TD4 Ex 1 (1)

$F : \mathcal{A} \to \mathcal{B}$ additif $\mathcal{U}$ exact

$\mathcal{U}(L_i F) \simeq L_i(\mathcal{U}(F))$ ?

$L_i F)(M) = H_i(F(P_M))$ $P_M$ proj reso

so $\mathcal{U}(L_i F(M)) = \mathcal{U} H_i(F(P_M)) \simeq H_i(\mathcal{U}F(P_M))$

$= L_i(\mathcal{U}F)(M)$

we just need to check that exact functor commutes with homology

$H^i(X) \simeq \mathrm{Coker}(\mathrm{Im} d_{i+1} \to \ker d_i)$

so $\mathcal{U} H^i(X) \simeq \mathrm{Coker}(\mathcal{U}(\mathrm{Im} d_{i+1}) \to \mathcal{U}(\ker d_i))$

$\simeq \mathrm{Coker}(\mathrm{Im}\,\mathcal{U}d_{i+1} \to \ker(\mathcal{U}d_i)) \simeq H_i(\mathcal{U}X)$.

(2) (a) $F : A \to \mathcal{B}$ additive

(a) let $0 \to A \to B \to C \to 0$ ses then we have a lg exact seq

$$\cdots \to L_1 F(C) \to \underbrace{L_0 F(A) \to L_0 F(B) \to L_0 F(C) \to 0}$$

So $L_0(F)$ is right exact

(b) $L_m L_n(F)$ ? $M$ and $P_M \to M \to 0$ a proj resolution.

then $L_m(L_n F)(M) = H_m[L_n(P_M)]$

$$\cdots \to L_n F(P_1) \to L_n F(P_0) \to 0$$

if $n > 0$ then $L_n F(P) = 0$ $\forall$ proj hence no homology.

if $n = 0$ we have

$L_m(L_0 F)(M) = H_m(L_0 F(P_\cdot))$ $\cdots \to L_0 F(P_1) \to \underbrace{L_0 F(P_0) \to L_0 F(M) \to 0}_{\text{exact}}$

Hence $\boxed{L_0 L_0 F(M) \simeq H_0(L_0(F)(P_\cdot)) \simeq L_0 F(M)}$

~~and $L_0 F(P_0) =$~~

$n=0$ $\quad L_m L_0 F = ?$

$$\cdots \to P_1 \to P_0 \to M \to 0$$

$L_0 F$ : $\quad \cdots \to L_0F(P_1) \to L_0F(P_0) \to L_0F(M) \to 0$ exact

So $H_0(L_0F)(M) \cong L_0F(M)$ and moreover $L_0F(P_i) = F(P_i)$ because $0 \to P_i \to P_i \to 0$ is a proj res so

$L_\bullet(L_0F)(M)$ is the homology of $F(P_2) \to F(P_1) \to F(P_0) \to \cdots$

So by definition it is $L_m(L_0F)(M)$.

<u>Ex2</u> (1) $0 \to M \to P \to A \to 0$ exact with $P$ $F$-acyclic. Then

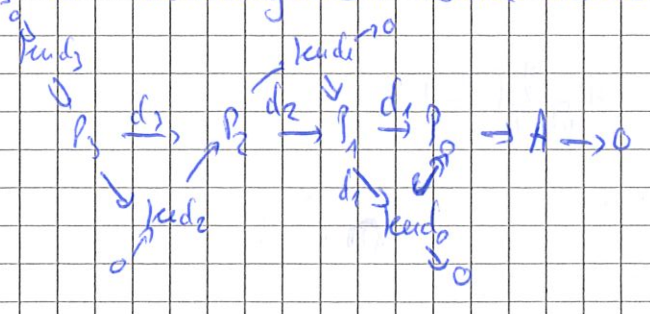
Apply $F$ and get

$$0 \to L_2F(A) \to L_1F(M) \to 0 \to L_1F(A) \to F(M) \to F(P) \to F(A) \to 0$$

So $L_iF(A) \cong L_{i-1}F(M)$ $\forall\, i \geq 2$

$L_1F(A) = \ker(F(M) \to F(P)) \to 0$

(2) The idea is to add many ses: same assume $P_i$ $F$ acyclic:




So we get $\quad 0 \to M_m \to P_m \to \ker d_{m-1} \to 0$

$\quad 0 \to \ker d_{m-1} \to P_{m-1} \to \ker d_{m-2} \to 0$

$\quad 0 \to \ker d_1 \to P_1 \to \ker d_0$

$\quad 0 \to \ker d_0 \to P_0 \to A \to 0$

$m+1$ ex sequences

<u>Try Q1</u>:

$L_i(F)(A) \simeq L_{i-1}F(\ker d_0) \simeq \cdots \simeq L_{i-m-1}F(M_n)$
if $i \geq m+2$

(a) For $i = m+1$ we have $L_{m+1}F(A) \simeq L_1 F(\ker d_{m-1})$
apply F to first res give
$0 \to L_1F(\ker d_{m-1}) \to F(M_m) \to F(P_m)$ exact.

(3) $Q_\bullet \xrightarrow{\Sigma} A$ F-acyclic resolution of $A$.

$0 \to \ker(d_{n-1}) \xhookrightarrow{\alpha} Q_{n-1} \to \cdots \to Q_0 \to A \to 0$ exact so with $Q_e$ we get $L_n F(A) \simeq \ker F(\alpha)$ and it remains to check that this is the nth homology of $F(Q_\bullet)$

We have
$$\begin{tikzcd} Q_{n+1} \arrow[r, "d_{n+1}"] & Q_n \arrow[r, "d_n"] \arrow[d, "\pi_n"'] & Q_{n-1} \arrow[r, "d_{n-1}"] & Q_{n-2} \\ & \ker d_{n-1} \arrow[ur, hook, "\alpha"'] \arrow[d] & & \\ & 0 & & \end{tikzcd}$$

as F is right exact it preserves epi so

$$\begin{tikzcd} F(Q_{n+1}) \arrow[r, "F(d_{n+1})"] \arrow[d, "0"] & F(Q_n) \arrow[r, "F(\pi)"] \arrow[d, "F(d_n)"] & F(\ker d_{n-1}) \arrow[r] \arrow[d, "F(\alpha)"] & 0 \\ 0 \arrow[r] & 0 \arrow[r, "0"] & F(Q_{n-1}) \arrow[r, "Id"'] & F(Q_{n-1}) \end{tikzcd}$$
exact

So apply snake lemma and we have $F(Q_{n+1}) \xrightarrow{F(d_{n+1})} \ker(F d_n) \to \ker F(\alpha) \to 0$

So $\ker(F(\alpha)) \simeq \dfrac{\ker F(d_n)}{\operatorname{Im} F(d_{n+1})} = H_n F(Q)$. □