show that $H^i(X^{0,\bullet}) \cong H^i(X^{\bullet,0})$ by diag chasing

$$\begin{tikzcd}
 & & & x^{0,i} \arrow[r] \arrow[d] & 0 \\
 & & x^{1,i-2} \arrow[r] & x^{1,i-1} & \\
 & x^{i-1,1} \arrow[r] \arrow[d] & \cdots & & \\
x^{i,0} \arrow[r, "d''"] \arrow[d, "d'"'] & x^{i,1} \arrow[d, dashed, "d''"] & & & \\
0 & {} & & &
\end{tikzcd}$$

Start with $x^{i,0} \in \ker(d'^{i})$ then $x^{i,1} = d''(x^{i,1}) \in \ker(d')$ since double complex. Exactness lead to $x^{i-1,1}$; $d'(x^{i-1,1}) = x^{i,1}$ we have $d'' x^{i-1,1} = x^{i-1,2} \in \ker(d')$ etc.

$\leadsto x^{0,i} \in \ker(d''^{i})$ and this is the construction of the isomorphism.

For details see: Weibel §2.7 + dual for left derived functors $\square$

2) Functor Ext

Def 6.2 (1) If $\mathcal{E}$ has enough projective (or enough injective) we denote by $\mathrm{Ext}^i(A,B)$ the derived functor $[R^i\,\mathrm{Hom}(-,B)](A)$ (resp $R^i\,\mathrm{Hom}(A,-)[B]$).

(2) If $\mathcal{E} = \mathrm{Mod}\,A$ $\mathrm{Ext}^i(A,B) \cong R^i\,\mathrm{Hom}(-,B)(A) \cong R^i\,\mathrm{Hom}(A,-)(B)$.

This makes sense since $\mathrm{Hom}(-,B)$ satisfies the hypothesis of Thm 6.1.

Examples $R = \mathbb{Z}$, M be an torsion abelian group, then to compute $\mathrm{Ext}^i(M,\mathbb{Z})$ one can
- find an ~~injective~~ proj ~~(x)~~ resolution of M
- or find an ~~proj~~ injective coresolution of $\mathbb{Z}$

We have $0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \to 0$ exact with $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ injective.

So we have a long exact sequence

$$0 \to \underset{=0}{\mathrm{Hom}(M,\mathbb{Z})} \to \underset{=0}{\mathrm{Hom}(M,\mathbb{Q})} \to \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1(M,\mathbb{Z})$$
$$\to \underset{=0}{\mathrm{Ext}^1(M,\mathbb{Q})} \to \underset{=0}{\mathrm{Ext}^1(M,\mathbb{Q}/\mathbb{Z})} \to \mathrm{Ext}^2(M,\mathbb{Z}) \to \underset{=0}{\mathrm{Ext}^2(M,\mathbb{Q})}$$

$\hookrightarrow$ $\begin{cases} \mathrm{Ext}^1(M,\mathbb{Z}) \simeq \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \\ \mathrm{Ext}^i(M,\mathbb{Z}) = 0 \quad \forall i \neq 0 \end{cases}$

Thm 6.3 (1) TFAE A mod

1. P is projective
2. $\mathrm{Hom}_A(P,-)$ is exact
3. $\forall i \geq 1$, $\forall B \in \mathrm{Mod}\,A$, $\mathrm{Ext}^i(P,B) = 0$
4. $\forall B \in \mathrm{Mod}\,A$, $\mathrm{Ext}^1(P,B) = 0$

(2) Dual for injective

Sketch (1) $\Leftrightarrow$ (2) by def (2) $\Rightarrow$ (3) clear (3) $\Rightarrow$ (2) long exact sequence

(3) $\Rightarrow$ (4) clear (4) $\Rightarrow$ (2) long exact sequence

Ex Ext^i can be computed using Ext^1 if $i \geq 2$.

3. Tor functor

Def 6.4 A k-alg $\mathcal{E} = \mathrm{Mod}\,A$ We denote by Tor_i^A the i-th derived functor of $- \otimes_A -$

That is $\mathrm{Tor}_i^A(M,N) = L_i(- \otimes_A N)(M) \simeq L_i(M \otimes_A -)$

Example B be an abelian gp. Compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/_{p\mathbb{Z}}, B)$
p prime

$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/_{p\mathbb{Z}} \to 0$ proj resolution of $\mathbb{Z}/_{p\mathbb{Z}}$. Look at long exact sequence.

Prop 6.5 TFAE for an A module B.

(1) B is flat

(2) $- \otimes_A B$ is exact

(3) $\mathrm{Tor}_i^A(-, B) = 0 \quad \forall i \geq 1$

(4) $\mathrm{Tor}_1^A(-, B) = 0$

4 - Application 1: universal coefficients theorems

If $(C_i, d_i)_i$, $(D_i, d_i')_i$ are two chain complexes for right A-modules and left A-modules: can construct the double complex

d_j' deg -1 passing 1 of degree i

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_i \otimes_A D_j \arrow[rr, "(-1)^i 1\otimes d_j'"] \arrow[d, "d_i \otimes 1"'] & & C_i \otimes_A D_{j-1} \arrow[r] \arrow[d, "d_i \otimes 1"] & \cdots \\
\cdots \arrow[r] & C_{i-1} \otimes_A D_j \arrow[rr, "(-1)^{i-1} 1\otimes d_j'"'] \arrow[d] & & C_{i-1} \otimes_A D_{j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Koszul sign convention if an object of degree i jump above an object of degree j $\rightsquigarrow$ multiply by $(-1)^{ij}$

Define the total complex $(C \otimes_A D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$ with differential

$$\partial_n (x \otimes y) = d_i^C(x) \otimes y + (-1)^i x \otimes d_j^D(y) \quad \text{if } \begin{array}{l} x \in C_i \\ y \in C_j \end{array}$$

Special case when D is concentrated in degree 0:

$$\cdots \to C_n \otimes D \xrightarrow[d^C \otimes 1]{} C_{n-1} \otimes D \to \cdots$$

Fact (1) $(C \otimes_A D, \partial)$ is a chain complex of abelian gp.

(2) There is an obvious map $\bigoplus_{i+j=n} H_i(C) \otimes H_j(D) \xrightarrow{*} H_n(C \otimes D)$
$[x] \otimes [y] \mapsto [x \otimes y]$

[$C_i \otimes D_{j+1} \xrightarrow{1 \otimes d_j^D} C_i \otimes D_j \xrightarrow{}$, $C_i \otimes D_j \xrightarrow{d_i^C \otimes 1}$ $x \in Z_i(C)$ then $\partial(x \otimes y) = 0$
$y \in Z_j(D)$

$x \in Z_i(C), y \in B_j(D)$, then $y = d_{j+1}^D(z)$ and $\partial(x \otimes z) = x \otimes d_D^{j+1}(z)$
$= x \otimes y$

So $x \otimes y$ is a boundary of $C \otimes D$. Similarly for $x \otimes y$ when $x \in B_i(C)$
$y \in Z_j(D)$ and when x and y are two boundaries since boundaries are cycles.
So the map $(*)$ is well defined.]

Thm 6.6 [Künneth Formula] If $Z_n(C)$ and $B_n(C)$ are flat $\forall n \in \mathbb{Z}$,
there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(C) \otimes H_j(D) \to H_n(C \otimes D) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^A(H_i(C), H_j(D)) \to 0$$

↳ if $H_i(C)$ or $H_j(D)$ is flat, proj, free for any i or j. Then the first map is an isomorphism.

Cor 6.7 (1) Let C be a chain complex of right A-modules such that $Z_n(C)$ and $B_n(C)$ are flat. And let M be a left A-module, then there is a short exact sequence

$$0 \to H_n(C) \otimes_A M \to H_n(C \otimes_A M) \to \mathrm{Tor}_1^A(H_{n-1}(C), M) \to 0$$

(2) If C is a chain complex of free abelian gps, then (1) apply. Moreover the short exact sequence splits.

<u>Proof</u>: we leave thm 6.6 as a (difficult/technical) exercise and we prove Coro 6.7.

We have $0 \to Z_n(C) \to C_n \to B_{n-1}(C) \to 0$ short <u>exact</u>
apply $-\otimes M$ leads to

$$\underbrace{\mathrm{Tor}_1(B_{n-1},M)}_{=0} \to Z_n(C)\otimes M \to C_n\otimes M \to B_{n-1}(C)\otimes M \to 0$$

since $B_{n-1}(C)$ is flat.

⇝ $0 \to Z_\bullet(C)\otimes M \to C_\bullet\otimes M \to B(C)\overset{\text{shift}}{[-1]}\otimes M \to 0$ ses of complexes.
so we have a long exact sequence in homology

$$\xrightarrow{\delta_{n+1}} H_n(Z\otimes M) \xrightarrow{\alpha_n} H_n(C\otimes M) \xrightarrow{\beta_n} H_n(B(C)[-1]\otimes M) \xrightarrow{\delta_n} H_{n-1}(Z(C)\otimes M) \to \cdots$$

Note that $Z_\bullet(C)$ and $B_\bullet(C)$ have zero differentials hence $H_n(Z(C)\otimes M) = Z_n(C)\otimes M$

so we get

$$\begin{array}{ccccccccc}
\cdots H_n(Z(C)\otimes M) & \xrightarrow{\alpha_n} & H_n(C\otimes M) & \xrightarrow{\beta_n} & H_n(B(C)[-1]\otimes M) & \xrightarrow{\delta_n} & H_{n-1}(Z(C)\otimes M) & \to \cdots \\
\| & & & & \| & & \| & \\
B_n(C)\otimes M \xrightarrow{\delta_{n+1}} Z_n(C)\otimes M & \to & H_n(C\otimes M) & \to & B_{n-1}(C)\otimes M & \xrightarrow{\delta_n} & Z_n(C)\otimes M & \to
\end{array}$$

check that $\delta_n = i\otimes M$ where $i : B_{n-1}(C) \hookrightarrow Z_n(C)$ is the inclusion.

↳ We have a short exact sequence

$$0 \to \ker(\beta_n) \to H_n(C\otimes M) \to \mathrm{Im}(\beta_n) \to 0$$

$$\ker(\beta_n) = \mathrm{Im}(\alpha_n) \cong \mathrm{Coker}(\delta_n), \qquad \mathrm{Im}(\beta_n) = \ker(\delta_n)$$

so $0 \to \mathrm{Coker}(\delta_{n+1}) \to H_n(C\otimes M) \to \ker(\delta_n) \to 0$

Finally look at: $0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$ and tensor with M

giving $0 \to \mathrm{Tor}_1(H_n(C), M) \to B_{n-1}(C) \otimes M \xrightarrow{\delta_{n+1}} Z_n(C) \otimes M \to H_n(C) \otimes M \to 0$

Hence $\mathrm{Coker}(\delta_{n+1}) = H_n(C) \otimes M$ and $\ker(\delta_n) = \mathrm{Tor}_1(H_n(C), M)$.

(2) C is a chain complex of free abelian gps then $B_n(C)$ and $Z_n(C)$ are free, so flat and can apply 1.

Moreover $0 \to Z_n(C) \hookrightarrow C_n \xrightarrow[d_n]{} B_n(C) \to 0$ splits ~~so $C_n \cong Z_n(C) \oplus B_n(C)$~~

Let R be a splitting of the inclusion $Z_n(C) \hookrightarrow C_n$. Check that

$H_n(C \otimes M) \to H_n(C) \otimes M$ is well defined and is a splitting of

$[x \otimes g] \mapsto [R(x)] \otimes g$

the natural map $H_n(C) \otimes M \to H_n(C \otimes M)$

[$R(x) \in Z_n(C)$ ✓ $\quad x \otimes g \in \mathrm{Im}(d_{n+1})$ iff $x \otimes g = d_{n+1}(x') \otimes g$

so $R(d_{n+1}(x')) = d_{n+1}(x')$ since $d_{n+1}(x') \in Z_n(C)$

so $[R(x)] = 0$.

$[x] \otimes g \to [x \otimes g] \to [R(x)] \otimes g = [x] \otimes g$ if $x \in Z_n(C)$]

□

Thm 6.8 [Universal coef thm for cohomology] Let C_n be a complex of chains of A-modules and $M \in A\mathrm{Mod}$. Assume that $Z_n(C)$ and $B_n(C)$ are projective $\forall n \in \mathbb{Z}$.

Then, there is a short exact sequence

$$0 \to \mathrm{Ext}^1_A(H_{n-1}(C), A) \to H^n \mathrm{Hom}_A(C, M) \to \mathrm{Hom}_A(H_n(C), A) \to 0$$

Proof is similar to the one of 6.6. □

↳ If $(C_n)_n$ is a chain complex of A-modules and M is an A-module

then $H_n(C \otimes_A M)$ are called the homology gp of C with coef in M

$H^n(C, M)$ — — coh — of C with coef in M

↳ Basically: it is enough to understand $H_n(C)$.