

Ref E. Riehl Category Theory in Context (Chap I)

Mac Lane Category for the Working Mathematician

Assem Introduction au langage catégorique (Chap I, II)

Motivation ~ 1945 Eilenberg - Mac Lane give the good formalism for "natural isomorphism" (General theory of natural equivalences)

V k.-vs of finite dim $V \cong V^*$ not natural but why?

$V \cong V^{**}$ natural

Turned out: solving this question gave a formalism that: ① Unified mathematical concepts

② New links

⚠ This is not a theory that trivializes mathematics

③ Give new questions!

↳ used today (almost) everywhere: AG, Algebra, rep theory, topology, combinatorics, ...

I. Categories and functors

Def 1.1 A category \mathcal{E} is the data of

- A collection of morphisms $\text{Mor}(\mathcal{E})$
- A collection of objects $\text{Ob}(\mathcal{E})$

s.t.

(1) Every morphism $f \in \text{Mor}(\mathcal{E})$ has a specified domain $x \in \text{Ob}(\mathcal{E})$ and codomain $y \in \text{Ob}(\mathcal{E})$. Write $f: x \rightarrow y$

(2) $\forall x \in \text{Ob}(\mathcal{E}), \exists 1_x (= \text{Id}_x) \in \text{Mor}(\mathcal{E})$

(3) $\forall f: x \rightarrow y, g: y \rightarrow z \in \text{Mor}(\mathcal{E}) \exists gf: x \rightarrow z \in \text{Mor}(\mathcal{E})$

satisfying

(Identity) $\forall f: x \rightarrow y \in \text{Mor}(\mathcal{E}) \quad 1_y f = f = f 1_x$

(Associativity) $\forall f, g, h \in \text{Mor}(\mathcal{E})$ "composable" $h(gf) = (hg)f$

Rem ① Collection: we don't want to worry about set theory

② If $\text{Mor}(\mathcal{E})$ is a set we say that \mathcal{E} is small

③ $\text{Hom}_{\mathcal{E}}(x, y) (= \mathcal{E}(x, y))$ the collection of $f: x \rightarrow y \in \text{Mor}(\mathcal{E})$

if $\forall x, y \text{ Hom}_E(x, y)$ is a set we say that E is locally small.

Examples 1.2 (concrete categories)

- (1) Set obj = sets Mor = functions
- (2) Top obj = topological spaces Mor = continuous functions
- (3) Grps, Ring, fields...
- (4) k -vs, R -modules $\text{Mod } R$ right R -modules
 $R\text{Mod}$ left "

...

Examples 1.3 (abstract categories)

- (1) k field Mat_k obj \mathbb{N} $\text{Hom}(m, n) = \text{Mat}_{nm}(k)$
- (2) G grps $BG = \bullet$ obj = $\{\bullet\}$ compo = product in G
 $\text{Hom}(\bullet, \bullet) = G \quad 1 = 1_G$

Et what is the minimal hypothesis on G to make this work?

- (3) (P, \leq) poset $\rightsquigarrow \hat{P}$ obj = P
 $|\text{Hom}(x, y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

TODO (1) check that this defines a category

(2) Minimal hypothesis on \leq ?

(4) HiTop obj topological spaces

$$\text{Hom}(x, y) = \text{Hom}_{\text{top}}(x, y) / \sim \text{homotopy}$$

...

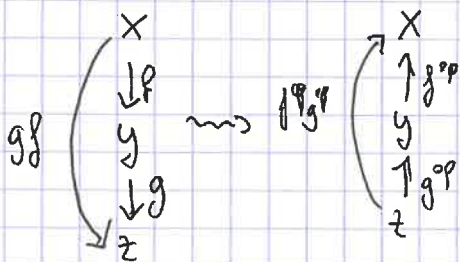
Examples 1.4 (Categories constructed from categories)

(1) E cat $\rightsquigarrow E^{op}$ the opposite category

$$\text{ob}(E^{op}) = \text{ob}(E)$$

$$\text{Hom}_{E^{op}}(x, x) = \text{Hom}_E(x, x)$$

$$f^{op} \longleftarrow f$$



Q? $(BG)^{op}$?, $(P, \leq)^{op}$

$\Delta (E_{\text{Set}})^{op}$ is not E_{Set} ! see $\forall D$.

② Notions of subcategory, product of category... look in the references.

Rem In a category \mathcal{E} the objects can be anything so for $x \in \text{ob}(\mathcal{E})$ the "notion $x \in X$ " ~~is not~~ doesn't make sense.

↳ Categorical notions must be defined using "arrows" not "elements"

Def 1.5 Let \mathcal{E} be a category.

(1) $f: x \rightarrow y \in \text{Mor}(\mathcal{E})$ is an isomorphism if $\exists g: y \rightarrow x \in \text{Mor}(\mathcal{E})$ s.t. $fg = 1_y$ and $gf = 1_x$.

(2) $f: x \rightarrow y$ is a monomorphism if $\forall W \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} x$ s.t. $fg = fh$ then $g = h$ ($W \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} x \xrightarrow{f} y$). We say that f is left cancellable.

(3) $f: x \rightarrow y$ is an epimorphism if $\forall Y \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} y$ s.t. $gf = hf$ then $g = h$. ($x \xrightarrow{f} y \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} z$). We also say that f is right cancellable.

⚠ Mono + epic \neq iso (see 1.10)

Def 1.6 \mathcal{E}, \mathcal{D} be two categories. A (covariant) functor (foncteur)

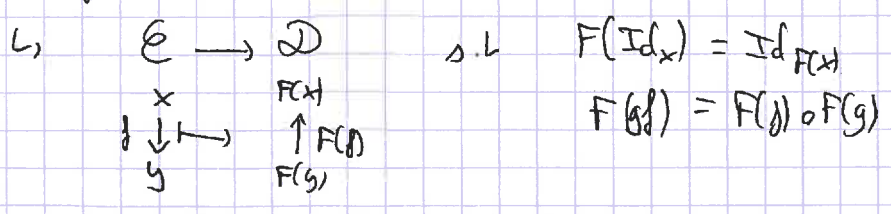
$F: \mathcal{E} \rightarrow \mathcal{D}$ is the data of:

- (•) $F(c) \in \text{ob}(\mathcal{D}) \quad \forall c \in \text{ob}(\mathcal{E})$
- (•) $F(f) \in \text{Hom}_{\mathcal{D}}(F(x), F(y)) \quad \forall f \in \text{Hom}_{\mathcal{E}}(x, y)$

s.t. (1) $F(\text{Id}_x) = \text{Id}_{F(x)}$

(2) $\forall f, g$ composable $F(g \circ f) = F(g) \circ F(f)$

Def 1.7 \mathcal{E}, \mathcal{D} categories. A contravariant functor is a functor from \mathcal{E}^{op} to \mathcal{D} .



Example 1.8 (1) $U: \text{Grp} \rightarrow \text{Set}$ functor that forgets the gp structure.

$$\begin{array}{ccc} (G, \cdot) & \mapsto & G \\ \downarrow f & & \downarrow f \\ (H, \cdot) & \mapsto & H \end{array}$$

There are many forgetful functors more or less trivial

(2) $U: \text{Ass} \rightarrow \text{Lie}$ from the category of associative algebras
 $(A, \cdot) \mapsto (A, \cdot, [-, \cdot])$ to the category of Lie algebras

where $[a, b] = ab - ba$

forgets the "associative structure of the algebra" but remembers more than an abelian gp!

(3) $F: \text{Set} \rightarrow \mathbb{Z}\langle X \rangle$ abelian gps

Where $\mathbb{Z}\langle X \rangle$ is the free abelian gp with basis X

$$\begin{array}{ccc} X & \mapsto & \mathbb{Z}\langle X \rangle \\ \downarrow f & & \downarrow \mathbb{Z}\langle f \rangle \\ Y & \mapsto & \mathbb{Z}\langle Y \rangle \end{array}$$

$\mathbb{Z}\langle X \rangle = \{ \text{formal linear combination of elements of } X \}$

$\cong \mathbb{Z}^{(X)} = \{ g: X \rightarrow \mathbb{Z} \text{ with finite support} \}$

Abelian gp $\sum_{x \in X} \lambda_x x + \sum_{y \in Y} \mu_y y = \sum_{x \in X} (\lambda_x + \mu_x) x$

$\mathbb{Z}\langle f \rangle: \mathbb{Z}\langle X \rangle \rightarrow \mathbb{Z}\langle Y \rangle$ (extend by linearity f)

$\sum_{x \in X} \lambda_x x \mapsto \sum_{x \in X} \lambda_x f(x)$

check the details!

(4) \mathcal{E} loc small $\forall x \in \mathcal{E}$ we have $\text{Hom}_{\mathcal{E}}(x, -): \mathcal{E} \rightarrow \text{Set}$

$x \xrightarrow{\alpha} y \xrightarrow{\beta} z$
 $\downarrow f \circ \alpha$

$y \mapsto \text{Hom}_{\mathcal{E}}(x, y)$
 $\downarrow f \circ -$
 $z \mapsto \text{Hom}_{\mathcal{E}}(x, z)$

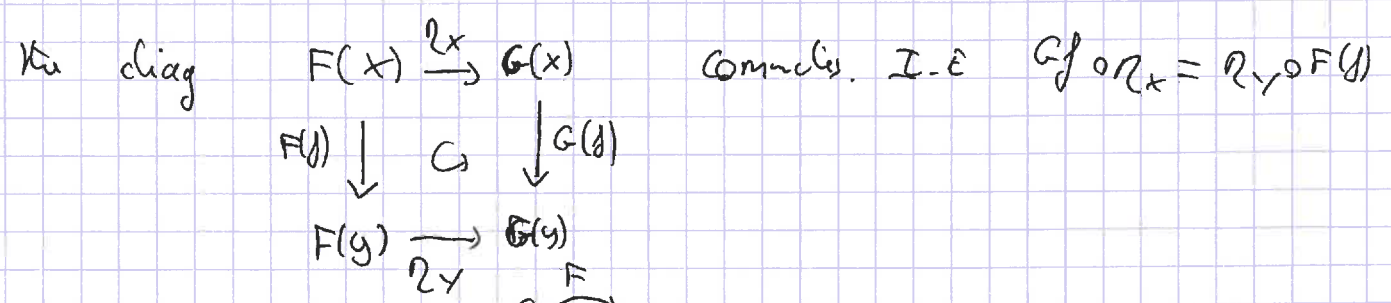
Similarly we have a contravariant functor $\text{Hom}_{\mathcal{E}}(-, x): \mathcal{E} \rightarrow \text{Set}$

$y \mapsto \text{Hom}_{\mathcal{E}}(y, x)$
 $\downarrow f \circ -$
 $z \mapsto \text{Hom}_{\mathcal{E}}(z, x)$

$\hookrightarrow \text{Hom}_E(-, -)$ is a bifunctor i.e. a functor from $E \times E^{\text{op}} \rightarrow \text{Ens}$

(5) $E \xrightarrow{F} \mathcal{D} \xrightarrow{G} E$ one can compose functors in the obvious sense.

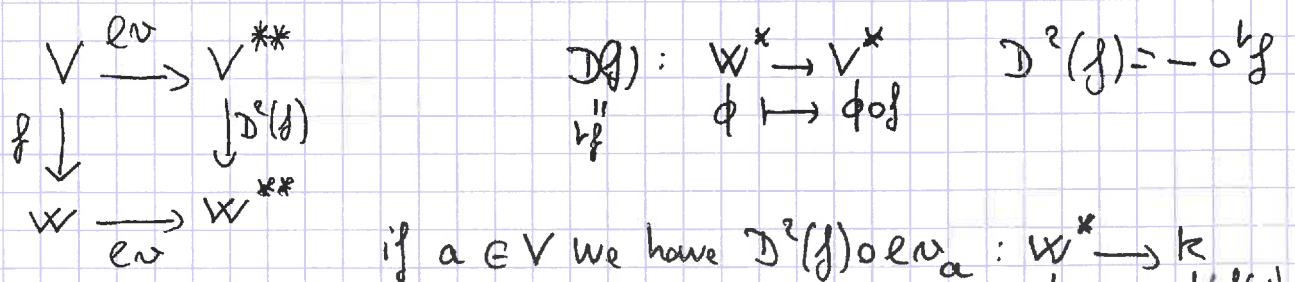
Def 1.9 Let $E \xrightleftharpoons[F]{G} \mathcal{D}$ be two functors. A natural transformation η from F to G is the data of $\eta_x : F(x) \rightarrow G(x) \in \text{Mor}(\mathcal{D}) \forall x \in \text{Ob}(E)$ s.t. $\forall f: x \rightarrow y \in E$



Notation $\eta: F \Rightarrow G$ or $E \xrightleftharpoons[\eta]{F, G} \mathcal{D}$

Example 1.10 V k -vector space. $eV: V \rightarrow V^{**} = \text{Hom}(\text{Hom}(V, k), k)$
 $v \mapsto e_v: \text{Hom}(V, k) \rightarrow k$
 $\phi \mapsto \phi(v)$

$\text{Id}_{\text{Vect}_k}$ and $D^2 = \text{Hom}_{\text{Vect}}(\text{Hom}_{\text{Vect}}(-, k), k)$ are two endofunctors of Vect_k



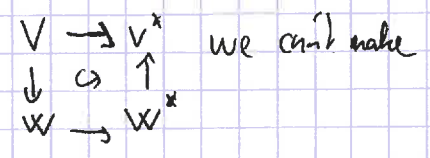
and on the other direction $eW \circ f(a) = eV_a$.

So it is a natural transformation.

However: there is no natural transformation from $\text{Id}_{\text{Vect}_k}$ and D

Problem 1 One is covariant and one is contravariant

but even if we modify the definition of naturality it works!



Def 1.11 A natural transformation η for $E \begin{matrix} \xrightarrow{F} \\ \Downarrow \eta \\ \xrightarrow{G} \end{matrix} D$ is a natural isomorphism if $\forall x \in \text{Ob}(E) \eta_x$ is an isomorphism.

Rem One can compose natural transformations "vertical composition" $E \begin{matrix} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{H} \end{matrix} D \quad (\beta \circ \alpha)_x : F(x) \xrightarrow{\alpha_x} G(x) \xrightarrow{\beta_x} H(x)$

But one can also compose natural transformations "horizontal composition" $E \begin{matrix} \xrightarrow{F_1} \\ \Downarrow \alpha_1 \\ \xrightarrow{F_2} \end{matrix} D \begin{matrix} \xrightarrow{G_1} \\ \Downarrow \alpha_2 \\ \xrightarrow{G_2} \end{matrix} E \quad \alpha_2 * \alpha_1 : F_2 \circ F_1 \Rightarrow G_2 \circ G_1$

there is another choice but gives the same result (why?) $F_1 \xrightarrow{(\alpha_2)_x} F_2 \xrightarrow{G_2} G_2(x)$

Def 1.12 ^{Prop} E, D be two categories. Then $\text{Fun}(E, D) (= \mathcal{D}^E)$ the category whose objects are functors from E to D and morphisms natural transformations is called the functor category from E to D .

Rem $\left\{ \begin{array}{l} \text{Categories} \\ \text{Functors} \\ \text{Natural Trans} \end{array} \right.$ is the prototypical example of a 2-category (tricalgebra)

2 - Equivalence of category

Def 1.13 Let E, D be two categories. An equivalence of category from E to D is the data of

(1) $F: E \rightarrow D$ and $G: D \rightarrow E$ two functors

(2) A natural isomorphism $\eta: 1_E \Rightarrow GF$ and a natural isomorphism

$\epsilon: FG \Rightarrow 1_D$ where 1_E and 1_D are the identity functors of E and D .

Rem (1) G is a quasi-inverse of F and if F is part of an equivalence we simply say that it is an equivalence.

(2) If F, G are contravariant: speak of duality

(3) If two categories are equivalent every property that can be expressed in "terms of arrows" is preserved.

(1) Most of the time we say that F is an equivalence if $\exists G; (F, G)$ is a equivalence.

Def 1.14 Let $F: \mathcal{E} \rightarrow \mathcal{D}$ a functor. Then

(1) F is faithful (fidèle) if $\forall x, y \in \text{ob}(\mathcal{E})$ $F: \text{Hom}_{\mathcal{E}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$
 $f \mapsto F(f)$

is injective

(2) F is full (plein) if it is surjective

(3) F is essentially surjective if $\forall y \in \text{ob}(\mathcal{D}) \exists x \in \text{ob}(\mathcal{E})$ s.t.
 $F(x) \simeq y$ in \mathcal{D} .

Thm 1.15 Let $F: \mathcal{E} \rightarrow \mathcal{D}$ be a functor. Then F is an equivalence of category iff F is fully faithful and essentially surjective

Proof \triangleleft Set theoretic issue \Rightarrow always ok
 \Leftarrow need choice axiom on the class $\text{Ob}(\mathcal{E})$

$$\Rightarrow F: \mathcal{E} \rightarrow \mathcal{D} \quad G: \mathcal{D} \rightarrow \mathcal{E} \quad \eta: 1_{\mathcal{E}} \Rightarrow GF \quad \epsilon: FG \Rightarrow 1_{\mathcal{D}}$$

(1) We have $\forall y \in \text{ob}(\mathcal{D}) \quad \epsilon_{G(y)}: FG(y) \xrightarrow{\sim} y$ so F is essentially surjective.

(2) In order to see that $F: \text{Hom}_{\mathcal{E}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ is bijective we

construct the inverse bijection by applying the "do what you can" rule

$$\phi: \text{Hom}_{\mathcal{D}}(F(x), F(y)) \xrightarrow{G} \text{Hom}_{\mathcal{E}}(GF(x), GF(y)) \xrightarrow{\text{Hom}(F_x, GF_y)} \text{Hom}_{\mathcal{E}}(x, GF(y)) \xrightarrow{\text{Hom}(1_x, \eta_y^{-1})} \text{Hom}_{\mathcal{E}}(x, y)$$

$$F(x) \xrightarrow{F} F(y) \xrightarrow{G} GF(x) \xrightarrow{GF} GF(y) \xrightarrow{\eta_y} y$$

Let us look at the commo $\text{Hom}_{\mathcal{E}}(x, y) \xrightarrow{F} \text{Hom}_{\mathcal{D}}(F(x), F(y)) \xrightarrow{\phi} \text{Hom}_{\mathcal{E}}(x, y)$

$$f \mapsto x \xrightarrow{\eta_x} GF(x) \xrightarrow{GF} GF(y) \xrightarrow{\eta_y^{-1}} y$$

$$\begin{array}{ccc} \downarrow & \downarrow GF & \downarrow \eta_y \\ y & \xrightarrow{GF} & GF(y) \end{array}$$

(com η laws)

so the commo is indeed the identity $\Rightarrow \phi \circ F = \text{Id}_{\text{Hom}_{\mathcal{E}}(x, y)}$ so F is faithful.

Now let us look at the other composition

$$\text{Hom}(F(X), F(Y)) \xrightarrow{\phi} \text{Hom}(X, Y) \xrightarrow{F} \text{Hom}(F(X), F(Y))$$

$$\alpha \longmapsto (X \xrightarrow{\alpha} GF(X) \xrightarrow{G(\alpha)} GF(Y) \xrightarrow{\alpha'} Y) \xrightarrow{F(\alpha' \circ G(\alpha) \circ \alpha)} F(X) \xrightarrow{\alpha} F(Y)$$

We have two diagrams:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & GF(X) \\ \phi(\alpha) \downarrow & \cong & \downarrow G(\alpha) \\ Y & \xrightarrow{\alpha'} & GF(Y) \\ & \cong & \\ & \alpha' & \end{array} \quad \textcircled{1}$$

$$\text{and} \quad \begin{array}{ccc} X & \xrightarrow{\alpha} & GF(X) \\ \phi(\alpha) \downarrow & \cong & \downarrow GF(\phi(\alpha)) \\ Y & \xrightarrow{\alpha'} & GF(Y) \\ & \cong & \\ & \alpha' & \end{array} \quad \textcircled{2}$$

The first commutes by definition and the second by naturality of α .

Since we have $F(\alpha' \circ \phi(\alpha)) \stackrel{\textcircled{1}}{=} F(G(\alpha) \circ \alpha)$
 $\stackrel{\textcircled{2}}{=} F(GF(\phi(\alpha)) \circ \alpha)$

Since F is faithful we have $G(\alpha) \circ \alpha = GF(\phi(\alpha)) \circ \alpha$ and α invertible gives $G(\alpha) = GF(\phi(\alpha))$. By the previous point applied to G , G is also faithful so $\alpha = F(\phi(\alpha))$ \square

Goal construct G

(1) $\forall D \in \text{Ob}(\mathcal{D}) \exists X \in \mathcal{E}$ and $\psi_D : D \xrightarrow{\sim} F(X)$. set $G(D) := X$

(2) On the morphisms?

$$\begin{array}{ccc} D & \xrightarrow{\psi_D} & FG(D) \\ \downarrow & & \downarrow FG(\alpha) \\ D' & \xrightarrow{\psi_{D'}} & FG(D') \end{array}$$

Since $F: \text{Hom}(GD, GD') \rightarrow \text{Hom}(FGD, FG D')$ is ~~fully~~ bijective $\exists!$ morphism $G(\alpha) \in \text{Hom}(GD, GD')$ making the diagram commute

check ① G is a functor

② G is an equiv quasi-inverse of F

details in the literature \square

Ex $\text{Vect}_k \xrightarrow{\sim} \text{Mat}_k$ is fully faithful

$$\begin{array}{ccc} k^n & \xrightarrow{I_n} & k^n \\ \downarrow A & & \downarrow A \\ k^m & \xrightarrow{\quad} & k^m \end{array}$$

(~~also~~ linear morphism is determined by its matrix)
 + ess. surjective. \square

Chapter 2 Universal properties

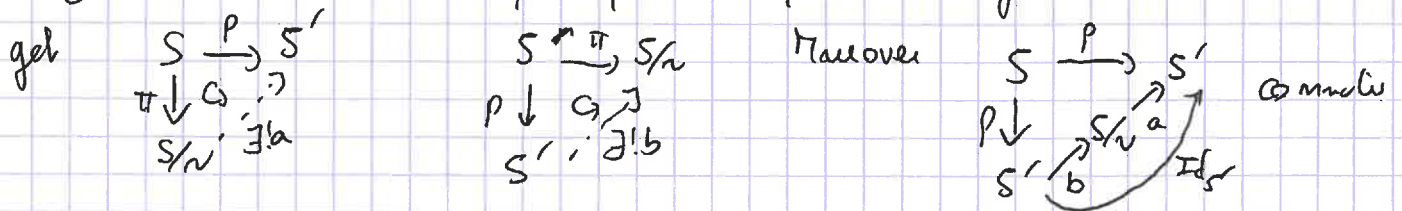
Ref: Riehl Chap 2,3,4
 MacLane III, IV, V
 ASSEM III, IV, V

Motivation S rel \sim equiv relation on S $\rightarrow S/\sim$ quotient

$\forall f: S \rightarrow X$ compatible with \sim we have $S \xrightarrow{f} X$ where $S \xrightarrow{\pi} S/\sim$ is the quotient map

Say that $S \xrightarrow{\pi} S/\sim$ is a solution for the universal problem posed by the compatible maps

easy such solution is unique up to (unique) iso: if $S \xrightarrow{p} S'$ is another



indeed $abp = a\pi = p$ but identity of S' also makes this diagram commute so by unicity we have $ab = Id_{S'}$ and similarly $ba = Id_{S/\sim}$

I Initial and final objects

Def 2.1 Let E be a category. An object $c \in \text{Ob}(E)$ is initial (final) if $\forall d \in \text{Ob}(E)$ we have $|\text{Hom}_E(c,d)| = 1$ ($|\text{Hom}_E(d,c)| = 1$).

Prop 2.2 If these objects exist they are unique up to unique isomorphism

Proof If c and c' are two initial objects get $c \xrightarrow{\exists! a} c'$ and $c' \xrightarrow{\exists! b} c$ and $c \xrightarrow{a} c' \xrightarrow{b} c$ $ba = Id_c$ since $|\text{Hom}_E(c,c)| = 1$. \square

- Example 2.3
- (1) \emptyset is initial in Ens and any singleton is final
 - (2) $\{0\}$ is both initial and final in Vect_K ($R\text{Mod}$)
 - (3) Category of fields does not have initial/final objects (think characteristic!)

Want to say that a universal object is an initial or final object! A category only have at most 2 so seem too restrictive ... but:

Def 2.4 ^{Prop} Let $F: \mathcal{E} \rightarrow \text{Set}$ a functor. Let $\int F$ the category

$$\text{ob } \int F = \{ (c, x); c \in \text{ob}(\mathcal{E}) \text{ and } x \in F(c) \}$$

$$\text{Hom}((c, x), (c', x')) = \{ f \in \text{Hom}(c, c'); F(f)(x) = x' \}$$

$$\text{Compo} = \text{compo in } \mathcal{E}$$

$$\text{Id} = \text{Id in } \mathcal{E}$$

There is an obvious forgetful functor: $\pi: \int F \rightarrow \mathcal{E}$

$$(c, x) \mapsto c$$

$$\begin{array}{ccc} \downarrow f & & \downarrow f \\ (d, y) & \mapsto & d \end{array}$$

② If F is contravariant let $\int F$ obj $\{ (c, x) \in \mathcal{E}; x \in F(c) \}$

$$\text{Hom}((c, x), (c', x')) = \{ f: c \rightarrow c'; F(f)(x) = x' \}$$

Still have a forgetful functor $\pi: \int F \rightarrow \mathcal{E}$.

Example: S set \sim equiv relation

$$F: \text{Ens} \rightarrow \text{Ens}$$

$$x \mapsto F(x) = \{ f: S \rightarrow x; x \sim y \Rightarrow f(x) = f(y) \}$$

$$\begin{array}{ccc} \alpha \downarrow & \alpha \downarrow & S \rightarrow x \\ y \mapsto F(y) & & \downarrow \alpha \\ & & S \end{array}$$

$\int F$ has few objects $(x, S \xrightarrow{f} x)$; f compatible

$$\text{Morphism } \alpha: \begin{array}{ccc} S & \xrightarrow{f'} & x' \\ f \downarrow \alpha & \searrow \alpha & \\ x & & \end{array}$$

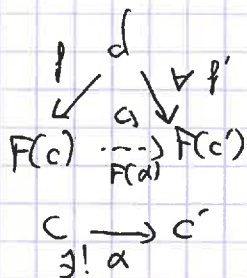
easy $\pi: S \rightarrow S/\sim$ is an initial object of $\int F$.

Def 2.5 $F: \mathcal{E} \rightarrow \text{Set}$ a functor. A universal element for F is an initial object of $\int F$

\hookrightarrow it is a pair (c, x) $c \in \text{ob}(\mathcal{E})$ $x \in F(c)$ s.t. $\forall (d, y) d \in \mathcal{E}, y \in F(d)$
 $\exists! \alpha: c \rightarrow d$ s.t. $y = F(\alpha)(x)$

Def 2.6 $F: \mathcal{E} \rightarrow \mathcal{D}$ be a functor, $d \in \text{ob}(\mathcal{D})$ a universal arrow from d to F is a pair (c, f) where $c \in \text{ob}(\mathcal{E}), f \in \text{Hom}_{\mathcal{D}}(d, F(c))$

s.t. $\forall (c', f') c' \in \text{ob}(\mathcal{E}) f': d \rightarrow F(c') \exists! \alpha \in \text{Hom}_{\mathcal{E}}(c, c')$ s.t. $F(\alpha) \circ f = f'$



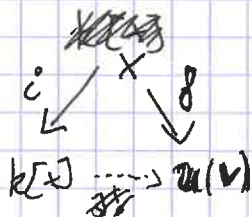
exercise Define a category $d \downarrow F$ s.t. universal arrow is an initial object of $d \downarrow F$.

Example 2.7 $\mathcal{U}: \text{Vect}_k \rightarrow \text{Set}$ forgetful functor. Let $X \in \text{Set}$. A universal arrow from $X \rightarrow \mathcal{U}$ is the "bank" V_X k -v.s with $X \rightarrow V_X$

Set $V_X = k[X]$ k -vector space with basis X

$X \xrightarrow{\text{inc}} V_X$
 $x \mapsto x$

then



f extend by linearity as a linear map $k[X] \xrightarrow{f} V$

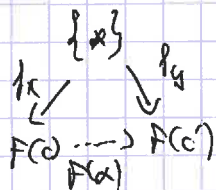
and this map makes the diagram commutative. Moreover if α is another map then α and f coincide on the basis X so they are equal.

Prop 2.8 Universal elements and arrows are two equivalent notions

Proof (1) If $F: E \rightarrow \text{Set}$ and (c, x) is a universal element,

Consider $f_x: \{*\} \rightarrow F(c)$. Then (c, f_x) is a universal arrow

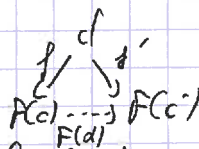
$* \rightarrow F$.



$y \in F(c') \exists \alpha: c \rightarrow c'; F(\alpha)(x) = y$

$\Rightarrow F(\alpha) \circ f_x = f_y$

(2) Conversely if $F: E \rightarrow \mathcal{D}$, $d \in \mathcal{D}$, a universal arrow (c, f) $d \rightarrow F$. If \mathcal{D} is locally small we have



$\text{Hom}_{\mathcal{D}}(d, F(-)): E \rightarrow \text{Set}$ a set valued functor.

$x \mapsto \text{Hom}_{\mathcal{D}}(d, F(x))$

Then $f \in \text{Hom}_{\mathcal{D}}(d, F(c))$ is a universal element for this functor. \square

II - Representable functors

Def 2.9 E category (loc small) $F: E \rightarrow \text{Ens}$ functor

(1) F is representable if $\exists c \in \text{Ob}(E)$ and a natural isomorphism between F and $\text{Hom}_E(c, -)$ ($\text{Hom}_E(-, c)$ when F is contravariant)

(2) A representation of F is the data of $c \in \text{Ob}(E)$ and an iso $\eta: \text{Hom}(c, -) \Rightarrow F$.

Example $\mathcal{U}: \text{Grp} \rightarrow \text{Set}$ is representable since $\text{Hom}_{\text{Grp}}(\mathbb{Z}, -) \simeq \mathcal{U}$

$(G, \cdot) \mapsto G$

$\left(\begin{array}{ccc} \text{Hom}_{\text{Grp}}(\mathbb{Z}, G) & \xrightarrow{\alpha} & \mathcal{U}(G) \\ \alpha & \mapsto & \alpha(1) \end{array} \right.$ natural + iso since α is characterized by $\alpha(1)$

Question How to find $\alpha: \text{Hom}_E(C, -) \Rightarrow F$ in general?

Thm 2.10 [Yoneda lemma] Let $F: E \rightarrow \text{Set}$ a functor with E local and $c \in \text{ob}(E)$. Then

$$\text{Nat}(\text{Hom}_E(C, -), F) \cong F(c)$$

$$\alpha \longmapsto \alpha_c(\text{Id}_c)$$

Moreover this isomorphism is natural in c and in F .

Proof (•) $\alpha: \text{Hom}_E(C, -) \Rightarrow F$

We have $\alpha_x: \text{Hom}_E(C, x) \rightarrow F(x)$
 $f: C \rightarrow x \longmapsto \alpha_x(f)$

but $f: C \rightarrow x$ induces a commutative diagram (nat of α !)

$$\begin{array}{ccc} \text{Hom}_E(C, c) & \xrightarrow{\alpha_c} & F(c) \\ \text{Hom}_E(C, f) \downarrow & & \downarrow F(f) \\ \text{Hom}(C, x) & \xrightarrow{\alpha_x} & F(x) \end{array}$$

at Id_c we have

$$\alpha_x(f) \in F(x) = F(f)(\alpha_c(\text{Id}_c))$$

So α is completely determined by ~~$F(f)$~~ $\alpha_c(\text{Id}_c)$. $\alpha \longmapsto \alpha_c(\text{Id}_c)$ is injective.

(•) Conversely if $e \in F(c) \rightsquigarrow \alpha: \text{Hom}(C, -) \Rightarrow F$ defined by

$$\alpha_x: \text{Hom}_E(C, x) \rightarrow F(x)$$

$$f \longmapsto F(f)(e)$$

- check
- (1) α is a natural transformation
 - (2) $e \longmapsto \alpha_c$ is the inverse bijection
 - (3) Naturality.

Universal II

Rem $\forall (1) F: \mathcal{E} \rightarrow \mathbf{Ens}$ Then (c, x) is universal element for F iff the natural map $\alpha_x: \text{Hom}(c, -) \Rightarrow F$ induced by x is an isomorphism

Indeed α_x iso iff $\forall c' \in \mathcal{E}$ $(\alpha_x)_{c'}: \text{Hom}(c, c') \rightarrow F(c')$
 $f \mapsto F(f)(x)$
 is bijective iff $\forall c' \in \mathcal{E} \forall y \in F(c') \exists! f: c \rightarrow c'; F(f)(x) = y$.

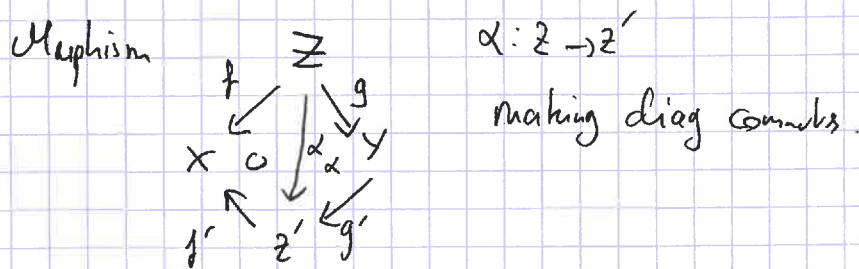
(2) for universal arrow: use $\mathcal{D}(x, F(-))$ as before.

III Examples of objects defined by universal properties

1) Product, Coproduct

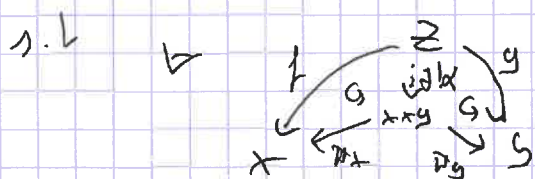
Let \mathcal{E} be a small category and $X, Y \in \text{ob}(\mathcal{E})$

Exig category objects $\text{tuples } (z, f, g); z \in \text{ob}(\mathcal{E}) f \in \text{Hom}(X, z) g \in \text{Hom}(z, Y)$



Def 2.11 A product of X and Y is a final object in Exig

\hookrightarrow unwrapped: It is an object $X \times Y$ with two maps $\pi_X: X \times Y \rightarrow X$
 $\pi_Y: X \times Y \rightarrow Y$



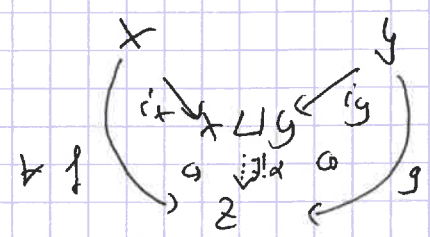
Coro if it exists it is unique up to unique isomorphism in Exig .

Example In Set the cartesian product $X \times Y = \{(x,y); x \in X, y \in Y\}$ together with the canonical projections is a product.

Dually

Def 2.11 A coproduct of X and Y is a product in E^{op}

unwrapped satisfy: universal property



Example In Set the coproduct of X and Y is the disjoint union together with its canonical inclusions

$$X \cup Y := X \times \{0\} \cup Y \times \{1\} \quad i_x: X \rightarrow X \cup Y \quad \leftarrow Y$$

$$x \mapsto (x, 0) \quad (0, 1) \leftarrow 1_y$$

(2) Equalizer and coequalizer

Def 2.12 Let E be a category and $X, Y \in \text{ob}(E)$ $f, g: X \rightarrow Y \in \text{mor}(E)$

Consider the contravariant functor $F: E \rightarrow \text{Set}$

$$c \mapsto \{\alpha: c \rightarrow X \in \text{mor}(E); f\alpha = g\alpha\}$$

$$d \mapsto \{\beta: d \rightarrow X; f\beta = g\beta\}$$

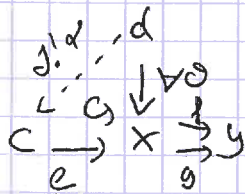
$\uparrow \beta \circ d$
 $\uparrow \beta \circ d$
 $\uparrow \beta \circ d$

An equalizer in E is a representation of the contravariant functor F

\leadsto unwrapped Yoneda $\text{Hom}(-, c) \cong F \iff$ element of $F(c)$

so a representation is a pair (c, e) $c \in \text{ob}(E)$ $e \in F(c)$ s.t. Yoneda nat has is an iso!

concretely $\forall d \in \text{ob}(\mathcal{C})$ $\text{Hom}_{\mathcal{C}}(d, d) \xrightarrow{\cong} F(d) \xrightarrow{\cong} F(d)$ is an iso
 $f \mapsto F(f)(e)$

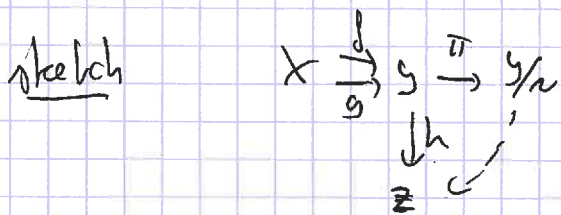


Example In Set $E = \{x \in X; f(x) = f(y)\} \hookrightarrow X$ is an equalizer.
 $x \xrightarrow{f} y$

Dually:

Def 2.13 dual notion of coequalizer: $x \xrightarrow{f} y \xrightarrow{g} z$; $\pi f = \pi g$
 $\downarrow \pi$

Example In Set $x \xrightarrow{f} y \xrightarrow{g} z$ Consider \sim the equivalence relation on y generated by $f(x) \sim g(x)$ (smallest equiv relation with this property). Then $y \xrightarrow{\pi} y/\sim$ is a coequalizer



by construction get $\pi f = \pi g$

$hf = hg \Rightarrow h$ compatible with \sim so done by univ prop of quotient.

III - Adjoint functors

Kan 1958

Def 2.14 An adjunction (G, D) is a pair of functors $G: \mathcal{C} \rightarrow \mathcal{D}$ and $D: \mathcal{D} \rightarrow \mathcal{C}$ together with an isomorphism $\text{Hom}_{\mathcal{D}}(G(c), d) \cong \text{Hom}_{\mathcal{C}}(c, D(d))$ which is natural in both variables

Not $G \dashv D$ and we say that G is a left-adjoint for D and D is a right-adjoint for G