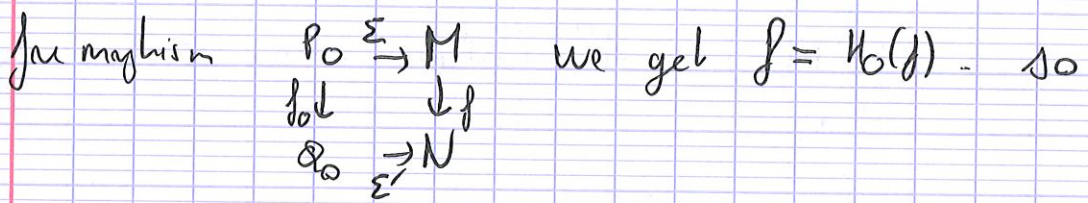


(b) -  $H_i(P_M) = 0$  for  $i \neq 0$  because of exactness

-  $H_0(P_M) = \ker(d_0) / \text{Im}(d_1) = P_0 / \text{Im}(d_1) \cong M$  nice

the complex  $\dots \rightarrow P_0 \xrightarrow{\epsilon} M \rightarrow 0$  is exact



we have  $H_0 \circ P_1 \cong \text{Id}_M$  □

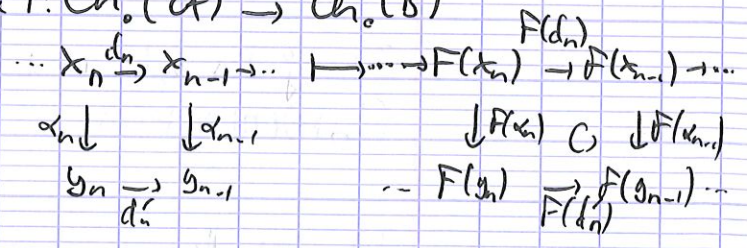
Rq

It is easy to see that  $M$  is quasi-isomorphic to  $P_M$  when  $P_M$  is a projective resolution of  $M$ . But  $M$  is not homotopy equivalent to  $P_M$ .

no It is better to work in a category where  $\text{qis}$  become isomorphisms  
 $\hookrightarrow$  The derived category.

If  $F: \mathcal{A} \rightarrow \mathcal{B}$  is an additive functor between abelian categories

then  $F$  induces a functor  $F: \text{Ch}_0(\mathcal{A}) \rightarrow \text{Ch}_0(\mathcal{B})$



and also a functor from  $F: \mathcal{K}_0(\mathcal{A}) \rightarrow \mathcal{K}_0(\mathcal{B})$

Def 5.26 Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor from  $\mathcal{A}$  to  $\mathcal{B}$  two ~~are~~ abelian categories such that  $\mathcal{A}$  has enough projectives.

The  $n$ th left derived functor  $L_n F$  of  $F$  is the composite

$$\mathcal{A} \xrightarrow{P_R} \mathcal{K}_+(\text{Proj } \mathcal{A}) \xrightarrow{F} \mathcal{K}_+(\mathcal{B}) \xrightarrow{H_n} \mathcal{B}$$

Concretely:  $L_n F(X) = H_n(F(P_X))$  where  $P_X$  is a projective resolution of  $X$  and  $L_n F(Y) = H_n(F(Y))$   $\kappa$  induced on Proj  $R^F$

Def 5.24 If  $\mathcal{A}$  has enough injectives, the  $n$ th right derived functor of  $F$  is the composite  $\mathcal{A} \xrightarrow{I_X} \mathcal{K}^+(\text{Inj } \mathcal{A}) \xrightarrow{F} \mathcal{K}^+(B) \xrightarrow{H^n} B$   $R^n F$

$\hookrightarrow R^n F(X) = H^n(F(I_X))$  where  $I_X$  is an injective resolution of  $X$ .

Thm 5.28 We assume that  $\mathcal{A}$  has enough projectives.

(1)  $L_n^R F$  is additive  $\forall n \geq 0$ .

(2) If  $F$  is right exact then  $L_0^R F \cong F$  and  $F$  is right exact

(3) If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a ses in  $\mathcal{A}$  then we have a long exact sequence

$$\dots \rightarrow L_2^R F(C) \rightarrow L_1^R F(A) \rightarrow L_1^R F(B) \rightarrow L_1^R F(C) \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

(4)  $L_n^R F(P) = 0 \forall P$  projective

Moreover this is natural in the ses.

sketch of proof: (1) One has to check that  $L_n^R F$  is additive

(2) If  $F$  is right exact  $F(P_0) \rightarrow F(M) \rightarrow 0$  is exact hence

$$(L_0^R F)(M) \cong F(M)$$

$\pi = \text{coker}(d)$   $F$  right exact

$$\text{give } P \otimes M = \text{coker}(F(d)) = L_0^R F(M)$$

(3) (a) there is a short exact sequence  $0 \rightarrow P_0 \rightarrow Q_0 \rightarrow R_0 \rightarrow 0$  of complexes of projective objects and morphisms

$$0 \rightarrow P_0 \rightarrow Q_0 \rightarrow R_0 \rightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

where each vertical map is a projective resolution and each row is exact.

(So called Horseshoe Lemma)

if  $0 \rightarrow P_i \rightarrow Q_i \rightarrow R_i \rightarrow 0$  is exact with projective then it splits: so need to choose proj resolutions for  $A$  and  $C$  then the middle term is  $P_i \oplus R_i$   $\square$

since  $F$  is additive and each rows of  $0 \rightarrow P_i \rightarrow Q_i \rightarrow R_i \rightarrow 0$  splits the sequence  $0 \rightarrow F(P_i) \rightarrow F(Q_i) \rightarrow F(R_i) \rightarrow 0$  is exact hence by the long exact sequence in homology we have

$$\begin{array}{ccccccc} \dots & \rightarrow & H_1 F(P_i) & \rightarrow & H_0 F(Q_i) & \rightarrow & H_0 F(R_i) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & L_1 F(A) & \rightarrow & L_0 F(B) & \rightarrow & L_0 F(C) \rightarrow 0 \\ \dots & \rightarrow & L_1 F(A) & \rightarrow & L_1 F(B) & \rightarrow & L_1 F(C) \rightarrow 0 \end{array}$$

(4) is clear since  $\dots \rightarrow P \rightarrow P \rightarrow 0$  is a projective resolution of  $P$   $\square$

Thm 5.29 If  $\mathcal{A}$  has enough injectives

- (1)  $R^n F$  is additive  $\forall n \geq 0$
- (2) If  $F$  is left exact then  $R^0 F = F$
- (3) If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is exact and  $F$  is left exact, then we have a long exact sequence.

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow R^1 F(A) \rightarrow R^1 F(B) \rightarrow R^1 F(C) \rightarrow R^2 F(A) \rightarrow \dots$$

Case of contravariant functors :  $F: \mathcal{E} \rightarrow \mathcal{D}$  contravariant  
 $\Leftrightarrow F: \mathcal{E}^{op} \rightarrow \mathcal{D}$  covariant

hence it is exact on the left if

$$\forall 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \text{ we have } 0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \text{ exact}$$

Moreover proj in  $\mathcal{E} \Leftrightarrow$  inj in  $\mathcal{E}^{op}$  so

Def 5.30 (1)  $F: \mathcal{A} \rightarrow \mathcal{B}$  contravariant exact on the left. Then

$$R^i F(A) = H^i(F(P)) \text{ where } P \text{ is a proj reso of } A$$

when  $F$  is right exact.

similarly  $\forall I_i \text{ inj}$  of  $B$ ,  $H_i(F(D)) = 0$  where  $I$  is an inj coresol of  $B$ .

## Chapter 6 Ext and Tor

Here we work with  $A \text{ Mod}$  or  $\text{Mod } A$  for  $A$   $k$ -alg

~~Ext~~

### 1) Deriving bifunctors

$F: E \times E' \rightarrow E''$  left exact functor between abelian categories, then for  $x \in E, y \in E'$  one can construct  $R^j F(x, -)$  and  $R^j F(-, y)$

Thm 6.1 Assume that  $\forall I$  inj of  $E$ , the functor  $F(I, \cdot): E \rightarrow E''$  is exact and  $\forall I'$  inj of  $E'$ ,  $F(\cdot, I'): E \rightarrow E''$  is exact.

Then  $\forall j \in \mathbb{Z}, \forall x \in E, \forall y \in E'$  there is a binatural isomorphism

$$R^j F(x, \cdot)(y) \cong R^j F(\cdot, y)(x)$$

Sketch

$0 \rightarrow x \rightarrow I_x^0$  inj resolutions

$0 \rightarrow y \rightarrow I_y^0$

apply  $F$  gives a double complex

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & 0 & \rightarrow & F(I_x^0, y) & \rightarrow & F(I_x^1, y) \rightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & F(x, I_y^0) & \rightarrow & F(I_x^0, I_y^0) & \rightarrow & F(I_x^1, I_y^0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & F(x, I_y^1) & \rightarrow & F(I_x^0, I_y^1) & \rightarrow & \dots
 \end{array}$$

cohomology of first row  $\cong R^j F(\cdot, y)(x)$

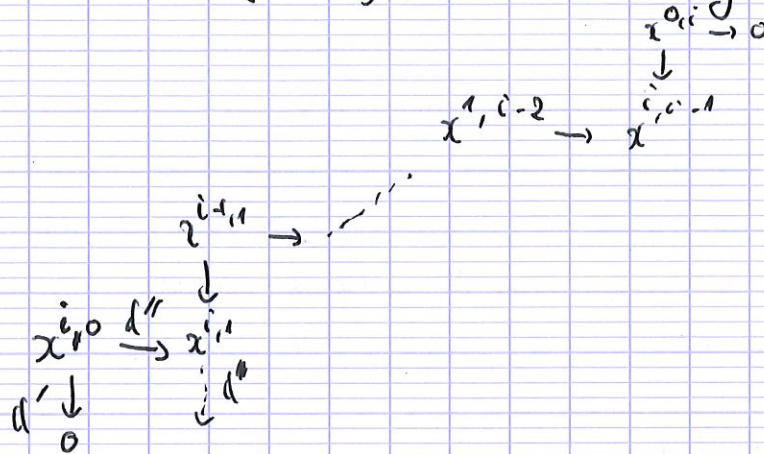
hit at  $\sim R^j F(x, \cdot)(y)$

$x^{0,0}$

$x^{0,0}$

all other rows and col are exact

show that  $H^i(X^{\bullet, \bullet}) \cong H^i(X^{\bullet, 0})$  by diag chasing



Start with  $x^{i, 0} \in \ker(d''^i)$  then  $x^{i, -1} = d''(x^{i, 0}) \in \ker(d')$   
 since double complex. Exactness lead to  $x^{i-1, -1}$ ;  $d'(x^{i-1, -1}) = x^{i, 0}$   
 we have  $d''(x^{i-1, -1}) = x^{i, -2} \in \ker(d')$  etc.  
 $\rightarrow x^{i, 0} \in \ker(d''^i)$  and this is the construction of the isomorphism.

For details see: Weibel § 2.7 + deal for left derived functors  $\square$

2) Funder Ext

Def 6.2 (1) If  $\mathcal{E}$  has enough projective (or enough injective)  
 we denote by  $\text{Ext}^i(A, B)$  the derived functors  
 $[R^i \text{Hom}(-, B)](A)$  (resp  $R^i \text{Hom}(A, -)[B]$ ).

(2) If  $\mathcal{E} = \text{Mod } A$   $\text{Ext}^i(A, B) \cong R^i \text{Hom}(-, B)(A) \cong R^i \text{Hom}(A, -)(B)$ .

This makes sense since  $\text{Hom}(-, B)$  satisfies the hypothesis of Thm 6.1.

Examples  $R = \mathbb{Z}$ ,  $M$  be an <sup>Torsion</sup> abelian group, then to compute  
 $\text{Ext}^i(M, \mathbb{Z})$  one can  
 - find an ~~injective~~ <sup>proj</sup> resolution of  $M$   
 - or find an ~~proj~~ <sup>inj</sup> injective coresolution of  $\mathbb{Z}$

We have  $0 \rightarrow \mathcal{Z} \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}/\mathcal{Z} \rightarrow 0$  exact with  $\mathcal{Q}$  ad  $\mathcal{Q}/\mathcal{Z}$  injective.

So we have a long exact sequence

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Hom}(\pi, \mathcal{Z}) & \rightarrow & \text{Hom}(\pi, \mathcal{Q}) & \rightarrow & \text{Hom}(\pi, \mathcal{Q}/\mathcal{Z}) \rightarrow \text{Ext}^1(\pi, \mathcal{Z}) \\
 & & \uparrow \cong & & \uparrow \cong & & \\
 & & 0 & & 0 & & \\
 & & \text{Ext}^1(\pi, \mathcal{Q}) & \rightarrow & \text{Ext}^1(\pi, \mathcal{Q}/\mathcal{Z}) & \rightarrow & \text{Ext}^2(\pi, \mathcal{Z}) \rightarrow \text{Ext}^2(\pi, \mathcal{Q}) \\
 & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
 & & 0 & & 0 & & 0
 \end{array}$$

$\hookrightarrow \begin{cases} \text{Ext}^1(\pi, \mathcal{Z}) \cong \text{Hom}(\pi, \mathcal{Q}/\mathcal{Z}) \\ \text{Ext}^i(\pi, \mathcal{Z}) = 0 \quad \forall i \neq 0 \end{cases}$

### Thm 6.3 (1) $\forall$ FAE $A$ mod

- 1  $P$  is projective
- 2  $\text{Hom}_A(P, -)$  is exact
- 3  $\forall i \geq 1, \forall B \in \text{Mod } A, \text{Ext}^i(P, B) = 0$
- 4  $\forall B \in \text{Mod } A, \text{Ext}^1(P, B) = 0$

(2) Dual for injective

Sketch (1)  $\Leftrightarrow$  (2) by def (2)  $\Rightarrow$  (3) clear (3)  $\Rightarrow$  (2) long exact sequence

(3)  $\Rightarrow$  (4) clear (4)  $\Rightarrow$  (2) long exact sequence

Ex  $\text{Ext}^i$  can be computed using  $\text{Ext}^1$  if  $i \geq 2$ .

### 3. The functors

Def 6.4 A  $k$ -alg  $E = \text{Mod } A$  we denote by  $\text{Tor}_i^A$  the  $i$ -th derived functor of  $-\otimes_A -$

that is  $\text{Tor}_i^A(M, N) = L_i(-\otimes_A N)(M) \cong L_i(M \otimes_A -)$