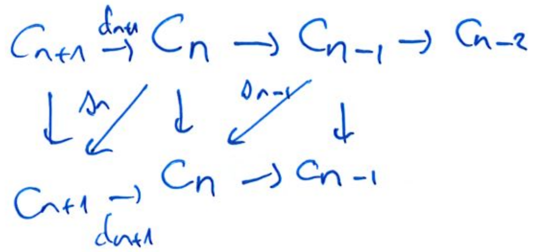
TD2 — **TD2 (5) + (6)**

Exercice 5



$d_{n+1} = d_{n+1} \circ s_n \circ d_{n+1}$

(1) If $Id \sim 0$ then C is split + exact

the homotopy $(s_n)_n$ gives $\quad Id = s_{n-1} d_n + d_{n+1} s_n$

hence $d_n = d_n s_{n-1} d_n + d_n d_{n+1} s_n$ (the last term is 0)

so the complex is split.

Moreover C is homotopy equivalent to 0 hence they have same homology hence C is exact

(By hand: $x \in \ker d_n$ hence $x = s_{n-1} d_n(x) + d_{n+1} s_n(x)$
$\in \operatorname{Im} d_{n+1}$.

(2) <u>Conversely if C is split exact</u> : we have $s_n : C_n \to C_{n+1}$; $d_n s_{n-1} d_n = d_n$

$B_n = \operatorname{Im}(d_{n+1}) \subseteq C_n$

$Z_n = \ker(d_n) \subseteq C_n$

We have $\quad 0 \to Z_n \hookrightarrow C_n \overset{d_n}{\underset{h_n}{\rightleftarrows}} B_{n-1} \to 0$ exact and

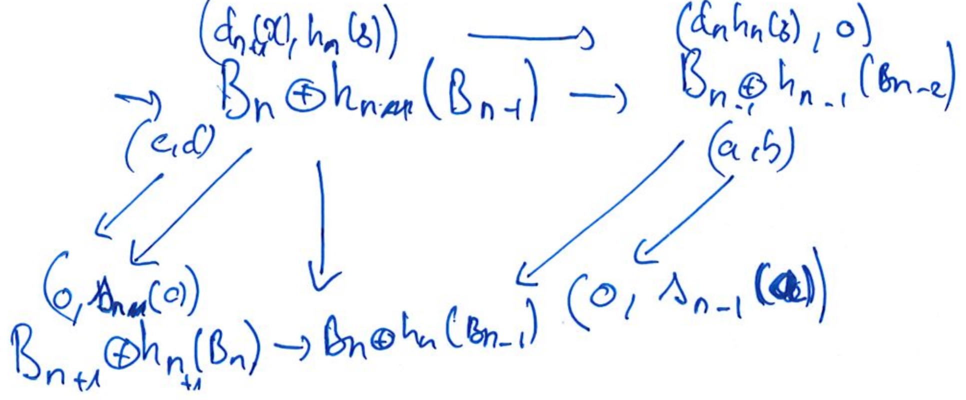
$h_n = s_{n-1}$ is a splitting of d_n hence $C_n = Z_n \oplus h_n(B_{n-1}) \simeq Z_n \oplus B_{n-1}$

now the complex C is exact, hence $B_n = Z_n$

hence $\boxed{C_n = B_n \oplus h_n(B_{n-1})}$ and $C_\bullet = \; \to B_n \oplus h_n(B_{n-1}) \xrightarrow{\begin{pmatrix} 0 & d_n \\ 0 & 0 \end{pmatrix}} B_{n-1} \oplus h_{n-1}(B_{n-2}) \to$

Since $d_n(x) = 0$ when $x \in B_n$ and $d_n h_n(y) = d_n s_{n-1} d_n(y) = d_n(y)$

Now let us build the homotopy



~~well defined $h_{n+1}(c) \in h_{n+1}(B_n)$ by contradic~~

and If $a \in B_{n-1}$ then $a = d_n(x)$ so $s_{n-1} d_n(x) = s_{n-1}(a) \in h_n(B_{n-1})$, with $s_{n-1} d_n(x) = h_n(x)$

Moreover letting h be this homotopy we have $hd + dh(d_{n+1}(x), h_n(3))$

$= (d_{n+1} s_n d_{n+1}(x) = d_{n+1}(x),\ s_{n-1} d_n h_n(3)$
$= s_{n-1} d_n s_{n-1} d_n(3)$
$= s_{n-1} d_n(3)$
$= h_n(3))$

<u>finish the proof.</u>

(2) $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ exact non contractile

(3) $C_\bullet =$ $C_n \to C_{n-1} \to \cdots C_1 \xrightarrow{d_1} C_0 \to 0$ exact with C_i free

($\Rightarrow$ projectives) since $C_1 \underset{s_0}{\overset{d_1}{\rightleftarrows}} C_0 \to 0$ is surjective and C_0 projective it splits

we have s_0 s.t. $d_1 s_0 = Id_{C_0}$ $\Rightarrow$ $\boxed{d_1 s_0 d_1 = d_1}$

<u>De plus</u> $0 \to \ker(d_1) \to C_1 \underset{s_0}{\overset{d_1}{\rightleftarrows}} C_0 \to 0$ se scinde et

$C_1 = s_0(C_0) \oplus \ker(d_1)$

on a alors que $\ker d_1 | C_1$ est projectif et

Comme le complexe est exact, on a $C_1 = \mathrm{Im}\, d_2 \oplus s_0(C_0)$

d'où $\ker d_2 \to C_2 \xrightarrow{d_2} \mathrm{Im}(d_2) \to 0$ se scinde et ($\tilde{s}_1$)

on pose $s_1 : C_1 \to C_2$ par $s_1(d_2(x), s_0(a)) = \tilde{s}_1 d_2(x)$

et on a $d_2 s_1 d_2 = d_2$

+ induction.

(4) Il faut nécessairement un complexe infini des deux côtés

$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \to \cdots$$

est un complexe de $\mathbb{Z}/4$-modules libres qui est exact car $\mathrm{Im}(\times 2) = \ker(\times 2)$

mais n'est pas scindé

car $s : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ be a splitting then $d_n = d_n s_{n-1} d_n = d_n^2 s_n = 0$ since s_n has to be a morphism of $\mathbb{Z}/4\mathbb{Z}$-modules

Ex 6 $f : C_\bullet \to D_\bullet$ morphism of chain complexes.

$(C_\bullet[1])_n = C_{n-1}$ $\quad d = -d^C$

$\hookrightarrow C_0 = C_1$

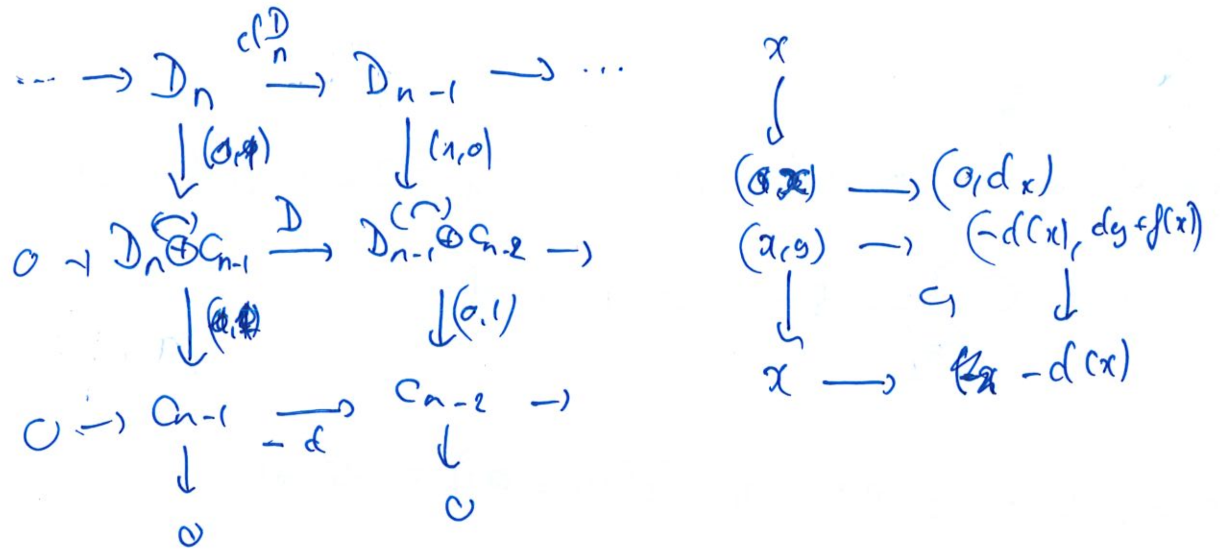
$C \qquad C_1 \to C_0 \to C_{-1}$

$C[1] \quad \cdots \to C_1 \xrightarrow{-d_1} C_0 \xrightarrow{-d_0} C_{-1} \xrightarrow{-d_{-1}} C_{-2} \quad \cdots$

$$\begin{array}{ccc} \mathrm{Cone}(f)_n = & C_{n-1} \oplus D_n \\ \downarrow & -d^C_{n-1} \downarrow \quad \searrow f \quad \downarrow d^D_n \\ \mathrm{Cone}(f)_{n-1} = & C_{n-2} \oplus D_{n-1} \end{array}$$

(1) clear that it is a complex

$$\begin{array}{ccccc} C_{n-1} & \xrightarrow{-d} & C_{n-2} & \xrightarrow{-d} & C_{n-3} \\ \oplus & \searrow f & \oplus & \searrow f & \oplus \\ D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & D_{n-2} \end{array} \quad \begin{cases} d^2 = 0 \\ df - fd = 0 \\ d^2 = 0 \end{cases}$$



just need to check the details.

(2) Apply H_n gives long exact sequence

$$\to H_n(D) \to H_n(\mathrm{Cone}(f)) \to H_n(C[1]) \xrightarrow{\delta} H_{n-1}(D) \to H_{n-1}(\mathrm{Cone}(f)) \to$$

if $x \in H_n(C[1]) \simeq H_{n-1}(C)$

we have $[\delta(x)] = (0,\varphi)^{-1} \circ D \circ (0,1)^{-1}(x)$

$$= (0,\varphi)^{-1} \circ D(0,x)$$

$$= (0,1)^{-1}(f(x)) = [f(x)]$$

hence $\delta = H_{n-1}(f)$

$$\hookrightarrow H_n(C) \xrightarrow{H_n(f)} H_n(D) \xrightarrow{a} H_n(\mathrm{Cone}(f)) \xrightarrow{b} H_{n-1}(C) \xrightarrow{H_{n-1}(f)} H_{n-1}(D) \to H_{n-1}(\mathrm{Cone}(f))$$

$$\to H_{n-2}(C) \xrightarrow{H_{n-2}(f)} \cdots$$

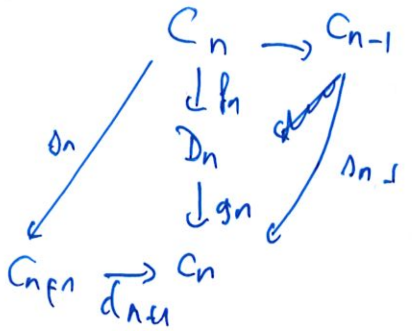
So f qis $\Rightarrow H_n(f)$ iso so $\ker(a) = H_n(D) \to \boxed{a=0}$

$\mathrm{Im}(b) = 0 \Rightarrow \boxed{b=0}$

exactness $\ker(b) = H_n(\mathrm{Cone} f) = \mathrm{Im}(a) \Rightarrow H_n(\mathrm{Cone}(f)) = 0.$

Conversely If Cone(f) is exact, then $H_n(f)$ iso $\forall n$.

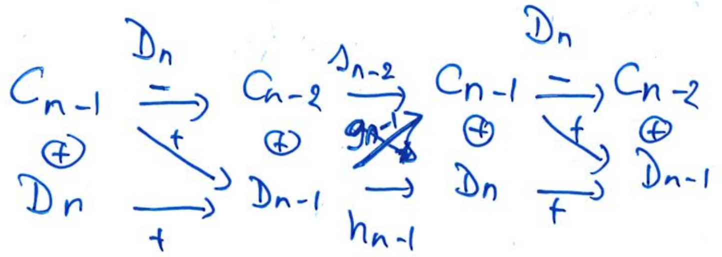
(3) f with g s.t $fg \sim Id$ and $gf \sim Id$



$g_n f_n = Id + s_{n-1} d^C_n + d_{n+1} s_n$

$d^C_{n+1} s_n + s_{n-1} d^C_n = g_n f_n - Id$

de même on a h_n des homotopies pour D_n



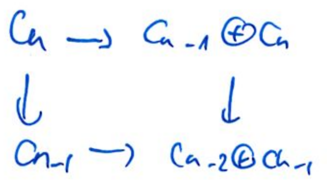
devrait être un splitting

De plus f equiv d'homotopie $\Rightarrow$ f qis $\Rightarrow$ cone exact dans

le cone est contractible d'après l'exercice d'avant.

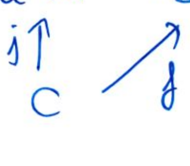
(4) clair

(5) $j : C \to Cone(C)$ l'application canonique

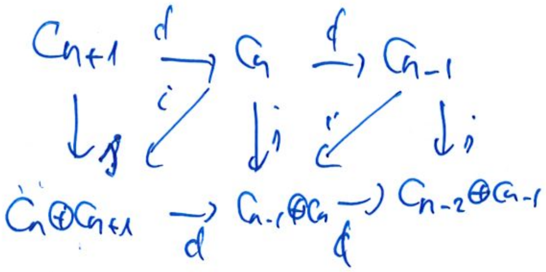


[⇒] alors $f \sim 0$ on pose $s_n : Cone(C)_n \to D_n$

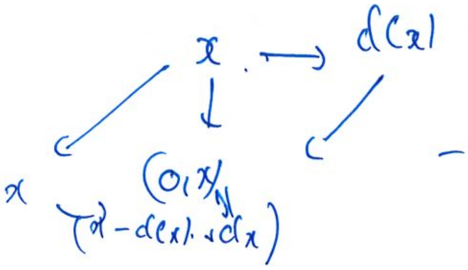
[⇐] Cone (C) $\xrightarrow{?}$ D



on montre $j \sim 0$ le résultat suit



$i : C_n \longrightarrow C_n$



$\square$