

Prop/Def 4.18 Soit \mathcal{E} une catégorie additive, la catégorie homologique de \mathcal{E} , noté $k(\mathcal{E})$ a pour objet: les objets de $Ch(\mathcal{E})$

morphismes: $Hom_{k(\mathcal{E})}(X, Y) \cong$

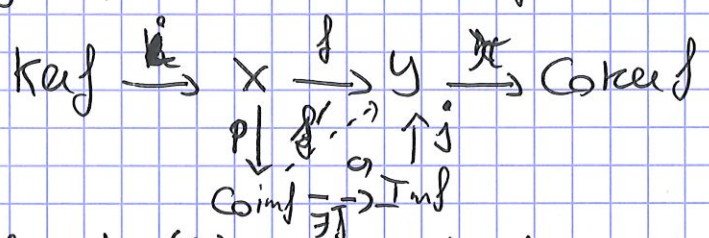
Démo il faut vérifier que c'est une catégorie: on utilise le lemme 4.17 \square

Rem (1) on a des versions bornées de $k(\mathcal{E})$: $k_+(E), k_-(E), k_b(E)$
 (2) $k(\mathcal{E})$ est un exemple catégorie triangulée.

Chapitre 5 Catégories abéliennes

\mathcal{E} being additive category in which every morphism has a kernel and a cokernel ($ker(f) = eq(f, 0)$ $Coker(f) = coeq(f, 0)$)

\hookrightarrow every morphism has a canonical factorization

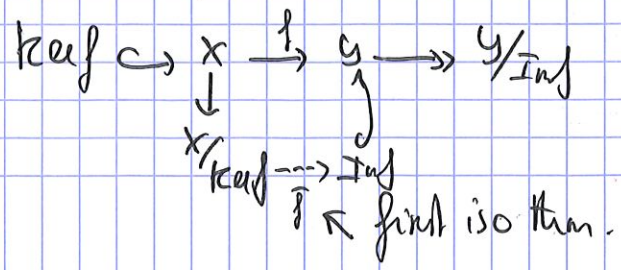


Let $Im f := ker(\pi)$ where $\pi: Y \rightarrow Coker f$ is a coker of f
 $Coim f := Coker(i)$ — $i: ker f \hookrightarrow X$ is a kernel of f

On a $f \circ i = 0$ donc par la pte universelle de $Coim f \exists f': Coim f \rightarrow Y$
 tq $f' \circ p = f$

Alors $\pi \circ f' \circ p = \pi \circ f = 0$ et π epi donne $\pi \circ f' = 0$. Donc $\exists j: Coim f' \rightarrow Im f$ qui fait commuter le carré via la pte universelle du noyau.

Exemple $\mathcal{E} = AMod$



Def 5.1 Soit \mathcal{E} une catégorie additive. Alors \mathcal{E} est abélienne si

(1) tout morphisme de \mathcal{E} admet un noyau et un conoyau

(2) $\forall f: X \rightarrow Y$ l'application $\bar{f}: \text{Coim} f \rightarrow \text{Im} f$ est un isomorphisme

Ex 5.2 (1) A est un anneau, alors $A\text{-Mod}$ est abélienne. Si A est un anneau, $A\text{-Mod}$ est abélienne.

(2) Il ya des catégories avec (1) mais pas (2) par exemple la catégorie des groupes abéliens topologiques séparés (T_2 \odot \odot) voir AMM page 279.

(3) \mathcal{E} est un \mathcal{E}^{op} abélien

Prop 5.3 (1) Soit \mathcal{E} une catégorie abélienne et \mathcal{I} (small) catégorie. Alors $\text{Fun}(\mathcal{I}, \mathcal{E})$ est abélien

(2) The categories of chain complexes and cochain complexes $\text{Ch}_0(\mathcal{E})$ and $\text{Ch}^0(\mathcal{E})$ are abelian

Proof (1) Sketch $F \xrightarrow{\eta} G$ nat maps $\ker(\eta) = ?$

Skip $\forall i \in \mathcal{I}$ $F(i) \xrightarrow{\eta_i} G(i) \in \mathcal{E}$ so can let $\ker(\eta)(i) = \ker(\eta_i)$

Fundamental:

$$\begin{array}{ccc} i & \xrightarrow{\eta_i} & F(i) \xrightarrow{\eta_i} G(i) \\ \downarrow \alpha & & \downarrow F(\alpha) \quad \downarrow G(\alpha) \\ j & \xrightarrow{\eta_j} & F(j) \xrightarrow{\eta_j} G(j) \end{array}$$

we have

$$\eta_j \circ F(\alpha) = G(\alpha) \circ \eta_i$$

$$G(\alpha) \circ \eta_i \circ \alpha = 0$$

\hookrightarrow univ prop of kernel gives the result: $\ker(\eta)(\alpha) = \alpha$

check details + check cokernel + $\text{Coim} f \rightarrow \text{Im} f$ is an iso since it is at every evaluation

(2) $\text{Ch}_0(\mathcal{E}) \subseteq \text{Fun}(\mathbb{Z}, \mathcal{E})$ hence $f: C_0 \rightarrow D_0$ has a kernel and cokernel in $\text{Fun}(\mathbb{Z}, \mathcal{E})$ then check that $\ker(f)$ is a chain complex + same for coker. \square

(2) There is another definition of abelian category see Assem $\in \mathbb{D}_5$ (21)

Rem

(1) Abelian category has finite limits and colimits

(2) $f \in \mathcal{E}$ Abelian $\begin{cases} \ker f = 0 \Leftrightarrow f \text{ mono} \\ \operatorname{coker} f = 0 \Leftrightarrow f \text{ epi} \end{cases}$

(3) $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $gf = 0$ in an abelian category.

then we get

$$\begin{array}{ccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \text{so } gf=0 \Rightarrow g \circ \alpha \circ \underbrace{\beta \circ \pi}_{\text{epi}} = 0 \\ \pi \downarrow & & \uparrow \alpha & & & \Rightarrow g \circ \alpha = 0 \\ \operatorname{Coker} f & \xrightarrow{\beta} & \operatorname{Im} f & & & \end{array}$$

$$\text{So } \operatorname{ker} g \rightarrow Y \xrightarrow{g} Z \rightsquigarrow$$

$$\begin{array}{ccc} & \uparrow \alpha & \\ \operatorname{Im} f & \xrightarrow{\beta} & \operatorname{Im} g \end{array}$$

\hookrightarrow if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a complex, there is a canonical map $\operatorname{Im}(f) \rightarrow \operatorname{ker}(g)$

Def 5.4 (i) A complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if the canonical map $\operatorname{Im}(f) \rightarrow \operatorname{ker}(g)$ is an isomorphism.

(ii) More generally a complex (C_i, d_i) is exact if $\operatorname{Im}(d_i) \xrightarrow{\cong} \operatorname{ker}(d_{i+1})$
 $\forall i \in \mathbb{Z}$

(iii) A short exact sequence is an exact complex of the form

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

Rem 5.5 There is a theorem (difficult) of Freyd-Mitchell that says that any abelian category (small) can be ~~embedded~~ seen as a full subcategory of the category $A\text{-Mod}$ for some ring A in such a way that the abelian structure is induced by the one of $A\text{-Mod}$.

\hookrightarrow To simplify we will be sloppy and think that we work with modules!

Most of the arguments work in the generality but require extra care.

Prop 5.6 $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be a SES in an abelian category

VFAE

(1) $\exists \sigma: Z \rightarrow Y; g\sigma = 1_Z$

(2) $\exists R: Y \rightarrow X; Rf = 1_X$

(3) ~~exists isomorphism~~ $\exists h: X \rightarrow X \oplus Y$ s.t.

$$\begin{array}{ccccccc}
 0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z \rightarrow 0 \\
 & & \parallel & & \downarrow h & & \parallel \\
 0 & \rightarrow & X & \rightarrow & X \oplus Y & \rightarrow & Z \rightarrow 0
 \end{array}$$

Soit un iso de complex

(4) ~~at hom~~ at homotopy an complex nul.

Proof exercise.

Def 5.7 In this case we say that the sequence splits.

Ex In Vect_K all sequences split. What about $\mathbb{Z} \text{ Mod}$?

Def 5.8 $F: \mathcal{E} \rightarrow \mathcal{D}$ be a functor between abelian categories. Then

- (1) F is left exact if it commutes with finite limits
- (2) F is right exact if it commutes with finite colimits
- (3) F is exact if it is left and right exact

Lems 9 $F: \mathcal{E} \rightarrow \mathcal{D}$ be an additive functor between abelian categories

VFAE (1) F left exact

(2) F commutes with kernels: $f: x \rightarrow y, F(\ker f) \xrightarrow{\cong} \ker(Ff)$

(3) $\forall 0 \rightarrow x \xrightarrow{\alpha} x' \xrightarrow{\beta} x'' \in \mathcal{E}$, the sequence $0 \rightarrow F(x) \rightarrow F(x') \rightarrow F(x'')$

is exact

(4) $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ exact the sequence

$0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z)$ exact

+ dual for right exact.

Proof maybe skip? \Leftarrow YES

(1) \Rightarrow (2) clear since kernel is an equalizer hence a limit.

(2) \Rightarrow (4) F is additive hence commutes with finite \oplus

F preserves kernel hence preserves eq $f \circ g = \ker(f \circ g)$ so it preserves finite limits with chapter 2.

(2) \Rightarrow (3): F preserves kernel, hence mono so $F(\alpha)$ is a mono.

mono and we have $\alpha \circ \ker(\beta)$

Cor 5.10 \forall FAE for an additive functor

(1) F exact

(2) $\forall 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ res in \mathcal{A} , the sequence $0 \rightarrow F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \rightarrow 0$ is exact.

Prop 5.11 (1) \mathcal{E} be an abelian category. Then $\text{Hom}_{\mathcal{E}}(-, -): \mathcal{E}^{\text{op}} \times \mathcal{E} \rightarrow \text{Mod } \mathcal{A}$ is left exact in each variable

(2) $\text{Hom}_{\mathcal{E}}(-, -): \text{Mod } \mathcal{A} \times \text{Mod } \mathcal{E} \rightarrow \text{Mod } \mathcal{A}$ is right-exact in each variable

(3) F is a left adjoint of G, then F is right-exact and G left-exact

Proof (3) is clear from chapter 2, (2) follows by (3) + Cartan formula

(1) was also proved (in TD?). Can also be easily checked by hand (Do it!) \square

Rem write exactness on the left and right for contravariant functors!

2. Chain complexes in an abelian category.

Def 5.12 \mathcal{A} be an abelian category $(x_n, d_n) \in \text{Ch}_0(\mathcal{A})$. Then for $n \in \mathbb{Z}$

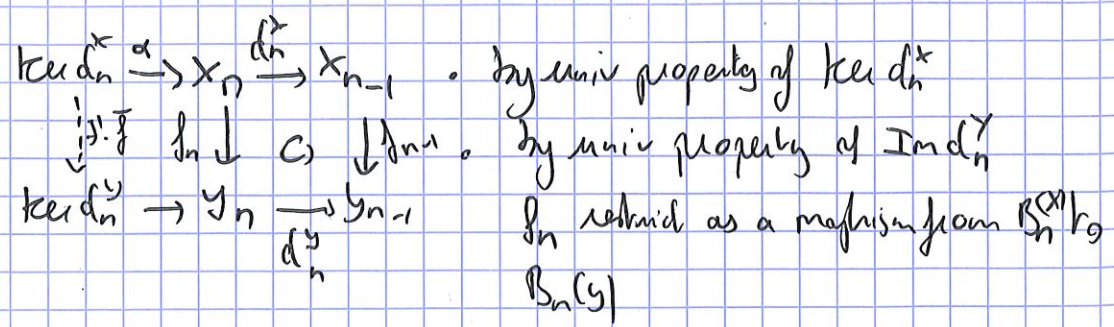
let: $Z_n(x) = \ker(d_n)$ n -cycles

$B_n(x) = \text{Im}(d_{n+1})$ n -boundaries

$H_n(x) = Z_n(x) / B_n(x)$ n^{th} homology of x_0

When Cochains: speak about cocycles, coboundaries and cohomology.

$\exists f: x_0 \rightarrow y_0$ is a morphism then



and

\hookrightarrow Get $H_n(f): H_n(x) \rightarrow H_n(y)$
 $[x] \mapsto [f(x)]$

we have: H_n functor and $H_n(x \oplus y) \simeq H_n(x) \oplus H_n(y)$. Hence it is an additive functor $H_n: \text{Ch}_0(\mathcal{A}) \rightarrow \mathcal{A}$.

Def 5.13 A morphism $f_0: X_0 \rightarrow Y_0$ is a quasi-isomorphism (= qis) if $H_n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.

Prop 5.14 Let \mathcal{A} be an abelian category and $f, g: X_0 \rightarrow Y_0 \in \text{Mor}(\text{Ch}_0(\mathcal{A}))$

(1) If f and g are homotopic, then $H_n(f) = H_n(g) \forall n \in \mathbb{Z}$

Proof (2) If two chain complexes are homotopic, then they are quasi-isomorphic. Two if $f = ds + ad$ so $f_n = H_n(f) = [ds + ad] = [d_n] \in \text{Im}(f)$ hence $= 0$.

Def 5.15 (1) C_0 is contractible if C is homotopy equivalent to 0

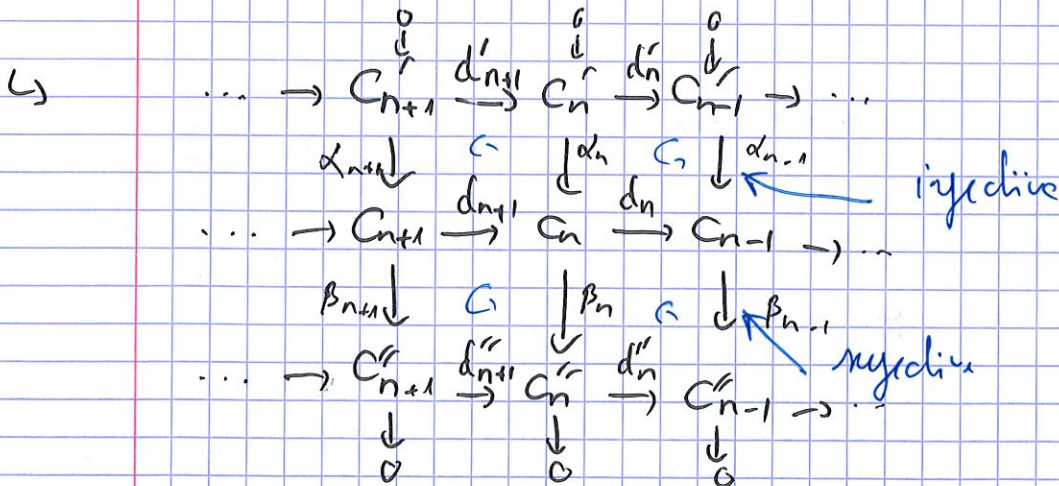
(2) C_0 is acyclic if C is qis to 0

\hookrightarrow Contractible \Rightarrow acyclic (obviously see (2)).

Recall that a short exact sequence of complexes $0 \rightarrow C'_0 \xrightarrow{\alpha} C_0 \xrightarrow{\beta} C''_0 \rightarrow 0$

is the data of two morphisms of chain complexes α, β s.t. $\forall n$

$0 \rightarrow C'_n \rightarrow C_n \rightarrow C''_n \rightarrow 0$ is short exact.



Thm 5.16 [Long exact sequence] A short exact sequence $0 \rightarrow C'_0 \xrightarrow{\alpha} C_0 \xrightarrow{\beta} C''_0 \rightarrow 0$

gives rise to a long exact sequence

$$\rightarrow H_n(C') \xrightarrow{H_n(\alpha)} H_n(C) \xrightarrow{H_n(\beta)} H_n(C'') \xrightarrow{\delta_n} H_{n-1}(C) \rightarrow \dots$$

Proof (1) Construction of \mathcal{S} : diag chaining + check details
 as (2) use snake lemma

Consider

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \ker d_n' & \rightarrow & \ker d_n & \rightarrow & \ker d_n'' & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & C_n' & \rightarrow & C_n & \rightarrow & C_n'' & \rightarrow 0 \\
 & \downarrow d_n' & & \downarrow d_n & & \downarrow d_n'' & \\
 0 \rightarrow & C_{n-1}' & \rightarrow & C_{n-1} & \rightarrow & C_{n-1}'' & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \text{Coker}(d_n') & \rightarrow & \text{Coker}(d_n) & \rightarrow & \text{Coker}(d_n'') & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

(*)

and consider: $\bar{d}_n : C_n / \text{Im}(d_{n+1}) \rightarrow \text{Im}(d_n) \subseteq \ker(d_{n+1})$

leads to

$$\begin{array}{ccccccc}
 C_n / \text{Im}(d_{n+1}) & \xrightarrow{\alpha_n} & C_n / \text{Im}(d_n) & \xrightarrow{\beta_n} & C_n'' / \text{Im}(d_{n+1}) & \rightarrow & 0 \\
 \bar{d}_n \downarrow & & \downarrow d_n & & \downarrow d_n'' & & \\
 0 \rightarrow & \ker(\bar{d}_n) & \xrightarrow{\alpha_{n-1}} & \ker(d_{n-1}) & \xrightarrow{\beta_{n-1}} & \ker(d_{n-1}'') &
 \end{array}$$

(**)

exactitude follow from snake lemma applied to (*).

Now $\ker(\bar{d}_n) = \frac{\ker(d_n)}{\text{Im}(d_{n+1})} = H_n(C)$

$\text{Coker}(\bar{d}_n) = \frac{\ker(d_{n+1})}{\text{Im}(d_n)} = H_{n-1}(C)$

so snake lemma applied to (**) gives the result: $\boxed{\mathcal{S} = \alpha_{n-1}' \circ \bar{d}_n \circ \beta_n^{-1}}$ \square

Rem \mathcal{S} is natural in ses!

$$\begin{array}{ccccccc}
 0 \rightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \rightarrow 0 \\
 & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \beta & \\
 0 \rightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \rightarrow 0
 \end{array}$$

ms

$$\begin{array}{ccccccc}
 H_n(A) & \rightarrow & H_n(B) & \rightarrow & H_n(C) & \xrightarrow{\mathcal{S}} & H_{n-1}(B) \rightarrow \dots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_n(A') & \rightarrow & H_n(B') & \rightarrow & H_n(C') & \rightarrow & H_{n-1}(A')
 \end{array}$$

just seeing that H_n functors

(*) follows from \square

\square

3. Projective, injective ~~and~~ objects

Def 5.17 \mathcal{E} be an abelian category. Then an object

- (1) $I \in \text{ob}(\mathcal{E})$ is injective if $\text{Hom}_{\mathcal{E}}(\cdot, I): \mathcal{E}^{\text{op}} \rightarrow \text{Ab}$ is exact
- (2) $P \in \text{ob}(\mathcal{E})$ is projective if $\text{Hom}_{\mathcal{E}}(P, \cdot): \mathcal{E} \rightarrow \text{Ab}$ is exact
- (3) \mathcal{E} has enough projectives (injectives) if $\forall X \in \text{ob}(\mathcal{E}), \exists f: P \rightarrow X$ epi with P proj ($g: X \rightarrow I$ mono with I injective)

Prop 5.18 (1) $I \in \mathcal{E}$ is injective iff \forall $\begin{array}{ccc} \alpha & X & \xrightarrow{f} Y \\ & \alpha \downarrow & \downarrow g \\ & I & \xleftarrow{f'} Z \end{array} \quad \exists \beta: \alpha = \beta f'$

(2) $P \in \mathcal{E}$ is projective iff \forall $\begin{array}{ccc} & P & \\ \downarrow f' & & \downarrow f \\ M & \rightarrow & N \rightarrow 0 \end{array} \quad ; f = g \circ f'$

Proof (2) P proj iff $\text{Hom}_{\mathcal{E}}(P, -)$ is exact
 iff $\text{Hom}_{\mathcal{E}}(P, -)$ is right exact
 iff $\forall M \rightarrow N \rightarrow 0$ ex. the induced sequence
 $\text{Hom}(P, M) \rightarrow \text{Hom}(P, N) \rightarrow 0$ is exact
 iff the ~~image~~ f lifts \uparrow along g .

(1) is dual. □

Thm 5.19 \mathcal{E} be an abelian category.

- (1) Let $(P_{\lambda})_{\lambda \in \Lambda}$ be a family of objects of \mathcal{E} . Then $\coprod_{\lambda \in \Lambda} P_{\lambda}$ is projective iff P_{λ} is projective $\forall \lambda$
- (2) P is projective iff $\forall X \xrightarrow{f} P$ epi $\exists \Delta: P \rightarrow X; f \Delta = 1_P$
 + dual for injective

Proof (1) just play with univ property of coproduct.

(2) If P proj look at $\begin{array}{ccc} & & \leftarrow f \\ & & \downarrow \\ X & \xrightarrow{f} & P \end{array}$

Conversely: