THÉORIE DE L'HOMOLOGIE
EXAMEN PREMIÈRE SESSION

Les exercices 1, 2 et 3 sont indépendants. L'exercice 4 utilise l'exercice 3 et l'exercice 5 dépend des exercices 3 et 4.

Exercice 1 - Centre d'une catégorie

Le centre $Z(\mathcal{C})$ d'une (petite) catégorie $\mathcal{C}$ est l'ensemble des endomorphismes naturels du foncteur $\mathrm{Id}_{\mathcal{C}}$. Dans cet exercice, on ignore les éventuels problèmes ensemblistes.

(1) Montrer que la composition des transformations naturelles fait du centre de $\mathcal{C}$ un monoïde commutatif.

(2) Soit G un groupe et $\mathbf{B}(G)$ la catégorie avec G comme endomorphismes d'un unique objet $\bullet$. Quel est le centre de $\mathbf{B}(G)$?

(3) Lorsque A est un anneau associatif unitaire, démontrer que le centre de $\mathrm{Mod}\,A$ est isomorphe au centre de l'anneau A. On attend ici une réponse détaillée.

(4) Quel est le centre de la catégorie des ensembles ?

Correction

(1) The first task is to unwrap the definition. An endomorphism of the identity functor of $\mathcal{C}$ is a collection $\eta = (\eta_c)_{c\in Ob(\mathcal{C})}$ where $\eta_c \in \mathrm{Hom}_{\mathcal{C}}(c,c)$ such that for every $f : c \to c' \in \mathrm{Mor}(\mathcal{C})$, we have a commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ c' & \xrightarrow[\eta_{c'}]{} & c' \end{array}$$

If α and β are two natural transformations of the identity functor. Then for each object c, we have $\alpha_c : c \to c$ and $\beta_c : c \to c$. Hence, the naturality applied to $\eta = \alpha$ and $f = \beta_c$ gives, $\beta_c \circ \alpha_c = \alpha_c \circ \beta_c$.

(2) Since $\mathbf{B}(G)$ has only one object, an endomorphism of the identity functor is an element $h \in G$ such that $\forall g \in G$, we have $gh = hg$. That is an element of the center of G.

(3) If η is a natural transformation of the identity of $\mathrm{Mod}\,A$, then $\eta_A \in \mathrm{End}_A(A)$, so $\eta_A(1) \in A$. Moreover, for $a \in A$, the left multiplication l_a by a on A is a morphism of right A-modules. Hence, the naturality of η gives $l_a \circ \eta_A = \eta_A \circ l_a$. On the element $1 \in A$, we get $a\eta_A(1) = \eta_A(a) = \eta_A(1 \cdot a) = \eta_A(1)a$. So $\eta_A(1) \in Z(A)$. Hence we have a morphism $\Psi : Z(\mathrm{Mod}\,A) \to Z(A)$. It is a ring homomorphism, since $\Psi(Id) = 1$, $\Psi(\eta + \gamma) = \eta_A(1) + \gamma_A(1)$ and $\psi(\eta\gamma) = \eta_A(\gamma_A(1)) = \eta_A(1 \cdot \gamma_A(1)) = \eta_A(1)\gamma_A(1)$.

Conversely, if $z \in Z(A)$, we construct a family of morphisms $R_z = (R_{z,M})_{M\in\mathrm{Mod}\,A}$, where $R_{z,M} : M \to M$ is the right multiplication by z. Since $z \in Z(A)$, this is a morphism of right A-modules. Moreover, it is a natural transformation of the identity functor of $\mathrm{Mod}\,A$. Indeed, if $f : M \to N$ is a morphism of right A-modules, we have $\big(f \circ R_{z,M}\big)(m) = f(mz) = f(m)z = \big(R_{z,N} \circ f\big)(m)$. This gives a morphism Φ from $Z(A)$ to $Z(\mathrm{Mod}\,A)$.

It is clear that $\Psi \circ \Phi(z) = z$. If $\eta \in Z(\mathrm{Mod}\,A)$, we have that $\Phi \circ \Psi(\eta)_M$ is the right multiplication on M by $\eta_A(1)$. If $m \in M$, let $f : A \to M$ be the morphism of right-modules such that $f(1) = m$. By naturality of η, we have a commutative square :

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[\eta_M]{} & M \end{array}$$

So, $f(\eta_A(1)) = \eta_M(f(1)) = \eta_M(m)$. And $f(\eta_A(1)) = f(1 \cdot \eta_A(1)) = f(1)\eta_A(1) = m\eta_A(1)$. Hence η_M is the right multiplication by $\eta_A(1)$ and this finishes the proof.

(4) The identity functor of the category of sets is isomorphic to $\mathrm{Hom}(1,-)$ where 1 is a set with one element. By Yoneda Lemma we have that $Z(Set) \cong End(\mathrm{Hom}(1,)) \cong \mathrm{Hom}(1,1)$. This is a group with only one element.

Exercice 2 - Homologie des ensembles ordonnés

Soit $(P,\leq)$ un ensemble (partiellement) ordonné. On appelle m-chaine de P un sous-ensemble totalement ordonné de P contenant exactement $m+1$ éléments. Les 0-chaines correspondent donc aux éléments de P, et les 1-chaines sont les couples (x,y) avec $x \leq y \in P$ et $x \neq y$. Pour $n \in \mathbb{N}$, on pose $C_n(P)$ le groupe abélien libre sur l'ensemble des n-chaines de P. Pour $n \in \mathbb{N}^*$, on pose $d_n : C_n(P) \to C_{n-1}(P)$ l'application définie sur une n-chaine $(x_0 < \cdots < x_n)$ par

$$d_n(x_0 < \cdots < x_n) = \sum_{i=0}^{n} (-1)^i (x_0 < \cdots < \widehat{x_i} < \cdots x_n),$$

où $(x_0 < \cdots < \widehat{x_i} < \cdots x_n)$ est la $(n-1)$-chaine obtenue en retirant x_i.

(1) Montrer que $C(P)_\bullet = (C_n(P), d_n)_{n\in\mathbb{N}}$ est un complexe de chaines de groupes abéliens. Son homologie s'appelle l'homologie de l'ensemble ordonné P.

(2) On considère $\epsilon : C_0(P) \to \mathbb{Z}$ l'application définie sur une 0-chaine $x \in P$ par $\epsilon(x) = 1$. Justifier que l'on peut augmenter le complexe $C_\bullet(P)$ en posant $C_{-1} = \mathbb{Z}$ et $d_0 = \epsilon$. On note alors $\widetilde{C}_\bullet$ ce complexe.

(3) Pour $x, y \in P$, on dit que x et y sont *comparables* si $x \leq y$ ou $y \leq x$. On dit que P est *connexe* si pour tout $x, y \in P$, il existe $x_0, x_1, \cdots, x_s \in P$ avec $x_0 = x$, $x_s = y$ et x_i, x_{i+1} sont comparables pour tout $i \in \{0, \cdots, s-1\}$. Décrire l'homologie de degré 0 de P, lorsque P est connexe. Quelle est l'homologie $H_0(P)$ lorsque P n'est pas connexe ? (pour cette seconde partie, on ne demande pas une démonstration détaillée).

(4) On considère l'ensemble $X = \{1,2,3\}$ et $P = (\mathcal{P}(X), \subset)$ l'ensemble des parties de X ordonnées par inclusion. On considère $\overline{P} = P \setminus \{\emptyset, X\}$ ordonné par inclusion. Calculer l'homologie de $\overline{P}$.

(5) On suppose P possède un plus petit élément 0 (i-e tel que $0 \leq x \ \forall x \in P$). Montrer que $\widetilde{C}_\bullet$ est contractile. En déduire l'homologie de P.

Correction

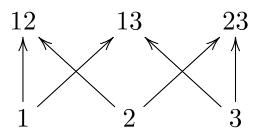
(1) This is clear.

(2) This is also clear since $d_1(x,y) = y - x \in \ker(\epsilon)$.

(3) Let $x \in P$ be a fixed element. For $y \in P$ y not equal to x, there is a sequence $x_0 = x, x_1, \cdots, x_s = y$ with x_i comparable to x_{i+1}. So, the chain (x_i, x_{i+1}) or the chain (x_{i+1}, x_i) is in P and not both. We let $w_i = (x_i, x_{i+1})$ in the first case, and $w_i = -(x_{i+1}, x_i)$ in the second case. In any case we have $d_1(w_i) = x_{i+1} - x_i$. Let $w_y = \sum_{i=0}^{s-1} w_i$. We have, $d_1(w_y) = -x_0 + x_1 - x_1 + \cdots - x_{s-1} + x_s = -x + y$. Hence $[x] = [y]$ in $\mathbb{Z}[P]/Im(d_1)$. Moreover $\epsilon(x) \neq 0$, hence $[x] \neq 0$. In other words, we have $H^0(P) \cong \mathbb{Z}$.

The transitive closure of 'being comparable' is an equivalence relation and the equivalence classes are called the connected components of P. Hence, if P is not connected, it is a disjoint union of connected components. Moreover, if $P = \sqcup_\alpha P_\alpha$, it is clear that $C_\bullet(P) \cong \bigoplus_\alpha C_\bullet(P_\alpha)$. It follows that $H_0(P)$ is the free abelian group with rank the number of connected components of P.

(4) The poset looks like :



There are 6 elements in P and there are 6 1-chains. There are no larger chains, so the complex $C_\bullet(P)$ is concentrated in degrees 0 and 1. The poset is connected hence the homology in degree 0 is $\mathbb{Z}$. The kernel of d_1 is a submodule of a finitely generated free $\mathbb{Z}$-module, hence it is a free. Since, the map d_1 has rank 5, we see that the kernel is $\mathbb{Z}$. Hence P has homology $\mathbb{Z}$ in degree 0 and 1 and all the other homology groups are zero. Of course one can also compute the matrix of d_1 and check the details.

(5) Let $s_{-1} : \mathbb{Z} \to C_0(P)$ be the map defined by $s_{-1}(1) = 0$. We have $\epsilon \circ s_{-1}(1) = \epsilon(0) = 1$. For $n \geq 0$, let $s_n : C_n(P) \to C_{n+1}(P)$ be the map defined on an n-chain by

$$s_n(x_0, \cdots, x_n) = \begin{cases} (0, x_0, \cdots, x_n) & \text{if } x_0 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have to check that $Id_{C_n(P)} = s_{n-1}d_n + d_{n+1}s_n$. There is a special case when $n = 0$: if $x = 0$, then $s_0(x) = 0$, hence $(sd+ds)(x) = s_{-1}(1) = 0$. If $x \neq 0$, we have $sd+ds(x) = d_1(0,x) + s_{-1}(1) = x - 0 + \hat{0} = x$. The general case is similar. If $(x_0, \cdots, x_n)$ is an n-chain with $x_0 = 0$, then it is killed by s_n and the factor of $d_n(x_0, \cdots, x_n)$ which is not killed by s_{n-1} is the one obtained by removing x_0. Hence we have $s_{n-1}d_n(0, x_1, \cdots, x_n) = s_{n-1}(x_1, \cdots, x_n) = (0, x_1, \cdots, x_n)$. When $x_0 \neq 0$, then we have :

$$\begin{aligned} d_{n+1}s_n(x_0, \cdots, x_n) &= d_{n+1}(0, x_0, \cdots, x_n) \\ &= (x_0, \cdots, x_n) + \sum_{i=1}^{n+1} (-1)^i (0, x_0, \cdots, \widehat{x_{i-1}}, \cdots, x_n) \\ &= (x_0, \cdots, x_n) - s_{n-1}d_n(x_0, \cdots, x_n). \end{aligned}$$

The result follows. Hence the homology of the augmented complex is zero and the homology of P is $\mathbb{Z}$ in degree 0 and 0 in other degrees.

Exercice 3 - Algèbres de dimension finie Soit A une algèbre de dimension finie sur un corps k. On admet que les A-modules de type fini sont exactement les A-modules de dimension finie sur le corps k. On note $\operatorname{mod} A$ la catégorie des A-modules (à droite) de type fini.

(1) Montrer que la catégorie $\operatorname{mod} A$ est abélienne.

(2) Démontrer que la catégorie $\operatorname{mod} A$ possède assez de projectifs.

(3) Démontrer que $\operatorname{mod} A$ possède assez d'injectifs. Indication, on pourra utiliser le foncteur $D = \operatorname{Hom}_k(-, k)$.

Correction

(1) The key is that $\operatorname{mod} A$ is a full subcategory of $\operatorname{Mod} A$. Since $\operatorname{Mod} A$ is abelian, it follows that $\operatorname{mod} A$ is an preadditive category. The coproduct (in $\operatorname{Mod} A$) of two finite dimensional modules is finite dimensional, hence it is in $\operatorname{mod} A$. Now, if $f : M \to N$ is a morphism between finite dimensional modules, then its kernel and cokernel (in $\operatorname{Mod} A$) are also finite dimensional. Since $\operatorname{Hom}_{\operatorname{mod} A}(M, N) = \operatorname{Hom}_{\operatorname{Mod} A}(M, N)$, it is clear that the kernel of f is a kernel in $\operatorname{mod} A$. Hence, $\operatorname{mod} A$ is a preabelian category. It remains to look at the canonical morphism from the coimage to the image. Since this is exactly the same morphism as in $\operatorname{Mod} A$, it is an isomorphism.

(2) Let X be an A-module generated by $x_1, \cdots, x_n$. Let $\phi_i : A \to X$ the morphism defined by $\phi_i(1) = x_i$. Then $f : A^n \to X$ the map defined by $f(a_i) = \sum_{i=1}^n \phi_i(a_i) = \sum_{i=1}^n a_i x_i$ is an epimorphism. Hence, $\operatorname{mod} A$ has enough projectives.

(3) Using the right A-module structure on M, we see that $D = \operatorname{Hom}_k(M, k)$ is an A-module on the left. Concretely for $\phi \in D(M)$, and $a \in A$, the map $a \cdot \phi$ sends m to $\phi(ma)$. Hence D is a functor from $\operatorname{mod} A$ to $A \operatorname{mod}$. Similarly, if we start with a left A-module, $D(M)$ is a right A-module. Moreover $D^2 \cong Id$, hence D is a contravariant equivalence from $\operatorname{mod} A$ to $A \operatorname{mod}$. In other words, an equivalence from $(\operatorname{mod} A)^{op}$ to $A \operatorname{mod}$. As an equivalence between abelian categories it sends projective object to projective object and epimorphism to epimorphism. By Question 2, $\operatorname{mod} A$ has enough projectives. So if $M \in \operatorname{mod} A$, then there is a projective module P in $A \operatorname{mod}$ and an epimorphism $\pi : P \to D(M)$. Applying D we have in $(\operatorname{mod} A)^{op}$ an epimorphism $D(\pi) : D(P) \to DDM \cong M$. So in $\operatorname{mod} A$, we have a monomorphism $M \to D(P)$. Moreover $D(P)$ is projective in $(\operatorname{mod} A)^{op}$, hence it is injective in $\operatorname{mod} A$.

Exercice 4 - Dimension globale

Soit $\mathcal{A}$ une catégorie abélienne avec assez de projectifs. On appelle *dimension globale* de $\mathcal{A}$:

$$\operatorname{gldim}(\mathcal{A}) = \sup\{n \in \mathbb{N} \mid \exists A, B \in Ob(\mathcal{A}) \ : \operatorname{Ext}^n_{\mathcal{A}}(A, B) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Si $M \in \mathcal{A}$, on appelle *dimension projective* de M notée $\operatorname{pdim}(M)$, le plus petit n tel qu'il existe une résolution projective de M de la forme

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

S'il n'existe pas de résolution projective finie on pose $\operatorname{pdim}(M) = \infty$.

On dit que $\mathcal{A}$ est *semisimple* si $\operatorname{gldim}(\mathcal{A}) = 0$ et $\mathcal{A}$ est *héréditaire* si $\operatorname{gldim}(A) \leq 1$. On peut supposer que les objets de $\mathcal{A}$ sont des modules (à droite) sur un anneau unitaire associatif A.

(1) Montrer que les assertions suivantes sont équivalentes.
  (a) $\mathcal{A}$ est semisimple.
  (b) Tout objet de $\mathcal{A}$ est projectif.
  (c) Tout objet de $\mathcal{A}$ est injectif.
  (d) Toute suite exacte courte est scindée.

(2) Donner un exemple d'anneau dont la catégorie des modules est semisimple.

(3) Soit $M \in \mathcal{A}$, démontrer que les assertions suivantes sont équivalentes :
  (a) $\operatorname{pdim}(M) \leq n$.
  (b) $\operatorname{Ext}^i_{\mathcal{A}}(M, X) = 0$ pour tout $i > n$ et tout $X \in \mathcal{A}$.
  (c) $\operatorname{Ext}^{n+1}_{\mathcal{A}}(M, X) = 0$ pour tout $X \in \mathcal{A}$.
  (d) Si $0 \to M_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ est une suite exacte avec P_i projectifs pour tout i, alors M_n est projectif.

(4) Justifier que $\operatorname{gldim}(\mathcal{A}) = \sup\{\operatorname{pdim}(M) \mid M \in \mathcal{A}\}$.

(5) Ici on suppose que les objets de $\mathcal{A}$ sont des modules. Démontrer que $\mathcal{A}$ est héréditaire si et seulement si tout sous-module d'un module projectif est projectif.

(6) Soit A une k-algèbre de dimension finie. Montrer que la catégorie des A-modules de dimension finie est héréditaire si et seulement si tout idéal à droite de A est projectif.

(7) Une k-algèbre A de dimension finie sur un corps k est dite *auto-injective* si le module régulier $A \in \operatorname{mod} A$ est injectif. Montrer que dimension globale de $\operatorname{mod} A$ est alors 0 ou ∞.

(8) Soient $n \geq 1$ et k un corps. Montrer que $A = k[X]/(X^n)$ est auto-injective et en déduire la dimension globale de $\operatorname{mod} A$. Indication, on pourra utiliser le critère de Baer.

Correction For this exercise there are many different ways of solving it and we do not claim to have the most efficient one. Actually, we try to use as little results from the class as possible, so this makes some proof slightly longer.

(1) (b) implies (a). If every object is projective, we have $0 \to X \to 0$ is a projective resolution of X, hence the nth right derived functor of $Hom(X, -)$ vanishes except when $n = 0$. (a) implies (b) because if P is an object, and $0 \to L \to M \to N \to 0$ is a short exact sequence, we have a long exact sequence $0 \to \operatorname{Hom}(P, L) \to \operatorname{Hom}(P, M) \to \operatorname{Hom}(P, N) \to \operatorname{Ext}^1(P, L) = 0$. Hence, the functor $\operatorname{Hom}(P, -)$ is exact and P is projective. By a dual argument, we have $(a) \Leftrightarrow (c)$. Clearly, (b) implies (d) since an epi toward a projective object splits. And if every short exact sequence splits, consider the diagram :

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

with P projective. Adding the kernel of the bottom map, leads to

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

and the splitting of the short exact sequence allows to lift the morphism $P \to N$ along $M \to N$. So (d) implies (b). Alternatively, any additive functor is exact on split exact sequences, hence (d) implies (b) or P is projective if and only if any epi toward P splits, hence (d) implies (b)...

(2) If k is a field, then the category of k-modules is semisimple, since any short exact sequence splits, or any modules is free (hence projective)...

(3) (a) implies (b) is clear, (b) implies (c) also. Moreover (d) implies (a) is also clear in view of the inductive construction of projective resolution. (c) implies (d) is an argument of 'décalage' as in TD4 Ex2. If d_i is the differential map from P_i to P_{i-1} and $\pi : P_0 \to M$, then $M_n \cong \operatorname{Ker}(d_{n-1})$. We have n short exact sequences :

$$\begin{aligned} & 0 \to \operatorname{Ker}(\pi) \to P_0 \to M \to 0 \\ & 0 \to \operatorname{Ker}(d_1) \to P_1 \to \operatorname{Ker}(\pi) \to 0 \\ & 0 \to \operatorname{Ker}(d_2) \to P_2 \to \operatorname{Ker}(d_1) \to 0 \\ & \quad \cdots \\ & 0 \to \operatorname{Ker}(d_{n-1}) \to P_{n-1} \to \operatorname{Ker}(d_{n-2}) \to 0. \end{aligned}$$

Moreover if $0 \to L \to P \to N \to 0$ is a short exact sequence with P projective, applying $\operatorname{Hom}(-, X)$ gives a long exact sequence, from which we see $\operatorname{Ext}^i(L, X) \cong \operatorname{Ext}^{i-1}(L, N)$, for $i \geq 2$. Hence, we have $\operatorname{Ext}^{n+1}(M, X) \cong \operatorname{Ext}^1(M_n, X)$. Condition (c) implies that $\operatorname{Ext}^1(M_n, X) = 0$ for all X, hence M_n is projective.

(4) By Question 3, if $\operatorname{Ext}^{n+1} = 0$, every module have projective dimension at most n. Conversely if every module have projective dimension at most n, then $\operatorname{Ext}^{n+1}(-, X) = 0$ for all X. So $(a) = (b)$.

(5) Assume that $\mathcal{A}$ is hereditary and let M be a submodule of a projective P_0. Then, we have a short exact sequence $0 \to M \to P_0 \to P_0/M \to 0$. By Question 3 (d), we have that M is projective. Conversely, if X is an object in $\mathcal{A}$, let $\pi : P \to X$ an epimorphism with P projective (it exists since $\mathcal{A}$ has enough projective). This gives a short exact sequence $0 \to \operatorname{Ker}(\pi) \to P \to X \to 0$. Since $\operatorname{Ker}(\pi)$ is a submodule of P, it is projective, hence X has a projective resolution of length at most 1.

(6) Since a submodule of A is a right ideal by (5) we have that hereditary implies ideals are projective. Conversely, let M be a submodule of a finitely generated projective. Since such finitely generated projective is a direct summands of a free module, we can assume that M is a submodule of A^n and we prove the result by induction on n. If $n = 1$, we are done since the submodules of A are the right ideals. If $n \geq 1$, let $\pi : A^n \to A$ the projection onto the last coordinates. Then we have a short exact sequence

$$0 \to \operatorname{Ker}(\pi|_M) \to M \to Im(\pi|_M) \to 0.$$

The image of $\pi|_M$ is a subomodule of $Im(\pi) = A$. Hence it is a projective module and the short exact sequence splits and $M \cong \operatorname{Ker}(\pi|_M) \oplus Im(\pi|_M)$. Now, $\operatorname{Ker}(\pi|_M) = \operatorname{Ker}(\pi) \cap M$ is a submodule of $\operatorname{Ker}(\pi) \cong A^{n-1}$. By induction, we have that $\operatorname{Ker}(\pi) \cap M$ is projective, hence M is projective as direct sums of two projective modules.

Remark : a submodule of a finitely generated free module is not necessarily a direct sum of right ideals!

(7) Since A is injective, then a finite direct sum (which is also a product) of copies of A is injective and a direct summand of an injective module is injective, hence every finitely generated projective module is also injective. Let

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be a finite projective resolution of the finite dimensional module M. Then $d_n : P_n \to P_{n-1}$ is injective and we have $P_{n-1}/d_n \cong P_{n-1}/\operatorname{Ker}(d_{n-1}) \cong Im(d_{n-1})$. Hence we have a short exact sequence :

$$0 \to P_n \to P_{n-1} \to Im(d_{n-1}) \to 0.$$

Since P_n is injective, the sequence splits and $Im(d_{n-1})$ is a direct summands of a projective, so it projective. Hence

$$0 \to Im(d_{n-1}) \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution of M. By induction on the length, this implies that M is projective. Hence the only modules with finite projective dimension are the projective modules.

(8) In order to show that $A = k[X]/(X^n)$ is an injective module, we take an ideal I of A and a map $f : I \to A$ and we have to show that it extends as a map from A to A. The ideals of $k[X]/(X^n)$ are the ideals $J/(X^n)$ for an ideal J of $k[X]$. The ring $k[X]$ is principal, hence the ideals of A are the $I_i := (X^i)/(X^n)$ for $i \in \{0, \cdots, n-1\}$. A map from I_i to A is determined by $f(X^i)$ which has to be an element $\overline{Q}$ of A such that $X^{n-i}\overline{Q} = 0$. This means that X^n divides $X^{n-i}Q$ in $k[X]$. So Q is of the form X^iR for a polynomial R. We define $g : A \to A$ to be the map sending 1 to $\overline{R}$. Then $g(X^i) = X^i\overline{R} = \overline{Q} = f(X^i)$. By Baer's criterion, we obtain that $k[X]/(X^n)$ is injective. It remains to see that the algebra is not semisimple when $n \geq 2$. By Question (1) it is enough to find a non split exact sequence. The evaluation at 0 gives a surjective map $A \to k \cong k[X]/(X)$ and the kernel is $(X)/(X^n)$. If there is a splitting s of the evaluation, s is determined by $s(1) = \overline{Q}$. Since s is a morphism of modules, we have $0 = s(X \cdot 1) = X \cdot \overline{Q}$. So X^n divides XQ, hence X divides Q and the evaluation at 0 of Q is zero, hence $ev_0 \circ s \neq Id_k$. Alternatively, we can see that $k[X]/(X^n)$ is indecomposable (the matrix of the action is a Jordan block of size n) hence the sequence cannot split.

Exercice 5 - Un exemple non commutatif de dimension globale 2 Soit k un corps et A une k-algèbre de dimension finie. On considère la catégorie $\operatorname{mod} A$ des A-modules à droite de dimension finie. On suppose que cette catégorie possède un *générateur additif* M. C'est-à-dire que pour tout A-module X, il existe $n \in \mathbb{N}$ et Y un A-module tel que $X \oplus Y = M^n$. En d'autres termes, tout A-module est facteur direct d'une somme directe finie de copies de M. Le théorème de réduction de Jordan permet de montrer (on ne demande pas de le faire) que pour $n \geq 1$, l'algèbre $\mathbb{C}[X]/(X^n)$ possède cette propriété. On pose $\Gamma_M = \operatorname{End}_A(M)$.

(1) Justifier que la fonctorialité de $\operatorname{Hom}_A(M, X)$ par rapport à la première variable permet de munir $\operatorname{Hom}_A(M, X)$ d'une structure de Γ_M-module à droite. On en déduit que $F = \operatorname{Hom}_A(M, -)$ est un foncteur de la catégorie $\operatorname{Mod} A$ vers la catégorie $\operatorname{Mod} \Gamma_M$.

(2) Montrer que F se restreint en un foncteur de $\operatorname{mod} A$ vers $\operatorname{proj} \Gamma_M$, où $\operatorname{proj} \Gamma_M$ est la catégorie des Γ_M-modules projectifs de dimension finie.

(3) (Bonus) Montrer que $F : \operatorname{mod} A \to \operatorname{proj} \Gamma_M$ est une équivalence de catégorie. On pourra montrer que c'est un foncteur pleinement fidèle, en commençant par regarder le module M. Puis voir qu'il est essentiellement surjectif.

(4) Montrer que la dimension globale de $\operatorname{mod} \Gamma_M$ est au plus 2. Indication, utiliser la question 3.

(5) Un module N non nul est *indécomposable* s'il n'est pas somme directe de deux sous-modules non nuls. C'est-à-dire pour tous sous-modules X, Y de N tels que $N = X \oplus Y$, alors ($X = 0$ et $Y = N$) ou ($X = N$ et $Y = 0$). Un A-module est simple si ses seuls sous-modules sont $\{0\}$ et lui même.

  (a) Si P est un A-module indécomposable, montrer que $F(P)$ est un Γ_M-module projectif indécomposable.

  (b) Si A n'est pas semisimple, on admet [1] qu'il existe un module simple S non projectif, un module projectif indécomposable P et un épimorphisme $\pi : P \to S$. Montrer que $\operatorname{Hom}(M, \pi)$ n'est ni un épimorphisme ni un monomorphisme. En déduire que la dimension globale de Γ_M est exactement 2 dans ce cas.

(6) Que se passe-t-il lorsque A est semisimple ?

Correction

1. On pourrait montrer que si tous les modules simples sont projectifs, alors l'algèbre est semisimple. De plus, dans la catégorie des $\operatorname{mod} A$ il existe toujours un projectif indécomposable qui se surjecte sur un module simple donné.

(1) Let $\Phi \in \Gamma_M$, then $\mathrm{Hom}_A(\phi, X) : \mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(M, X)$. We get a structure of Γ_M-module on the right, by setting $\alpha \cdot \phi = \mathrm{Hom}_A(\phi, X)(\alpha)$. Then $(\alpha \cdot \phi_1) \cdot \phi_2 = \mathrm{Hom}_A(\phi_2, X)[\mathrm{Hom}_A(\phi_1, X)(\alpha)] = \mathrm{Hom}_A(\phi_1\phi_2, X)(\alpha) = \alpha \cdot (\phi_1\phi_2)$. And $\alpha \cdot Id_M = \alpha$. Note that the action is exactly the one you think about : $\alpha \cdot \phi = \alpha \circ \phi$.

(2) The image of M by F is Γ_M, so by additivity of F, the image of a direct sum of M is a free Γ_M-module. A direct summand of M^n is then sent to a direct summand of a free module, in other words to a projective module. Since every finite dimensional A-module is a direct summand of a finite direct sum of M, the result follows.

(3) (Bonus) It is easy to see that $\mathrm{End}_A(M) \cong \mathrm{End}_{\Gamma_M}(F(M))$. By additivity of F and the fact that any A-module is direct summands of finite direct sum of M, we have that F is fully-faithful. If P is a projective Γ_M-module, it is a direct summand of Γ_M^n for some n. Hence, there is an idempotent $e \in \mathrm{End}_{\Gamma_M}(F(M)^n)$ whose kernel is P. The functor F is full, hence there is $\alpha \in \mathrm{End}_A(M^n)$ such that $F(\alpha) = e$. $F(\alpha \circ \alpha) = e$, hence $\alpha^2 = \alpha$ since F is faithful. Now $F(\mathrm{Ker}(\alpha)) \cong \mathrm{Ker}(F(\alpha)) \cong P$, so F is essentially surjective.

(4) For X a Γ_M-module, let P_0 and P_1 such that $P_1 \to P_0 \to X$ is the beginning of a projective resolution. By the previous question, we can choose $P_0 = F(N_0)$ and $P_1 = F(N_1)$ for two A-modules N_0 and N_1. Moreover $\mathrm{Hom}(F(N_1), F(N_0)) \cong \mathrm{Hom}_A(N_1, N_0)$ hence the morphism between the two projectives is of the form $F(\alpha)$ for $\alpha : N_1 \to N_0$. We have an exact sequence of A-modules :

$$0 \to \mathrm{Ker}(\alpha) \to N_1 \xrightarrow{\alpha} N_0,$$

since $\mathrm{Hom}_A(M, -)$ is left-exact, we deduce an exact sequence of Γ_M-modules

$$0 \to \mathrm{Hom}(M, \mathrm{Ker}(\alpha)) \to \mathrm{Hom}(M, N_1) \xrightarrow{F(\alpha)} \mathrm{Hom}(M, N_0) \to X,$$

and X has a projective resolution of length 2.

(5) (a) If $F(P) = V \oplus W$, then by applying the quasi-inverse equivalence G we have $P \cong GF(P) \cong G(V) \oplus G(W)$. Hence, $G(V) = 0$, or $G(W) = 0$, say $G(V) = 0$. Applying F,we get $V = FG(V) = F(0) = 0$, so $F(P)$ is indecomposable.

(b) A simple module is certainly indecomposable. Hence $Hom(M, P)$ and $Hom(M, S)$ are two projective indecomposable Γ_M-module. We have a short exact sequence of A-modules :

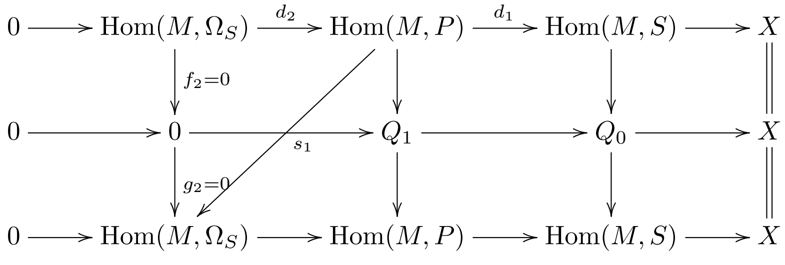
$$0 \to \Omega_S \to P \to S \to 0,$$

where $\Omega_S = \mathrm{Ker}(\pi)$. Since $\mathrm{Hom}(M, -)$ is left exact, we have an exact sequence of Γ_M-modules :

$$0 \to \mathrm{Hom}(M, \Omega_S) \to \mathrm{Hom}(M, P) \to \mathrm{Hom}(M, S).$$

The module M is an additive generator, hence $\mathrm{Hom}(M, \Omega_S)$ is non-zero, so $\mathrm{Hom}(M, \pi)$ is not a monomorphism. If it is an epimorphism, it spits, since $\mathrm{Hom}(M, S)$ is projective. So $\mathrm{Hom}(M, P) \cong \mathrm{Hom}(M, S) \oplus \mathrm{Hom}(M, \Omega_S)$ and this is a contradiction since $\mathrm{Hom}(M, P)$ is indecomposable. This exact sequence is a projective resolution of length 2 of the cokernel X of $\mathrm{Hom}(M, \pi)$. It remains to show that one cannot find a resolution of length 1 of this module X. This is mostly because $\mathrm{Hom}(M, P)$ is indecomposable.

Indeed, assume now that it also has a projective resolution $Q_\bullet$ of length at most 1, $Q_1 \to Q_0 \to X \to 0$. Since two projective resolutions of the same module are homotopy equivalent, there is $f : \mathcal{P} \to Q$ and $g : Q \to \mathcal{P}$ such that $fg \sim id_Q$ and $gf \sim id_{\mathcal{P}}$.



Since $Q_2 = 0$, we have $f_2 = 0$ and $g_2 = 0$. The existence of the homotopy s implies that $Id_{\mathrm{Hom}(M,\Omega_N)} = s_1 \circ d_2$. Hence the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, \Omega_S) \xrightarrow{d_2} \mathrm{Hom}(M, P) \longrightarrow \mathrm{Coker}(d_2) \longrightarrow 0$$

splits and $\mathrm{Hom}(M, P) \cong \mathrm{Hom}(M, \Omega_S) \oplus \mathrm{Coker}(d_2)$. We already saw that $\mathrm{Hom}(M, \Omega_N) \neq 0$. By exactness of the initial sequence we have $\mathrm{Coker}(d_2) \cong \mathrm{Hom}(M, P)/\mathrm{Ker}(d_1) \cong Im(d_1)$. The morphism $d_1 = \mathrm{Hom}(M, \pi)$ is non-zero, since it is the image of a non-zero morphism by an equivalence of categories. So $\mathrm{Hom}(M, P)$ is decomposable and this is a contradiction.

(6) In this case, M is projective, hence is a progenerator. So A and Γ_M are Morita equivalent, so $\operatorname{mod} A$ is equivalent to $\operatorname{mod} \Gamma_M$ and it is not very difficult to check that two equivalent abelian categories have same global dimension. Hence, Γ_M is also semisimple.

The exercises 1, 2 and 3 are independents. The exercises 4 requires Exercise 3 and Exercise 5 depends on Exercises 3 and 4.

Exercise 1 - Center of a category

The center $Z(\mathcal{C})$ of a (small) category $\mathcal{C}$ is the set of all natural endomorphisms of the identity functor of $\mathcal{C}$. Here we ignore all possible set theoretical issues.

(1) Show that the composition of the natural transformations endows $Z(\mathcal{C})$ of a structure of commutative monoid.

(2) Let G be a group and $\mathbf{B}(G)$ be the category having G as endomorphisms of a unique object $\bullet$. What is the center of $\mathbf{B}(G)$?

(3) Let A be an associative ring with unit. Prove that the center of $\operatorname{Mod} A$ is isomorphic to the usual center of the ring A. We expect a detailed answer.

(4) What is the center of the category of sets ?

Exercise 2 - Homology of partially ordered sets

Let $(P, \leq)$ be a partially ordered set. An m-chain in P is a totally ordered subset of P containing exactly $m+1$ elements. The 0-chains are in bijection with the elements of P and the 1-chains are the pairs (x,y) with $x \leq y$ and $x \neq y$. For $n \in \mathbb{N}$, we set $C_n(P)$ the free abelian group on the set of n-chains of P. For $n \in \mathbb{N}^*$, let $d_n : C_n(P) \to C_{n-1}(P)$ be the map defined on an n-chain $(x_0 < \cdots < x_n)$ by

$$d_n(x_0 < \cdots < x_n) = \sum_{i=0}^{n} (-1)^i (x_0 < \cdots < \widehat{x_i} < \cdots x_n),$$

where $(x_0 < \cdots < \widehat{x_i} < \cdots x_n)$ is the $(n-1)$-chain obtained by removing x_i.

(1) Show that $C(P)_\bullet = (C_n(P), d_n)_{n\in\mathbb{N}}$ is a chain complex of abelian groups. Its homology is called the homology of P.

(2) Let $\epsilon : C_0(P) \to P$ be the morphism defined on a 0-chain x by $\epsilon(x) = 1$. Justify that we can augment the complex $C_\bullet(P)$ by setting $C_{-1} = \mathbb{Z}$ and $d_0 = \epsilon$. To avoid confusion, we denote this complex by $\widetilde{C}_\bullet$.

(3) For $x, y \in P$ we say that x and y are *comparable* if $x \leq y$ or $y \leq x$. The poset P is said to be *connected* if for every $x, y \in P$, there exist $x_0, \cdots, x_s \in P$ such that $x_0 = x$, $x_s = y$ and x_i, x_{i+1} are comparable for $i \in \{0, \cdots, s-1\}$. Compute the homology of degree 0 of a connected poset P. What is $H_0(P)$ when P is not connected ? (For the second part of the question, we only expect a sketch of proof).

(4) Let $X = \{1,2,3\}$ and $P = (\mathcal{P}(X), \subset)$ be the set of subsets of X ordered by inclusion. We consider $\overline{P} = P \setminus \{\emptyset, X\}$ ordered by inclusion. What is the homology of $\overline{P}$?

(5) We assume that P has a *smallest element* 0 (i-e such that $0 \leq x \ \forall x \in P$). Prove that $\widetilde{C}_\bullet$ is contractible and deduce the homology of P.

Exercise 3 - Finite dimensional algebras Let A be a finite dimensional algebra over a field k. We recall that the finitely generated A-modules are exactly the A-modules which are finite dimensional over the field k. Let $\operatorname{mod} A$ be the category of finitely generated right A-modules.

(1) Prove that $\operatorname{mod} A$ is an abelian category.

(2) Prove that it has enough projectives.

(3) Prove that $\operatorname{mod} A$ has enough injectives. Hint, we can use the functor $D = \operatorname{Hom}_k(-, k)$.

Exercise 4 - Global dimension

Let $\mathcal{A}$ be an abelian category with enough projectives. The *global dimension* of $\mathcal{A}$ is :

$$\operatorname{gldim}(\mathcal{A}) = \sup\{n \in \mathbb{N} \mid \exists A, B \in Ob(\mathcal{A}) \ : \operatorname{Ext}^n_{\mathcal{A}}(A,B) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Let $M \in \mathcal{A}$, the *projective dimension* of M denoted by $\operatorname{pdim}(M)$, is the smallest integer n such that there exists a projective resolution of M of the form

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

If M does not admit a finite projective resolution we let $\operatorname{pdim}(M) = \infty$.

We say that $\mathcal{A}$ is *semisimple* if $\operatorname{gldim}(\mathcal{A}) = 0$ and $\mathcal{A}$ is *hereditary* if $\operatorname{gldim}(A) \leq 1$. We can assume that the objects of $\mathcal{A}$ are right modules over a ring A.

(1) Prove that the following are equivalent

  (a) $\mathcal{A}$ is semisimple.

  (b) Every object of $\mathcal{A}$ is projective.

  (c) Every object of $\mathcal{A}$ is injective.

   (d) Every short exact sequence in $\mathcal{A}$ splits.

(2) Give an example of a semisimple category.

(3) Let $M \in \mathcal{A}$. Prove that the following are equivalent

   (a) $\mathrm{pdim}(M) \leq n$.

   (b) $\mathrm{Ext}^i_{\mathcal{A}}(M, X) = 0$ for all $i > n$ and all $X \in \mathcal{A}$.

   (c) $\mathrm{Ext}^{n+1}_{\mathcal{A}}(M, X) = 0$ for all $X \in A$.

   (d) If $0 \to M_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ is an exact sequence with P_i projective for all i, then M_n is also projective.

(4) Justify that $\mathrm{gldim}(\mathcal{A}) = \sup\{\mathrm{pdim}(M) \mid M \in \mathcal{A}\}$.

(5) Here we assume that the objects of $\mathcal{A}$ are modules. Prove that $\mathcal{A}$ is hereditary if and only if every submodule of a projective is projective.

(6) Let A be a finite dimensional k-algebra. Prove that the category of finite dimensional right A-modules is hereditary if and only if every right ideal of A is projective.

(7) A finite dimensional k-algebra over a field k is *self-injective* if the regular module $A \in \mathrm{mod}\, A$ is injective. Prove that the global dimension of $\mathrm{mod}\, A$ is either 0 or ∞.

(8) Let $n \geq 1$ and k be a field. Prove that $A = k[X]/(X^n)$ is self-injective and compute its global dimension. Hint one can use Baer's criterion.

Exercise 5 - A non-commutative example of global dimension 2 Let k be a field and A be a finite dimensional k-algebra. Let $\mathrm{mod}\, A$ be the category of finite dimensional right A-modules. We assume that there is an *additive generator* M in $\mathrm{mod}\, A$, meaning that for every A-module X, there are $n \in \mathbb{N}$ and $Y \in \mathrm{mod}\, A$ such that $X \oplus Y = M^n$. In other words, every A-module is a direct summand of a finite direct sum of copies of M. Jordan's reduction theorem can be used to show (we do not ask for a proof) that for $n \geq 1$, the algebra $\mathbb{C}[X]/(X^n)$ has an additive generator. Let $\Gamma_M = \mathrm{End}_A(M)$.

(1) Justify that the functoriality of $\mathrm{Hom}_A(M, X)$ with respect to the first variable induces a structure of Γ_M-module on $\mathrm{Hom}_A(M, X)$. It follows that $F = \mathrm{Hom}_A(M, -)$ is a functor from $\mathrm{Mod}\, A$ to $\mathrm{Mod}\, \Gamma_M$.

(2) Show that F restricts to a functor from $\mathrm{mod}\, A$ to $\mathrm{proj}\, \Gamma_M$, where $\mathrm{proj}\, \Gamma_M$ is the category of finite dimensional projective Γ_M-modules.

(3) (Bonus) Prove that $F : \mathrm{mod}\, A \to \mathrm{proj}\, \Gamma_M$ is an equivalence of categories. We can prove that it is a fully-faithful functor by first looking at the module M and then prove that it is essentially surjective.

(4) Prove that the global dimension of $\mathrm{mod}\, \Gamma_M$ is at most 2. Hint use Question 3.

(5) A non-zero module N is *indecomposable* if it is not equal to the direct sum of two submodules. In other words for all X, Y submodules of N such that $N = X \oplus Y$, we have ($X = 0$ and $Y = N$) or ($X = N$ and $Y = 0$). An A-module is simple if its only submodules are $\{0\}$ and itself.

   (a) If N is an indecomposable A-module, prove that $F(N)$ is a projective indecomposable Γ_M-module.

   (b) If A is not semisimple, we admit that there exist : a non-projective simple module S, an indecomposable projective module P and an epimorphism $\pi : P \to S$. Show that $\mathrm{Hom}(M, \pi)$ is not an epimorphism and not a monomorphism. Prove that the global dimension of Γ_M is exactly 2.

(6) What is happening when A is semisimple ?