Prop/Def 4.18 Soit $\mathcal{E}$ une catégorie additive, la catégorie homotopique de $\mathcal{E}$, noté $K(\mathcal{E})$ a pour objets : les objets de $Ch(\mathcal{E})$

morphismes : $\mathrm{Hom}_{Ch(\mathcal{E})}(X,Y)/\sim$

Démo il faut vérifier que c'est une catégorie : on utilise le lemme 4.17 □

Rem (1) On a des versions bornées de $K(\mathcal{E})$: $K_+(\mathcal{E})$, $K_-(\mathcal{E})$, $K_b(\mathcal{E})$

(2) $K(\mathcal{E})$ est un exemple catégorie triangulée.

Chapitre 5 Catégories abéliennes

$\mathcal{E}$ be an additive category in which every morphism has a kernel and a cokernel ($\mathrm{Ker}(f) = eq(f,0)$ $\mathrm{Coker}(f) = coeq(f,0)$)

↳ every morphism has a canonical factorization

$$\begin{tikzcd}
\mathrm{Ker} f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "\pi"] & \mathrm{Coker} f \\
& \mathrm{Coim} f \arrow[ur, dashed, "f'"] \arrow[r, dashed, "\exists \bar{f}"'] & \mathrm{Im} f \arrow[u, "j"'] &
\end{tikzcd}$$

Let $\mathrm{Im} f := \mathrm{Ker}(\pi)$ where $\pi : Y \to \mathrm{Coker} f$ is a coker of f

$\mathrm{Coim} f := \mathrm{Coker}(i)$ — $i : \mathrm{Ker} f \hookrightarrow X$ is a kernel of f

On a $f \circ i = 0$ donc par la pté universelle de $\mathrm{Coim} f$ $\exists f' : \mathrm{Coim} f \to Y$ tq $f'p = f$

Alors $\pi \circ f' \circ p = \pi f = 0$ et p épi donne $\pi \circ f' = 0$. Donc $\exists \bar{f}$ $\mathrm{Coim} f' \to \mathrm{Im} f$ qui fait commuter le carré via la pté universelle du noyau.

Example $\mathcal{E} = A\mathrm{Mod}$

$$\begin{tikzcd}
\mathrm{Ker} f \arrow[r, hook] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, two heads] & Y/\mathrm{Im} f \\
& X/\mathrm{Ker} f \arrow[r, dashed, "\bar{f}"'] & \mathrm{Im} f \arrow[u, hook] &
\end{tikzcd}$$

$\bar{f}$ ← first iso thm.

Def 5.1 Soit $\mathcal{E}$ une catégorie additive. Alors $\mathcal{E}$ est abélienne si :

(1) tout morphisme de $\mathcal{E}$ admet un noyau et un conoyau

(2) $\forall f : x \to y$ l'application $\bar{f} : \mathrm{Coim} f \to \mathrm{Im} f$ est un isomorphisme

Ex 5.2 (1) A est un anneau, alors $A\mathrm{Mod}$ est abélienne. Si A noethérien $A\mathrm{mod} = $ fg modules est abélienne.

(2) Il y a des catégories avec (1) mais pas (2) par exemple la catégorie des groupes abéliens topologiques séparés (T_2 ⊙⊙) voir Assem page 279.

(3) $\mathcal{E}$ ab then $\mathcal{E}^{op}$ abelian

Prop 5.3 (1) Let $\mathcal{E}$ be an abelian category and I (small) category. Then $\mathrm{Fun}(I, \mathcal{E})$ is abelian

(2) The categories of chain complexes and cochain complexes $Ch_\bullet(\mathcal{E})$ and $Ch^\bullet(\mathcal{E})$ are abelian

Proof (1) Sketch (skip) $F \overset{\eta}{\Rightarrow} G$ nat trans $\ker(\eta) = ?$

$\forall i \in I$ $F(i) \xrightarrow{\eta_i} G(i) \subset \mathcal{E}$ so can let $\ker(\eta)(i) = \ker(\eta_i)$

Functoriality:

$$\begin{array}{ccc} i & & \ker \eta_i \xrightarrow{\iota} F(i) \xrightarrow{\eta_i} G(i) \\ \downarrow \alpha & & \downarrow \tilde{\alpha} \qquad \downarrow F(\alpha) \qquad \downarrow G(\alpha) \\ j & & \ker \eta_j \to F(j) \xrightarrow[\eta_j]{} G(j) \end{array}$$

we have $\eta_j \circ F(\alpha) \circ i$
$\qquad \qquad \qquad \|$
$G(\alpha) \eta_i \circ i = 0$

↳ univ prop of kernel gives the result: $\ker(\eta)(\alpha) = \tilde{\alpha}$.

check details + check cokernel + $\mathrm{Coim} f \to \mathrm{Im} f$ is an iso since it is at every evaluation

(2) $Ch_\bullet(\mathcal{E}) \subseteq \mathrm{Fun}(\mathbb{Z}, \mathcal{E})$ hence $f : C_\bullet \to D_\bullet$ has a kernel and cokernel in $\mathrm{Fun}(\mathbb{Z}, \mathcal{E})$ then check that $\ker(f)$ is a chain complex + same for coker. ∎

(0) There is another definition of abelian category see Assem & D's

Rem (1) Abelian category has finite limits and colimits

(2) $f \in \mathcal{E}$ $\mathcal{E}$ abelian $\begin{cases} \ker f = 0 \Leftrightarrow f \text{ mono} \\ \operatorname{Coker} f = 0 \Leftrightarrow f \text{ epi} \end{cases}$

(3) $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $gf = 0$ in an abelian category.

then we get

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \arrow[r, "g"] & Z \\ \operatorname{Coim} f \arrow[r, "\bar{f}"'] & \operatorname{Im} f \arrow[u, "\alpha"'] \end{tikzcd}$$

so $gf = 0 \Rightarrow g \circ \alpha \circ \underbrace{\bar{f} \circ \pi}_{\text{epi}} = 0$

$\Rightarrow g \circ \alpha = 0$

So
$$\begin{tikzcd} \ker g \arrow[r] & Y \arrow[r, "g"] & Z \\ & \operatorname{Im} f \arrow[ul, dashed, "\exists ! \gamma"] \arrow[u, "\alpha"'] \end{tikzcd}$$

↳ if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a complex, there is a canonical map $\operatorname{Im}(f) \longrightarrow \ker(g)$

Def 5.4 (i) A complex $X^{\bullet} \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if the canonical map $\operatorname{Im}(f) \to \ker(g)$ is an isomorphism.

(ii) More generaly a complex $(C_\bullet, d_\bullet)$ is exact if $\operatorname{Im}(d_i) \xrightarrow{\sim} \ker(d_{i+1})$ $\forall i \in \mathbb{Z}$

(iii) A short exact sequence is an exact complex of the form

$$0 \xrightarrow{0} X^{\bullet} \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{0} 0$$

Rem 5.5 There is a theorem (difficult) of Freyd-Mitchell that says that any abelian category (small) can be ~~embedded~~ seen as a full subcategory of the category A-Mod for some ring A in such a way that the abelian structure is induced by the one of A-Mod

↳ To simplify we will be sloppy and think that we work with modules!
Most of the arguments work in the generality but require extra care.

Prop 5.6 $0 \to X' \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ (*) be a ses in an abelian category

TFAE

(1) $\exists\, s : Z \to Y$; $gs = 1_Z$

(2) $\exists\, R : Y \to X$; $Rf = 1_X$

(3) $\exists\, h : Y \to X \oplus Y$ s.t.

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \to & X & \to & X \oplus Z & \to & Z & \to & 0 \end{array}$$

soit un iso de complex

(4) (*) est homotope au complex nul.

Proof exercise.

Def 5.7 In this case we say that the sequence splits.

Ex In $Vect_K$ all sequences split. What about $\mathbb{Z}$ Mod?

Def 5.8 $F : \mathcal{E} \to \mathcal{D}$ be a functor between abelian categories. Then

(1) F is left exact if it commutes with finite limits

(2) right — colimits

(3) F is exact if it is left and right exact

Lem 5.9 $F : \mathcal{E} \to \mathcal{D}$ be an additive functor between abelian categories

TFAE (1) F left exact

(2) F commutes with kernels : $f : X \to Y$, $F(\ker f) \xrightarrow{\sim} \ker(F(f))$

(3) $\forall\ 0 \to X' \xrightarrow[\alpha]{} X \xrightarrow[\beta]{} X'' \in \mathcal{E}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact

(4) $\forall\ 0 \to X \to Y \to Z \to 0$ exact the sequence $0 \to F(X) \to F(Y) \to F(Z)$ exact

\+ dual for right exact.

Proof maybe skip? $\leftarrow$ GES

(1) $\Rightarrow$ (2) clear since kernel is an equalizer hence a limit.

(2) $\Rightarrow$ (1) F is additive hence commutes with finite $\oplus$
F preserves kernel hence preserves $eq(f,g) = \ker(f-g)$ so it preserves finite limits with chapter 2.

(2) $\Rightarrow$ (3) : F preserves kernel, hence mono so $F(\alpha)$ is a mono.

~~mon~~ and we have $\boxed{\alpha = \ker(\beta)}$

Coro 5.10 TFAE for an additive functor

(1) F exact

(2) $\forall\ 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ ses in $\mathcal{A}$, the sequence $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$ is exact.

Prop 5.11 (1) $\mathcal{E}$ be an abelian category. Then $\mathrm{Hom}_{\mathcal{E}}(-,-) : \mathcal{E}^{op} \times \mathcal{E} \to \mathrm{Mod}\,\mathbb{Z}$ is left exact in each variable

(2) $- \otimes_A - : \mathrm{Mod}\,A \times A\mathrm{Mod} \to \mathbb{Z}\mathrm{Mod}$ is right-exact in each variable

(3) F is a left adjoint of G, then F is right-exact and G left-exact

Proof (3) is clear from chapter 2, (2) follows by (3) + Cartan formula

(1) was also proved (in TD?). Can also be easily checked by hand (Do it!) $\square$

Rem write exactness on the left and right for contravariant functors!

2. Chain complexes in an abelian category.

Def 5.12 $\mathcal{A}$ be an abelian category $(X_\bullet, d_\bullet) \in Ch_\bullet(\mathcal{A})$. Then for $n \in \mathbb{Z}$

let : $Z_n(X) = \ker(d_n^X)$ n-cycles

$B_n(X) = \mathrm{Im}(d_{n+1}^X)$ n-boundaries

$H_n(X) = Z_n(X) / B_n(X)$ $\boxed{n^{th} \text{ homology of } X_\bullet}$

when cochains: speak about cocycles, coboundaries and cohomology.

If $X_\bullet \xrightarrow{f} Y_\bullet$ is a morphism then

$$\begin{tikzcd}
\ker d_n^X \arrow[r, "\alpha"] \arrow[d, dashed, "\exists!\, f"] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & X_{n-1} \arrow[d, "f_{n-1}"] \\
\ker d_n^Y \arrow[r] & Y_n \arrow[r, "d_n^Y"'] & Y_{n-1}
\end{tikzcd}$$

- by univ property of $\ker d_n^X$
- by univ property of $\mathrm{Im}\, d_n^Y$

f_n restricted as a morphism from $B_n^{(X)}$ to $B_n(Y)$

and

$\hookrightarrow$ Get $H_n(f) : H_n(X) \to H_n(Y)$

$[x] \mapsto [f(x)]$

we have: H_n functor and $H_n(X \oplus Y) \simeq H_n(X) \oplus H_n(Y)$. Hence it is an additive functor $H_n : Ch_\bullet(\mathcal{A}) \to \mathcal{A}$.

Def 5.13 A morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a quasiisomorphism ($=$ qis) if $H_n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.

Prop 5.14 Let $\mathcal{A}$ be an abelian category and $f, g : X_\bullet \to Y_\bullet \in Mor(Ch_\bullet(\mathcal{A}))$

(1) If f and g are homotopic, then $H_n(f) = H_n(g)$ $\forall n \in \mathbb{Z}$

(2) If two chain complexes are homotopic, then they are quasi isomorphic

Proof $f \sim 0$ if $f = ds + sd$ so $f_* = H_n(f) = [ds + \underbrace{sd}_{0}] = [ds] \in Im(d)$ hence $= 0$. $\square$

Def 5.15 (1) $C_\bullet$ is contractible if C is homotopy equivalent to 0

(2) $C_\bullet$ is acyclic if C is qis to 0

↳ Contractible $\Rightarrow$ acyclic (obviously see TD).

Recall that a short exact sequence of complexes $0 \to C'_\bullet \xrightarrow{\alpha} C_\bullet \xrightarrow{\beta} C''_\bullet \to 0$ is the data of two morphisms of chain complexes α, β s.t. $\forall n$ $0 \to C'_n \to C_n \to C''_n \to 0$ is short exact.

↳
$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "\alpha_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C'_n \arrow[r, "d'_n"] \arrow[d, "\alpha_n"] \arrow[dr, phantom, "\circlearrowleft"] & C'_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\beta_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n \arrow[r, "d_n"] \arrow[d, "\beta_n"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[r] \arrow[d, "\beta_{n-1}"] & \cdots \\
\cdots \arrow[r] & C''_{n+1} \arrow[r, "d''_{n+1}"] \arrow[d] & C''_n \arrow[r, "d''_n"] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

injective

surjective

Thm 5.16 [long exact sequence] A short exact sequence $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ gives rise to a long exact sequence

$$\to H_n(C') \xrightarrow{H_n(\alpha)} H_n(C) \xrightarrow{H_n(\beta)} H_n(C'') \xrightarrow{\delta_n} H_{n-1}(C') \to \cdots$$

<u>Proof</u> (1) Construction of δ: diag chasing + check details

or (2) use snake lemma

<u>Consider</u>

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \ker d_n' \arrow[r] \arrow[d] & \ker d_n \arrow[r] \arrow[d] & \ker d_n'' \arrow[d] & \\
0 \arrow[r] & C_n' \arrow[r] \arrow[d, "d_n'"] & C_n \arrow[r] \arrow[d, "d_n"] & C_n'' \arrow[r] \arrow[d, "d_n''"] & 0 \\
0 \arrow[r] & C_{n-1}' \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & C_{n-1}'' \arrow[r] \arrow[d] & 0 \\
& \mathrm{Coker}(d_n') \arrow[r] \arrow[d] & \mathrm{Cok}(d_n) \arrow[r] \arrow[d] & \mathrm{Cok}(d_n'') \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \quad (\circledast)$$

and consider: $\overline{d_n} : C_n/_{\mathrm{Im}(d_{n+1})} \to \mathrm{Im}(d_n) \subseteq \ker(d_{n-1})$

leads to:

$$\begin{tikzcd}
& C_n'/_{\mathrm{Im}(d_{n+1}')} \arrow[r, "\alpha_n"] \arrow[d, "\overline{d_n'}"'] \arrow[dr, phantom, "(1)"] & C_n/_{\mathrm{Im}(d_{n+1})} \arrow[r, "\beta_n"] \arrow[d, "d_n"] \arrow[dr, phantom, "(2)"] & C_n''/_{\mathrm{Im}(d_{n+1}'')} \arrow[r] \arrow[d, "d_n''"] & 0 \\
0 \arrow[r] & \ker(d_{n-1}') \arrow[r, "\alpha_{n-1}"'] & \ker(d_{n-1}) \arrow[r, "\beta_{n-1}"'] & \ker(d_{n-1}'') &
\end{tikzcd} \quad (**)$$

exactitude follow from snake lemma applied to $\circledast$.

Now $\ker(\overline{d_n}) = \dfrac{\ker(d_n)}{\mathrm{Im}(d_{n+1})} = H_n(C)$

$\mathrm{Coker}(\overline{d_n}) = \dfrac{\ker(d_{n-1})}{\mathrm{Im}(d_n)} = H_{n-1}(C)$

so snake lemma applied to $(**)$ gives the result: $\boxed{\delta = \alpha_{n-1}^{-1} \circ d_n \circ \beta_n^{-1}}$ □

<u>Rem</u> δ is natural in ses!

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "h"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"'] & B' \arrow[r, "\beta'"'] & C' \arrow[r] & 0
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_n(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_n(C) \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft \ \otimes"] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\delta'"'] & H_{n-1}(A') &
\end{tikzcd}$$

just saying that H_n functor

$\otimes$ follows from □. □

3. Projective, injective ~~and flat~~ objects

Def 5.17 $\mathcal{E}$ be an abelian category. Then an object

(1) $I \in ob(\mathcal{E})$ is injective if $Hom_{\mathcal{E}}(\bullet, I): \mathcal{E}^{op} \to Ab$ is exact

(2) $P \in ob(\mathcal{E})$ is projective if $Hom_{\mathcal{E}}(P, \bullet): \mathcal{E} \to Ab$ is exact

(3) $\mathcal{E}$ has enough projectives (injectives) if $\forall X \in ob(\mathcal{E}), \exists f: P \twoheadrightarrow X$ epi with P proj ($g: X \hookrightarrow I$ mono with I injective)

Prop 5.18 (1) $I \in \mathcal{E}$ is injective iff $\forall$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[dl, dashed, "\exists \bar{\alpha}"] \\ & I & \end{tikzcd}$$

$\exists$; $\alpha = \bar{\alpha} f$

(2) $P \in \mathcal{E}$ is projective iff $\forall$

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "\exists \bar{f}"'] & \\ M \arrow[r, two heads, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

; $f = g \circ \bar{f}$

Proof (2) P proj iff $Hom_{\mathcal{E}}(P, -)$ is exact

iff $Hom_{\mathcal{E}}(P, -)$ is right exact

iff $\forall$ $M \xrightarrow{g} N \to 0$ ex. the induced sequence

$Hom(P, M) \xrightarrow{g_*} Hom(P, N) \to 0$ is exact

iff ~~that means~~ f lifts along g.

(1) is dual. $\square$

Thm 5.19 $\mathcal{E}$ be an abelian category.

(1) Let $(P_\lambda)_{\lambda \in \Lambda}$ be a family of objects of $\mathcal{E}$. Then $\coprod_{\lambda \in \Lambda} P_\lambda$ is projective iff P_λ is projective $\forall \lambda$

(2) P is projective iff $\forall$ $X \overset{f}{\twoheadrightarrow} P$ epi $\exists s: P \to X$; $fs = 1_P$.

+ dual for injective

Proof (1) just play with univ property of coproduct.

(2) If P proj look at

$$\begin{tikzcd} & P \arrow[d, "1"] \arrow[dl, dashed] \\ X \arrow[r, two heads] & P \end{tikzcd}$$

Conversely: