CATÉGORIES, ÉQUIVALENCES, FONCTEURS ADJOINTS

Les exercices 1,4,5,6 et 9 sont à travailler en priorité. Les exercices avec $\star$ sont plus difficiles, plus exotiques ou en dehors du programme et peuvent être ignorés en première lecture.

1. Catégories et équivalences

Exercice 1 -

(1) Quels sont les monomorphismes, les épimorphismes et les isomorphismes dans la catégorie des ensembles ? Même question pour la catégorie des espaces topologiques.

(2) Quels sont les monomorphismes et les épimorphismes($\star$) dans la catégorie des groupes ?

(3) Montrer que l'inclusion $\mathbb{Z} \to \mathbb{Q}$ est un monomorphisme et un épimorphisme dans la catégorie des anneaux unitaires. Est-ce un isomorphisme ?

$\star$ Le foncteur qui oubli la structure multiplicative de la catégorie des anneaux unitaires vers celle des groupes abéliens est-il plein ? fidèle ? essentiellement surjectif ? Mêmes questions pour le foncteur qui envoie un anneau unitaire sur son groupe des unités et pour le foncteur qui oubli l'unité de la catégorie des anneaux unitaires vers celle des anneaux non unitaires. Ici on attend une réponse intuitive plutôt que très précise, car certaines questions sont difficiles.

Correction

(1) It is easy to check that an injective map is a monomorphism and a surjective map is an epimorphism. Assume that f is a monomorphism and let $x, y \in X$ such that $f(x) = f(y)$. We let $W = \{0\}$ be a set with one element and $1_x : W \to X$, $1_y : W \to X$ the maps sending 0 to x and y. Then we have $f \circ 1_x = f \circ 1_y$, so $1_x = 1_y$ because f is a monomorphism, so $x = y$ and f is injective.

If f is a surjective map, consider $Z = \{0, 1\}$ a set with two elements. Let $g : Y \to Z$ be the map sending $y \in Y$ to 1 and $h : Y \to Z$ the map defined by :

$$h(y) = \begin{cases} 1 & \text{if } y \in f(X), \\ 0 & \text{otherwise.} \end{cases}$$

We have $h \circ f(x) = 1 = g \circ f(x)$, so $h = g$ since f is an epimorphism, so $f(X) = Y$ and f is surjective.

For the topological spaces we can use the same arguments however we need to choose a topology on the sets such that all maps are continuous. For W we can choose the discrete topology (largest topology) and for Z the trivial topology (smallest topology).

(2) Since a group is a set with an extra property (in fancy words the category of groups is a concrete category) injective group morphisms are monomorphisms and surjective group morphisms are epimorphisms. However, the two constructions of the maps g and h in the first question do not give group morphisms hence we need to be more careful.

(a) If $f : G \to H$ is a monomorphism, let $x, y \in G$ such that $f(x) = g(y)$. Now if $z \in G$, there is a group homomorphism $f_z : \mathbb{Z} \to G$ defined by $f_z(n) = z^n := z \times z \times \cdots \times z$, n times. Here we use the convention that $f_z(0) = 1_G$ and for $n < 0$, $f_z(-n) = (z^{-1})^n$. In categorical notation we have $\mathrm{Hom}_{Gr}(\mathbb{Z}, G) \cong G$. Then we have $f \circ f_x = f \circ f_y$ so $f_x = f_y$ and $x = y$.

(b) If $f : G \to H$ is an epimorphism we let $K = f(G)$. We consider the left action of H on the cosets H/K. This gives a group homomorphism $g : H \to Sym(H/K)$ where $g(x)$ sends yK to xyK. Now add an extra element ∞ to H/K and we extend g as a group homomorphism from H to $Sym(H/K \sqcup \{\infty\})$ by setting $g(\infty) = \infty$. Finally we set $h = (K, \infty) \circ g \circ (K, \infty)$ where (K, ∞) is the transposition exchanging the trivial coset and the extra element ∞.

For $x \in G$, we have :
— For $y \notin K$, $(g \circ f(x))(yK) = f(x)yK$.
— $(g \circ f(x))(K) = K$ and $(g \circ f(x))(\infty) = \infty$.
— $(h \circ f(x))(K) = \big((K, \infty)g(f(x))(K, \infty)\big)(K) = K$.
— $(h \circ f(x)(\infty) = \big((K, \infty)g(f(x))\big)(K) = \infty)$.
— For $y \notin K$, $(g \circ h(x))(yK) = f(x)yK$.

Since f is an epimorphism, we have $g = h$. If $x \in H$ we have $g(x)(K) = xK$ and $h(x)(K) = \big((K, \infty) \circ g(x) \circ (K, \infty)\big)(K) = K$. To have equality we must have $x \in K$, so $K = H$ and the map f is surjective.

(3) The inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is an injective map so it is a monomorphism. Let $g, h : \mathbb{Q} \to A$ be two ring homomorphisms such that $g \circ i = h \circ i$. Then g and h coincides over $\mathbb{Z}$. Since $g(p/q) = g(p) * g(q)^{-1}$, we see that $g = h$. So i is both a monomorphism and an epimorphism but $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic in the category of rings. For example : if α is an inverse isomorphism, then $\alpha(1/2)$ is an inverse of 2 in $\mathbb{Z}$.

(4) The forgetful functor from the category of unitary rings to the category of abelian group is :

(a) Faithful, since the functor is the identity on morphism.

(b) Not full since a morphism of unitary rings must send 1 to 1 and this is not the case for a group homomorphism.

(c) This question is tricky since if the rings are not required to be unitary we can always define $a \cdot b = 0$ and so every abelian group can be seen as the additive group of a ring (with no units). If the rings are unitary then the functor is not essentially surjective. Consider the additive group $\mathbb{Q}/\mathbb{Z}$. If has the property that every element has finite order but there is no upper bound on the order. Let us assume that there is a ring structure on it when a unit element 1. Let k be its order and $a \in \mathbb{Q}/\mathbb{Z}$, then $a + a + \cdots + a = (1 + 1 + \cdots + 1) \cdot a = (k \cdot 1) \cdot a = 0 \cdot a = 0$ so the order of a is bounded by k and this is a contradiction.

The functor sending a ring to its group of units is :

(a) not full. The idea is : by additivity a ring homomorphism is fixed on the subgroup generated by 1. This is a strong property so the functor is probably not full : for example compare $\mathrm{End}_{Ring}(\mathbb{F}_5)$ with $\mathrm{End}_{Gr}(\mathbb{F}_5^{\times})$.

(b) not faithful. The idea is : the group of units can be very small so 'to coincide' on a small part is probably not strong enough to coincide. For example compare $\mathrm{End}_{Ring}(\mathbb{Z}/2 \times \mathbb{Z}/2)$ and $\mathrm{End}_{Gr}((\mathbb{Z}/2 \times \mathbb{Z}/2)^{\times})$. The identity and the map sending (x, y) to (y, x) coincide on the unit $(1, 1)$.

(c) This is a really hard question.
See `https://math.stackexchange.com/questions/384422/which-finite-groups-are-the-group-of-units-of-some-ring`!

The functor from unitary ring to ring is :

(a) Faithful : this is clear since the functor is the identity on morphisms.

(b) Not full : The map from $\mathbb{Z} \to \mathbb{Z}$ sending 1 to -1 is not in the image of the functor.

(c) It is not essentially surjective : if a ring without unit is isomorphic via f to a ring with unit, then $f(1)$ is a unit...

You can of course continue to play this game with you favorite examples!

Exercice 2 - $\star$ Un *groupoïde* est une catégorie dans laquelle tout morphisme est inversible.

(1) Soit $\mathcal{C}$ une catégorie. Justifier qu'il existe un groupoïde qui est sous-catégorie de $\mathcal{C}$ ayant les mêmes objets. On l'appelle le *coeur* de $\mathcal{C}$.

(2) Si $(P, \leq)$ un préordre. Rappeler de quelle façon P peut se voir comme une catégorie et décrire son coeur.

(3) Soit $\mathcal{G}$ un groupoïde. Vérifier que pour tout objet X de $\mathcal{G}$, l'ensemble $\mathrm{Hom}_{\mathcal{G}}(X, X)$ est un groupe.

(4) Soient X et Y sont deux objets de $\mathcal{G}$ tels que $\mathrm{Hom}_{\mathcal{G}}(X, Y)$ est non-vide. Montrer que $\mathrm{End}_{\mathcal{G}}(X)$ et $\mathrm{End}_{\mathcal{G}}(Y)$ sont des groupes isomorphes. Si tous les $\mathrm{Hom}_{\mathcal{G}}(X, Y)$ sont non vides, on dira que le groupoïde est *connexe*.

(5) Soit $\mathcal{G}$ un groupoïde connexe. Montrer que $\mathcal{G}$ est équivalent à la catégorie $\mathbf{B}G$ d'un groupe G. Construire une équivalence quasi-inverse.

(6) Soit M un espace topologique. On appelle $\pi(M)$ la catégorie dont les objets sont les éléments de M et les morphismes sont les classes d'homotopie des chemins. Justifier que $\pi(M)$ est un groupoïde.

(7) Justifier l'emploi du mot "connexe" ci-dessus. Si $x \in M$, qu'est-ce que le groupe $\mathrm{End}_{\pi(M)}(x)$?

Exercice 3 - Montrer que la catégorie des préordres finis est équivalente à la catégorie des espaces topologiques finis. Si $(X, \leq)$ est un préordre, on pourra utiliser les parties fermées vers le haut pour construire une topologie et inversement si $(X, \mathcal{T})$ est un espace topologique on peut poser $x \leq y$ si x est dans la fermeture de $\{y\}$.

Correction A preorder is a set X with a binary relation $\leq$ which is reflexive and transitive. The category of preorders has for objects the preorders and for morphisms the increasing maps.

A topological space is a set X with a topology $\mathcal{T} \subseteq \mathcal{P}(X)$ such that $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$ and $\mathcal{T}$ is closed under unions and finite intersections.

If $(X, \leq)$ is a preorder we say that $U \subseteq X$ is upper-closed if $x \in U$ and $x \leq y$ implies that $y \in U$. Let $\mathcal{T}_{\leq} = \{U \in \mathcal{P}(X) \mid U \text{ is upper-closed}\}$. It is immediate to check that $\mathcal{T}_{\leq}$ is a topology. Moreover if $f : (X, \leq) \to (Y, \leq)$ is an increasing map, then $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. So $F : (X, \leq) \mapsto (X, \mathcal{T}_X)$ is a functor from the category of preorder to the category of topological spaces.

Conversely if $(X, \mathcal{T})$ is a topological space we set $\leq_{\mathcal{T}}$ the binary relation defined by $x \leq_{\mathcal{T}} y$ if and only if $x \in cl(y)$ where $cl(y)$ is the smallest closed subset containing y. We have $x \in cl(x)$ so the relation is reflexive. If $x \in cl(y)$ and $y \in cl(z)$, then $x \in cl(y) \subseteq cl(z)$ so the relation is transitive. If $f : X \to Y$ is a continuous map one has to check that f is increasing with respect to the preorder relations. Let $x \leq y \in X$, and consider a closed subset F of Y containing $f(y)$. Then $y \in f^{-1}(F)$ which is a closed subset, so $cl(y) \subseteq f^{-1}(F)$ so $x \in f^{-1}(F)$, so $f(x) \in F$. It follows that $f(x) \in cl(f(y))$. In conclusion, we have a functor $G : (X, \mathcal{T}) \mapsto (X, \leq_{\mathcal{T}})$ from the category of topological spaces to the category of preorders. Moreover we are in a particularly nice situation : the two functors are the identity on morphisms.

It remains to check that when we restrict to finite preorders and finite topological spaces these two functors are two equivalences of categories quasi-inverse of each other.

(1) We have $x \leq_{\mathcal{T}_{\leq}} y$ if and only if x is in every closed set containing y. This is equivalent to y is in every open set containing x. Since the open sets for $\mathcal{T}_{\leq}$ are the upper-closed sets, this is equivalent to y is in the intersection of all upper-closed sets containing x. Considering $(x) = \{z \in X \mid x \leq z\}$ we have $x \leq y$. Conversely if $x \leq y$, then y is in all upper-closed sets containing x, so $\leq = \leq_{\mathcal{T}_{\leq}}$.

(2) Now we consider the topology $\mathcal{T}_{\leq_{\mathcal{T}}}$ and we have to prove that this is equal to $\mathcal{T}$. Recall that $x \leq_{\mathcal{T}} y$ if and only if y is in every open set containing x. If U is an open set containing x and $x \leq_{\mathcal{T}} y$, then $y \in U$ so U is upper-closed for $\leq_{\mathcal{T}}$. We denote by $O(x)$ the intersection of all open sets of $\mathcal{T}$ containing y. Since our topological space is finite this intersection is finite and this is an open set. Moreover if U is upper-closed for $\leq_{\mathcal{T}}$ and $y \in O(x)$ we have $x \leq_{\mathcal{T}} y$, so $O(x) \subseteq U$. Finally we obtain that $U = \bigcup_{x \in U} O(x)$ is an open set of $\mathcal{T}$ as an union of open sets.

(3) These facts prove that FG and GF are the identity functors of their respective categories.

Exercice 4 - La catégorie des ensembles est-elle équivalente à sa catégorie opposée ? On pourra, par exemple utiliser le fait que l'ensemble vide possède la propriété suivante : si $f : X \to \varnothing$ est une application, alors f est un isomorphisme.

Correction

Let us start with a quick (slightly hand waving) solution : the property holds for the category of sets and it is purely defined in categorical terms it is preserved by equivalence of categories. So if Set and Set^{op} are equivalent, the category Set^{op} has such an object. Unwrapping the definition, this means that there is an object Z in $Sets$ such that any map with domaine Z is an isomorphism. Since we can always add a new element in Z there is no such set.

Let us move to a formalization of this argument : let $F : Set \to Set^{op}$ be an equivalence of category with quasi-inverse G and let $Z = F(\varnothing)$. Let $f \in \mathrm{Hom}_{Set^{op}}(X, Z)$. Then $G(f) : G(X) \to G(Z) \cong \varnothing$ is an isomorphism. So $G(f)$ is an isomorphism and $FG(f)$ is an isomorphism because a functor preserves isomorphisms. We conclude that f is an isomorphism because $FG \cong Id$.

Going back to Set, we have an object Z such that any morphism from $Z \to X$ is an isomorphism. As explained above this is a contradiction. Actually one can say more about the object Z. Since the empty set is an initial object in Set, its image by an equivalence of category is an initial object in Set^{op}, hence a final object in Set. So Z is just a singleton and the contradiction is even easier.

Exercice 5 - Donner un exemple de catégorie équivalente à sa catégorie opposée.

Correction The category of finite dimensional vector spaces. Check the details !

Exercice 6 - Centre d'une catégorie, Exam 2022.

Le centre $Z(\mathcal{C})$ d'une (petite) catégorie $\mathcal{C}$ est l'ensemble des endomorphismes naturels du foncteur $\mathrm{Id}_{\mathcal{C}}$. Dans cet exercice, on ignore les éventuels problèmes ensemblistes.

(1) Montrer que la composition des transformations naturelles fait du centre de $Z(\mathcal{C})$ un monoïde commutatif.

(2) Soit G un groupe et $\mathbf{B}(G)$ la catégorie avec G comme endomorphismes d'un unique objet $\bullet$. Quel est le centre de $\mathbf{B}(G)$?

(3) Lorsque A est un anneau, démontrer que le centre de $\mathrm{Mod}\,A$ est isomorphe au centre de l'anneau A. On attend ici une réponse détaillée.

(4) Quel est le centre de la catégorie des ensembles ?

Correction

(1) The first task is to unwrap the definition. An endomorphism of the identity functor of $\mathcal{C}$ is a collection $\eta = (\eta_c)_{c \in Ob(\mathcal{C})}$ where $\eta_c \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ such that for every $f : c \to c' \in \mathrm{Mor}(\mathcal{C})$, we have a commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ c' & \xrightarrow[\eta_{c'}]{} & c' \end{array}$$

If α and β are two natural transformations of the identity functor. Then for each object c, we have $\alpha_c : c \to c$ and $\beta_c : c \to c$. Hence, the naturality applied to $\gamma = \alpha$ and $f = \beta_c$ gives, $\beta_c \circ \alpha_c = \alpha_c \circ \beta_c$.

(2) Since $\mathbf{B}(G)$ has only one object, an endomorphism of the identity functor is an element $h \in G$ such that $\forall g \in G$, we have $gh = hg$. That is an element of the center of G.

(3) If η is a natural transformation of the identity of $\operatorname{Mod} A$, then $\eta_A \in \operatorname{End}_A(A)$, so $\eta_A(1) \in A$. Moreover, for $a \in A$, the left multiplication l_a by a on A is a morphism of right A-modules. Hence, the naturality of η gives $l_a \circ \eta_A = \eta_A \circ l_a$. On the element $1 \in A$, we get $a\eta_A(1) = \eta_A(a) = \eta_A(1 \cdot a) = \eta_A(1)a$. So $\eta_A(1) \in Z(A)$. Hence we have a morphism $\Psi : Z(\operatorname{Mod} A) \to Z(A)$. It is a ring homomorphism, since $\Psi(Id) = 1$, $\Psi(\eta + \gamma) = \eta_A(1) + \gamma_A(1)$ and $\psi(\eta\gamma) = \eta_A(\gamma_A(1)) = \eta_A(1 \cdot \gamma_A(1)) = \eta_A(1)\gamma_A(1)$.

Conversely, if $z \in Z(A)$, we construct a family of morphisms $R_z = (R_{z,M})_{M \in \operatorname{Mod} A}$, where $R_{z,M} : M \to M$ is the right multiplication by z. Since $z \in Z(A)$, this is a morphism of right A-modules. Moreover, it is a natural transformation of the identity functor of $\operatorname{Mod} A$. Indeed, if $f : M \to N$ is a morphism of right A-modules, we have $\big(f \circ R_{z,M}\big)(m) = f(mz) = f(m)z = \big(R_{z,N} \circ f\big)(m)$. This gives a morphism Φ from $Z(A)$ to $Z(\operatorname{Mod} A)$.

It is clear that $\Psi \circ \Phi(z) = z$. If $\eta \in Z(\operatorname{Mod} A)$, we have that $\Phi \circ \Psi(\eta)_M$ is the right multiplication on M by $\eta_A(1)$. If $m \in M$, let $f : A \to M$ be the morphism of right-modules such that $f(1) = m$. By naturality of η, we have a commutative square :

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[\eta_M]{} & M \end{array}$$

So, $f(\eta_A(1)) = \eta_M(f(1)) = \eta_M(m)$. And $f(\eta_A(1)) = f(1 \cdot \eta_A(1)) = f(1)\eta_A(1) = m\eta_A(1)$. Hence η_M is the right multiplication by $\eta_A(1)$ and this finishes the proof.

(4) The identity functor of the category of sets is isomorphic to $\operatorname{Hom}(1, -)$ where 1 is a set with one element. By Yoneda Lemma we have that $Z(Set) \cong End(\operatorname{Hom}(1,)) \cong \operatorname{Hom}(1, 1)$. This is a group with only one element.

2. Catégories de modules

Exercice 7 - Lemme des cinq Considérons le diagramme commutatif de R-modules

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 & \xrightarrow{\alpha_4} & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \xrightarrow[\beta_1]{} & N_2 & \xrightarrow[\beta_2]{} & N_3 & \xrightarrow[\beta_3]{} & N_4 & \xrightarrow[\beta_4]{} & N_5 \end{array}$$

où les lignes sont exactes. Montrer que :

(1) Si f_1 est surjective et f_2 et f_4 sont injectives alors f_3 est injective.

(2) Si f_5 est injective et f_2 et f_4 sont surjectives alors f_3 est surjective.

(3) En déduire que si f_1, f_2, f_4 et f_5 sont des isomorphismes, alors f_3 aussi

Correction This can be done by 'diagram chasing'. We only solve the first question, the rest being similar. Let $m \in M_3$ such that $f_3(m) = 0$. We have $0 = \beta_3 f_3(m) = f_4\alpha_3(m)$. Since f_4 is injective we have $m \in Ker(\alpha_3) = Im(\alpha_2)$. So there is $m_2 \in M_2$ with $m = \alpha_2(m_2)$ and $0 = f_3(m) = f_3\alpha_2(m_2) = \beta_2 f_2(m)$. Hence $f_2(m) \in Ker(\beta_2) = Im(\beta_1)$. So $f_2(m_2) = \beta_1(n)$ for some $n \in N_1$. Since f_1 is surjective there is $m_1 \in M_1$ such that $f_2(m_2) = \beta_1 f_1(m_1) = f_2\alpha_1(m_1)$. Since f_2 is injective we have $m_2 = \alpha_1(m_1)$ and $m = \alpha_2 \circ \alpha_1(m_1) = 0$.

Exercice 8 - Lemme du serpent

(1) Montrer qu'étant donné un carré commutatif de R-modules

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A' & \longrightarrow & B' \end{array}$$

on peut le compléter d'une seule manière en un diagramme commutatif

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\ker\alpha & \longrightarrow & \ker\beta \\
\downarrow & & \downarrow \\
A & \longrightarrow & B \\
\downarrow \alpha & & \downarrow \beta \\
A' & \longrightarrow & B' \\
\downarrow & & \downarrow \\
\operatorname{Coker}\alpha & \longrightarrow & \operatorname{Coker}\beta \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

(2) Soit un diagramme commutatif de R-modules

$$\begin{array}{ccccccc}
 & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\
 & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\
0 \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' &
\end{array}$$

où les lignes sont exactes. Montrer qu'il existe un morphisme $\delta : \ker\gamma \longrightarrow \operatorname{Coker}\alpha$ qui rend la suite

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma$$

exacte.

(3) Montrer que si de plus $A \longrightarrow B$ est injective alors $\ker\alpha \longrightarrow \ker\beta$ l'est aussi, et si $B' \longrightarrow C'$ est surjective alors $\operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma$ l'est aussi.

Exercice 9 - Suites scindées

(1) Soit $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ une suite exacte courte de R-modules. Montrer que les propositions suivantes sont équivalentes :

(a) f admet une rétraction (i.e. il existe $B \xrightarrow{r} A$ tel que $rf = id_A$).

(b) g admet une section (i.e. il existe $C \xrightarrow{s} B$ tel que $gs = id_C$).

(c) f admet une rétraction r et g une section s telles que $fr + sg = id_B$.

(d) Il existe un isomorphisme $B \xrightarrow{h} A \oplus C$ qui rend le diagramme suivant commutatif

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \simeq\downarrow h & & \| & & \\
0 & \longrightarrow & A & \xrightarrow[i_A]{} & A \oplus C & \xrightarrow[\pi_C]{} & C & \longrightarrow & 0
\end{array}$$

Lorsque ces propositions sont satisfaites, la suite est dite *scindée.*

(2) Montrer que toute suite exacte courte d'espaces vectoriels est scindée.

(3) Montrer que la suite exacte courte de $\mathbb{Z}$-modules $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ n'est pas scindée.

(4) Déterminer toutes les suites exactes courtes de $\mathbb{Z}$-modules $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\alpha} M \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$. Sont-elles scindées ?

3. Foncteurs adjoints et limites

Exercice 10 -

(1) Soit R un anneau commutatif et M un R-module. Montrer que les foncteurs $- \otimes_R M$ et $\hom_R(M, -)$ de la catégorie des R-modules dans elle-même forment une paire adjointe. On donnera l'unité et la counité de l'adjonction.

(2) ⋆ Soit V un espace vectoriel. Montrer que le foncteur de la catégorie des espaces vectoriels dans elle-même donné par $W \mapsto W \otimes V$ a un adjoint à gauche et un adjoint à droite si et seulement si V est de dimension finie. Observer que ces adjoints sont canoniquement isomorphes dans ce cas. On pourra utiliser qu'un adjoint à droite preserve les limites, et on pourra montrer que $V^* \otimes_k - \cong \mathrm{Hom}(V, -)$ quand V est de dimension finie.

(3) Montrer que le foncteur évident de la catégorie des groupes abéliens dans la catégorie des groupes a un adjoint à gauche.

(4) Parmi vos constructions mathématiques favorites, trouvez en qui viennent d'un foncteur. Est-il représentable ? A-t-il des adjoints ?

Correction

(1) This was done in class or it will be !

(2) This question is more difficult but let us try ☺. First assume that $W \otimes_k -$ has a left adjoint, then $W \otimes_k -$ is a right adjoint so its preserves arbitrary limits, in particular it preserves arbitrary products. There is always a canonical map $can : W \otimes_k \prod_{i \in I} V_i \to \prod_{i\in I} W \otimes_k V_i$. Indeed in the left handside we have the collection of maps $id_W \otimes \pi_i$ where π_i is the canonical projection of the product onto the ith components. The universal property of the product gives the map can. This map is easy to describe, it sends $w \otimes (v_i)_i$ to $(w \otimes v_i)_i$. The functor $W \otimes_k -$ preserves products if this canonical map is an isomorphism. We apply it to $I = W$ and $V_i = k$. We have :
$$W \otimes_k \big(\prod_{w\in W} k\big) \to \prod_{w\in W} W \otimes_k k \cong \prod_{w \in W} W.$$
On the right hand-side we have the element $\prod_{w\in W} w$. Because the map is an isomorphism, there is an element $x = \sum_{j=1}^{n} w_j \otimes (a^i_j)_{i\in W}$ such that $can(x) = \prod_{w\in W} w$. Hence $\prod_{w\in W} w = \prod_{w\in W} (\sum_{j=1}^n a^w_j w_j)$. This says that the family $(w_1, \cdots, w_n)$ is a generating family for the vector space W. Hence W is finite dimensional.

Conversely if W is finite dimensional we use the second part of the question as a hint : the left adjoint of $W \otimes_k -$ should be $\mathrm{Hom}_k(W, -)$! let us try to prove this, if we can do it by uniqueness of adjoint the second part of the question follows.

To that extends we show that $V^* \otimes_k -$ is naturally isomorphic to $\mathrm{Hom}(V, -)$. Let W be a vector space, there is a natural map $\phi_W : V^* \otimes W \to \mathrm{Hom}(V, W)$ defined by $\phi_W(\alpha \otimes w)(v) = \alpha(v)w$. It is easy to see that these maps form a natural transformation from $V^* \otimes_k -$ to $\mathrm{Hom}(V, -)$. Since V is finite dimensional, one can choose a basis $v_1, \cdots, v_n$ with dual basis $v_1^*, \cdots, v_n^*$. Then let $\psi_W : \mathrm{Hom}(V, W) \to V^* \otimes_k W$ defined by $\psi_W(f) = \sum_{i=1}^n v_i^* \otimes f(v_i)$ and check that this is an inverse isomorphism of ϕ_V.

It follows that $V \otimes_k - \cong \mathrm{Hom}(V^*, -)$. Hence by the classical adjunction $\mathrm{Hom}(V, -) \cong V^* \otimes_k -$ is a left adjoint to $V \otimes_k -$.

(3) If G is a group, we denote by $D(G)$ the subgroup generated by the commutators. Then $G/D(G)$ is called the 'abelianization" of G. One has to check that this is indeed a functor and that this is a left adjoint to the forgetful functor. We leave the details to the reader.

(4) If you do Lie theory : there is a forgetful functor from the category of associative unital algebras to the category of Lie algebras. It sends $(A, +, *)$ to $(A, +, [-,-])$ with $[a, b] = ab - ba$. What is its left adjoint ?

Exercice 11 - Décrire, à l'aide de propriétés universelles, toutes les limites et colimites classiques du cours : (co)produit, co(égalisateur), produit fibré, somme amalgamée, etc...

Exercice 12 - ⋆

(1) Montrer que la catégorie des ensembles est complète et cocomplète.

(2) Soient $\mathcal{C}$ une catégorie localement petite et J une petite catégorie. On suppose que $\mathcal{C}$ admet des J-limites. Alors montrer que pour tout $X \in \mathcal{C}$ et $F : J \to \mathcal{C}$, il y a un isomorphisme fonctoriel en X et en F :
$$\lim \mathrm{Hom}_{\mathcal{C}}(X, F) \cong \mathrm{Hom}_{\mathcal{C}}(X, \lim F).$$

(3) Montrer que la catégorie des espaces topologiques est complète et cocomplète.

(4) Il est facile de construire des catégories qui ne sont ni complètes ni cocomplètes, donner quelques exemples. Qu'en est-il de la catégorie des corps ?

Correction For (1) and (3) See (for example) Propositions 3.51 and 3.52 of Riehl's book. The idea is a category is (co)complete if and only if it admits all (co)products and (co)equalizers. Describe them in the category of sets and choose the suitable topology on them. For (4) there is not field morphism between fields of different characteristic, hence if K and L have different characteristic there is no (co)product of K and L.