, we apply the algorithm to the two pairs. The root T_R is labelled by 2. Then, we see that the leaves of the right-subtree T'_R of T_R have positions $(2, 3, 5)$. Hence the output of the algorithm is $(T'_R, (4, 5, 10))$ and $(T'_L, (7))$ where T'_L is the left subtree of T_R. For the left subtree T_L, the root is labelled by 3. Since the left subtree is empty, the only output is $(T''_R, (6, 8, 9))$ where T''_R is the right-subtree of T_L. The result of the procedure is illustrated in Figure 7.

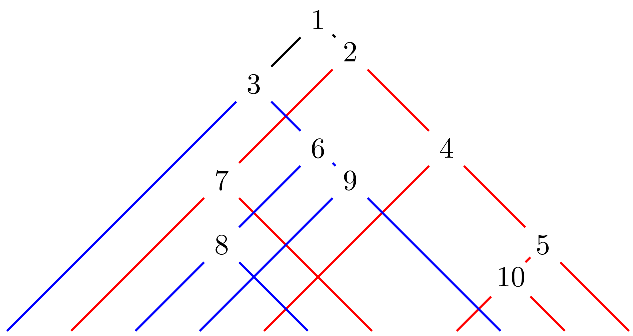


FIGURE 7. Labelling of the vertices of the interleaved tree of Figure 5.

If T is an interleaved tree we construct a function f_T as follows. First label the vertices of T by the procedure described above. If v is a vertex labelled by i we let $f_T(i) = j$ where j is the position of the right-most leaf of the subtree with root i in T.

In terms of faithfully balanced modules the function f is obtained by taking the index of the simple top of each of the indecomposable summand of the module in a suitable total ordering of the indecomposable summands.

For example, if T is the interleaved tree of Figure 5, then the function f_T is:

(8.1)

i	1	2	3	4	5	6	7	8	9	10
$f_T(i)$	1	1	3	1	1	3	5	6	3	2

Before proving that the function f_T is self-bounded, we need a technical lemma comparing the labels in a tree and the labels in the left and right subtrees. If T_R is the right subtree of an interleaved tree T and v is a vertex of T_R, then v has a label in T obtained by applying the labelling procedure to $(T, (1, 2, \cdots, n)$ and a label in T_R obtained by applying the labelling procedure to $(T_R, (1, 2, \cdots, n_R))$. Similarly, if v is a vertex of T_L it has a label in T and a label in T_L.

Lemma 8.12. *Let $T = (r, T_R, T_L, \mathrm{lea}_R)$ be an interleaved tree and f_T its corresponding function. We let f_{T_R} and f_{T_L} be the functions corresponding to T_R and T_L.*

(1) *Let $v \in T_R$. Then v is labelled by i in T_R if and only if it is labelled by $\mathrm{lea}_R(i+1)$ in T. Moreover, we have*
$$f_{T_R}(i) = j \Leftrightarrow f_T(\mathrm{lea}_R(i+1)) = \mathrm{lea}_R(j),$$
with the convention that $\mathrm{lea}_R(1) = 1$.

(2) *Let $v \in T_L$. Then v is labelled by i in T_L if and only if it is labelled by $\mathrm{lea}_L(i)$ in T. Moreover, we have*
$$f_{T_L}(i) = j \Leftrightarrow f_T(\mathrm{lea}_L(i)) = \mathrm{lea}_L(j).$$

Proof. By construction, if $v \in T_R$ is labelled by i (in T_R), then it is labelled by $\mathrm{lea}_R(i+1)$ in T. Moreover, we have $f_{T_R}(i) = j$ if and only if the right-most leaf of the subtree with root i in T_R is j. In the tree T, the corresponding vertex has right-most leaf $\mathrm{lea}_R(j)$ with the convention that $\mathrm{lea}_R(1) = 1$. The second statement is similar. □

Lemma 8.13. *Let T be an interleaved tree. Then the function f_T is self-bounded.*

Proof. Using Lemma 8.12, we will prove the result by induction on the number of vertices of the trees.

Let v be a vertex of T. If v is the root of T, it is labeled by 1 and $f_T(1) = 1$. If $v \in T_R$. Then, let i be its label in T_R. If we set $j = f_{T_R}(i)$, then we have by induction that $j \leq i$. The function lea_R is increasing, hence $\mathrm{lea}_R(j) < \mathrm{lea}_R(i+1)$ and we have $\mathrm{lea}_R(j) = f_T(\mathrm{lea}_R(i+1)) < \mathrm{lea}_R(i+1)$.

Similarly, if $v \in T_L$ is labeled by i in T_L, then by induction $j = f_{T_L}(i) \leq i$. Since the function lea_L is increasing, we get $\mathrm{lea}_L(j) = f_T(\mathrm{lea}_L(i)) \leq \mathrm{lea}_L(i)$.

□

In order to show that the map $T \mapsto f_T$ is a bijection between interleaved trees and self-bounded function, we show that the two sets have the same grammar. For that, we will define a *right* and a *left sub-function* as well as a partition of $\{2, \dots, n\}$ into two sequences F_R and F_L.

The sequence F_R is the (totally ordered) sequence inductively constructed as follows. Let i_1 be the smallest integer such that $i_1 \neq 1$ and $f(i_1) = 1$. If there is no such integer then F_R is the empty sequence, otherwise $F_R = (i_1)$. For $i = i_1 + 1, \dots, n$, if $f(i) = 1$ or $f(i) \in F_R$, then add i to F_R. The sequence F_L is the sequence inductively constructed as follows. Let i be the smallest integer such that $f(i) = i$ and for $i = 2, \dots, n$ if $f(i) \in F_L$ or $f(i) = i$, then add i to F_L.

The sequences F_R and F_L are recording the positions of the leaves of the left and right subtrees of the (yet) hypothetical interleaved tree T such that $f = f_T$. Hence, looking at Lemma 8.12, we guess how to define the left and right sub-functions of f. The right sub-function f_R is defined by $f_R(i) = j$ if and only if $f(w_i) = w_{j-1}$ where $F_R = (w_1, \dots, w_{n_R})$ with the convention that $w_0 = 1$. The left sub-function f_L is defined by $f_L(i) = j$ if and only if $f(u_i) = u_j$ where $F_L = (u_1, \dots, u_{n_L})$.

In the case of the function (8.1), we have $F_R = (2,4,5,7,10)$ and $F_L = (3,6,8,9)$. The sub-functions are

i	1	2	3	4	5
$f_R(i)$	1	1	1	4	2

i	1	2	3	4
$f_L(i)$	1	1	2	1

Using this decomposition of a self-bounded function we can inductively construct an interleaved tree: the root corresponds to $f(1) = 1$ and the interleaving function is defined by setting $\mathrm{lea}_R(i+1)$ to be the ith element of F_R. The right subtree corresponds to f_R and the left subtree corresponds to f_L. The only function on the empty set corresponds to the trivial interleaved tree and the unique self-bounded function on a set with one element corresponds to the unique interleaved tree with one vertex.

Lemma 8.14. *Let T be an interleaved tree with n vertices and f its self-bounded function. Then $\mathrm{Im}(\mathrm{lea}_R) = F_R$ and $\mathrm{Im}(\mathrm{lea}_L) = F_L$.*

Proof. Let us recall that the vertices of T_R are labelled by the sequence $(\mathrm{lea}_R(i))_{i\in\{2,\cdots,n_R+1\}}$.

Let i_1 be the first element of F_R. Then $f(i_1) = 1$, so i_1 is the label of the right child of the root of T_R and we have $i_1 = \mathrm{lea}_R(2)$. If x is such that $f(x) = 1$, then x is in T_R. Hence it is labelled by an element of $\mathrm{Im}(\mathrm{lea}_R)$. If x is such that $f(x) = y$ with $y \in F_R$, then by induction y is the label of a leaf of T_R. So the vertex labelled by x is a parent of a leaf of T_R, so it is in T_R and we have, $F_R \subseteq \mathrm{Im}(\mathrm{lea}_R)$.

Conversely if $x \in \mathrm{Im}(\mathrm{lea}_R)$, then it labels a vertex of T_R and we have $f(x) = 1$ or $f(x) = y$ where y labels a leaf of T_R. Looking at the proof of Lemma 8.12, we see that $y < f(x)$. So $x \in F_R$

The proof of the other case is left to the reader. □

Theorem 8.15. *The map sending an interleaved tree T to the function f_T is a bijection between the set of interleaved trees with n vertices and the set of self-bounded functions on $\{1,2,\ldots,n\}$.*

Proof. Using Lemmas 8.12 and 8.14, the result follows by induction on n. □

Remark 8.16. This bijection is not the composition of the untangling procedure and the bijection between increasing binary trees and self-bounded functions given in Section 4 of [8].

In the classical case of binary trees, the bijection restricts to a bijection between the set of binary trees with n vertices and the set of non-decreasing self-bounded functions on $\{1,2,\ldots,n\}$. These functions are known to be counted by the Catalan numbers (See e.g. part (s) of Exercise 6.19 of [19]).

Proposition 8.17. *Let T be an interleaved tree with n vertices and f_T its self-bounded function. Then T is a binary tree if and only if f_T is such that $f_T(1) \leq f_T(2) \leq \cdots \leq f_T(n)$.*

Proof. If T is a binary tree, its labelling is well-ordered, so it follows that $f_T(i) \leq f_T(i+1)$. Conversely, if $f_T(1) = 1 \leq f_T(2) \leq \cdots \leq f_T(n)$, then the sequence F_R is of the form $(2,3,\ldots,k)$ because if y is the smallest integer which is not in this sequence, then $f_T(y) = y$. Since $y \leq f_T(y+1)$, the value of $f_T(y+1)$ is either y or $y+1$. This implies that $y+1$ is also in F_L and we see that $F_L = \{y, y+1, \ldots, n\}$. So the interleaving function of T is trivial and the left and right sub-functions both satisfy the non-decreasing property of the Lemma. The result follows by induction. □

Proof of Theorem 1.5. The bijection between (i) and (ii) is given by Proposition 8.6, between (ii) and (iii) by Proposition 8.9, and between (ii) and (iv) by Theorem 8.15 and Proposition 8.17. □

For M a faithfully balanced module for Λ_n with n summands, we define $\chi(M) = \sum_i n_i(i-1) \in \mathbb{N}$, where n_i is the number of indecomposable summands of M in row i of the Young diagram, or equivalently with top $S[i]$, so $\mathrm{top}\, M \cong \bigoplus_{i=1}^n S[i]^{n_i}$. See [7] for the notion of a 'mahonian statistic'.

Proposition 8.18. *The mapping $\chi : fb(n) \to \mathbb{N}$ is a mahonian statistic, that is,*

$$\sum_{M\in fb(n)} q^{\chi(M)} = [n]_q!$$

Proof. By Theorem 1.5, the faithfully balanced modules M with n summands correspond to self-bounded functions f, and by the discussion after Definition 8.11, $\chi(M) = \sum_{i=1}^n (f(i)-1)$. Thus

$$\sum_{M\in fb(n)} q^{\chi(M)} = \sum_f q^{\sum_{i=1}^n (f(i)-1)} = \prod_{i=1}^n \Bigg(\sum_{f(i)=1}^{i} q^{f(i)-1} \Bigg) = [n]_q!. \qquad \square$$

Using Theorem 1.1 and Lemma 8.1, we see that a faithfully balanced module with exactly n summands for Λ_n corresponds to the data of vertices in the Young diagram of staircase shape satisfying the following two conditions

(1) There is a vertex in the top left box of the diagram.
(2) Each vertex or leaf has a vertex on its left in the same row or above it in the same column but not both.

This is very similar to the definition of *tree-like* tableaux in the sense of [6]. If there is an empty row or an empty column in the faithfully balanced module M, we can simply remove it and shrink the diagram. We denote the result by $sh(M)$.

Proposition 8.19. *The map sending M to $sh(M)$ is a bijection between the set of faithfully balanced modules with exactly n summands and the tree-like tableaux with n pointed cells.*

Proof. Since we will not need this bijection, we only sketch the proof. We label the leaves of the Young diagram $(n, n-1, \ldots, 1)$ from 1 to $n+1$ starting at the top right and finishing at the bottom left. The southeast border of a tree-like tableaux can be seen as a path formed by vertical and horizontal steps. It has exactly $n+1$ steps that we label from 1 to $n+1$ starting at the top right and finishing at the bottom left. In both cases, the labelling induces a labelling of the rows and the columns of the diagram. The vertex at the intersection of the row i and the column j is said to have coordinates (i, j).

Let T be a tree-like tableau with n pointed cells. We can construct a configuration of vertices in the Young tableau $(n, n-1, \ldots, 1)$ by sending the pointed cell with coordinates (i, j) to the vertex with same coordinates in the Young tableaux of staircase shape.

It is straightforward to check that the result is a faithfully balanced module and that this map is a bijection which is inverse to $M \mapsto sh(M)$. □

Remark 8.20. Tree-like tableaux are known to be counted by $n!$, so this gives another easy bijective proof for the cardinality of $fb(n)$. However, it is not completely obvious that there are $n!$ tree-like tableaux with n pointed cells. Proposition 8.19 relates $fb(n)$ with other fillings of Young tabeaux such as *permutation tableaux* (see e.g. [21]) and *alternating tableaux* (see e.g. [15]).

Finally let us remark that there is a bijection Φ_2 between tree-like tableaux and increasing binary trees that can be found in [6]. Composing it with the bijection of Proposition 8.19, we have another bijection between the set $fb(n)$ and the set of increasing binary trees with n vertices. The two bijections give the same underlying tree but the labellings are quite different.

9. On partial orders

Let Λ be a finite-dimensional Nakayama algebra. We define a relation $\trianglelefteq$ on minimal faithfully balanced modules by

$$N \trianglelefteq M \;\Leftrightarrow\; \operatorname{cogen}(N) \subseteq \operatorname{cogen}(M) \text{ and } \operatorname{gen}(N) \supseteq \operatorname{gen}(M).$$

It is clearly reflexive and transitive, and by Theorem 5.11 it is also antisymmetric, so a partial order. The relation $\trianglelefteq$ has a smallest element given by Λ and a largest element given by $\mathrm{D}\Lambda$, so its (finite) Hasse diagram is connected. As before, our main interest is in the restriction of the partial order to $fb(n)$ for the algebra Λ_n.

Remark 9.1. If Λ is hereditary and M, N are cotilting modules (implying $\operatorname{cogen}(X) = \operatorname{cogen}^1(X)$, $\operatorname{gen}(X) = \operatorname{gen}_1(X)$ for $X = M, N$), then the following are equivalent: (a) $N \trianglelefteq M$, (b) $\operatorname{cogen}(N) \subseteq \operatorname{cogen}(M)$, (c) $\operatorname{gen}(N) \supseteq \operatorname{gen}(M)$, and (d) $\operatorname{Ext}^1(N, M) = 0$. This suggests many possible partial orders generalizing the usual partial order on tilting modules for hereditary algebras (cf. [10]). For example we can consider the partial order $\leq$ given by

$$N \leq M \;\Leftrightarrow\; \operatorname{cogen}^1(N) \subseteq \operatorname{cogen}^1(M) \text{ and } \operatorname{gen}_1(N) \supseteq \operatorname{gen}_1(M).$$

In Figure 8 we show the Hasse diagrams for $fb(3)$. The poset induced by the relation $\trianglelefteq$ seems to be the most interesting, since the other two do not give lattice structures on $fb(n)$ when $n \geq 4$.

Definition 9.2. Using the canonical isomorphism of K-algebras $\varphi\colon \Lambda_n^{op} \to \Lambda_n$, the dual $\mathrm{D}M$ of any left Λ_n-module M can be considered as a left Λ_n-module, which we denote by M°. This defines a

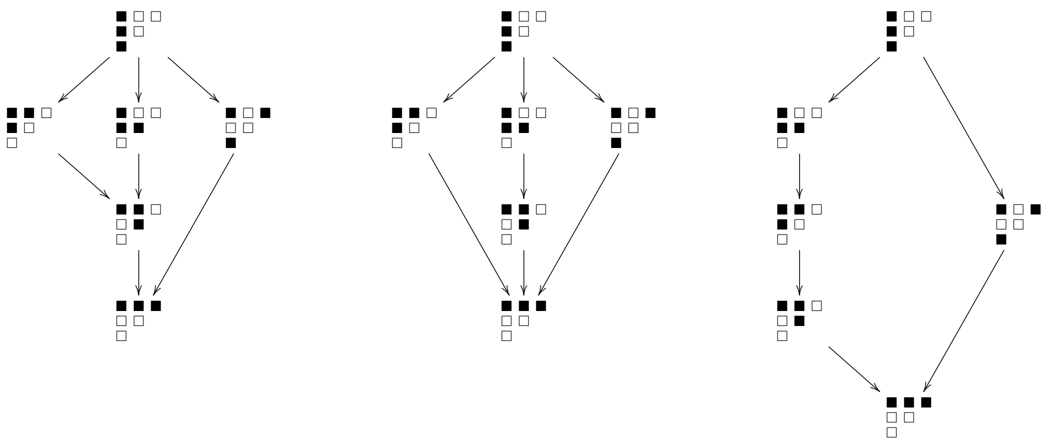


FIGURE 8. Hasse diagrams for $fb(n)$ with respect to inclusion of cogen^1-categories (left), $\leq$ (middle) and $\trianglelefteq$ (right). The largest element is at the bottom.

duality on the category Λ_n-mod which preserves faithfully balancedness, so we have an involution $fb(n) \to fb(n)$ mapping $M \mapsto M^\circ$.

Remark 9.3. For any basic module M, the module M° can be found by reflecting the Auslander Reiten quiver along the symmetry axis passing through $M_{1,n}, M_{2,n-1}, M_{3,n-2}, \ldots$ and these are the only indecomposable modules X with $X^\circ \cong X$. In $fb(3)$, we find two modules with $M^\circ \cong M$ and looking at Figure 9, one can see $M \neq M^\circ$ for all $M \in fb(4)$. We have $(\mathrm{cogen}^i(M))^\circ \cong \mathrm{gen}_i(M^\circ)$ for every $i \geq 0$. It follows that $M \trianglelefteq N \Leftrightarrow N^\circ \trianglelefteq M^\circ$ for all $M, N \in fb(n)$.

Consider the poset $(fb(n), \trianglelefteq)$. A module L is a common lower bound of M and N in $(fb(n), \trianglelefteq)$ if and only if $\mathrm{cogen}(L) \subseteq \mathrm{cogen}(M) \cap \mathrm{cogen}(N)$ and $\mathrm{gen}(L) \supseteq \mathrm{gen}(M) \cup \mathrm{gen}(N)$. For any two elements M and N in $(fb(n), \trianglelefteq)$ the module Λ_n is always a common lower bound of them.

Proposition 9.4. *The poset $(fb(n), \trianglelefteq)$ is a lattice for all $n \geq 1$.*

Proof. Since the poset is finite with a greatest element it is enough to show that it is a meet semi-lattice (see e.g. [20, Proposition 3.3.1]). Thus we need to show that any two elements $M, N \in fb(n)$ have a meet.

By Lemma 3.3, an indecomposable module is in $\mathrm{cogen}(M) \cap \mathrm{cogen}(N)$ if and only if it embeds in an indecomposable summand of M and in an indecomposable summand of N. Let C be the basic module such that $\mathrm{add}(C)$ is the minimal cocover of $\mathrm{cogen}(M) \cap \mathrm{cogen}(N)$, see [4, §2]. It follows that for each simple $S[i]$, there is at most one indecomposable summand X of C with socle $S[i]$, and if it occurs, it is a summand of M or N.

Let G be the basic module such that $\mathrm{add}(G)$ is the minimal cover of $\mathrm{add}(\mathrm{gen}(M) \cup \mathrm{gen}(N))$. Again, for each simple $S[i]$ there is at most one indecomposable summand of G with top $S[i]$. Clearly we have $\mathrm{cogen}(C) = \mathrm{cogen}(M) \cap \mathrm{cogen}(N)$ and $\mathrm{gen}(G) = \mathrm{gen}(\mathrm{gen}(M) \cup \mathrm{gen}(N))$. Moreover, since the indecomposable direct summands of C are in $\mathrm{add}(M) \cup \mathrm{add}(N)$, we have $C \in \mathrm{gen}(\mathrm{gen}(M) \cup \mathrm{gen}(N)) = \mathrm{gen}(G)$.

We claim that $|C| \leq n$ and equality holds if and only if $C = \mathrm{D}\Lambda_n = M = N$. To see this, note that if $|C| = n$, then M and N have summands with socle $S[i]$, for all i. But since they are in $fb(n)$, Theorem 1.1 implies they are isomorphic to $\mathrm{D}\Lambda_n$. There is nothing to prove if $M = N$, so we may assume that $|C| = t < n$. We write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_s$ where the tops of $G_1, \ldots, G_s$ are $S[i_1], \ldots, S[i_s]$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$. Since M_{1n} is a summand of G, we have $i_1 = 1$. For each $2 \leq \alpha \leq s$, let H_α be the indecomposable module with $\mathrm{top}(H_\alpha) = S[i_\alpha]$ and having the following properties:

(P1) H_α is a proper submodule of an indecomposable summand of C,
(P2) G_α is a quotient of H_α,
(P3) H_α has minimal length with respect to (P1) and (P2).

(Observe that the projective cover of $S[i_\alpha]$ satisfies (P1) and (P2) since it properly embeds in $M_{1,n}$, which is a direct summand of C.) Define $H = \bigoplus_{2\leq\alpha\leq s} H_\alpha$ and $L = C \oplus H$. We have $C \in \text{gen}(G) \subseteq \text{gen}(H \oplus M_{1n})$.

First we show that the module L is basic. We have $\text{add}(H) \cap \text{add}(C) = \{0\}$ since every summand of C is splitting injective in $\text{add}(C)$ and every indecomposable summand of H embeds properly into one of the summands of C by (P1). Since H is basic, because the H_α have different tops, and $\text{add}(C) \cap \text{add}(H) = \{0\}$, we have that L is basic.

Then, we show that L is faithfully balanced. The indecomposable direct summands of H are proper submodules of C by construction. Moreover an indecomposable direct summand X of C is a quotient of some G_α since we have $C \in \text{gen}(G)$. By construction we have $G_\alpha \in \text{gen}(H_\alpha)$, so we see that X is a proper quotient of a module H_α. Hence the module L satisfies (FB0) and (FB1) in Theorem 1.1. Consider $\text{cohook}(i, i-1)$ for $2 \leq i \leq n$. If $S[i-1] \in \text{cogen}(L)$, then we have $\text{cohook}(i, i-1) \cap \text{add}(L) \neq \emptyset$. If $S[i-1] \notin \text{cogen}(L) = \text{cogen}(C)$, then it is either not in $\text{cogen}(M)$ or not in $\text{cogen}(N)$. Without loss of generality we may assume $S[i-1] \notin \text{cogen}(M)$, in which case we must have $S[i] \in \text{gen}(M)$ since M is faithfully balanced. Thus we have $S[i] \in \text{gen}(H) \subseteq \text{gen}(L)$ and $\text{cohook}(i, i-1) \cap \text{add}(L) \neq \emptyset$. This proves that L also satisfies (FB2), so it is a faithfully balanced module.

Now we show that $|L| = n$, so $L \in fb(n)$. The virtual cohook $(i, i-1)$ of L is non-empty and it has three possible shapes according to the following two conditions: $S[i-1]$ is or is not a submodule of L and $S[i]$ is or is not a quotient of L. Let us denote by u the number of virtual cohooks $(i, i-1)$ for which $S[i-1]$ is a submodule of L and $S[i]$ is a quotient of L. Applying the inclusion-exclusion principle to virtual cohooks yields $0 = (n-1) - (t-1) - (s-1) + u$. If $u \neq 0$, then by the construction of H we know that there exists some i such that $\text{cohook}(i, i-1)$ contains an indecomposable summand of C and an indecomposable summand of G. But this contradicts the fact that $M, N \in fb(n)$. Hence we have $u = 0$ and $|L| = t + (s-1) = n$, as desired.

By construction $\text{cogen}(L) = \text{cogen}(C)$ and $\text{gen}(L) = \text{gen}(H \oplus M_{1n})$, which implies that L is a common lower bound of M and N.

Assume $L' \in fb(n)$ is also a common lower bound of M, N. This means that $\text{cogen}(L') \subseteq \text{cogen}(M) \cap \text{cogen}(N) = \text{cogen}(L) = \text{cogen}(C)$ and $\text{gen}(G) = \text{gen}(\text{gen}(M) \cup \text{gen}(N)) \subseteq \text{gen}(L')$. We have to show that $\text{gen}(L) \subseteq \text{gen}(L')$ to prove that $L' \trianglelefteq L$. Since L' is in $\text{cogen}(L') \subseteq \text{cogen}(C)$, we see that every indecomposable direct summand of L' is a submodule of C. On the other hand, $G_\alpha \in \text{gen}(G) \subseteq \text{gen}(L')$. This means that there is an indecomposable direct summand H'_α of L' such that $G_\alpha \in \text{gen}(H'_\alpha)$. By minimality of H_α, we see that $H_\alpha \in \text{gen}(H'_\alpha)$. It follows that $\text{gen}(L) \subseteq \text{gen}(L')$. □

Example 9.5. The following table gives two examples of the construction above for $n = 4$.

M	N	C	G	H	L
■□□□ ■■□ ■□ □	■■■□ □□■ □□ □	■□□□ □■□ □□ □	■□□□ ■□□ ■□ □	□□□□ ■□□ ■□ □	■□□□ ■■□ ■□ □
■■□■ □□□ □■ □	■■■□ □□■ □□ □	■■□□ □□□ □□ □	■□□□ □□■ □■ □	□□□□ □■□ □■ □	■■□□ □■□ □■ □

Figure 9 shows the Hasse diagram of $(fb(4), \trianglelefteq)$. The underlying graph of the Hasse diagram can be visualized as a truncated octahedron with two dissected hexagons, as in Figure 2.

Remark 9.6. We can consider the poset of minimal faithfully balanced modules for the algebra Λ_n, which is a stricly larger poset when $n \geq 4$, and we may wonder if it is also a lattice. It turns out that the construction of the module L of Proposition 9.4 still makes sense, however it does not always give a minimal faithfully balanced module. We cannot really improve our argument since the poset fails to be a lattice for $n = 5$.

Given $M = X \oplus U \in fb(n)$ with X indecomposable, $X \neq M_{1n}$, we want to describe the possible indecomposable modules Z such that $Z \oplus U \in fb(n)$. By Lemma 5.10, the module X is either splitting projective or splitting injective in $\text{add}(M)$ and not both.

Assume $X = M_{ij}$ is a splitting projective module. If $\text{gen}(X) \cap \text{add}(U) \neq \{0\}$, we pick the unique $X_0 = M_{it} \in \text{gen}(X) \cap \text{add}(U)$ of maximal length. If $\text{gen}(X) \cap \text{add}(U) = \{0\}$, we let $t = i - 1$. we define

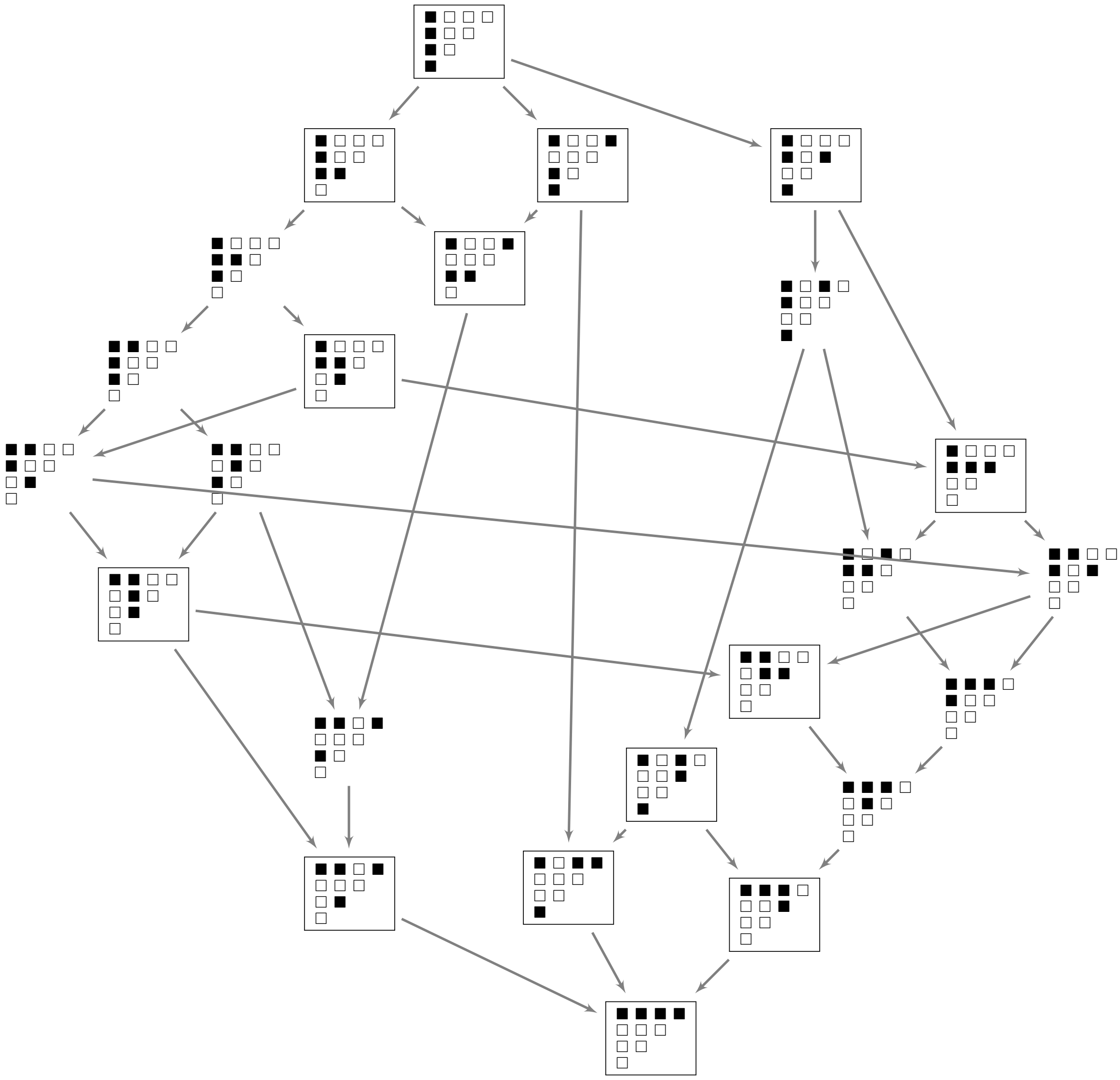


FIGURE 9. Hasse diagram of $(fb(4), \trianglelefteq)$. The cotilting modules are in boxes.

the *internal cohook* as

$$\mathrm{cohook}_M(X) = \mathrm{cohook}(i,t) \cap (\mathrm{gen}(U) \cup \mathrm{cogen}(U)).$$

Assume $X = M_{ij}$ is a splitting injective module. If $\mathrm{cogen}(X) \cap \mathrm{add}(U) \neq \{0\}$, we pick the unique $X_0 = M_{vj} \in \mathrm{cogen}(X) \cap \mathrm{add}(U)$ of maximal length. If $\mathrm{cogen}(X) \cap \mathrm{add}(U) = \{0\}$ we let $v = j+1$. We define the *internal cohook* as

$$\mathrm{cohook}_M(X) = \mathrm{cohook}(v,j) \cap (\mathrm{gen}(U) \cup \mathrm{cogen}(U)).$$

Furthermore, we define a total order $\preceq$ on $\mathrm{cohook}_M(X)$ generated by the following covering relations: $A \preceq B$ if there is a map $A \to B$ which is irreducible in $\mathrm{cohook}_M(X)$ (not necessarily irreducible in Λ_n-mod), or if A and B are both of minimal length in $\mathrm{cohook}_M(X)$ and $\mathrm{Ext}^1(B, A) \neq 0$.This restricts to a total order on any subset. The module X_0 (or the leaf) is called the *corner* of the internal cohook.

Proposition 9.7. *Let $M = X \oplus U \in fb(n)$ where X is an indecomposable module, $X \neq M_{1n}$. For every indecomposable Z the following are equivalent:*

(1) $Z \oplus U \in fb(n)$
(2) $Z \in \mathrm{cohook}_M(X)$.

In particular, there is always an indecomposable injective I and an indecomposable projective P such that $I \oplus U, P \oplus U \in fb(n)$. Assume now

$$\mathrm{cohook}_M(X) = \{P = Z_1 \preceq Z_2 \preceq \cdots \preceq Z_m = I\}$$

then we have in $fb(n)$

$$Z_1 \oplus U \trianglelefteq Z_2 \oplus U \trianglelefteq \cdots \trianglelefteq Z_m \oplus U.$$

Proof. Denote by c the corner of the internal cohook $\mathrm{cohook}_M(X)$. Let us recall that by the proof of Lemma 5.10 and by Lemma 8.1 (c), the cohook of an indecomposable summand of M and the virtual cohooks have an empty row or an empty column. Hence X is the only indecomposable summand of M in $\mathrm{cohook}(c)$.

If $Z \notin \mathrm{cohook}_M(X)$, then we have $U \oplus Z \notin fb(n)$ since in this case $\mathrm{cohook}(c) \cap \mathrm{add}(U \oplus Z) = \emptyset$. If $Z \in \mathrm{cohook}_M(X)$, then by the construction of the internal cohook we have $\mathrm{cohook}(c) \cap \mathrm{add}(U \oplus Z) \neq \emptyset$ and $\mathrm{cohook}(Z) \cap \mathrm{add}(U) \neq \emptyset$. It follows that $U \oplus Z$ satisfies (FB1) and (FB2) in Theorem 1.1, so it is in $fb(n)$.

Since M_{1n} is a summand of U, $\mathrm{cohook}_M(X)$ contains a unique indecomposable projective module P and a unique indecomposable injective module I. Observe that moving to the left in the row of $\mathrm{cohook}_M(X)$ shrinks the gen-category and stabilizes the cogen-category; going up in the column of $\mathrm{cohook}_M(X)$ grows the cogen-category and stabilizes the gen-category; moving from the left of c to the top of c grows the cogen-category and shrinks the gen-category. Now, the last statement follows. □

Lemma 9.8. *Let $N, M \in fb(n)$ such that $N \trianglelefteq M$ and $N \neq M$. Then there is an $N' \in fb(n)$ or an $M' \in fb(n)$ such that*

(a) *$N \trianglelefteq N' \trianglelefteq M$ and $|\mathrm{add}(N) \cap \mathrm{add}(N')| = n - 1$, or*
(b) *$N \trianglelefteq M' \trianglelefteq M$ and $|\mathrm{add}(M) \cap \mathrm{add}(M')| = n - 1$*

is fulfilled.

Proof. The proof is purely combinatorial and requires a careful checking of the different possibilities. We write $M = Z \oplus V$ and $N = Z \oplus W$ where $\mathrm{add}(V) \cap \mathrm{add}(N) = \{0\}$ and $\mathrm{add}(W) \cap \mathrm{add}(M) = \{0\}$.

First assume that there is an indecomposable summand X of V that is splitting projective in $\mathrm{add}(M)$. Since $X \in \mathrm{gen}(M) \subseteq \mathrm{gen}(N)$ there is an indecomposable summand Y of N such that $X \in \mathrm{gen}(Y)$. We choose it with minimal length with respect to this property and we let U be such that $M = U \oplus X$. Since X is splitting projective, we see that $Y \in \mathrm{add}(W)$. Moreover, $Y \in \mathrm{cogen}(N) \subseteq \mathrm{cogen}(M)$. So $Y \in \mathrm{cogen}(U)$ and we see that

- $Y \in \mathrm{cohook}_M(X)$ and $Y \preceq X$. Proposition 9.7 tells us that $M' = U \oplus Y \in fb(n)$.
- By Proposition 9.7 we have $M' \trianglelefteq M$.
- $\mathrm{cogen}(M) = \mathrm{cogen}(M')$ because X and Y are in $\mathrm{cogen}(U)$.
- $U \in \mathrm{gen}(M) \subseteq \mathrm{gen}(N)$ and $Y \in \mathrm{gen}(N)$.

The third point implies that $\mathrm{cogen}(N) \subseteq \mathrm{cogen}(M) = \mathrm{cogen}(M')$. The fourth point implies that $\mathrm{gen}(M') \subseteq \mathrm{gen}(N)$. In other words, we have $N \trianglelefteq M' \trianglelefteq M$.

If there is an indecomposable summand Y of W that is splitting injective in $\mathrm{add}(N)$ we can construct N' such that $N \trianglelefteq N' \trianglelefteq M$ by dualizing the previous argument.

Now we assume that all the summands of V are splitting injective in $\mathrm{add}(M)$ and all the summands of W are splitting projective in $\mathrm{add}(N)$. We choose the indecomposable $X \in \mathrm{add}(V)$ maximal with respect to the index of its socle and then with respect to its top. As before, we write $M = U \oplus X$. Combinatorially, the column containing X is the first (reading from left to right) that contains an element of $\mathrm{add}(V)$ and the module is the lowest element of $\mathrm{add}(V)$ in its column. In order to help the comprehension of the proof we draw in Figure 10 the shape of the Young diagram containing M and N in the neighborhood of X. We first explain the figure for the module M. Since X is a splitting injective, there is $z \in \mathrm{add}(M)$ such that $X \in \mathrm{gen}(z)$ and there are no summands of M in its column above it. We represent this by dashed horizontal lines. We denote by c the corner of $\mathrm{cohook}_M(X)$. The hypothesis on X implies that $z \in \mathrm{add}(Z)$ and that $c \in \mathrm{add}(Z)$ or is a leaf. Since M is a faithfully balanced module with exactly n summands, there is no module in the same row as c on its left. We represent this by using dashed vertical lines.

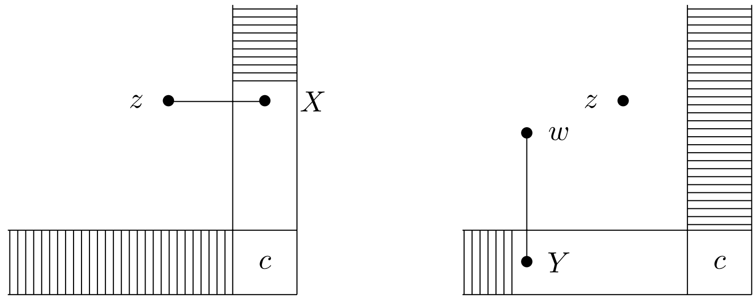


FIGURE 10. On the left the module M on the right the module N.

Let us move to the module N. By assumption, we have $\mathrm{cogen}(N) \subseteq \mathrm{cogen}(M)$ and X is splitting injective, so, there is no indecomposable summand I of N such that $X \in \mathrm{cogen}(I)$. Combinatorially this means that the column of c in N is empty at X and above. If there is an indecomposable summand I of N such that $c \subsetneq I \subsetneq X$, then by definition of X, this module is not a direct summand of M, so it is a splitting projective in $\mathrm{add}(N)$. But if $I \in \mathrm{add}(N)$ with $c \subsetneq I \subsetneq X$ of maximal length it has to be a splitting injective. Since this is not possible, there is no module above c in its column. Since N is a faithfully balanced module, there is $Y \in \mathrm{add}(N)$ which is on the left of c. There are no indecomposable summands of Z on the left of c, so $Y \in \mathrm{add}(W)$ and by assumption it is a splitting projective module in $\mathrm{add}(N)$. So there is $w \in \mathrm{add}(N)$ such that Y is a proper submodule of w. Since $\mathrm{cogen}(N) \subseteq \mathrm{cogen}(M)$, we have $Y \in \mathrm{cogen}(U)$. In conclusion, we have:

- $Y \in \mathrm{cohook}_M(X)$, so $M' = U \oplus Y \in fb(n)$ and $M' \trianglelefteq M$.
- $U \in \mathrm{gen}(M) \subseteq \mathrm{gen}(N)$ and $Y \in \mathrm{add}(N)$, so $\mathrm{gen}(M') \subseteq \mathrm{gen}(N)$.
- By construction if I is an indecomposable module with $\mathrm{soc}(I) \neq \mathrm{soc}(c)$, then $I \in \mathrm{cogen}(M)$ if and only if $I \in \mathrm{cogen}(M')$. It follows that $\mathrm{cogen}(N) \subseteq \mathrm{cogen}(M')$.

It follows that $N \trianglelefteq M' \trianglelefteq M$. □

Corollary 9.9. *If $N, M \in fb(n)$ are neighbours in the Hasse diagram of $\trianglelefteq$, then $|\mathrm{add}(N) \cap \mathrm{add}(M)| = n - 1$.*

Recall, whenever $N \trianglelefteq M$ is a covering relation, we draw an arrow $N \to M$ in the Hasse diagram.

Corollary 9.10. *Let $M \in fb(n)$.*

(1) *Let $X \in \mathrm{add}(M)$ be an indecomposable module. Assume that $X = Z_i$ in its internal cohook and we write $M = U \oplus X$. Then there is a cover relation $U \oplus X \trianglelefteq U \oplus Y$ in $(fb(n), \trianglelefteq)$ if and only if*
- *X is not injective.*
- *Z_{i+1}, the successor of X in its internal cohook, is not in $\mathrm{add}(M)$.*
- *$Y = Z_{i+1}$.*

(2) *In the Hasse diagram of $\trianglelefteq$, the number of arrows starting (resp. ending) at M is smaller than or equal to the number of non-injective (resp. non-projective) indecomposable summands of M.*

Proof. If $M \trianglelefteq N$ is a cover relation, then the two modules differ by exactly one indecomposable summand. Say that $M = U \oplus X$ and $N = U \oplus Y$. By Proposition 9.7, we see that Y must be in the internal cohook of X. Say that $X = Z_i$ in this cohook. Then $Y = Z_j$ for $i < j$. Because it is a cover relation, j is the smallest integer such that $Z_j \notin \mathrm{add}(U)$. If $Z_{i+1} \in \mathrm{add}(U)$, then we write $M = V \oplus X \oplus Z_{i+1}$. Then $M \trianglelefteq N$ is not a cover relation because it factorizes as $M \trianglelefteq V \oplus X \oplus Z_j \trianglelefteq V \oplus Z_{i+1} \oplus Z_j = N$. □

Proof of Theorem 1.6. The first point is proved in Proposition 9.4. The third point is proved in Corollary 9.10. The Tamari lattice is isomorphic to the poset of basic cotilting modules for Λ_n with partial order given by $T_1 \leq T_2$ if and only if $\mathrm{Ext}^1(T_1, T_2) = 0$. Looking at Remark 9.1 we see that it is a subposet of $(fb(n), \trianglelefteq)$. Recall that in the poset of cotilting modules the meet of two cotilting modules T_1 and T_2 is a cotilting module T_3 such that $\mathrm{cogen}(T_3) = \mathrm{cogen}(T_1) \cap \mathrm{cogen}(T_2)$ (see for example Section 11 of [24] for the dual statement on tilting modules). By construction, the meet L of two faithfully balanced modules M and N is such that $\mathrm{cogen}(L) = \mathrm{cogen}(M) \cap \mathrm{cogen}(N)$, so in order to prove the second point it is enough to show that L is a cotilting module when M and N are cotilting modules. Since L has exactly n non-isomorphic indecomposable summands, it is enough to show that it has no self-extensions.

Let M and N be two cotilting modules and denote by L their meet in $(fb(n), \trianglelefteq)$. Recall from the proof of Proposition 9.4 that $L = C \oplus H$ where every indecomposable summand of C is a direct summand of M or N and every indecomposable summand of H is a submodule of an indecomposable summand of C. Also from the proof of Proposition 9.4, we recall that we considered a basic module G such that $\operatorname{add}(G)$ is the minimal cover of $\operatorname{add}(\operatorname{gen}(M) \cup \operatorname{gen}(N))$. If X is an indecomposable summand of G, then X is an indecomposable summand of M or of N.

Let us recall that for a cotilting module T we have $\operatorname{cogen}(T) = \{Y \in \Lambda_n\text{-mod} \mid \operatorname{Ext}^1_{\Lambda_n}(Y, T) = 0\}$ (see for example the dual of Theorem 2.5, in Chapter VI of [3]). Since each indecomposable summand of C is a summand of M or of N, we have $\operatorname{Ext}^1_{\Lambda_n}(Y, C) = 0$ for every $Y \in \operatorname{cogen}(C) = \operatorname{cogen}(M) \cap \operatorname{cogen}(N)$. This applies in particular to $Y = L$. Hence, we have $\operatorname{Ext}^1_{\Lambda_n}(L, C) = 0$.

To finish the proof we need to show $\operatorname{Ext}^1_{\Lambda_n}(L, H) = 0$. We claim that it's enough to show $\operatorname{Ext}^1_{\Lambda_n}(C, H) = 0$, and we postpone its proof. For if H_α is an indecomposable direct summand of H, then by construction it is a submodule of an indecomposable summand C' of C. Hence, we have a short exact sequence $0 \to H_\alpha \to C' \to C'/H_\alpha \to 0$, and we apply $\operatorname{Hom}_{\Lambda_n}(-, H)$ to it. This leads to an exact sequence $\cdots \to \operatorname{Ext}^1_{\Lambda_n}(C', H) \to \operatorname{Ext}^1_{\Lambda_n}(H_\alpha, H) \to 0$ since Λ_n is hereditary. Since we will see later that $\operatorname{Ext}^1_{\Lambda_n}(C', H) = 0$, we get $\operatorname{Ext}^1_{\Lambda_n}(H_\alpha, H) = 0$.

Now we prove by contradiction that $\operatorname{Ext}^1_{\Lambda_n}(C, H) = 0$. Assume there is an indecomposable summand M_{ac} of C and an indecomposable summand M_{bd} of H with $\operatorname{Ext}^1_{\Lambda_n}(M_{ac}, M_{bd}) \neq 0$. By [11, Lemma 8.1] this is equivalent to $a < b \leq c + 1 \leq d$. By construction M_{bd} surjects onto an indecomposable summand M_{bg} of G, so a summand of $T = M$ or N. The minimality condition (P3) imposed on the summands of H implies that $c < g \leq d$. This implies that $\operatorname{Ext}^1_{\Lambda_n}(M_{ac}, M_{bg}) \neq 0$. If M_{ac} is a direct summand of T, then we have a contradiction because T is a cotilting module. If M_{ac} is not a direct summand of T (so M_{ac} and M_{bg} are summands of different cotilting modules), since $M_{ac} \in \operatorname{cogen}(T)$, there is an indecomposable summand $M_{a'c}$ of T having M_{ac} as a proper submodule. Hence, $a' < a$ and we have $\operatorname{Ext}^1_{\Lambda_n}(M_{a'c}, M_{bg}) \neq 0$ which again contradicts the fact that T is a cotilting module.

□

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