DERIVED FUNCTORS

Exercice 1 -

(1) Soit $F : \mathcal{A} \to \mathcal{B}$ un foncteur additif entre catégories abéliennes et $U : \mathcal{B} \to \mathcal{C}$ un foncteur additif et exact entre catégories abéliennes. Montrer que $U(L_i F)$ est isomorphe à $L_i(U(F))$ et que $U(R^i F)$ est isomorphe à $R^i(U(F))$.

(2) Soit $F : \mathcal{A} \to \mathcal{B}$ un foncteur additif entre catégories abéliennes.

 (a) Montrer que $L_0 F$ est exact à droite et que $R^0 F$ est exact à gauche.

 (b) Montrer que

$$L_m L_n F = \begin{cases} L_m F, & \text{si } n = 0 \\ 0, & \text{sinon.} \end{cases}$$

(3) Si F est exact, calculer ses foncteurs dérivés gauches et droits.

Correction

(1) Exactness of U says that it preserves kernel and images and quotient, hence U commutes with (co)homology.

(2a) This is the long exact sequence for derived functors.

(2b) If $n > 0$, then $L_n F(P) = 0$ for every projective module. So $L_m L_n F(X) = H_m(L_n F(P_\bullet)) = H_m(0) = 0$ where $P_\bullet$ is a projective resolution of X. If $n = 0$ consider $0 \to \Omega_X \to P \to X \to 0$ a short exact sequence with P projective. Then by the long exact sequence for derived functors we have :

$$0 = L_1 F(P) \to L_1 F(X) \to L_0 F(\Omega_X) \to L_0 F(P) \to L_0 F(X) \to 0.$$

So $L_1 F(X)$ is the kernel of the map $L_0 F(\Omega_X) \to L_0 F(P)$. We have a similar exact sequence for the derived functors of $L_0 F$ however L_0 is right exact so $L_0 L_0 F = L_0 F$. Hence we have

$$0 = L_1 L_0 F(P) \to L_1 L_0 F(X) \to L_0 F(\Omega_X) \to L_0 F(P) \to L_0 F(X) \to 0.$$

So $L_1 L_0 F(X)$ is the kernel of $L_0 F(\Omega_X) \to L_0 F(P)$ so it is equal to $L_1 F(X)$. This holds for any module X, so using a décalage argument, we have $L_m F = L_m L_0 F$.

(3) If $P_\bullet$ is a projective resolution, then $F(P_\bullet)$ is exact except in degree 0, so all the derived functors vanish expect the ones of degree 0. In degree zero we have $L_0 F = R^0 F = F$.

Exercice 2 -

Soit F un foncteur additif entre deux catégories abéliennes, exact à droite.

(1) Montrer que si $0 \to M \to P \to A \to 0$ est exacte avec P projectif alors $L_i(F(A))$ est isomorphe à $L_{i-1}(F(M))$ pour tout $i \geq 2$. Montrer que $L_1 F(A)$ est le noyau de $F(M) \to F(P)$.

(2) On se donne une suite exacte

$$0 \to M_m \to P_m \to P_{m-1} \to \ldots \to P_0 \to A \to 0$$

où les P_i sont projectifs. Montrer que $L_i(F(A))$ est isomorphe à $L_{i-m-1}F(M_m)$ pour tout $i \geq m+2$ et que $L_{m+1}F(A)$ est le noyau de $F(M_m) \to F(P_m)$.

(3) On dit qu'un objet Q est F-acyclique si $L_i(F(Q)) = 0$ pour tout $i \geq 1$.
Montrer que si $Q \to A$ est une résolution de A telle que Q_n est F-acyclique pour tout $n \geq 0$ alors $L_i F(A) = H_i(F(Q)), \forall i \geq 0$.

Correction

(1) We saw the solution in the first exercise.

(2) The long exact sequence gives $m+1$ short exact sequences $0 \to M_0 \to P_0 \to A \to 0$, $0 \to M_1 \to P_1 \to M_0 \to 0, \cdots, 0 \to M_m \to P_m \to M_{m-1} \to 0$. Hence applying the previous question we have for $i \geq m+2$, $L_i F(A) \cong L_{i-1}F(M_0) \cong L_{i-2}F(M_1) \cdots \cong L_{i-m-1}F(M)$ and $L_{m+1}(F(A)) = ker(F(M_m) \to F(P_m))$.

(3) We can easily see that the décalage argument will work with any short exact sequence with an F-acyclic middle term. Using this adapted version of Question 2, we see that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(Q_m)$ with $M_m = \ker(d_m)$ whith $d_m : Q_m \to Q_{m-1}$. It remains to identify this as the homology of $F(Q)$. We have $M_m = ker(d_m) : Q_m \to Q_{m-1}$. Moreover by exactness, we have $Im(d_{m+1}) = ker(d_m)$, so $M_m = Im(d_{m+1}) = Q_{m+1}/ker(d_{m+1}) = coker(d_{m+2})$. Since F is right exact, it preserves cokernels and $F(M_m) = Coker(F(d_{m+2}))$. Hence $L_{m+1}F(A) = ker(F(Q_{m+1})/Im(F(d_{m+2})) \to F(Q_m)) = H_m(F(Q))$.

Exercice 3 - Extrait de l'examen du 4 novembre 2016 Soit R un anneau et B un R-module à gauche. Pour $r \in R$, on note ${}_rB = \{b \in B | r \cdot b = 0\}$ et $r \cdot B = \{r \cdot b, b \in B\}$.

(1) On pose $R = \mathbb{Z}/m\mathbb{Z}$ et $A = \mathbb{Z}/d\mathbb{Z}$ avec d divise m et $m \geq 2$. Montrer que, pour tout R-module B, on a :

$$\begin{cases} \mathrm{Ext}^0_R(A, B) = {}_dB \\ \mathrm{Ext}^i_R(A, B) = ({}_{\frac{m}{d}}B)/(d \cdot B), & \text{si } i \text{ est impair,} \\ \mathrm{Ext}^i_R(A, B) = ({}_dB)/(\frac{m}{d} \cdot B), & \text{si } i \geq 2 \text{ est pair.} \end{cases}$$

(2) * Donner une extension représentative pour chaque classe d'équivalence d'extensions de $\mathbb{Z}/4\mathbb{Z}$ par $\mathbb{Z}/2\mathbb{Z}$ en tant que $\mathbb{Z}/8\mathbb{Z}$-module.

Correction

(1) We start by constructing a projective resolution of A. It is not too hard to see that it has a periodic projective resolution of the form

$$\cdots \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cdot \frac{m}{d}} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0.$$

We have $\mathrm{Ext}^0(A, B) = \mathrm{Hom}(\mathbb{Z}/d\mathbb{Z}, B) = {}_bB$. For the other extension groups we apply $Hom(-, B)$ to the projective resolution together with the isomorphism $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, B) \cong {}_mB$ to get :

$$0 \longrightarrow {}_mB \xrightarrow{\cdot d} {}_mB \xrightarrow{\cdot \frac{m}{d}} {}_mB \xrightarrow{\cdot d} \cdots$$

the result follows by taking homology.

(2) By the previous result there are two equivalence classes of such extensions. Hence we only need to find a split short exact sequence (equivalence class of zero) and a non split one. For the non split one, we can look at the canonical surjection from $\mathbb{Z}/8$ to $\mathbb{Z}/4$. The kernel is isomorphic to $\mathbb{Z}/2$ and it is not split.

Exercice 4 -

Let m, n be two integers and B an abelian group.

(1) Compute $\mathrm{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all $i \geq 0$.

(2) Same question for $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all $i \geq 0$.

(3) If $d \mid n$, compute $\mathrm{Tor}^{\mathbb{Z}/n\mathbb{Z}}_i(\mathbb{Z}/d\mathbb{Z}, B)$ for all $i \geq 0$.

Correction

(1) Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ where the first map is the multiplication by n. This gives a projective of Z/n. It is immediate that $Tor_i(\mathbb{Z}/n, -)$ vanishes for $i \geq 2$. We can checked that $\mathrm{Tor}_1(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathrm{Tor}_0(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/gcd(m, n)$.

(2) The answer is the same for Ext.

(3) For this question we can do as we dit in Exercise 3. There will be a value for $n = 0$ and one for odd integers and one for non zero even integers.

Exercice 5 - Group cohomology Let G be a group.

(1) * Show that $\mathrm{Fun}(G, \mathbb{Z}\,\mathrm{Mod})$ is equivalent to $\mathbb{Z}[G]\,\mathrm{Mod}$, where $\mathbb{Z}[G]$ is the group algebra of G over $\mathbb{Z}$.

(2) Let $\mathbb{Z}$ be the trivial $\mathbb{Z}[G]$-module, i-e every element of G acts as the identity on $\mathbb{Z}$. For $M \in \mathbb{Z}[G]\,\mathrm{Mod}$, we set $H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.

   (a) Let $M^G = \{m \in M;\ gm = m\ \forall g \in G\}$. Show that $H^0(G, M) \cong M^G$.

   (b) Make M^G functorial in M and check that $H^n(G, M)$ is the nth right derived functor of $M \mapsto M^G$.

(3) Let $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ be the map defined by $\epsilon(\sum_{g \in G} \lambda_g g) = \sum_{g \in G} \lambda_g$ and $I(G) = \ker(\epsilon)$. Show that $I(G)$ is a free $\mathbb{Z}$-module with basis $\{g - 1;\ g \in G\}$.

(4) Let $\mathrm{Der}(G, M) = \{f : G \to M;\ f(gh) = gf(h) - f(g)\ \forall g, h \in G\}$. Let $Inn(G, M) = \{f : G \to M\ ;\ \exists m \in M, f(g) = gm - m\ \forall g \in G\}$.

   (a) If $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(I(G), M)$, let $D_\phi : G \to M$ be the map defined by $D_\phi(g) = \phi(g - 1)$. Show that $D_\phi \in \mathrm{Der}(G, M)$.

   (b) Show that $\phi \mapsto D_\phi$ is an isomorphism between $Hom_{\mathbb{Z}[G]}(I(G), M)$ and $\mathrm{Der}(G, M)$.

(5) Show that there is an exact sequence $0 \to M^G \to M \to \mathrm{Der}(G, M) \to H^1(G, M) \to 0$, where the map $M \to \mathrm{Der}(G, M)$ sends m to D_m the derivation defined by $D_m(g) = gm - m$.

(6) Conclude that $H^1(G, M) \cong \mathrm{Der}(G, M)/Inn(G, M)$.

(7) Let F_n be the free $\mathbb{Z}$-module of with $\mathbb{Z}$-basis consisting of the $n+1$-tuples $(g_0, \cdots, g_n)$. This is a $\mathbb{Z}[G]$ module for the action induced by the diagonal left multiplication by the elements of G. One can see that it is also a free $\mathbb{Z}[G]$-module with basis $\{(1, g_1, \cdots, g_n) \ ; \ g_i \in G\}$. Check that the face maps $d_i : G^{n+1} \to G^n$ defined by $d_i(g_0, \cdots, g_n) = (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots g_n)$ endow $(F_n)_n$ of a structure of chain complexes of abelian groups.

(8) Show that F_n is a free resolution, as $\mathbb{Z}[G]$-modules, of the trivial $\mathbb{Z}[G]$-module. Hint : check that the augmented complex, using $\epsilon : \mathbb{Z}[G] \to Z$ is exact, by showing that it is contractible. One can use $s_n : F_n \to F_{n+1}$ defined by $s_n(g_0, \cdots, g_n) = (1, g_0, \cdots, g_n)$.

(9) Let us denote by $[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, g_1g_2g_3, \cdots, g_1g_2\cdots g_n) \in G^{n+1}$. Then $(1, g_1, \cdots, g_n) = [g_1|g_1^{-1}g_2|\cdots|g_{n-1}^{-1}g_n]$. As a consequence F_n is a free $\mathbb{Z}[G]$-module with basis the set $\{[g_1|g_2|\cdots|g_n] \ ; \ g_i \in G\} =: \underline{G}_n$. Check that with this notation, the face maps are :

$$\delta_i([g_1|\cdots|g_n]) = \begin{cases} g_1[g_2|\cdots|g_n] & \text{if } i = 0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n] & \text{if } 1 \le i \le n-1 \\ [g_1|\cdots|g_{n-1}] & \text{if } i = n. \end{cases}$$

Note that F_0 has basis $[]$ the empty symbol. So $\epsilon : F_0 \to \mathbb{Z}$ is defined by $\epsilon([]) = 1$. This is the so-called bar resolution.

(10) If M is a $\mathbb{Z}[G]$-module, show that $\mathrm{Hom}_{\mathbb{Z}[G]}(F_n, M) \cong \mathrm{Hom}_{Set}(\underline{G}_n, M) =: C^n(G, M)$. Deduce that the cohomology $H^n(G, M)$ is isomorphic to the cohomology of a complex of cochains constructed with the $C^n(G, M)$.

(11) Can you recover the result of Question 6 ?

(12) Describe the 2-cocycles and the 2-coboundaries of $C^\bullet(G, M)$.

(13) If G is finite, show that every element in $H^2(G, M)$ as order dividing $|G|$. Can you generalize to higher cohomology groups ?

(14) Conclude, that if M and G are two finite groups such that $1 = gcd(|G|, |M|)$, then $H^2(G, M) = 0$.

(15) Show that when G is finite and M finitely generated, then the abelian groups $H^n(G, M)$ are finite for $n \neq 0$.

Exercice 6 - *Cohomologie des groupes cycliques

Soit $G = \mathbb{Z}/n\mathbb{Z} = \{\omega^i, 0 \le i \le n-1\}$.
Dans $\mathbb{Z}[G]$ on considère les éléments $T = \sum_{i=0}^{n-1} \omega^i$ et $N = \omega - 1$.

(1) Montrer que le complexe

$$\ldots \to \mathbb{Z}[G] \overset{\times T}{\to} \mathbb{Z}[G] \overset{\times N}{\to} \ldots \to \mathbb{Z}[G] \overset{\times T}{\to} \mathbb{Z}[G] \overset{\times N}{\to} \mathbb{Z}[G]$$

donne une résolution libre de $\mathbb{Z}$ comme $\mathbb{Z}[G]$-module trivial.

(2) En déduire le calcul de $H_k(G, \mathbb{Z})$ et $H^k(G, \mathbb{Z})$ pour $k \ge 0$.

(3) En déduire le calcul de $H_*(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ et $H^*(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$.

Correction

(1) There is an augmentation morphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ sending $g \in G$ to 1. It is easy to see that the set of $g - 1_G$ generates the kernel of ϵ and this is in fact a basis. Hence, sending $1 \in \mathbb{Z}[G]$ to $N = w - 1$ induces a map of $\mathbb{Z}[G]$-modules from $\mathbb{Z}[G]$ to the kernel. This map is nothing by the multiplication by N. Since $\sum_{k=0}^{i} w^i$ is sent to $w^{i+1} - 1$, the map is surjective. The kernel of the multiplication by N consists of the set of $\sum_i \lambda_i w^i$ such that $\sum \lambda_i(w^{i+1} - w_i) = 0$. This leads to $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1}$. Hence this is the image of the multiplication by T. Similarly if $\sum \lambda_i w_i$ is in the kernel of T, we have

$$\begin{aligned} 0 &= (\sum \lambda_i w^i) \cdot \sum_j w^j \\ &= \sum_{i,j} \lambda_i w^{i+j}. \end{aligned}$$

The coefficient of w^{n-1} is $\sum_i \lambda_i$. Hence $\sum \lambda_i = 0$ and the kernel of T is the kernel of ϵ and we have the required resolution.

(2) By definition $H_k(G, \mathbb{Z}) = \mathrm{Tor}_k^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$. This can be computed by tensoring the projective resolution and then taking homology. After tensoring we have the complex :

$$\cdots \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \xrightarrow{T \otimes 1} \mathbb{Z}[G] \otimes \mathbb{Z} \xrightarrow{N \otimes 1} \mathbb{Z}[G] \otimes \mathbb{Z} \longrightarrow 0$$

We have $\mathbb{Z}[G] \otimes \mathbb{Z} \cong \mathbb{Z}$ moreover since $\mathbb{Z}$ is the trivial module the action of N is zero and the action of T is n. Hence this complex is isomorphic to

$$\cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

The homology is $\mathbb{Z}/n\mathbb{Z}$ in odd degrees and 0 in even degrees except in degree zero where it is $\mathbb{Z}$. The computation of the cohomology is similar.

(3) We assume that there is a Künneth formula for group homology and cohomology. See Weibel Chapter 6 for more details. That is there is a split short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(G,\mathbb{Z}) \otimes H_q(H,\mathbb{Z}) \to H_n(G \times H,\mathbb{Z}) \to \bigoplus_{p+q=n-1} Tor_1^{\mathbb{Z}}(H_p(G,\mathbb{Z}), H_q(H,\mathbb{Z})) \to 0.$$

Hence we need the second question and also Exercise 4 in order to finish the computation. The computation can be done by looking at the double complex $H(G) \otimes H(H)$. The total complex of this double complex gives the first term of the short exact sequence. After computation the answer is : $H_0(G \times H) = \mathbb{Z}$, if n is even non zero, we have

$$H_n(G \times H) = (\mathbb{Z}/\gcd(m,n)\mathbb{Z})^{\oplus \frac{n}{2}},$$

and if n is odd

$$H_n(G \times H) = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \oplus (\mathbb{Z}/\gcd(m,n)\mathbb{Z})^{\oplus \frac{n-1}{2}}$$

Exercice 7 - Dimension globale, Exam 2023

Soit $\mathcal{A}$ une catégorie abélienne avec assez de projectifs. On appelle *dimension globale* de $\mathcal{A}$:

$$\text{gldim}(\mathcal{A}) = \sup\{n \in \mathbb{N} \mid \exists A, B \in Ob(\mathcal{A}) \ : \text{Ext}^n_{\mathcal{A}}(A,B) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Si $M \in \mathcal{A}$, on appelle *dimension projective* de M notée $\text{pdim}(M)$, le plus petit n tel qu'il existe une résolution projective de M de la forme

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

S'il n'existe pas de résolution projective finie on pose $\text{pdim}(M) = \infty$.

On dit que $\mathcal{A}$ est *semisimple* si $\text{gldim}(\mathcal{A}) = 0$ et $\mathcal{A}$ est *héréditaire* si $\text{gldim}(A) \leq 1$. On peut supposer que les objets de $\mathcal{A}$ sont des modules (à droite) sur un anneau unitaire associatif A.

(1) Montrer que les assertions suivantes sont équivalentes.
  (a) $\mathcal{A}$ est semisimple.
  (b) Tout objet de $\mathcal{A}$ est projectif.
  (c) Tout objet de $\mathcal{A}$ est injectif.
  (d) Toute suite exacte courte est scindée.

(2) Donner un exemple d'anneau dont la catégorie des modules est semisimple.

(3) Soit $M \in \mathcal{A}$, démontrer que les assertions suivantes sont équivalentes :
  (a) $\text{pdim}(M) \leq n$.
  (b) $\text{Ext}^i_{\mathcal{A}}(M,X) = 0$ pour tout $i > n$ et tout $X \in \mathcal{A}$.
  (c) $\text{Ext}^{n+1}_{\mathcal{A}}(M,X) = 0$ pour tout $X \in \mathcal{A}$.
  (d) Si $0 \to M_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ est une suite exacte avec P_i projectifs pour tout i, alors M_n est projectif.

(4) Justifier que $\text{gldim}(\mathcal{A}) = \sup\{\text{pdim}(M) \mid M \in \mathcal{A}\}$.

(5) Ici on suppose que les objets de $\mathcal{A}$ sont des modules. Démontrer que $\mathcal{A}$ est héréditaire si et seulement si tout sous-module d'un module projectif est projectif.

(6) Soit A une k-algèbre de dimension finie. Montrer que la catégorie des A-modules de dimension finie est héréditaire si et seulement si tout idéal à droite de A est projectif.

(7) Une k-algèbre A de dimension finie sur un corps k est dite *auto-injective* si le module régulier $A \in \text{mod}\, A$ est injectif. Montrer que dimension globale de $\text{mod}\, A$ est alors 0 ou ∞.

(8) Soient $n \geq 1$ et k un corps. Montrer que $A = k[X]/(X^n)$ est auto-injective et en déduire la dimension globale de $\text{mod}\, A$. Indication, on pourra utiliser le critère de Baer.

Correction For this exercise there are many different ways of solving it and we do not claim to have the most efficient one. Actually, we try to use as little results from the class as possible, so this makes some proof slightly longer.

(1) (b) implies (a). If every object is projective, we have $0 \to X \to 0$ is a projective resolution of X, hence the nth right derived functor of $Hom(X, -)$ vanishes except when $n = 0$. (a) implies (b) because if P is an object, and $0 \to L \to M \to N \to 0$ is a short exact sequence, we have a long exact sequence $0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to \mathrm{Ext}^1(P, L) = 0$. Hence, the functor $\mathrm{Hom}(P, -)$ is exact and P is projective. By a dual argument, we have $(a) \Leftrightarrow (c)$. Clearly, (b) implies (d) since an epi toward a projective object splits. And if every short exact sequence splits, consider the diagram :

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

with P projective. Adding the kernel of the bottom map, leads to

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

and the splitting of the short exact sequence allows to lift the morphism $P \to N$ along $M \to N$. So (d) implies (b). Alternatively, any additive functor is exact on split exact sequences, hence (d) implies (b) or P is projective if and only if any epi toward P splits, hence (d) implies (b)...

(2) If k is a field, then the category of k-modules is semisimple, since any short exact sequence splits, or any modules is free (hence projective)...

(3) (a) implies (b) is clear, (b) implies (c) also. Moreover (d) implies (a) is also clear in view of the inductive construction of projective resolution. (c) implies (d) is an argument of 'décalage' as in TD4 Ex2. If d_i is the differential map from P_i to P_{i-1} and $\pi : P_0 \to M$, then $M_n \cong \mathrm{Ker}(d_{n-1})$. We have n short exact sequences :

$$\begin{aligned} &0 \to \mathrm{Ker}(\pi) \to P_0 \to M \to 0 \\ &0 \to \mathrm{Ker}(d_1) \to P_1 \to \mathrm{Ker}(\pi) \to 0 \\ &0 \to \mathrm{Ker}(d_2) \to P_2 \to \mathrm{Ker}(d_1) \to 0 \\ &\quad \cdots \\ &0 \to \mathrm{Ker}(d_{n-1}) \to P_{n-1} \to \mathrm{Ker}(d_{n-2}) \to 0. \end{aligned}$$

Moreover if $0 \to L \to P \to N \to 0$ is a short exact sequence with P projective, applying $\mathrm{Hom}(-, X)$ gives a long exact sequence, from which we see $\mathrm{Ext}^i(L, X) \cong \mathrm{Ext}^{i-1}(L, N)$, for $i \geq 2$. Hence, we have $\mathrm{Ext}^{n+1}(M, X) \cong \mathrm{Ext}^1(M_n, X)$. Condition (c) implies that $\mathrm{Ext}^1(M_n, X) = 0$ for all X, hence M_n is projective.

(4) By Question 3, if $\mathrm{Ext}^{n+1} = 0$, every module have projective dimension at most n. Conversely if every module have projective dimension at most n, then $\mathrm{Ext}^{n+1}(-, X) = 0$ for all X. So $(a) = (b)$.

(5) Assume that $\mathcal{A}$ is hereditary and let M be a submodule of a projective P_0. Then, we have a short exact sequence $0 \to M \to P_0 \to P_0/M \to 0$. By Question 3 (d), we have that M is projective. Conversely, if X is an object in $\mathcal{A}$, let $\pi : P \to X$ an epimorphism with P projective (it exists since $\mathcal{A}$ has enough projective). This gives a short exact sequence $0 \to \mathrm{Ker}(\pi) \to P \to X \to 0$. Since $\mathrm{Ker}(\pi)$ is a submodule of P, it is projective, hence X has a projective resolution of length at most 1.

(6) Since a submodule of A is a right ideal by (5) we have that hereditary implies ideals are projective. Conversely, let M be a submodule of a finitely generated projective. Since such finitely generated projective is a direct summands of a free module, we can assume that M is a submodule of A^n and we prove the result by induction on n. If $n = 1$, we are done since the submodules of A are the right ideals. If $n \geq 1$, let $\pi : A^n \to A$ the projection onto the last coordinates. Then we have a short exact sequence

$$0 \to \mathrm{Ker}(\pi|_M) \to M \to Im(\pi|_M) \to 0.$$

The image of $\pi|_M$ is a subomodule of $Im(\pi) = A$. Hence it is a projective module and the short exact sequence splits and $M \cong \mathrm{Ker}(\pi|_M) \oplus Im(\pi|_M)$. Now, $\mathrm{Ker}(\pi|_M) = \mathrm{Ker}(\pi) \cap M$ is a submodule of $\mathrm{Ker}(\pi) \cong A^{n-1}$. By induction, we have that $\mathrm{Ker}(\pi) \cap M$ is projective, hence M is projective as direct sums of two projective modules.

Remark : a submodule of a finitely generated free module is not necessarily a direct sum of right ideals!

(7) Since A is injective, then a finite direct sum (which is also a product) of copies of A is injective and a direct summand of an injective module is injective, hence every finitely generated projective module is also injective. Let

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be a finite projective resolution of the finite dimensional module M. Then $d_n : P_n \to P_{n-1}$ is injective and we have $P_{n-1}/d_n \cong P_{n-1}/\operatorname{Ker}(d_{n-1}) \cong Im(d_{n-1})$. Hence we have a short exact sequence :

$$0 \to P_n \to P_{n-1} \to Im(d_{n-1}) \to 0.$$

Since P_n is injective, the sequence splits and $Im(d_{n-1})$ is a direct summands of a projective, so it projective. Hence

$$0 \to Im(d_{n-1}) \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution of M. By induction on the length, this implies that M is projective. Hence the only modules with finite projective dimension are the projective modules.

(8) In order to show that $A = k[X]/(X^n)$ is an injective module, we take an ideal I of A and a map $f : I \to A$ and we have to show that it extends as a map from A to A. The ideals of $k[X]/(X^n)$ are the ideals $J/(X^n)$ for an ideal J of $k[X]$. The ring $k[X]$ is principal, hence the ideals of A are the $I_i := (X^i)/(X^n)$ for $i \in \{0, \cdots, n-1\}$. A map from I_i to A is determined by $f(X^i)$ which has to be an element $\overline{Q}$ of A such that $X^{n-i}\overline{Q} = 0$. This means that X^n divides $X^{n-i}Q$ in $k[X]$. So Q is of the form X^iR for a polynomial R. We define $g : A \to A$ to be the map sending 1 to $\overline{R}$. Then $g(X^i) = X^i\overline{R} = \overline{Q} = f(X^i)$. By Baer's criterion, we obtain that $k[X]/(X^n)$ is injective. It remains to see that the algebra is not semisimple when $n \geq 2$. By Question (1) it is enough to find a non split exact sequence. The evaluation at 0 gives a surjective map $A \to k \cong k[X]/(X)$ and the kernel is $(X)/(X^n)$. If there is a splitting s of the evaluation, s is determined by $s(1) = \overline{Q}$. Since s is a morphism of modules, we have $0 = s(X \cdot 1) = X \cdot \overline{Q}$. So X^n divides XQ, hence X divides Q and the evaluation at 0 of Q is zero, hence $ev_0 \circ s \neq Id_k$. Alternatively, we can see that $k[X]/(X^n)$ is indecomposable (the matrix of the action is a Jordan block of size n) hence the sequence cannot split.