Recollection and explaination

Historical motivation: History of homological algebra
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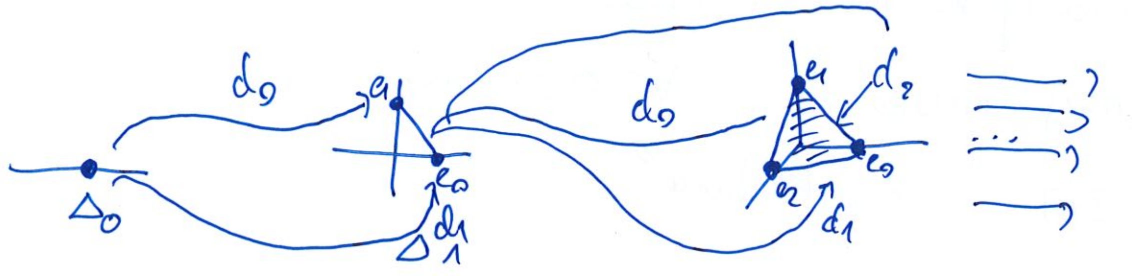
(1) Forget about simplicial sets: Singular chain complexes:

Main object: Standard n-simplices: $\Delta_n = \left\{ (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} x_i \geq 0 \ \forall i \\ \sum x_i = 1 \end{array} \right\}$



↳ these are topological spaces so if $X \in Top$ one can look at

$\mathrm{Hom}_{Top}(\Delta_i, X) = \{ f : \Delta_i \to X \text{ continuous} \}$

Now standard n-simplex is embedded $(n+1)$ times as face of standard $(n+1)$-simplex

denote $d^i_n : \Delta_n \longrightarrow \Delta_{n+1}$

$(x_0, \cdots, x_n) \longmapsto (x_0, \cdots, x_{i-1}, \underset{\substack{\downarrow \\ i}}{0}, x_i, \cdots, x_n)$

goal use this to construct a chain complex

Step 1 $C_n^{sing}(X) = \mathbb{Z}[\mathrm{Hom}_{Top}(\Delta_i, X)]$ take a free abelian gp on $\mathrm{Hom}_{Top}(\Delta_i, X)$

Step 2 differential $d^i : \Delta_n \to \Delta_{n+1}$ induces $\mathrm{Hom}(\Delta_{n+1}, X) \to \mathrm{Hom}(\Delta_n, X)$

$\sigma \longmapsto \Delta_n \xrightarrow{d_i} \Delta_{n+1} \xrightarrow{\sigma} X$

$\longmapsto \sigma \circ d_i$

hence $\mathrm{Hom}(\Delta_{n+1}, X) \longrightarrow \mathbb{Z}[\mathrm{Hom}(\Delta_n, X)]$

$\sigma \longmapsto \sum_{i=0}^{n+1} (-1)^i \sigma \circ d_i =: d(\sigma)$

by univ prop of free abelian gps: d extend as a gp homomorphism from $C_{n+1}^{sing}(X)$ to $C_n^{sing}(X)$

Step 3 $\boxed{\text{check } d^2 = 0}$ only uses (*) $\boxed{d^j \circ d^i = d^i \circ d^{j-1}}$

for $i < j$

Step 4 Abstract all this

replace $\{\Delta_n\}_n$ by the category Δ obj $= \{[n] = \{0,1,\dots,n\}\}$

$\mathrm{Hom}([n],[m]) = \{ f : [n] \to [m] ;\ x \leq y \Rightarrow f(x) \leq f(y)\}$

Combinatorial version of the standard simplices

$d_n^i : [n] \to [n+1]$ "misses i" $\forall\ 0 \leq i \leq n+1$

$s^i : [n+1] \to [n]$ "i is hit twice"

$\begin{matrix} i \\ i+1 \end{matrix} \rightrightarrows i$

Thm Every morphism in Δ is a composition of d^i's and s^i's and ~~the~~ the d^i and s^i are subject to many relations : the so-called simplicial relations eg

$$\begin{cases} d_{n+1}^j \circ d_n^i = d_{n+1}^i \circ d_n^{j-1} & i<j \quad (1) \\ s_j^i \circ s_j = s_j \circ s_{i-1} & i>j \quad (2) \\ + d^i \circ s^j \dots & (3) \end{cases}$$

This is a presentation by generators and relations of Δ.

Cd : To define a functor F from Δ to $\mathcal{E}$ it is enough to define $F(d^i)$, $F(s^j)$ and show that it respects the simplicial relations

~~For our homology?~~

For us it is actually enough to only keep the d^i that generate the full subcategory of Δ consisting of injective maps.

↳ Enough to define $F(d^i)$ and check that it respects (1)

Thm if $F : \Delta_{inj}^{op} \to \mathcal{A}b$ (semisimplicial abelian grp) is a functor then $(F([n]), d)$ where $d : F([n]) \to F([n-1])$, $x \mapsto \sum_{i=0}^{n} F(d^i)(x)$ is a chain complex of abelian grps. Proof = the one (*) since it was only using the relation (1)

Step 5 Other examples

(1) Bar resolution or standard resolution:

G be a finite gp ~~R commutative ring~~ ($= \mathbb{Z}$ eg)

Let F_n = Free ~~R-module~~ abelian gp on $\{(g_0, \dots, g_n) ; g_i \in G\} = \underbrace{G \times G \times \dots \times G}_{n+1}$

F_n has a structure of $\mathbb{Z}[G]$-module for the action $g \cdot (g_0, \dots, g_n) = (g g_0, \dots, g g_n)$

↳ hence $F_n \in \mathbb{Z}[G]\mathrm{Mod}$.

For n we let $\partial_i : F_n \to F_{n-1}$
$(g_0, \dots, g_n) \mapsto (g_0, \dots, g_{i-1}, \check{g_i}, g_{i+1}, \dots, g_n)$

we clearly have for $i<j$ $\quad \partial_i \circ \partial_j (g_0, \dots, g_n) = (g_0, \dots, \check{g_i}, \dots, \check{g_j}, \dots g_n)$

et $\quad \partial_{j-1} \circ \partial_i (g_0, \dots, g_n) = \partial_j (g_0, \dots, \check{g_i}, \dots g_j, \dots g_n)$
$= (g_0, \dots g_{i-1}, \dots \check{g_i}, \dots \check{g}_{j-1}, \dots g_n)$

donc $F : \Delta_{inj}^{op} \to \mathbb{Z}[G]\mathrm{Mod}$ est un foncteur (F est un $\mathbb{Z}[G]$-module semi-simplicial)

$$\begin{array}{ccc} [n] & \longmapsto & F_n \\ \downarrow d_i & & \uparrow \partial^i \\ [n+1] & \longmapsto & F_{n+1} \end{array}$$

donc (F_n, ∂) où $\partial : F_n \to F_{n-1}$
$(g_0, \dots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i \partial_i (\text{—})$

est un complexe de chaine

(2) ~~Koszul complex~~ Koszul complex (Voir TD)

R commutative ring E free module of rank R over R

$\bigwedge^i E$ the i-th exterior product of E

(3) Hochschild complex k com. ring A associative k-algebra

${}_A M_A$ be a A-A-bimodule. Then $A^{\otimes n} = \underbrace{A \otimes_A A \otimes_A \cdots \otimes_A A}_{n \text{ times}}$

$C_n(A,M) := M \otimes_A A^{\otimes n}$

$d_i : C_n(A,M) \to C_n(A,M)$

$m \otimes a_0 \otimes \cdots \otimes a_n \mapsto m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n$

$d_0(m \otimes \cdots \otimes a_n) = m a_1 \otimes a_2 \cdots \otimes a_n$

$d_n(m \otimes a_0 \otimes \cdots \otimes a_n) = a_n m \otimes a_0 \otimes \cdots \otimes a_{n-1}$

check that this is a simplicial A-A-bimodule.

↳ chain complex of A-A-bimodules.

il faut donc vérifier la relation (1)

$i < j$

$$\begin{aligned} d_i \circ d_j (m \otimes a_1 \otimes \cdots \otimes a_n) &= d_i(m \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n) \\ &= (m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots a_n) \\ &= d_{j-1} \circ d_i (-) \text{ donc} \end{aligned}$$

cas $j = n$? $d_i \circ d_n = d_i(a_n m \otimes a_1 \cdots \otimes a_{n-1})$

$= (a_n m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots)$

$= d_{n-1} \circ d_i (-)$.

etc. (4) simplicial homology ...

Going back to singular homology:

Functoriality of $C^{sing}(X)$ $\qquad f : X \to Y$ continuous

↳

$$\begin{array}{ccc} C_n^{sing}(X) = \mathbb{Z}[\mathrm{Hom}(\Delta_n, X)] & & \sigma : \Delta_n \to X \\ \downarrow C_n(f) & & \downarrow \\ C_n^{sing}(Y) = \mathbb{Z}[\mathrm{Hom}(\Delta_n, Y)] & & f \circ \sigma : \Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y \end{array}$$

check morphism of chain complexes

$$\begin{array}{ccc} C_n^{sing}(X) & \xrightarrow{d_n^X} & C_{n-1}^{sing}(X) \\ \downarrow & & \downarrow \\ C_n^{sing}(Y) & \xrightarrow{d_n^Y} & C_{n-1}^{sing}(X) \end{array}$$

$$C_{n-1}(f)\, d_n^X(\sigma) = \sum_{i=0}^{n+1} (-1)^i\, \sigma \circ d_i)$$

$$= \sum_{i=0}^{n+1} (-1)^i f \circ \sigma \circ d_i = \sum_{i=0}^{n+1} (-1)^i (f \circ \sigma) \circ d_i = d_n^Y(C_n(f)\ldots$$

<u>Remark</u> on $C^{sing}(X)$: it is a ~~[illegible]~~ huge free abelian gp

(2) look at what d_n is doing when n small

$C_1^{sing}(X) \xrightarrow{d_1} C_0^{sing}(X)$

$\sigma : \Delta_1 \to X \rightsquigarrow \underbrace{\sigma \circ d^0 - \sigma \circ d^1}$

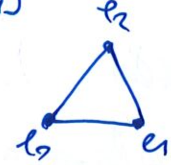
$\qquad = \sigma_{|[\hat{e}_0, e_1]} - \sigma_{|[e_0, \hat{e}_1]}$

$\qquad \text{"}=\text{"}\ \sigma(e_1) - \sigma(e_0)$

$\sigma : \Delta_1 \longrightarrow X$

$\sigma \circ d^0 :$ 

$\qquad = \sigma_{|[\hat{e}_0, e_1]}$

$\sigma :$  $\longrightarrow X$

$C_2^{sing}(X) \xrightarrow{d_2} C_1^{sing}(X) \qquad d_2 \sigma = \sigma_{|[\check{e}_0, e_1, e_2]} - \sigma_{|[e_0, \check{e}_1, e_2]} + \sigma_{|[e_0, e_1, \check{e}_2]}$

so $d_2(\sigma)$ is (up to a sign) the map σ restricted to the boundary of Δ_2

(the sign is in relation with the orientation of Δ_2 )

↳ guess what d_n is doing : just restrict σ to each faces of Δ_{n-1} up to coherent signs.

<u>Def 4.15</u> Let $\mathcal{E}$ be an additive category and $f, g \in \mathrm{Hom}_{Ch_\bullet(\mathcal{E})}(X, Y)$

Then (1) A homotopy from f to g is the data of $(s_i)_{i \in \mathbb{Z}}$ where $s_i \in \mathrm{Hom}_{\mathcal{E}}(X_i, Y_{i+1})$ such that $f - g = d_{i+1}^Y s_i + s_{i-1} d_i^X$

("$f - g = ds + sd$")

$$\begin{tikzcd}
X_{i+1} \arrow[r] \arrow[d, "f_{i+1}"'] & X_i \arrow[r] \arrow[d, "f_i"] \arrow[dl, "s_i"'] & X_{i-1} \arrow[d, "f_{i-1}"] \arrow[dl, "s_{i-1}"'] \\
Y_{i+1} \arrow[r] & Y_i \arrow[r] & Y_{i-1}
\end{tikzcd}$$

we then say that f and g are homotope and write $f \sim g$

(2) $f : X_\bullet \to Y_\bullet$ and $g : Y_\bullet \to X_\bullet$ are homotopy equivalence inverse of each other if fg is homotope to $Id_{Y_\bullet}$ and gf is homotope to $Id_{X_\bullet}$.

<u>Motivation</u> It comes from topology.

<u>Recall</u> $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ continuous map between f and g. A homotopy h from f to g is a continuous map $h : X \times I \to Y$ s.t $\begin{cases} h(-,0) = f \\ h(-,1) = g \end{cases}$ $\quad I = [0,1]$

"time"

"continuous deformation from f to g".

<u>Thm 4.16</u> Let X, Y be two topological spaces $f, g : X \to Y$ continuous and h a homotopy from f to g. Then $C_\bullet^{sing}(f)$ and $C_\bullet^{sing}(g)$ are homotopic in the sense of definition 4.15

<u>Proof</u> <u>Can skip</u> but here is the idea.

$$\begin{array}{ccccccccc}
C_2(X) & \xrightarrow{d_2^X} & C_1(X) & \xrightarrow{d_1^X} & C_0(X) & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & ? & \downarrow & 0 & & 0 & \\
C_2(Y) & \xrightarrow{d_2^Y} & C_1(Y) & \xrightarrow{d_1^Y} & C_0(Y) & \to & 0 & \to & 0
\end{array}$$

$s_0 : C_0(X) \to C_1(Y)$ need to send $\sigma : \Delta_0 \to X$ to $s_0(\sigma) : \Delta_1 \to X$

<u>Look at</u>

$$\begin{array}{c}
\Delta_1 \\
\downarrow \cong \alpha \\
\Delta_0 \times I \\
\downarrow \sigma \times Id_I \\
X \times I \\
\downarrow h \\
Y
\end{array} \qquad s_0(\sigma)$$

$\Delta_0 = \bullet$ hence $\Delta_0 \times I \cong I \cong \Delta_1$

so $\underbrace{h \circ (\sigma \times Id_I) \circ \alpha}_{=: s_0(\sigma)}$ is a map from Δ_1 to Y (and it is continuous)

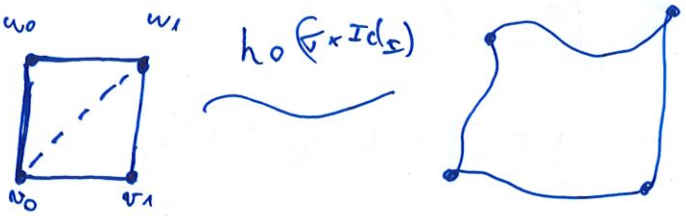
$$d_1^Y(s_0(\sigma)) = s_0(\sigma)\circ d_1 = s_0(\sigma)|_{[\hat{e}_0,e_1]} - s_0(\sigma)|_{[e_0,\hat{e}_1]}$$
$$= g\sigma - f\sigma$$

$$\begin{array}{ll} \Delta_1 & e_1 \\ \downarrow & \downarrow \\ \Delta_0\times I & (\bullet,1) \\ \sigma\downarrow & \downarrow \\ X\times I & (\sigma(\bullet),1) \\ h\downarrow & \downarrow \\ Y & h(\sigma(\bullet),1) = g\sigma(\bullet) \end{array}$$

cannot do

let us do $\underline{n=1}$ $\quad \sigma:\Delta_1\to X \overset{?}{\rightsquigarrow} s_1(\sigma):\Delta_2\to X$

pb $\Delta_1\times I$ ↕ ≠ Δ_2



$$\begin{array}{c} \Delta_1\times I \\ \downarrow{\scriptstyle \sigma\times Id_I} \\ X\times I \\ \downarrow{\scriptstyle h} \\ Y \end{array}$$

see that à $t=0$ $\quad h\circ(\sigma\times Id_I)(-,0) = h(\sigma(-),0) = f\circ\sigma$
$\qquad\qquad\quad t=1$ $\quad h\circ(\sigma\times Id)(-,1) = g\circ\sigma$

and $\begin{cases} h\circ(\sigma\times Id_I)|_{[w_0,\hat{v}_1]} = h(\sigma(w_0),-) \\ h\circ(\sigma\times Id_I)|_{[\hat{w}_0,w_1]} = h(\sigma(w_1),-) \end{cases}$

look at $s_0\circ d_1^X(\sigma) = s_0\left(\sigma|_{[\hat{v}_0,v_1]} - \sigma|_{[v_0,\hat{v}_1]}\right)$
$$= h\circ(\sigma|_{[\hat{v}_0,v_1]}\times Id) - h\circ(\sigma|_{[\hat{v}_0,v_1]}\times Id)$$

~~Now~~ So on the boundary of $\Delta_1\times I$ we understand $\underline{h\circ(\sigma\times Id)}$

Now define $s_1(\sigma) = h\circ(\sigma\times Id)|_{[v_0,w_0,w_1]} - h\circ(\sigma\times Id)|_{[v_0,v_1,w_1]}$

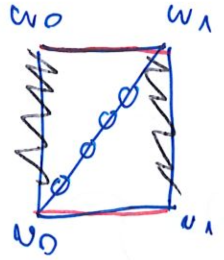
identifying Δ_2 with $[a,b,c]$ in the obvious way gives an element in $S_2^{sing}(Y)$

claim $(d_2 \circ s_1 - s_0 \circ d_1)(\sigma) = g\sigma - f\sigma$

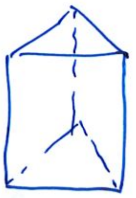
indeed $d_2 \circ s_1(\sigma) = d_2\left(h\circ(\sigma\times Id)_{|[v_0,w_0,w_1]} - h\circ(\sigma\times Id)_{|[v_0,v_1,w_1]}\right)$

$$= h\circ(\sigma\times Id)_{[\hat v_0,w_0,w_1]} - \cancel{h\circ(\sigma\times Id)_{|[v_0,\hat w_0,w_1]}} + h\circ(\sigma\times Id)_{|[v_0,w_0,\hat w_1]}$$

$$- h\circ(\sigma\times Id)_{|[\hat v_0,v_1,w_1]} + \cancel{h\circ(\sigma\times Id)_{[v_0,\hat v_1,w_1]}} - h\circ(\sigma\times Id)_{[v_0,v_1,\hat w_1]}$$



$$= g\sigma - f\sigma + s_0 d_1(\sigma).$$

n=3 $\Delta_2 \times I$  has to be splitted in 3 2-simplices

general formula. $s_n(\sigma) = \sum_{i=0}^{n} (-1)^i h\circ(\sigma\times Id)_{|[v_0,\dots,v_i,w_i,\dots,w_n]}$

□

Rem This is in fact an application of the so called methods of acyclic models. (See eg Rotman algebraic topology or MacLane Homology)

Lem 4.17 $f: X_\bullet \to Y_\bullet$, $g: Y_\bullet \to Z_\bullet$ two morphisms of chain complexes. If $f\sim 0$ then $g\circ f \sim 0$ and $f\circ h\sim 0$ $\quad h: S_\bullet \to X_\bullet$

Proof

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d, "f_i"] & X_{i-1} \arrow[dl, "s_{i-1}"'] \arrow[d, "f_{i-1}"] \\
Y_i \arrow[r] \arrow[d, "g_i"] & Y_{i-1} \arrow[d, "g_{i-1}"] \\
Z_i \arrow[r] & Z_{i-1}
\end{tikzcd}$$

$S_i = g_i \circ s_{i-1}$

check details ...

□

<u>Prop/Def</u> 4.18 $\mathcal{E}$ additive category. The homotopy category of $\mathcal{E}$ denoted by $K_\bullet(\mathcal{E})$ has for objects: • $ob(Ch_\bullet(\mathcal{E}))$

• $\mathrm{Hom}_{K_\bullet(\mathcal{E})}(X,Y) = \mathrm{Hom}_{Ch_\bullet(\mathcal{E})}(X,Y)/\sim$

<u>Rem</u> (1) One has to check that this is a category : use lemma 4.17

(2) We have bounded version for $K_\bullet(\mathcal{E})$

(3) $K_\bullet(\mathcal{E})$ is a complicated object: this is a <u>triangulated category</u>

On peut maintenant définir S :

$$S : \mathrm{Top} \to s\mathrm{Set}$$

$$\begin{array}{ccc} X & \longmapsto & \mathrm{Hom}_{\mathrm{Top}}(\Delta(-),X) \\ \downarrow f & & \downarrow f_* \\ Y & \longmapsto & \mathrm{Hom}_{\mathrm{Top}}(\Delta(-),Y) \end{array} \quad : \quad \begin{array}{l} \Delta \longrightarrow \mathrm{Top} \longrightarrow \mathrm{Set} \\ n \longmapsto \Delta_n \longmapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta_n, X) \end{array}$$

On vérifie facilt que S est un foncteur : en fait on a fait ça dans le Chapitre II.

Rem Pour des liens plus fort entre sAb et $Ch_+(\mathbb{Z})$ voir Dold-Kan correspondance

Def 4.12 Soit X un espace topologique. On note $C^{sing}(X)$ le complexe obtenu via Moore $\circ F \circ S(X)$.

$d_n : C_n(X) \to C_{n-1}(X)$ est donné par :

Concrètement : Si X est un espace topologique on a

$$C^{sing}(X) \qquad \cdots \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \to 0$$

où $C_i(X) = \mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}}(\Delta_n, X)]$ le gpe abélien libre de base les applications continues $\sigma : \Delta_n \to X$

Def 4.13 $\sigma : \Delta_n \to X$ est un n-simplexe singulier

Ex $\Delta_0 = \bullet$ donc un 0-simplexe singulier est un point de X

$\Delta_1 = \bullet\!\!-\!\!\bullet \simeq [0,1] = I$ donc un 1 simplexe singulier est un chemin dans X

$\sigma_2 : \Delta_2 \to X$ commence à être compliqué

donc $C_0(X) \simeq \mathbb{Z}[X]$ $\quad C_1(X) \simeq \mathbb{Z}(\text{chemin de } X)$

↳ $C_i(X)$ est très gros !

Prop 4.14 $C^{sing} : \mathrm{Top} \to Ch_+(\mathrm{Ab})$ est un foncteur.

démo Evident car C^{sing} est une composition de foncteurs.

Remarque action de d_i pour i petit.

$\sigma : \Delta_1 = \underset{0}{\bullet}\!\!-\!\!\underset{1}{\bullet} \to X$ alors $d_1(\sigma) = \sigma(1) - \sigma(2)$

$\sigma : \Delta_2 \to X$

$d_1(\sigma) = \sigma_{|[1,2]} - \sigma_{|[0,2]} + \sigma_{|[0,1]}$

Def 4.15 Soit $\mathcal{E}$ une catégorie additive et soient $f, g \in \mathrm{Hom}_{Ch(\mathcal{E})}(X_\bullet, Y_\bullet)$.
Alors

(1) Une <u>homotopie</u> S de f à g est la donnée de morphismes $s_i : X_i \to Y_{i+1}$ dans $\mathcal{E}$ tels que $f_i - g_i = d^Y_{i+1} s_i + s_{i-1} d^X_i$

$$\begin{tikzcd}
X_{i+1} \arrow[r, "d^X_{i+1}"] \arrow[d, "f_{i+1}-g_{i+1}"'] & X_i \arrow[r, "d^X_i"] \arrow[d, "f_i-g_i"] \arrow[dl, dashed, "s_i"'] & X_{i-1} \arrow[d, "f_{i-1}-g_{i-1}"] \arrow[dl, dashed, "s_{i-1}"'] \\
Y_{i+1} \arrow[r, "d^Y_{i+1}"'] & Y_i \arrow[r, "d^Y_i"'] & Y_{i-1}
\end{tikzcd}$$

" $f - g = sd + ds$ "

On dit alors que f et g sont homotopes

(2) $f : X_\bullet \to Y_\bullet$ et $g : Y_\bullet \to X_\bullet$ sont des <u>équivalences d'homotopies</u> si fg est homotope à Id_Y et gf est homotope à l'identité de X.

Motivation <u>vient bien sûr de la topologie</u>

Rappels : Soient X et Y des espaces topologiques et $f, g : X \to Y$ des applications continues. On dit que f et g sont <u>homotopes</u> s'il existe $F : X \times I \to Y$ continue avec $F(-,0) = f$ et $F(-,1) = g$. L'application F est appelée homotopie de f à g.

Thm 4.16 : Soient X, Y deux espaces topologiques et $f, g : X \to Y$ des applications homotopes. Alors $C^{sing}(f)$ et $C^{sing}(g) : C^{sing}(X) \to C^{sing}(Y)$ sont homotopes au sens de la définition 4.15.

Démo On construit par récurrence une homotopie S :

<u>0</u> $s_0 : C_0(X) \to C_1(Y)$; $d_1 s_0 = g_* - f_*$

Soit $\sigma : \Delta_0 \to X$ un 0-simplexe singulier

Alors on a $s_0(\sigma) : \Delta_1 \simeq \Delta_0 \times \Delta_1 \simeq \Delta_0 \times I \xrightarrow{\sigma \times Id} X \times I \xrightarrow{h} Y$

ainsi cette composée est un 1 simplexe singulier dans Y. et pour $t \in \Delta_1 \simeq I$.

On a $s_0(\sigma)(t) = h(\sigma(\bullet), t)$

Donc $$d_1 \circ s_0(\sigma) = s_0(\sigma)(1) - s_0\sigma(0)$$
$$= h(\sigma(\bullet), 1) - h(\sigma(\bullet), 0)$$
$$= g(\sigma(\bullet)) - f(\sigma(\bullet))$$

D'où $d_1 \circ s_0 = g_* - f_*$

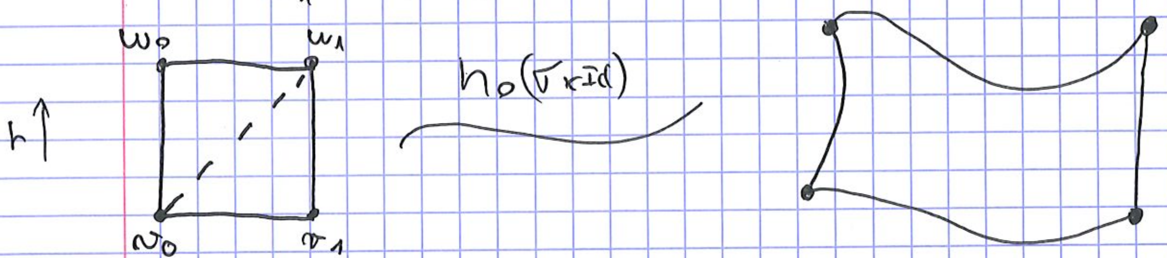
<u>n=1</u>

$$\begin{tikzcd}
 & C_1(X) \arrow[r, "d_1"] \arrow[d, "g_* - f_*"] \arrow[dl, dashed, "\exists ?\, s_1"'] & C_0(X) \\
C_2(Y) \arrow[r] & C_1(Y) \arrow[r] & C_0(Y)
\end{tikzcd}$$

on a $\Delta_1 \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{h} Y$ pour σ ~~elem~~ $: \Delta_1 \to X$

Problème $\Delta_1 \times I \not\simeq \Delta_1$.

(*)



Alors pour $t=0$ on a $h \circ (\sigma \times Id)_{|t=0} = h(\sigma(-), 0) = f \circ \sigma$

$t=1$ on a $h \circ (\sigma \times Id)_{|t=1} = h(\sigma(-), 1) = g \circ \sigma$

en $[v_0, \hat{v}_1]$: on a $h(\sigma_{|[v_0, \hat{v}_1]}, -)$

$[\hat{v}_0, v_1]$ $\quad h(\sigma_{|[\hat{v}_0, v_1]}, -)$

ces deux applications sont obtenues via $s_0 \circ d_1(\sigma)$ en effet :

$s_0 \circ d_1(\sigma) = h(\sigma_{|[\hat{v}_0, v_1]}, -) - h(\sigma_{|[v_0, \hat{v}_1]}, -)$.

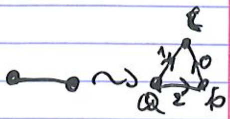
Pour avoir un 2-simplexe singulier depuis (*) on coupe le carré en deux ~~deux~~ 2-simplexes et on pose :

$$s_1(\sigma) = h\circ(\sigma\times Id)|_{[v_0,w_0,w_1]} - h\circ(\sigma\times Id)|_{[v_0,v_1,w_1]}$$

en identifiant Δ_2 avec $[a,b,c]$ de façon évidente, en suivant l'ordre, on a $s_1(\sigma)\in C_2(Y)$.

$e_0\to a$
$e_1\to b$
$e_2\to c$

claim $(d_2\circ s_1 - s_0\circ d_1)(\sigma) = g_1 - f_1$:

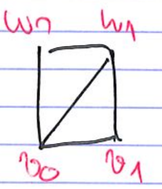


$$d_2\circ s_1(\sigma) = s_1(\sigma)|_{[\hat{a},b,c]} - s_1(\sigma)|_{[a,\hat{b},c]} + s_1(\sigma)|_{[a,b,\hat{c}]}$$

$$= \underbrace{h\circ(\sigma\times Id)|_{[\hat{v}_0,w_0,w_1]}}_{g} - \cancel{h\circ(\sigma\times Id)|_{[v_0,\hat{w}_0,w_1]}} + h\circ(\sigma\times Id)|_{[v_0,w_0,\hat{w}_1]}$$

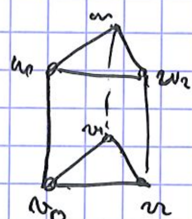
$$- h\circ(\sigma\times Id)|_{[\hat{v}_0,v_1,w_1]} + \cancel{h\circ(\sigma\times Id)|_{[v_0,\hat{v}_1,w_1]}} - \underbrace{h\circ(\sigma\times Id)|_{[v_0,v_1,\hat{w}_1]}}_{f}$$

diag



$$= g\circ\sigma - f\circ\sigma + d_1 s_1(\sigma)$$

n=3 $\Delta_2\times I$



que l'on coupe en 3 2-simplexes

en général on pose $P_n(\sigma) = \sum_{i=0}^{n}(-1)^i h\circ(\sigma\times Id)|_{[v_0,\dots,v_i,w_i\dots w_n]}$

□

↳ C'est en fait une application de la méthode des modèles acycliques.

Lem 4.17 : Soient $f: X\to Y$ et $g: Y\to Z$ des morphismes de $C_\bullet(\mathcal{E})$ si $f\sim 0$ alors $g\circ f\sim 0$

démo

$$\begin{tikzcd}
X_{i+1} \arrow[r] \arrow[d, "f_{i+1}"'] & X_i \arrow[r, "d_i^X"] \arrow[d, "f_i"] \arrow[dl, "h_i"'] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] \arrow[dl, "h_{i-1}"'] & {} \\
Y_{i+1} \arrow[r] \arrow[d, "g_{i+1}"'] & Y_i \arrow[r] \arrow[d, "g_i"] & Y_{i-1} \arrow[d, "g_{i-1}"] \\
Z_{i+1} \arrow[r, "d_{i+1}^Z"'] & Z_i \arrow[r, "d_i^Z"'] & Z_{i-1}
\end{tikzcd}$$

$g_{i+1}\circ h_i$, $g_i\circ h_{i-1}$

alors $d^Z_{i+1}\circ(g_{i+1}\circ h_i) + g_i\circ h_{i-1}\circ d^X_i$

$= g_i\circ d^Y_{i+1}\circ h_i + g_i\circ -$

$= g_i\circ f_i$

□