Quasi-hereditary property of double Burnside algebras. Propriété quasi-héréditaire des algèbres de Burnside doubles.

Baptiste Rognerud

7 septembre 2018

Abstract

In this short note we investigate some consequences of the *vanishing* of simple biset functors. As corollary, if there is no non-trivial vanishing of simple biset functors (e.g. if the group G is commutative), then we show that $kB(G,G)$ is a *quasi-hereditary* algebra in characteristic zero. In general, this not true without the non-vanishing condition, as over a field of characteristic zero, the double Burnside algebra of the alternating group of degree 5 has infinite global dimension.

Résumé

Dans cette note on s'intéresse à quelques conséquences du phénomène dit de *disparition* des foncteurs à bi-ensembles simples. On démontre que dans le cas où il n'y a pas de disparitions non triviales de foncteurs simples (par exemple si le groupe est commutatif) alors l'algèbre de Burnside double en caractéristique zéro est quasi-héréditaire. Sans l'hypothèse de non-disparitions triviales, ce résultat est en général faux. En effet, l'algèbre de Burnside double du groupe alterné de degré 5 en caractéristique zéro est de dimension globale infinie.

Key words: Finite group. Biset functor. Quasi-hereditary algebra.

A.M.S. subject classification: 19A22, 20C99, 16G10,18E10.

Notations. *Let k be a field. We denote by $\mathcal{C}_k$ the biset category. That is the category whose objects are finite groups and morphisms are given by the double Burnside module (see Definition* 3.1.1 *of [2]). For a finite group G, we denote by $\Sigma(G)$ the full subcategory of $\mathcal{C}_k$ consisting of the subquotients of G. If $\mathcal{D}$ is a k-linear subcategory of $\mathcal{C}_k$, we denote by $\mathcal{F}_{\mathcal{D},k}$ the category of k-linear functors from $\mathcal{D}$ to k-*Mod. *If L is a subquotient of K, we write $L \sqsubseteq K$ and if it is a proper subquotient, we write $L \sqsubset K$. If V and W are objects in the same abelian category, we denote by $[V : W]$ the number of subquotients of V isomorphic to W.*

1 Evaluation of functors.

Let us first recall some basic facts about the category of biset functors. Let $\mathcal{D}$ be an admissible subcategory of $\mathcal{C}_k$ in the sense of Definition 4.1.3 of [2]. The category $\mathcal{D}$ is a

skeletally small k-linear category, so the category of biset functors is an abelian category. The representable functors, also called Yoneda functors, are projective, so this category has *enough projective*. The simple functors are in bijection with the isomorphism classes of pairs (H, V) where H is an object of $\mathcal{D}$ and V is a $k\mathrm{Out}(H)$-simple module (see Theorem 4.3.10 of [2]).
A biset functor is called *finitely generated* if it is a quotient of a *finite* direct sum of representable functors. In particular, the simple biset functors and the representable functors are finitely generated. As in the case of modules over a ring, the choice axiom has for consequence the existence of maximal subfunctor for finitely generated biset functors. If F is a biset functor, the intersection of all its maximal subfunctors is called the radical of F and denoted $\mathrm{Rad}(F)$.
If G is an object of $\mathcal{D}$, then there is an evaluation functor $ev_G : \mathcal{F}_{\mathcal{D},k} \to \mathrm{End}_{\mathcal{D}}(G)$-Mod sending a functor to its value at G. It is obviously an exact functor and it is well known that it sends a simple functor to 0 or a simple $\mathrm{End}_{\mathcal{D}}(G)$-module. It turns out that the fact that a simple functor vanishes at G has some consequences for the functors having this simple as quotient.

Proposition 1.1. *Let $F \in \mathcal{F}_{\mathcal{D},k}$ be a finitely generated functor and let $G \in Ob(\mathcal{D})$. Then*

1. $\mathrm{Rad}(F(G)) \subseteq [\mathrm{Rad}(F)](G)$
2. *If none of the simple quotients of F vanishes at G, then* $\mathrm{Rad}(F(G)) = [\mathrm{Rad}(F)](G)$.

Proof. Let M be a maximal subfunctor of F. Then $M(G)$ is a maximal submodule of $F(G)$ if the simple quotient F/M does not vanish at G and $M(G) = F(G)$ otherwise. For the second part, if N is a maximal submodule of $F(G)$, let $\overline{N}$ be the subfunctor of F generated by N. There is a maximal subfunctor M of F such that $\overline{N} \subseteq M \subset F$. We have $\overline{N}(G) = N \subseteq M(G) \subset F(G)$. By maximality, $M(G) = N$. The result follows. □

Remark 1.2. In Section 9 of [3], the Authors gave some conditions for the fact that the evaluation of the radical of the so-called standard functor is the radical of the evaluation. The elementary result of Proposition 1.1 gives new lights on this section. Indeed, Proposition 9.1 [3] gives a sufficient condition for the non vanishing of the simple quotients of these standard functors.

Over a field, the category of finitely generated projective biset functors is Krull-Schmidt in the sense of [5] (Section 4), so every finitely generated biset functor has a projective cover.

Corollary 1.3. *Let $F \in \mathcal{F}_{\mathcal{D},k}$ be a finitely generated functor and let $G \in Ob(\mathcal{D})$. Then,*

1. *If F has a unique quotient S, and $S(G) \neq 0$, then $F(G)$ is an indecomposable* $\mathrm{End}_{\mathcal{D}}(G)$*-module.*
2. *If P is an indecomposable projective biset functor such that* $\mathrm{Top}(P)(G) \neq 0$*, then $P(G)$ is an indecomposable projective* $\mathrm{End}_{\mathcal{D}}(G)$*-module.*

2 Highest weight structure of the biset functors category.

Let us recall the famous theorem of Webb about the highest weight structure of the category of biset functors.

Theorem 2.1 (Theorem 7.2 [6]). *Let $\mathcal{D}$ be an admissible subcategory of the biset category. Let k be a field such that* $\mathrm{char}(k)$ *does not divide* $|\mathrm{Out}(H)|$ *for $H \in Ob(\mathcal{D})$. If $\mathcal{D}$ has a finite number of isomorphism classes of objects, then $\mathcal{F}_{\mathcal{D},k}$ is a highest weight category.*

The set indexing the simple functors is the set, denoted by Λ, of isomorphism classes of pairs (H, V) where $H \in Ob(\mathcal{D})$ and V is an $k\mathrm{Out}(H)$-simple module. Let H and K be two objects of $\mathcal{D}$. Then

$$\bigoplus_{\substack{X\in\mathcal{D}\\ X\sqsubset H}} \mathrm{Hom}_{\mathcal{D}}(X,K)\mathrm{Hom}_{\mathcal{D}}(H,X),$$

can be viewed as a submodule of $\mathrm{Hom}_{\mathcal{D}}(H,K)$ via composition of morphisms. We denote by $I_{\mathcal{D}}(H,K)$ this submodule and by $\overleftarrow{\mathrm{Hom}}_{\mathcal{D}}(H,K)$ the quotient $\mathrm{Hom}_{\mathcal{D}}(H,K)/I_{\mathcal{D}}(H,K)$. This is a natural right $k\mathrm{Out}(H)$-module. If V is an $k\mathrm{Out}(H)$-module, we denote by $\Delta^{\mathcal{D}}_{H,V}$ the functor

$$\Delta^{\mathcal{D}}_{H,V} := K \mapsto \overleftarrow{\mathrm{Hom}}_{\mathcal{D}}(H,K) \otimes_{k\mathrm{Out}(H)} V.$$

When the context is clear, we simply denote by $\Delta_{H,V}$ this functor. If $(H, V) \in \Lambda$, then $\Delta_{H,V}$ is a standard object of $\mathcal{F}_{\mathcal{D},k}$. The set Λ is ordered by $(H, V) < (K, W)$ if $K \sqsubset H$, that is if K is a strict subquotient of H. So the highest weight structure gives the fact that the projective indecomposable biset functors have a filtration by standard functors. This filtration has the following properties:

- If $P_{H,V}$ denotes a projective cover of the simple $S_{H,V}$, then $P_{H,V}$ is filtered by a finite number of standard functors. The first quotient is $\Delta_{H,V}$, which appears with multiplicity one. The other standard objects which appear as subquotients are some $\Delta_{K,W}$ for $K \sqsubset H$.
- Moreover the standard functors have finite length. The unique simple quotient of $\Delta_{H,V}$ is the simple functor $S_{H,V}$. The other simple functors which appear as composition factor of $\Delta_{H,V}$ are some $S_{K,W}$ for $H \sqsubset K$.

Definition 2.2. Let k be a field. Let G be a finite group. Let $\Sigma(G)$ be the full subcategory of $\mathcal{C}_k$ consisting of the subquotients of G. Then the group G is call a NV_k-group if the simple functors S of $\mathcal{F}_{\Sigma(G),k}$ do not vanish at G.

It is well known that commutative groups are NV_k-groups for every field k (see Proposition 3.2 of [4]), but there are non-commutative NV_k-groups.

Theorem 2.3. *Let G be a finite group. Let k be a field such that* $\mathrm{char}(k)$ *does not divide* $|\mathrm{Out}(H)|$ *for all subquotients H of G. If G is a* NV_k*-group, then $kB(G, G)$ is a quasi-hereditary algebra.*

Proof. By Corollary 3.3 of [3], the simple $kB(G,G)$-modules are exactly the evaluation at G of the simple biset functors $S_{H,V} \in \mathcal{F}_{\Sigma(G),k}$. Now by Corollary 1.3, if $P_{H,V}$ is a projective cover of $S_{H,V}$ in $\mathcal{F}_{\Sigma(G),k}$, then $P_{H,V}(G)$ is a projective cover of $S_{H,V}(G)$ as $kB(G,G)$-modules. Moreover since the standard functor $\Delta_{H,V} \in \mathcal{F}_{\Sigma(G),k}$ has a simple top, its evaluation at G is indecomposable.
Let $M_0 = 0 \subset M_1 \subset \cdots \subset M_n = P_{H,V}$ be a standard filtration of $P_{H,V}$ in $\mathcal{F}_{\Sigma(G),k}$. The evaluation functor is exact, so the $kB(G,G)$-modules $M_i(G)$ produce a filtration of the projective indecomposable module $P_{H,V}(G)$. Moreover the quotient $M_i(G)/M_{i-1}(G) = [M_i/M_{i-1}](G)$ is the evaluation at G of a standard functor indexed by a pair (K,W) such that $K \sqsubset H$. It remains to look at the composition factors of the $\Delta_{H,V}(G)$. We have:

$$\Delta_{H,V}(G)/\mathrm{Rad}(\Delta_{H,V}(G)) = [\Delta_{H,V}/\mathrm{Rad}(\Delta_{H,V})](G) = S_{H,V}(G).$$

Moreover by Proposition 3.5 of [3], a simple $kB(G,G)$-module $S_{K,W}(G)$ is a composition factor of $\Delta_{H,V}(G)$ if and only if $S_{K,W}$ is a composition factor of $\Delta_{H,V}$. As consequence $\Delta_{H,V}(G)$ has a simple top $S_{H,V}(G)$ and the other composition factors are some $S_{K,W}(G)$ for $H \sqsubset K$. This shows that $kB(G,G)$-Mod is a highest weight category in which the standard objects are the evaluation at G of the standard functors of $\mathcal{F}_{\Sigma(G),k}$. ☐

Remark 2.4. This result can be easily generalized to the algebra $\mathrm{End}_{\mathcal{D}}(G)$ when $\mathcal{D}$ a admissible subcategory of $\mathcal{C}_k$. If the simple functors of $\mathcal{F}_{\mathcal{D}\cap\Sigma(G),k}$ do not vanish at G, then $\mathrm{End}_{\mathcal{D}}(G)$ is a quasi-hereditary algebra over a suitable field.

As immediate corollary, for the double Burnside algebras, we have:

Corollary 2.5. *Let k be a field such that* $\mathrm{char}(k)$ *does not divide* $|\mathrm{Out}(H)|$ *for all subquotients H of a* NV_k*-group G. Then the global dimension of $kB(G,G)$ is finite.*

It should now be clear that the situation will not be that simple if some simple functors vanish at G. Indeed let $S_{H,V}$ be a simple functor of $\mathcal{F}_{\Sigma(G),k}$ such that $S_{H,V}(G) \neq 0$. If in a standard filtration of $P_{H,V}$ there is a standard functor $\Delta_{K,W}$ such that $S_{K,W}(G) = 0$, then $\Delta_{K,W}(G)$ is *not* in the set of standard modules for $kB(G,G)$ that we considered in the proof of Theorem 2.3. In the rest of this paper we look at the case of $G = A_5$. We first show that the situation described here actually happens for this group, and we show that there is no hope to choose a better filtration for the projective $kB(G,G)$-modules. The reason is that $kB(G,G)$ has infinite global dimension.

3 The example of A_5.

Let k be a field of characteristic different from $2, 3$ and 5. The double Burnside algebra $kB(A_5, A_5)$ is a rather complicated object. Unfortunately it seems to the Author that $G = A_5$ is the smallest (or one of the smallest) example where the situation described above can appear. Indeed, this situation requires the existence of enough non-split extensions between simple functors in $\mathcal{F}_{\Sigma(G),k}$. It is well known that this category is not semi-simple if there are some non-cyclic groups in $\Sigma(G)$ (Theorem 1.1 [1]), but as

it can be seen in Proposition 11.2 of [6], if the category $\Sigma(G)$ does not contain enough increasing chains (for the subquotient relation) of objects, then there are not so many non-split extensions in $\mathcal{F}_{\Sigma(G),k}$. Moreover A_5 is also one of the first groups where the evaluation of the radical of the standard functor is not the radical of the evaluation (see Example 13.5 of [3]), so it is a good candidate for our purpose.
In order to simplify the computations, we will use the following results.

- Let $P_{K,W}$ be a projective indecomposable functor in $\mathcal{F}_{\Sigma(G),k}$. Let $\Delta_{J,U}$ be a standard object in this category. Then

$$[P_{K,W} : \Delta_{J,U}] = [\nabla_{J,U} : S_{K,W}] = [\Delta_{J,U^*} : S_{K,W^*}]. \tag{1}$$

  Here $\nabla_{J,U}$ denotes the co-standard functor indexed by (J, U). The first equality is the so-called BGG-reciprocity and the last equality follows from the usual duality in the biset-functor category. See Paragraph 8 of [6] for more details. Note that for A_5 all the $k\mathrm{Out}(H)$-simple modules that we will consider are self-dual.
- If $\mathcal{D}$ is an admissible full-subcategory of $\Sigma(G)$, then there is a restriction functor from $\mathcal{F}_{\Sigma(G),k}$ to $\mathcal{F}_{\mathcal{D},k}$. By Proposition 7.3 of [6], if $H \in \mathcal{D}$, then we have:

$$[P^{\Sigma(G)}_{H,V} : \Delta^{\Sigma(G)}_{K,W}]_{\Sigma(G)} = [P^{\mathcal{D}}_{H,V} : \Delta^{\mathcal{D}}_{K,W}]_{\mathcal{D}}. \tag{2}$$

Lemma 3.1. *Let k_- be the non-trivial simple $k\mathrm{Out}(C_3) \cong k\mathrm{Out}(A_4) \cong kC_2$-module. There is a non-split exact sequence of functors of $\mathcal{F}_{\Sigma(A_5),k}$:*

$$0 \longrightarrow \Delta_{C_3,k_-} \longrightarrow P_{A_4,k_-} \longrightarrow \Delta_{A_4,k_-} \longrightarrow 0.$$

Proof. We know that P_{A_4,k_-} has a finite Δ-filtration with quotient Δ_{A_4,k_-}. We need to understand the other standard quotients of such a filtration. By the highest weight's structure of $\mathcal{F}_{\Sigma(A_5),k}$, such a standard quotient must be indexed by a subquotient of A_4. By using the BGG-reciprocity (1) and formula (2), a standard functor $\Delta_{H,V}$ appears in P_{A_4,k_-} if and only if S_{A_4,k_-} is a composition factor of $\Delta_{H,V}$ in $\mathcal{F}_{\Sigma(A_4),k}$. Using Proposition 3.5 of [3], this is equivalent to the fact that $S_{A_4,k_-}(A_4)$ is a composition factor of $\Delta_{H,V}(A_4)$. As immediate consequence we have:

- $\Delta_{1,k}$ is not in a Δ-filtration of P_{A_4,k_-}. Indeed $\Delta_{1,k}$ is isomorphic to kB, the usual Burnside functor. By the work of Bouc (see Section 5.4 and 5.5 of [2]), the simple subquotients of kB are the $S_{H,k}$ for a B-group H. As consequence, the simple functor S_{A_4,k_-} is not a subquotient of kB.
- $\Delta_{A_4,k}$ is not a subquotient of P_{A_4,k_-}. Indeed, the only composition factor of $\Delta_{A_4,k}$ with A_4 as minimal group if $S_{A_4,k}$.

We have the following: the subquotients of A_4 are : $A_4, V_4, C_3, C_2, 1$.

1. $\mathrm{Out}(C_2) \cong 1$ and we have $\Delta_{C_2,k}(A_4) \cong S_{C_2,k}(A_4)$.
2. $\mathrm{Out}(V_4) \cong S_3$. So there are three $k\mathrm{Out}(V_4)$-simple modules. We denote by k the trivial module and k_- the sign. Finally, we denote by V the simple module of dimension 2. Then we have: $\Delta_{V_4,k}(A_4) \cong S_{A_4,k}(A_4)$, $\Delta_{V_4,k_-}(A_4) \cong S_{A_4,k_-}(A_4)$ and $\Delta_{V_4,V}(A_4) = 0$.

3. $\mathrm{Out}(C_3) \cong C_2$ so there are two simple $k\mathrm{Out}(C_3)$-modules. We denote by k the trivial module and k_- the non trivial simple module. Then $\Delta_{C_3,k}(A_4)$ is a non-split extension between $S_{A_4,k}(A_4)$ and $S_{C_3,k}(A_4)$ and $\Delta_{C_3,k_-}(A_4)$ is a non-split extension between $S_{A_4,k_-}(A_4)$ and $S_{C_3,k_-}(A_4)$

So the only standard functors which appear in a standard filtration of P_{A_4,k_-} in $\mathcal{F}_{\Sigma(A_5),k}$ are Δ_{A_4,k_-} and Δ_{C_3,k_-}. The structure of the highest weight category implies that Δ_{C_3,k_-} must be a subfunctor of P_{A_4,k_-} and Δ_{A_4,k_-} must be a quotient of this functor. $\square$

Now we need to understand the evaluation at A_5 of P_{A_4,k_-}.

Lemma 3.2.
- *$\Delta_{A_4,k_-}(A_5) \cong S_{A_4,k_-}(A_5) \neq 0$.*
- *$\Delta_{C_3,k_-}(A_5) \cong S_{A_4,k_-}(A_5) \neq 0$.*

Proof. The first isomorphism follows from the fact that $\Delta_{A_4,k_-}(A_5)$ is one dimensional, with basis $\mathrm{Ind}_{A_4}^{A_5} \otimes 1$. So it is a simple $kB(A_5, A_5)$-module of the form $S_{H,V}(G)$. The element $\mathrm{Ind}_{A_4}^{A_5}\mathrm{Res}_{A_4}^{A_5}$ acts by 1 on $\Delta_{A_4,k_-}(A_5)$. So the minimal group H is smaller than A_4. By the highest-weight structure of $\mathcal{F}_{\Sigma(A_5),k}$, the only possibility is to have $H = A_4$ and $V = k_-$.
We know that Δ_{C_3,k_-} is a subquotient of P_{A_4,k_-}, so S_{A_4,k_-} is a composition factor of Δ_{C_3,k_-} by the BGG-reciprocity (1). Since $S_{A_4,k_-}(A_5) \neq 0$, this simple module is a composition factor of $\Delta_{C_3,k_-}(A_5)$. Since we have $\dim_k\Big(\Delta_{C_3,k_-}(A_5)\Big) = 1$, the result follows. $\square$

Proposition 3.3. *Let $G = A_5$ be the alternating group of degree 5. Let k be a field of characteristic different from 2, 3 and 5. Then $kB(G, G)$ has infinite global dimension. In particular $kB(G, G)$ is* not *a quasi-hereditary algebra.*

Proof. By using Lemma 3.1, we know that P_{A_4,k_-} has a Δ-filtration with Δ_{A_4,k_-} as quotient and Δ_{C_3,k_-} as subfunctor. Since the simple quotient of P_{A_4,k_-} does not vanish at G, then $P_{A_4,k_-}(A_5)$ is a projective cover of $S_{A_4,k_-}(A_5)$. By using Lemma 3.2, we see that this projective indecomposable functor is a non-split extension between $S_{A_4,k_-}(A_5)$ and itself. $\square$

References

[1] L. Barker. Rhetorical biset functors, rational p-biset functors and their semisimplicity in characteristic zero. *J. Algebra*, 319(9):3810 − 3853, 2008.

[2] S. Bouc. *Biset Functors for Finite Groups*, volume 1990 of *Lecture Notes in Mathematics*. Springer, 2010.

[3] S. Bouc, R. Stancu, and J. Thévenaz. Simple biset functors and double Burnside ring. *J. Pure Appl. Algebra*, 217(3):546–566, 2013.

[4] S. Bouc, R. Stancu, and J. Thévenaz. Vanishing evaluations of simple functors. *J. Pure Appl. Algebra*, 218(2):218–227, 2014.

[5] H. Krause. Krull-Remak-Schmidt categories and projective covers. `http://www.math.uni-bielefeld.de/~hkrause/krs.pdf`.

[6] P. Webb. Stratifications and Mackey functors II: globally defined Mackey functors. *J. K-Theory*, 6(1):99–170, 2010.

Baptiste Rognerud
EPFL / SB / MATHGEOM / CTG
Station 8
CH-1015 Lausanne
Switzerland
e-mail: baptiste.rognerud@epfl.ch