Chapter 7 Singular homology

1 Quick recollection

X topological space, R ring with 1, associative.

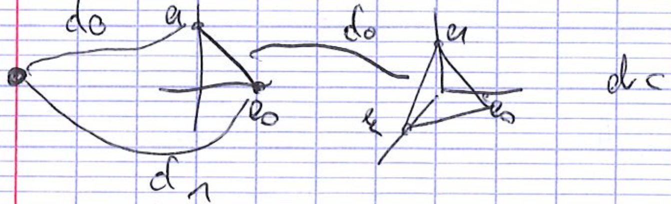
$$\Delta_n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{x_i \geq 0} x_i = 1 \right\}$$

free R-module

$$C_n^{sing}(X,R) = R \otimes_{\mathbb{Z}} C_n^{sing}(X) = R\left[\mathrm{Hom}_{Top}(\Delta_n, X)\right]$$

$d_n(\sigma) = \sum_{i=0}^{n} (-1) \sigma \circ d^i$ where $d^i : \Delta^{n-1} \longrightarrow \Delta^n$ is the ith face map

$$(x_0, \ldots, x_{n-1}) \longmapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$$



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Def 7.1 The singular homology of X with coef in R is $\underline{H_n(X,R) = H_n(C_n^{sing}(X,R))}$

where $X \neq \emptyset$

Rem1 There is an augmentation map $\varepsilon : C_0(X,R) \longrightarrow R$ and we denote
$\sigma \longmapsto 1_R$

by $\tilde{C}^{sing}(X,R)$ the augmented complex and $\tilde{H}_n(X,R)$ is called the reduced homology of X with coef in R

Rem2 by Künneth formula we have a ses

$$0 \longrightarrow H_n(X,\mathbb{Z}) \otimes_{\mathbb{Z}} R \longrightarrow H_n(X,R) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),R) \longrightarrow 0$$

2. Relative homology

Def 7.2 Let Top^2 be the category whose objects are pairs (X,A) where X is a topological space and $A \subseteq X$.
The morphisms from (X,A) to (Y,B) are the continuous maps $f : X \to Y$ s.t.

$f(A) \subseteq B$.

Def 7.3 Let $(X,A),(Y,B) \in Top^2$ and $f,g \in Hom_{Top^2}((X,A),(Y,B))$.

A homotopy from f to g is a morphism $H: (X\times I, A\times I) \to (Y,B)$ s.t. $H(-,0)=f$
$H(-,1)=g$

↳ It is a homotopy from ~~f~~ f to g such that $\forall i \in I, a\in A$ $H(a,i)\in B$.

Rem There is a notion of homotopy relative to A and this is not the same.

Lem 7.4 If $A \subseteq X$ then $C_*^{sing}(A,R) \subseteq C_*^{sing}(X,R)$

Proof $\sigma: \Delta_n \to A \longmapsto i\circ\sigma : \Delta_n \to A \overset{i}{\hookrightarrow} X$ is the map from $C_n^{sing}(A,R)$ to $C_n^{sing}(X,R)$. It is clearly injective since i is. □

Def 7.5 Let $C_*^{sing}((X,A)) = \dfrac{C_*^{sing}(X,R)}{C_*^{sing}(A,R)}$ ~~the relative homology~~ singular homology of X relatively to A

Rem (1) $A=\emptyset$ $C_n^{sing}(X,\emptyset) = C_n^{sing}(X)$
(2) Can put coefficients.

Thm 7.6 (Eilenberg - Steenrod axioms)
$(H_n)_{n\geq 0}$ is a family of functors from $Top^2 \to Ab$ together with natural transformations $\delta_n: H_n(X,A) \to \underset{\overset{\parallel}{H_{n-1}(A)}}{H_{n-1}(A,\emptyset)}$ satisfying

(1) [Dimension] $H_n(pt) = 0$ $\forall n>0$
(2) [Additivity] $X = \bigsqcup_{\alpha\in I} X_\alpha$ disjoint union, then $H_n(\bigsqcup_\alpha X_\alpha) \simeq \bigoplus_\alpha H_n(X)$
(3) [Long exact sequence] $\forall (X,A) \in Top^2$ we have a long exact sequence:

$$\xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(\pi)} H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A)$$

$j : A \hookrightarrow X$

$\pi : C_n(X) \to C_n(X,A)$

(4) [Homotopy Axiom] $f_0, f_1 : (X,A) \to (Y,B)$ are homotopic, then $H_n(f) = H_n(g)$ $\forall n \in \mathbb{N}$.

(5) [Excision] $\forall (X,A) \in \mathrm{Top}^2$ and $\mathcal{U} \subseteq A$ with $\overline{\mathcal{U}} \subseteq \mathring{A}$, the homomorphism $H_n(k): H_n(X\setminus\mathcal{U}, A\setminus\mathcal{U}) \to H_n(X,A)$ is an iso where $k : (X\setminus\mathcal{U}, A\setminus\mathcal{U}) \hookrightarrow (X,A)$ is the inclusion.

(•) We have a ses $0 \to C_\bullet^{sing}(A) \to C_\bullet^{sing}(X) \to C_\bullet^{sing}(X,A) \to 0$ which gives the long exact sequence of (3).

Rem Naturality: $(X,A) \xrightarrow{f} (Y,B)$ induces

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet^{sing}(A) \arrow[r] \arrow[d, "f_\#"] & C_\bullet^{sing}(X) \arrow[r] \arrow[d, "f_\#"] & C_\bullet^{sing}(X,A) \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & C_\bullet^{sing}(B) \arrow[r] & C_\bullet^{sing}(Y) \arrow[r] & C_\bullet^{sing}(Y,B) \arrow[r] & 0
\end{tikzcd}$$

and two long exact sequence with commutative squares.

(•) Lem 7.7 X topo space, $(X_\alpha)_{\alpha\in\Lambda}$ its path connected components. Then

$$H_n(X) \simeq \bigoplus_{\alpha\in\Lambda} H_n(X_\alpha)$$

Proof $\sigma : \Delta_n \to X$ then $\mathrm{Im}(\sigma)$ is path connected since Δ_n is so $\exists! \gamma \in \Lambda$: $\mathrm{Im}(\sigma) \subseteq X_\gamma$. By constructing σ we see that if $x \in C_n^{sing}(X)$ then $x = \sum_{\gamma\in\Lambda} \sum_i m_i \sigma_i$ with $\sigma_i : \Delta_n \to X_\gamma$

So $C_n(X) \simeq \bigoplus_{\alpha\in\Lambda} C_n(X_\alpha)$. The result follows from additivity of H_n □

Thm 7.8 (1) $X \neq \phi$ path connected, then $H_0(X) \simeq \mathbb{Z}$

(2) More generaly $H_0(X) \simeq \mathbb{Z}^{\text{Nb of path connected comp.}}$

Proof $S = \{ \sum_{x \in X} \lambda_x x ; \sum \lambda_x = 0 \} = \ker(\Sigma)$

(1) $S \subseteq B_0(X)$ if $\sum_{x \in X} \lambda_x x ; \sum \lambda_x = 0$ choose $x \in X$

$$\sum_{x\in X} \lambda_x x = \sum_{i=1}^{n} \lambda_i x_i$$

and $\sigma_i : x \rightsquigarrow x_i$

then $\sum \lambda_i \sigma_i \in CH_1(X)$ and $d^1(\sum \lambda_i \sigma_i) = \sum \lambda_i \sigma_i(1) - \sum \lambda_i \sigma_i(0)$

$$= \sum \lambda_i x_i - \underbrace{(\sum \lambda_i)}_{0} x$$

$$= \sum \lambda_i x_i .$$

(2) Conversely: $g \sum \lambda_i \sigma_i \in C H_1^{sing}(X)$ gives

$d(g) = \sum \lambda_i \sigma_i(1) - \sum \lambda_i \sigma_i(0)$

appears twice with opposite sign.

so $H_0(X) \simeq \frac{\mathbb{Z}X}{B_0(X)} \simeq \mathbb{Z}$ □

Thm 7.9 [Dimension axiom] $X = \{x\}$, then $H_n(X) = 0 \ \forall n > 0$

Proof just compute the chain complex. There is only one $\sigma_n : \Delta_n \to X$ $\forall n \in \mathbb{N}$ so $C_n^{sing}(X) = \mathbb{Z}[\sigma_n]$ and $\delta_n(\sigma_n) = \sum (-1)^n \sigma_{n-1}$

$$= \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & \text{otw} \end{cases}$$

so get $\mathbb{Z} \xrightarrow{iso} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{Iso} \mathbb{Z} \xrightarrow{0} \cdots \mathbb{Z} \xrightarrow{0} 0$

hence we have the result □

Homotopy axiom: saw that $f \sim g$ then $C^{sing}(f) \sim C^{sing}(g)$ as chain complexes.

Here need stronger version.

$f, g : (X, A) \longrightarrow (Y, B)$ $H : f \simeq g$ a homotopy $S_{n-1} : C^{n-1}(X) \to C^n(Y)$

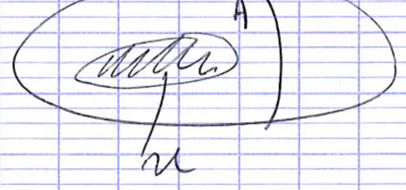
recall $\sigma \in C^{n-1}(X)$ $\Delta^n \longrightarrow \underbrace{\Delta^{n-1} \times I}_{\text{divided into } n \text{ copies of } \Delta^n} \xrightarrow{\sigma \times id} X \times I \xrightarrow{H} Y$

If $\sigma : \Delta_{n-1} \to A$, then $\Delta^{n-1} \times I \xrightarrow{\sigma \times id} A \times I \xrightarrow{H} B$ hence S_{n-1} maps $C^{n-1}(A)$ to $C^n(B)$ so induces a homotopy between the relative singular chain complexes.

Coro 7.10 Homotopy Axiom.

3 - Excision axiom

Theorality



can remove $\mathcal{U}$ without changing relative homology

Thm 7.11 (Excision II) X_1, X_2 subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$, then $j : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2) = (X_1, X_2)$ is a q.i.s.

Lem 7.12 Excision II is equivalent to I.

Proof $I \Rightarrow II$: $X = \mathring{X}_1 \cup \mathring{X}_2$ let $\mathcal{U} = X \setminus X_1$ $A = X_2$.

(o) Then $X \setminus X_1 \subset \underbrace{X \setminus \mathring{X}_1}$ ← closed

hence $\overline{\mathcal{U}} \subseteq X \setminus \mathring{X}_1 = (\mathring{X}_1 \cup \mathring{X}_2) \setminus \mathring{X}_1 \subseteq \mathring{X}_2 \setminus \mathring{X}_1 \subseteq \mathring{X}_2 = \mathring{A}$

(o) $X \setminus \mathcal{U} = X_1$

$A \setminus \mathcal{U} = X_2 \setminus (X \setminus X_1) = X_2 \cap (X_1^c)^c = X_2 \cap X_1$

$(X \setminus \mathcal{U}, A \setminus \mathcal{U}) = (X_1, X_1 \cap X_2)$

$(X, A) = (X, X_2)$.

$II \Rightarrow I$: let $\mathcal{U} \subset A$ with $\overline{\mathcal{U}} \subset \mathring{A}$ $X_2 = A$, $X_1 = X \setminus \mathcal{U}$, then

$\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathring{A} \Rightarrow X \setminus \mathring{A} \subset \underset{\substack{\nearrow \\ \text{open}}}{X \setminus \overline{\mathcal{U}}} \subset X \setminus \mathcal{U}$. Since $X \setminus \overline{\mathcal{U}}$ is open, we get

$X \setminus \bar{U} = (X - \bar{U})^\circ \supset X - \mathring{A}$, so $\mathring{X}_1 \cup \mathring{X}_2 = (X \setminus U)^\circ \cup A^\circ$

$\supset (X - \bar{U})^\circ \cup A^\circ$

$\supset (X - \mathring{A}) \cup A^\circ = X$ ◻

Lem 7.13 Let

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "a_n"] \arrow[d, "\alpha_n"] & B_n \arrow[r, "b_n"] \arrow[d, "\beta_n"] & C_n \arrow[r, "c_n"] \arrow[d, "\gamma_n"] & A_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_n' \arrow[r, "a_n'"'] & B_n' \arrow[r, "b_n'"'] & C_n' \arrow[r, "c_n'"'] & A_{n-1}' \arrow[r] & \cdots
\end{tikzcd}$$

be a commutative diagram with exact rows in an abelian category. If γ_n is an iso $\forall n$, we have a long exact sequence

$$\cdots \to A_n \xrightarrow{(a_n, \alpha_n)} B_n \oplus A_n' \xrightarrow{\begin{pmatrix}\beta_n \\ -a_n'\end{pmatrix}} B_n' \xrightarrow{c_n \gamma_n^{-1} b_n'} A_{n-1} \to \cdots$$

Proof of exactness diag chasing. ◻

Thm 7.14 (Mayer-Vietoris) $X_1, X_2 \subseteq X$ with $X = \mathring{X}_1 \cup \mathring{X}_2$

There is a long exact sequence

$$\cdots \to H_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\begin{pmatrix} j_* \\ -g_* \end{pmatrix}} H_n(X) \xrightarrow{\delta_n} H_{n-1}(X_1 \cap X_2) \to \cdots$$

where

$$\begin{tikzcd}
(X_1 \cap X_2, \phi) \arrow[r, "i_1"] \arrow[d, "i_2"] & (X_1, \phi) \arrow[r, "p"] \arrow[d, "g"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\
(X_2, \phi) \arrow[r, "j"'] & (X, \phi) \arrow[r, "q"'] & (X, X_2)
\end{tikzcd}$$

is a commutative diag in Top^2.

Proof apply H_* + Excision + Lem 7.13 ◻

4. Homology of Spheres

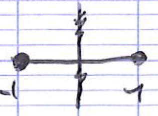
Thm 7.15 Let S^n be the nth sphere with $n \geq 0$ $(S^n = \{x \in \mathbb{R}^{n+1} ; \sum x_i^2$

Then $H_k(S^0) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 0 \\ 0 & \text{otw} \end{cases}$

for $n \geq 1$ $H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n \\ 0 & \text{otw} \end{cases}$

Rem we have an easier formula for the reduced homology $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } n=k \\ 0 & \text{o/w} \end{cases}$

Proof induction on n: S^0



hence $S^0 \simeq \{1\} \cup \{-1\}$, the result follows.

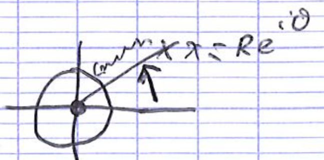
$n \geq 1$

a = north pôle b = south pôle.

$X_1 = S^n \setminus \{a\}$ $X_2 = S^n \setminus \{b\}$ are open and $X = \overset{\circ}{X_1} \cup \overset{\circ}{X_2}$

$X_i \underset{\text{homeo}}{\cong} \mathbb{R}^n$ via stereographic projection, hence are contractible.

$X \setminus \{a,b\} \cong \mathbb{R}^n \setminus \{pt\} \underset{\text{homo}}{\sim} S^{n-1}$



$\left(S^{n-1} \overset{i}{\hookrightarrow} \mathbb{R}^n \setminus \{pt\} \xrightarrow{\pi} S^{n-1} \right.$

$x \longmapsto \frac{x}{\|x\|}$

$\pi \circ i = Id_{S^{n-1}}$

$i \circ \pi \sim Id_{\mathbb{R}^n \setminus \{pt\}}$ via $H(x,t) = \frac{x}{\|x\|(1-t)+t}$.

Now we have a long exact sequence

$$\begin{tikzcd} H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] \arrow[d, equal] & H_n(X) \arrow[r, "\sim"] & H_{n-1}(X_1 \cap X_2) \arrow[r] & H_{n-1}(X_1) \oplus H_{n-1}(X_2) \arrow[d, equal] \\ & 0 & & & 0 \end{tikzcd}$$

Hence $H_n(S^n) \simeq H_{n-1}(S^{n-1})$ if $n \geq 2$. If $n=1$ we have to look at the tail of the complex $0 \to H_1(S^1) \to H_0(S^0) \to H_0(X_1) \oplus H_0(X_2) \to H_0(S^1) \to \mathbb{Z}$

leads to

$$\begin{tikzcd} 0 \arrow[r] & H_1(S^1) \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[dr] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\ & & 0 \arrow[r] & \mathbb{Z} \arrow[u] \arrow[r] & 0 & \end{tikzcd}$$

and $H_1(S^1) \simeq \mathbb{Z}$

□

Rem it is better to show that Thm 7.14 holds for reduced homology and then do not need to think about the case n=1.

Thm 7.16: (1) $m \neq n$ $\mathbb{S}^m \not\simeq \mathbb{S}^n$ not homeo nor h^{tpy} equiv

(2) $m \neq n$ $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeo

(3) $n \geq 0$ $\mathbb{S}^n$ is not contractible

(4) $f : \mathbb{D}^n \to \mathbb{D}^n$ continuous, then $\exists x \in \mathbb{D}^n$ with $f(x) = x$

Proof (1) clear (3) clear, (2) if they are then $\mathbb{R}^m \setminus \{pt\} \cong \mathbb{R}^n \setminus \{pt\}$ hence $\mathbb{S}^m \simeq \mathbb{S}^n$.

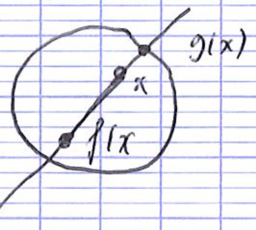
(4) if $n \geq 0$ $\quad \mathbb{S}^n \xhookrightarrow{i} \mathbb{D}^{n+1} \xrightarrow{\pi} \mathbb{S}^n$; $\pi \circ i \simeq Id$ does not ~~$i \circ \pi \simeq Id$~~

exists. Indeed, if so we have

$$\begin{tikzcd} H^n(\mathbb{S}^n) \arrow[r] \arrow[rr, bend left, "Id"] \arrow[d, equal] & H^n(\mathbb{D}^{n+1}) \arrow[r] \arrow[d, equal] & H^n(\mathbb{S}^n) \arrow[d, equal] \\ \mathbb{Z} & 0 & \mathbb{Z} \end{tikzcd}$$

Contradiction.

Now if $f(x) \neq x$ $\forall x \in \mathbb{D}^n$, $(x, f(x))$ is a line in $\mathbb{R}^{n+1}$ so it cut $\partial \mathbb{D}^{n+1}$.
Let $g(x) = \partial \mathbb{D}^{n+1} \cap [f(x), x)$



Then $x \longmapsto g(x)$ is a ~~homotopy~~ retract contradiction $\square$

Thm 7.17 [Eilenberg-Steenrod] Any two homology theories with isomorphic coefficient gps ($H_0(pts)$) are isomorphic as functors over nice category (compact polyhedral).

5. Behaviour of singular homology

Prop 7.18 $(X, A) \in Top^2$ and A contractible, then $H_n(X, A) \cong \tilde{H}_n(X)$ $\forall n \geq 0$

Proof long exact sequence associated with the pair

$$\underset{0}{\underset{\parallel}{H_n(A)}} \to H_n(X) \to H_n(X, A) \to \underset{0}{\underset{\parallel}{H_{n-1}(A)}}$$ only need to take care of the cases $n = 0, 1$.