(b) - $H_i(P_M) = 0$ for $i \neq 0$ because of exactness

- $H_0(P_M) = \ker(0)/\mathrm{Im}(d_1) = P_0/\mathrm{Im}(f_1) \underset{\varepsilon}{\cong} M$ since

the complex $\cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$ is exact

for morphism
$$\begin{tikzcd} P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & M \arrow[d, "f"] \\ Q_0 \arrow[r, "\varepsilon'"'] & N \end{tikzcd}$$
we get $f = H_0(f)$. so

we have $H_0 \circ P_1 \cong Id_{\mathcal{E}}$. □

Rq It is easy to see that M is quasiisomorphic to P_M when P_M is a projective resolution of M. But M is not homotopy equivalent to P_M.

⇝ It is better to work in a category where qis becomes isomorphisms
↳ the derived category.

If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories
then F induces a functor $F : Ch_\bullet(\mathcal{A}) \to Ch_\bullet(\mathcal{B})$

$$\begin{tikzcd} \cdots X_n \arrow[r, "d_n"] \arrow[d, "\alpha_n"'] & X_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r, mapsto] & \cdots \arrow[r] & F(X_n) \arrow[r, "F(d_n)"] \arrow[d, "F(\alpha_n)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X_{n-1}) \arrow[r] \arrow[d, "F(\alpha_{n-1})"] & \cdots \\ y_n \arrow[r, "d_n'"'] & y_{n-1} & & \cdots & F(y_n) \arrow[r, "F(d_n')"'] & F(y_{n-1}) \cdots \end{tikzcd}$$

and also a functor from $F : K_\bullet(\mathcal{A}) \to K_\bullet(\mathcal{B})$.

Def 5.26 Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor from $\mathcal{A}$ to $\mathcal{B}$ two abelian categories such that $\mathcal{A}$ has enough projectives.

The nth left derived functor L_nF of F is the composite
$$\mathcal{A} \xrightarrow{P_R} K_+(\mathrm{Proj}\,\mathcal{A}) \xrightarrow{F} K_+(\mathcal{B}) \xrightarrow{H_n} \mathcal{B}$$

Concretely: $L_nF(X) = H_n(F(P_X))$ where P_X is a projective resolution of X and $L_nF(f) = H_n(F(f_\bullet))$
↑ induced on Proj

Def 5.27 If $\mathcal{A}$ has enough injectives, the nth right derived functor R^nF of F is the composite $\mathcal{A} \xrightarrow{I_R} K^+(\mathrm{Inj}\,\mathcal{A}) \xrightarrow{F} K^+(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$

↳ $R^nF(X) = H^n(F(I_X))$ where I_X is an injective coresolution of X.

Thm 5.28 We assume that $\mathcal{A}$ has enough projectives.

(1) L_nF is additive $\forall n \geq 0$.

(2) If F is right exact then $L^0F \cong F$

(3) If $0 \to A \to B \to C \to 0$ is a ses in $\mathcal{A}$ and F is right exact then we have a long exact sequence

$$\cdots \to L_2F(C) \xrightarrow{\delta} L_1F(A) \to L_1F(B) \to (L_1F)(C) \xrightarrow{\delta} F(A) \to F(B) \to F(C) \to 0$$

(4) $L_nF(P) = 0$ $\forall P$ projective, if $n \geq 1$

Moreover this is natural in the ses.

sketch of proof: (1) One has to check that P_R is additive.

(2) If F is right exact $F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\pi)} F(M) \to 0$ is exact hence $(L_0F)(M) \cong F(M)$

$M = \mathrm{coker}(d_1)$, F right exact gives $F(M) = \mathrm{Coker}(F(d_1)) = L_0(F(M))$

(3) (a) There is a short exact sequence $0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$ of complexes of projective objects and morphisms

$$\begin{tikzcd}
0 \arrow[r] & P_\bullet \arrow[r] \arrow[d, "f"] & Q_\bullet \arrow[r] \arrow[d, "g"] & R_\bullet \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where each vertical map is a projective resolution and each row is exact.

(so called Horse shoe lemma)

if $0 \to P_i \to Q_i \to R_i \to 0$ is exact with projective then it splits : so need to chose proj resolutions for A and C then the middle term is $P_i \oplus R_i$ $\square$.

Since F is additive and each rows of $0 \to P_\cdot \to Q_\cdot \to R_\cdot \to 0$ splits the sequence $\boxed{0 \to F(P_\cdot) \to F(Q_\cdot) \to F(R_\cdot) \to 0 \text{ is exact}}$ hence by the long exact sequence in homology we have

$$\begin{tikzcd}[column sep=small]
 & & \cdots \arrow[r] & H_1F(R_\cdot) \arrow[r, "\delta"] \arrow[d, "\wr"] & H_0F(P_\cdot) \arrow[r] \arrow[d, "\wr"] & H_0F(Q_\cdot) \arrow[r] \arrow[d, "\wr"] & H_0F(R_\cdot) \arrow[r] \arrow[d, "\wr"] & 0 \\
 & & & L_1F(C) \arrow[r] & L_0F(A) \arrow[r] \arrow[d, "\wr"] & L_0F(B) \arrow[r] \arrow[d, "\wr"] & L_0F(C) \arrow[r] \arrow[d, "\wr"] & 0 \\
\cdots \arrow[r] & L_1F(A) \arrow[r] & L_1F(B) \arrow[r] & L_1F(C) \arrow[r] & F(C) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r] & 0
\end{tikzcd}$$

(4) is clear since $\cdots 0 \to P \to P \to 0$ is a projective resolution of P $\square$

Thm 5.29 If $\mathcal{A}$ has enough injective

(1) R^nF is additive $\forall n \geq 0$

(2) If F is left exact then $R^0F \cong F$

(3) If $0 \to A \to B \to C \to 0$ ~~exact then~~ and F is left exact, then we have a long exact sequence.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to$$

Case of contravariant functor : $F : \mathcal{E} \to \mathcal{D}$ contravariant

$\iff F : \mathcal{E}^{op} \to \mathcal{D}$ covariant

hence it is exact on the left if

$\forall$ $0 \to A \to B \to C \to 0$ we have $0 \to F(A) \to F(B) \to F(C)$ exact

Moreover proj in $\mathcal{E} \leftrightarrow$ inj in $\mathcal{E}^{op}$ so

Def 5.30 (1) $F : \mathcal{A} \to \mathcal{B}$ Contravariant exact on the left. Then

$$R^iF(A) = H^i(F(P)) \quad \text{where } P \text{ is a proj reso of } A$$

when F is right exact.

similarly $L_i F(B) = H_i(F(I))$ where I is an inj coresolution of B.

Chapter 6 Ext and Tor

Here we work with AMod and ModA for A k-alg

~~Def 6.1~~ 1) Deriving bifunctors

$F : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}''$ left exact functor between abelian categories, then for $x \in \mathcal{E}$, $y \in \mathcal{E}'$ one can construct $R^j F(x,-)$ and $R^j F(-,y)$

Thm 6.1 Assume that $\forall I$ inj of $\mathcal{E}$, the functor $F(I,\bullet) : \mathcal{E}' \to \mathcal{E}''$ is exact and $\forall I'$ inj of $\mathcal{E}'$, $F(\bullet, I') : \mathcal{E} \to \mathcal{E}''$ is exact.
Then $\forall j \in \mathbb{Z}$, $\forall x \in \mathcal{E}$, $\forall y \in \mathcal{E}'$ there is a binatural isomorphism

$$R^j F(x,\bullet)(y) \simeq R^j F(\bullet, y)(x)$$

Sketch $0 \to x \to I_x^\bullet$ inj coresolutions
$0 \to y \to I_y^\bullet$
apply F gives a double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F(I_x^0, y) \arrow[r] \arrow[d] & F(I_x^1, y) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & F(x, I_y^0) \arrow[r] \arrow[d] & F(I_x^0, I_y^0) \arrow[r] \arrow[d] & F(I_x^1, I_y^0) \arrow[d] & \\
0 \arrow[r] & F(x, I_y^1) \arrow[r] & F(I_x^0, I_y^1) \arrow[r] & \cdots &
\end{tikzcd}$$

Cohomology of first row $\sim R^j F(\bullet, y)(x)$ all others rows
first col $\sim R^j F(x,\bullet)(y)$ and col are exact

$X^{0,\bullet}$ $X^{\bullet,0}$

show that $H^i(x^{0,\bullet}) \simeq H^i(x^{\bullet,0})$ by diag chasing

$$\begin{array}{ccccccc}
 & & & & & x^{0,i} & \to 0 \\
 & & & & & \downarrow & \\
 & & & x^{1,i-2} & \to & x^{i,i-1} & \\
 & & \cdots & & & & \\
 & x^{i-1,1} & \to & & & & \\
 & \downarrow & & & & & \\
x^{i,0} \xrightarrow{d''} & x^{i,1} & & & & & \\
d' \downarrow & \vdots\, d'' & & & & & \\
0 & & & & & &
\end{array}$$

Start with $x^{i,0} \in \ker(d'^i)$ then $x^{i,1} = d''(x^{i,1}) \in \ker(d')$ since double complex. Exactness lead to $x^{i-1,1}$; $d'(x^{i-1,1}) = x^{i,1}$ we have $d'' x^{i-1,1} = x^{i-1,2} \in \ker(d')$ etc.
$\leadsto x^{0,i} \in \ker(d''^i)$ and this is the construction of the isomorphism.

For details see: Weibel §2.7 + dual for left derived functor □

2) Functor Ext

Def 6.2 (1) If $\mathcal{E}$ has enough projective (or enough injective) we denote by $\mathrm{Ext}^i(A,B)$ the derived functor $[R^i \mathrm{Hom}(-,B)](A)$ (resp $R^i \mathrm{Hom}(A,-)[B]$).

(2) If $\mathcal{E} = \mathrm{Mod}\, A$ $\mathrm{Ext}^i(A,B) \simeq R^i \mathrm{Hom}(-,B)(A) \simeq R^i \mathrm{Hom}(A,-)(B)$.

This makes sense since $\mathrm{Hom}(-,B)$ satisfies the hypothesis of Thm 6.1.

Examples $R = \mathbb{Z}$, M be an torsion abelian group, then to compute $\mathrm{Ext}^i(M,\mathbb{Z})$ on can
- find an ~~injective~~ proj ~~of~~ resolution of M
- or find an ~~proj~~ injective coresolution of $\mathbb{Z}$

We have $0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \to 0$ exact with $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ injective.

So we have a long exact sequence

$$0 \to \mathrm{Hom}(M,\mathbb{Z}) \to \underset{=0}{\mathrm{Hom}(M,\mathbb{Q})} \to \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1(M,\mathbb{Z})$$
$$\to \underset{=0}{\mathrm{Ext}^1(M,\mathbb{Q})} \to \underset{=0}{\mathrm{Ext}^1(M,\mathbb{Q}/\mathbb{Z})} \to \mathrm{Ext}^2(M,\mathbb{Z}) \to \underset{=0}{\mathrm{Ext}^2(M,\mathbb{Q})}$$

$\hookrightarrow \begin{cases} \mathrm{Ext}^1(M,\mathbb{Z}) \simeq \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \\ \mathrm{Ext}^i(M,\mathbb{Z}) = 0 \quad \forall i \neq 0 \end{cases}$

Thm 6.3 (1) TFAE A Mod

1. P is projective
2. $\mathrm{Hom}_A(P,-)$ is exact
3. $\forall i \geq 1$, $\forall B \in \mathrm{Mod}\,A$, $\mathrm{Ext}^i(P,B) = 0$
4. $\forall B \in \mathrm{Mod}\,A$, $\mathrm{Ext}^1(P,B) = 0$

(2) Dual for injective

Sketch (1) $\Leftrightarrow$ (2) by def (2) $\Rightarrow$ (3) clear (3) $\Rightarrow$ (2) long exact sequence
(3) $\Rightarrow$ (4) clear (4) $\Rightarrow$ (2) long exact sequence

Ex Ext^i can be computed using Ext^1 if $i \geq 2$.

3. Tor functor

Def 6.4 A k-alg $\mathcal{E} = \mathrm{Mod}\,A$ We denote by Tor_i^A the i-th derived functor of $- \otimes_A -$

That is $\mathrm{Tor}_i^A(M,N) = L_i(- \otimes_A N)(M) \simeq L_i(M \otimes_A -)$