Chapter I Introduction to Category theory

Ref E. Riehl Category theory in Context (Chap I)

Mac Lane Category for the working Mathematician

Assem Introduction au langage catégorique (Chap I, II)

Motivation ~ 1945 Eilenberg - MacLane give the good formalism for "natural isomorphism" (General theory of natural equivalences)

V k-vs of finite dim $V \simeq V^*$ not natural but why !

$V \simeq V^{**}$ natural

Turned out : solving this question gave a formalism that :

① Unified mathematical concepts

② New links

③ Give new questions !

⚠ This is not a theory that trivializes mathematics

↳ used today (almost) everywhere : AG, Algebra, rep theory, topology, combinatorics, ...

I. Categories and functors

Def 1.1 A category $\mathcal{C}$ is the data of

- A collection of morphisms $\mathrm{Mor}(\mathcal{C})$
- A collection of objects $\mathrm{Ob}(\mathcal{C})$

s.t.

(1) Every morphism $f \in \mathrm{Mor}(\mathcal{C})$ has a specified domain $X \in \mathrm{Ob}(\mathcal{C})$ and codomain $Y \in \mathrm{Ob}(\mathcal{C})$. Write $f : X \to Y$

(2) $\forall X \in \mathrm{Ob}(\mathcal{C})$, $\exists\ 1_X \ (= \mathrm{Id}_X) \in \mathrm{Mor}(\mathcal{C})$

(3) $\forall\ f : X \to Y,\ g : Y \to Z \in \mathrm{Mor}(\mathcal{C})$ $\exists\, g \circ f\ X \to Z \in \mathrm{Mor}(\mathcal{C})$

satisfying

(Identity) $\forall\ f : X \to Y \in \mathrm{Mor}(\mathcal{C})$ $1_Y f = f = f 1_X$

(Associativity) $\forall\ f, g, h \in \mathrm{Mor}(\mathcal{C})$ "composable" $h(gf) = (hg)f$

Rem ① Collection : we don't want to worry about set theory

② If $\mathrm{Mor}(\mathcal{C})$ is a set we say that $\mathcal{C}$ is small

(3) $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ $(= \mathcal{C}(X, Y))$ the collection of $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$

if $\forall x,y$ $\mathrm{Hom}_{\mathcal{E}}(x,y)$ is a set we say that $\mathcal{E}$ is locally small.

Examples 1.2 (Concrete categories)

(1) Set obj = sets Mor = functions

(2) Top obj = topological spaces Mor = continuous functions

(3) Grps, Ring, fields ...

(4) k-vs, R-modules $\underset{\text{RMod}}{\text{Mod}R}$ $\underset{\text{left} \;''}{\text{right } R\text{-modules}}$

...

Examples 1.3 (abstract categories)

(1) k field Mat_k obj $\mathbb{N}$ $\mathrm{Hom}(m,n) = \mathrm{Mat}_{n,m}(k)$

(2) G gps $BG = \underset{G}{\bullet}$ obj : $\{\bullet\}$ compo = product in G

$\mathrm{Hom}(\bullet,\bullet) = G$ $\quad 1 = 1_G$.

Ex what is the minimal hypothesis on G to make this works!

(3) $(P,\leq)$ poset $\rightsquigarrow$ $\hat{P}$ obj $= P$

$$|\mathrm{Hom}(x,y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otw} \end{cases}$$

TODO (1) check that this defines a category

(2) Minimal hypothesis on $\leq$?

(4) HTpy obj topo spaces

$\mathrm{Hom}(x,y) = \mathrm{Hom}_{\mathrm{Top}}(x,y) / \sim$ homotopy

...

Example 1.4 (Categories constructed from categories)

① $\mathcal{E}$ cat $\rightsquigarrow$ $\mathcal{E}^{op}$ the opposite category $\quad \mathrm{ob}(\mathcal{E}^{op}) = \mathrm{ob}(\mathcal{E})$

$\mathrm{Hom}_{\mathcal{E}^{op}}(y,x) = \mathrm{Hom}_{\mathcal{E}}(x,y)$

$f^{op} \longleftarrow f$

$$\begin{tikzcd}
X \arrow[d, "f"] \arrow[dd, bend right=60, "gf"'] & & & X \\
Y \arrow[d, "g"] & {} \arrow[r, squiggly] & f^{op}g^{op} & Y \arrow[u, "f^{op}"'] \\
Z & & & Z \arrow[u, "g^{op}"'] \arrow[uu, bend left=60]
\end{tikzcd}$$

Q? $(BG)^{op}$?, $(P,\leq)^{op}$

⚠ $\underset{\text{Set}}{(\text{Ens})}^{op}$ is not $\underset{\text{Set}}{\text{Ens}}$! see TD.

② Notions of subcategory, product of category ... look in the references.

Rem In a category $\mathcal{C}$ the objects can be anything so for $x \in \mathcal{O}b(\mathcal{C})$ the "notion $x \in X$" doesn't make sense.

↳ Categorical notions must be defined using "arrows" not "elements"

Def 1.5 Let $\mathcal{C}$ be a category.

(1) $f : x \to y \in \mathcal{M}or(\mathcal{C})$ is an isomorphism if $\exists\, g : y \to x \in \mathcal{M}or(\mathcal{C})$ st $fg = 1_y$ and $gf = 1_x$.

(2) $f : x \to y$ is a monomorphism if $\forall\ W \underset{h}{\overset{g}{\rightrightarrows}} x$ s.t $fg = fh$ then $g = h$ $\left(W \underset{h}{\overset{g}{\rightrightarrows}} x \xrightarrow{f} y \right)$. We say that f is left cancellable

(3) $f : x \to y$ is an epimorphism if $\forall\ Y \underset{h}{\overset{g}{\rightrightarrows}} z$ s.t $gf = hf$ then $g = h$. $\left(x \xrightarrow{f} y \underset{h}{\overset{g}{\rightrightarrows}} z \right)$. we also say that f is right cancellable.

⚠ Mono + epi $\neq$ iso (see TD).

Def 1.6 $\mathcal{C}, \mathcal{D}$ be two categories. A (covariant) functor (foncteur)

$F : \mathcal{C} \to \mathcal{D}$ is the data of:

(•) $F(c) \in \mathcal{O}b(\mathcal{D})$ $\forall\ c \in \mathcal{O}b(\mathcal{C})$

(•) $F(f) \in Hom_{\mathcal{D}}(F(x), F(y))$ $\forall\ f \in Hom_{\mathcal{C}}(x, y)$

s.t. (1) $F(Id_x) = Id_{F(x)}$

(2) $\forall\ f, g$ composable $F(gf) = F(g)\,F(f)$

Def 1.7 $\mathcal{C}, \mathcal{D}$ categories. A contravariant functor is a functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

↳ $$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{D} \\ x \arrow[d, "f"'] & F(x) \\ y \arrow[r, mapsto, from=2-1, shorten=0] & F(y) \arrow[u, "F(f)"'] \end{tikzcd}$$ s.t $F(Id_x) = Id_{F(x)}$

$F(gf) = F(f) \circ F(g)$

Example 1.8 (1) $\mathcal{U}: \mathrm{Grp} \longrightarrow \mathrm{Set}$ functor that forgets the gp structure.

$$\begin{tikzcd}
(G,\cdot) \arrow[r, mapsto] \arrow[d, "f"] & G \arrow[d, "f"] \\
(H,\cdot) \arrow[r, mapsto] & H
\end{tikzcd}$$

There are many forgetful functors more or less trivial

(2) $\mathcal{U}: \mathrm{Ass} \longrightarrow \mathrm{Lie}$ from the category of associative algebras

$(A,+,\cdot) \longmapsto (A,+,[-,-])$ to the category of Lie algebras

where $[a,b] = ab - ba$

forgets the "associative structure of the algebra" but remember more than an abelian gp!

(3) $F: \mathrm{Set} \longrightarrow \mathcal{A}b$ ← abelian gps

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathbb{Z}[X] \arrow[d, "\mathbb{Z}[f]"] \\
Y \arrow[r, mapsto] & \mathbb{Z}[Y]
\end{tikzcd}$$

Where $\mathbb{Z}[X]$ is the free abelian gps with basis X

$\mathbb{Z}[X] = \{$formal linear combination of elements of $X\}$

$\simeq \mathbb{Z}^{(X)} = \{g : X \to \mathbb{Z}$ with finite support$\}$

Abelian gp $\sum_{x \in X} \lambda_x x + \sum_{x \in X} \mu_x x = \sum_{x \in X} (\lambda_x + \mu_x) x$

$\mathbb{Z}[f] : \mathbb{Z}[X] \longrightarrow \mathbb{Z}[X]$ (extend by linearity f)

$\sum_{x \in X} \lambda_x x \longmapsto \sum_{x \in X} \lambda_x f(x)$

check the details!

(4) $\mathcal{E}$ loc small $\forall x \in \mathcal{E}$ we have $\mathrm{Hom}_{\mathcal{E}}(x,-) : \mathcal{E} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd}
x \arrow[r, "\alpha"] \arrow[rr, bend right, dashed, "f\circ\alpha"'] & y \arrow[r, "f"] & z
\end{tikzcd}$$

$$\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{E}}(x,y) \arrow[d, "f\circ -"] \\
z \arrow[r, mapsto] & \mathrm{Hom}_{\mathcal{E}}(x,z)
\end{tikzcd}$$

Similarly we have a contravariant functor $\mathrm{Hom}_{\mathcal{E}}(-,x) : \mathcal{E} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_{\mathcal{E}}(y,x) \\
z \arrow[r, mapsto] & \mathrm{Hom}_{\mathcal{E}}(z,x) \arrow[u, "-\circ f"']
\end{tikzcd}$$

↳ $\mathrm{Hom}_{\mathcal{E}}(-,-)$ is a bifunctor i.e a functor from $\mathcal{E}\times\mathcal{E}^{op} \to \mathrm{Ens}$

(5) $\mathcal{E}\xrightarrow{F}\mathcal{D}\xrightarrow{G}\mathcal{E}$ one can compose functors in the obvious sense.

Def 1.9 Let $\begin{tikzcd}\mathcal{E} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{D}\end{tikzcd}$ be two functors. A natural transformation η from F to G is the data of $\eta_x : F(x) \to G(x) \in \mathrm{Mor}(\mathcal{D})$ $\forall x \in \mathrm{Ob}(\mathcal{E})$ s.t. $\forall f : x \to y \in \mathcal{E}$

the diag

$$\begin{tikzcd}
F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\eta_y"'] & G(y)
\end{tikzcd}$$

commutes. I.e $Gf \circ \eta_x = \eta_y \circ F(f)$

Notation $\eta : F \Rightarrow G$ or

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "G"', ""{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

Example 1.10 V k-vector space. $ev\_ : V \to V^{**} = \mathrm{Hom}(\mathrm{Hom}(V,k),k)$

$v \mapsto ev_v : \mathrm{Hom}(V,k) \to k$

$\phi \mapsto \phi(v)$

$\mathrm{Id}_{\mathrm{Vect}_k}$ and $D^2 = \mathrm{Hom}_{\mathrm{Vect}}(\mathrm{Hom}_{\mathrm{Vect}}(-,k),k)$ are two endofunctors of Vect_k

$$\begin{tikzcd}
V \arrow[r, "ev"] \arrow[d, "f"'] & V^{**} \arrow[d, "D^2(f)"] \\
W \arrow[r, "ev"'] & W^{**}
\end{tikzcd}$$

$D(f) : W^* \to V^*$, $\phi \mapsto \phi\circ f$

$\overset{\|}{{}^tf}$

$D^2(f) = -\circ {}^tf$

if $a \in V$ we have $D^2(f)\circ ev_a : W^* \to k$

$\phi \mapsto \phi(f(a))$

and on the other direction $ev \circ f(a) = ev_{f(a)}$.

So it is a natural transformation.

However: there is no natural transformation from $\mathrm{Id}_{\mathrm{Vect}_k}$ and D

Problem 1 One is covariant and one is contravariant

but even if we modify the definition of naturality we can't make it works!

$$\begin{tikzcd}
V \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V^* \\
W \arrow[r] & W^* \arrow[u]
\end{tikzcd}$$

Def 1.11 A natural transformation for $$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{D} \end{tikzcd}$$ is a natural isomorphism if $\forall x \in Ob(\mathcal{E})$ η_x is an isomorphism.

Rem One can compose natural transformation $$\begin{tikzcd} \mathcal{E} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta"] & \mathcal{D} \end{tikzcd}$$ $$\begin{tikzcd} (\beta\circ\alpha)_x \arrow[d, phantom, "\parallel"] : F(x) \arrow[r, "\alpha_x"] & G(x) \arrow[d, "\beta_x"] \\ \beta_x \circ \alpha_x & H(x) \end{tikzcd}$$
"vertical composition"

But one can also compose natural transformation $$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow \alpha_1"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow \alpha_2"] & E \end{tikzcd}$$ $\alpha_2 * \alpha_1 : F_2 \circ F_1 \Rightarrow G_1 \circ G_2$

"horizontal composition" $\forall x \in \mathcal{E}$ $(\alpha_2 * \alpha_1)_x$: $$\begin{tikzcd} F_1 F_2(x) \arrow[r] \arrow[dr, "F_1((\alpha_2)_x)"'] & G_1 G_2(x) \\ & F_1 G_2(x) \arrow[u, "(\alpha_1)_{G_2(x)}"'] \end{tikzcd}$$.

There is another choice but gives the same result (why?)

Def Prop 1.12 $\mathcal{E}, \mathcal{D}$ be two categories. Then $Fun(\mathcal{E},\mathcal{D})$ $(=: \mathcal{D}^{\mathcal{E}})$ the category whose objects are functors from $\mathcal{E}$ to $\mathcal{D}$ and morphisms the natural transformation is called the functor category from $\mathcal{E}$ to $\mathcal{D}$.

Rem $\left\{\begin{array}{l}\text{Categories} \\ \text{Functors} \\ \text{Natural trans}\end{array}\right.$ is the prototypical example of a 2-category (bicategory)

2_ Equivalence of category

Def 1.13 Let $\mathcal{E}, \mathcal{D}$ be two categories. An equivalence of category from $\mathcal{E}$ to $\mathcal{D}$ is the data of

(1) $F : \mathcal{E} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ two functors

(2) A natural isomorphism $\eta : 1_{\mathcal{E}} \Rightarrow GF$ and a natural isomorphism $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ where $1_{\mathcal{E}}$ and $1_{\mathcal{D}}$ are the identity functors of $\mathcal{E}$ and $\mathcal{D}$.

Rem (1) G is called a quasi-inverse of F and ~~if F is part of an equivalence we simply say that it is an equivalence.~~

(2) If F, G are contravariant: speak of duality

(3) If two categories are equivalent ~~any thing those~~ every property that can be expressed in "terms of arrows" is preserved.

(1') Most of the time we say that F is an equivalence if $\exists G$; (F,G) is an equivalence.

Def 1.14 Let $F : \mathcal{C} \to \mathcal{D}$ a functor. Then

① F is faithful (fidèle) if $\forall x, y \in Ob(\mathcal{C})$ $F : Hom_{\mathcal{C}}(x,y) \longrightarrow Hom_{\mathcal{D}}(F(x),F$
$$f \longmapsto F(f)$$
is injective

(2) F is full (plein) if it is surjective

(3) F is essentially surjective if $\forall y \in Ob(\mathcal{D})$ $\exists x \in Ob(\mathcal{C})$ s.t. $F(x) \simeq y$ in $\mathcal{D}$.

Thm 1.15 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of category iff F is fully faithful and essentially surjective

Proof ⚠ Set theoretic issue $\Rightarrow$ always ok
$\Leftarrow$ need choice axiom on the class $Ob(\mathcal{C})$

$\boxed{\Rightarrow}$ $F : \mathcal{C} \to \mathcal{D}$ $G : \mathcal{D} \to \mathcal{C}$ $\eta : 1_{\mathcal{C}} \Rightarrow GF$ $\varepsilon : FG \overset{\simeq}{\Rightarrow} 1_{\mathcal{D}}$

(•) We have $\forall y \in Ob(\mathcal{D})$ $\varepsilon_{G(y)} : FG(y) \xrightarrow{\sim} y$ so F is essentially surjective.

(•) In order to see that $F : Hom_{\mathcal{C}}(x,y) \to Hom(Fx,Fy)$, $f \longmapsto F(f)$ is bijective we construct the inverse bijection by applying the "do what you can" rule

$$\phi : Hom(F(x),F(y)) \xrightarrow{G} Hom(GF(x),GF(y)) \xrightarrow{Hom(\eta_x, GFy)} Hom(x, GF(y)) \xrightarrow{Hom(x,\eta_y^{-1})} Hom(x,y)$$
$$F(x) \xrightarrow{g} F(y) \longmapsto g \; --- \longrightarrow x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(g)} GF(y) \xrightarrow{\eta_y^{-1}} y$$

Let us look at the compo $Hom(x,y) \xrightarrow{F} Hom(Fx,F(y)) \xrightarrow{\phi} Hom(x,y)$

$$\begin{tikzcd} f \arrow[r, mapsto] & x \arrow[r, "\eta_x"] \arrow[d, dashed, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & GF(x) \arrow[r, "GF(f)"] \arrow[d, dashed, "GF(f)"] & GF(y) \arrow[r, "\eta_y^{-1}"] \arrow[dl, phantom, "//"] & y \\ & y \arrow[r, "\eta_y"'] & GF(y) & & \end{tikzcd}$$

(car $\eta_\bullet$ trans nat

So the compo is indeed the identity $\Rightarrow$ $\phi \circ F = Id_{Hom(x,y)}$ so F is faithful.

Now let us look at the other composition

$$\mathrm{Hom}(F(x), F(y)) \xrightarrow{\phi} \mathrm{Hom}(x,y) \xrightarrow{F} \mathrm{Hom}(F(x), F(y))$$

$$\alpha \longmapsto \left(x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} GF(y) \xrightarrow{\eta_y^{-1}} y \right) \longrightarrow F(x) \xrightarrow{F(\eta_y^{-1} \circ G(\alpha) \circ \eta_x)} F(y)$$

we have two diagrams:

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[d, "\phi(\alpha)"'] \arrow[dr, phantom, "//"] & GF(x) \arrow[d, "G(\alpha)"] \\ y \arrow[r, "\eta_y"'] & GF(y) \end{tikzcd}$$

①

and

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[d, "\phi(\alpha)"'] \arrow[dr, phantom, "\circlearrowleft"] & GF(x) \arrow[d, "GF(\phi(\alpha))"] \\ y \arrow[r, "\eta_y"'] & GF(y) \end{tikzcd}$$

②

The first commutes by definition and the second by naturality of η.

Con. We have $F(\eta_y \circ \phi(\alpha)) \overset{①}{=} F(G(\alpha) \circ \eta_x)$

$$\underset{2}{=} F(GF(\phi(\alpha)) \circ \eta_x)$$

Since F is faithful we have $G(\alpha) \circ \eta_x = GF(\phi(\alpha)) \circ \eta_x$ and η_x invertible gives $G(\alpha) = GF(\phi(\alpha))$. By the previous point applied to G, G is also faithful so $\alpha = F(\phi(\alpha))$ □

$\Leftarrow$ Goal construct G

(1) $\forall D \in ob(\mathcal{D})$ $\exists$ $X \in ob\,\mathcal{C}$ and $\psi_D : D \xrightarrow{\sim} F(X)$. Set $G(D) := X$

(2) On the morphisms?

$$\begin{tikzcd} D \arrow[r, "\psi_D"] \arrow[d] & FG(D) \arrow[d, dashed, "FG(\alpha)"] \\ D' \arrow[r, "\psi_{D'}"'] & FG(D') \end{tikzcd}$$

Since $F : \mathrm{Hom}(GD, GD') \longrightarrow \mathrm{Hom}(FGD, FGD')$ is ~~fully faith~~ bijective $\exists!$ morphism $G(\alpha) \in \mathrm{Hom}(GD, GD')$ making the diagram commutes

check ① G is a functor

② G is an equiv quasi-inverse of F

details in the litterature □

Ex $\mathrm{Vect}_k \simeq \mathrm{Mat}_k$ is fully faithful (a linear morphism is determined by its matrix) + ess. surjective. □

$$\begin{tikzcd} k^n \arrow[d, "f_A"] & n \arrow[l] \arrow[d, "A"] \\ k^m & m \arrow[l] \end{tikzcd}$$

Chapter 2 Universal properties

Ref Riehl Chap 2,3,4
Maclane III, IV, V
ASGEM III, IV, V

Motivation S set $\sim$ equiv relation on S ⇝ $S/\sim$ quotient

∀ f compatible with $\sim$: $S \to X$ we have

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \\ S/\sim \arrow[ur, dashed, "\exists ! \bar{f}"'] & \end{tikzcd}$$

where $S \xrightarrow{\pi} S/\sim$ is the quotient map

Say that $S \xrightarrow{\pi} S/\sim$ is a solution for the universal problem posed by the compatible maps

easy such solution is unique up to (unique) iso: if $S \xrightarrow{p} S'$ is another get

in a sense

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & S' \\ S/\sim \arrow[ur, dashed, "\exists ! a"'] & \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & S/\sim \\ S' \arrow[ur, dashed, "\exists ! b"'] & \end{tikzcd}$$

Moreover

$$\begin{tikzcd} S \arrow[rr, "p"] \arrow[d, "p"'] & & S' \\ S' \arrow[r, "b"] \arrow[rru, bend right, "Id_{S'}"'] & S/\sim \arrow[ru, "a"] & \end{tikzcd}$$

commutes

indeed $abp = a\pi = p$ but identity of S' also makes this diagram commute so by unicity we have $ab = Id_{S'}$ and similarly $ba = Id_{S/\sim}$

I Initial and final objects

Def 2.1 Let $\mathcal{E}$ be a category. An object $c \in Ob(\mathcal{E})$ is initial (final) if $\forall d \in Ob(\mathcal{E})$ we have $|Hom_{\mathcal{E}}(c,d)| = 1$ $\left(|Hom_{\mathcal{E}}(d,c)| = 1\right)$.

Prop 2.2 If these objects exist they are unique up to unique isomorphism

Proof If c and c' are two initial objects get $c \xrightarrow{\exists ! a} c'$ and $c' \xrightarrow{\exists ! b} c$ and

$$\begin{tikzcd} c \arrow[r, "a"] \arrow[rr, bend right, "Id_c"'] & c' \arrow[r, "b"] & c \end{tikzcd}$$

$ba = Id_c$ since $|Hom_{\mathcal{E}}(c,c)| = 1$. □

Example 2.3 ① $\emptyset$ is initial in Ens and any singleton is final
(2) $\{0\}$ is both initial and final in $Vect_k$ (RMod)
(3) Category of fields does not have initial / final objects (think characteristic!)

Want to say that a universal object is an initial or final object! A category only have at most 2 so seem too restrictive ... but:

Def/Prop 2.4 ① Let $F : \mathcal{E} \to \mathrm{Set}$ a functor. Let $\int F$ the category

$$\text{ob } \int F = \{ (c,x) \,;\, c \in ob(\mathcal{E}) \text{ and } x \in F(c) \}$$

$$\mathrm{Hom}((c,x),(c',x')) = \{ f \in \mathrm{Hom}(c,c') \,;\, F(f)(x) = x' \}$$

Compo = compo in $\mathcal{E}$

Id = Id in $\mathcal{E}$

There is an obvious forgetful functor: $\pi : \int F \to \mathcal{E}$

$$\begin{tikzcd}
(c,x) \arrow[r, mapsto] \arrow[d, "f"] & c \arrow[d, "f"] \\
(d,y) \arrow[r, mapsto] & d
\end{tikzcd}$$

② If F is contravariant let $\int F$ obj $\{ (c,x) \in \mathcal{E} \,,\, x \in F(x) \}$

$$\mathrm{Hom}((c,x),(c',x')) = \{ f : c \to c' \,;\, F(f)(x') = x \}$$

Still have a forgetful functor $\pi : \int F \to \mathcal{E}$.

Example: S set $\sim$ equiv relation

$$F : \mathrm{Ens} \to \mathrm{Ens}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "\alpha"'] & F(X) = \{ f : S \to X \,;\, x \sim y \Rightarrow f(x) = f(y) \} \arrow[d, "\alpha \circ -"'] \\
Y \arrow[r, mapsto] & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r] & X \arrow[d, "\alpha"] \\
 & Y
\end{tikzcd}$$

$\int F$ has for objects $(X, S \xrightarrow{f} X)$; f compatible

Morphism α:
$$\begin{tikzcd}
S \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \\
X \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

easy $\pi : S \to S/\sim$ is an initial object of $\int F$.

Def 2.5 $F : \mathcal{E} \to Set$ a functor. A universal element for F is an initial object of $\int F$

↳ it is a pair (c, x) $c \in ob(\mathcal{E})$ $x \in F(c)$ s.t $\forall$ (d, y) $d \in \mathcal{E}, y \in F($
$\exists! \, \alpha : c \to d$ s.t $y = F(\alpha)(x)$

Def 2.6 $F : \mathcal{E} \to \mathcal{D}$ be a functor, $d \in ob(D)$ a universal arrow from d to F is a pair (c, f) where $c \in ob(\mathcal{E})$, $f \in Hom_{\mathcal{D}}(d, F(c))$ s.t. $\forall$ (c', f') $c' \in ob(\mathcal{E})$ $f' : d \to F(c')$ $\exists! \, \alpha \in Hom_{\mathcal{E}}(c, c')$ s.t $F(\alpha) \circ f = f'$

$$\begin{tikzcd}
& d \arrow[dl, "f"'] \arrow[dr, "\forall f'"] & \\
F(c) \arrow[rr, dashed, "c", "F(\alpha)"'] & & F(c')
\end{tikzcd}$$

$$\begin{tikzcd}
c \arrow[r, "\exists! \, \alpha"'] & c'
\end{tikzcd}$$

exercise Define a category $d \downarrow F$ s.t. universal arrow is an initial object of $d \downarrow F$.

Example 2.7 $\mathcal{U} : Vect_k \to Set$ forgetful functor. Let $X \in Set$. A universal arrow from $X \to \mathcal{U}$ is the "best" V_X k-v with $X \to V_X$

Set $V_X = k[X]$ k-vector space with basis X

$$\begin{tikzcd}
X \arrow[r, "inc"] & V_X \\
x \arrow[r, mapsto] & x
\end{tikzcd}$$

then

$$\begin{tikzcd}
& X \arrow[dl, "i"'] \arrow[dr, "f"] & \\
k[X] \arrow[rr, dashed] & & \mathcal{U}(V)
\end{tikzcd}$$

f extend by linearity as a linear map $k[X] \xrightarrow{\tilde{f}} V$

and this map makes the diagram commutes. Moreover if α is another map then α and $\tilde{f}$ coincide on the basis X so they are equal.

Prop 2.8 Universal elements and arrows are two equivalent notions

Proof (1) If $F : \mathcal{C} \to Set$ and (c, x) is an universal element.

Consider $f_x : \{*\} \to F(c)$, $* \mapsto x$. Then (c, f_x) is a universal arrow $* \to F$.

$$\begin{tikzcd} & \{*\} \arrow[dl, "f_x"'] \arrow[dr, "f_y"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

$y \in F(c')$ $\exists\, \alpha : c \to c'$; $F(\alpha)(x) = y$

$\iff F(\alpha) \circ f_x = f_y$.

(2) Conversely if $F : \mathcal{C} \to \mathcal{D}$, $d \in \mathcal{D}$, a universal arrow (c, f) $d \to F$. If $\mathcal{D}$ is locally small we have

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

$Hom_{\mathcal{D}}(d, F(-)) : \mathcal{C} \to Set$, $x \mapsto Hom_{\mathcal{D}}(d, F(x))$ a set valued functor.

Then $f \in Hom_{\mathcal{D}}(d, F(c))$ is a universal element for this functor. $\square$

II - Representable functors

Def 2.9 $\mathcal{C}$ category (loc small) $F : \mathcal{C} \to Ens$ functor

(1) F is representable if $\exists\, c \in Ob(\mathcal{C})$ and a natural isomorphism between F and $Hom_{\mathcal{C}}(c, -)$ ($Hom_{\mathcal{C}}(-, c)$ when F is contravariant)

(2) A representation of F is the data of $c \in Ob(\mathcal{C})$ and an iso $\eta : Hom(c, -) \Rightarrow F$.

Example $\mathcal{U} : Grp \to Set$, $(G, \cdot) \mapsto G$ is representable since $Hom_{Grp}(\mathbb{Z}, -) \simeq \mathcal{U}$

($Hom_{Grp}(\mathbb{Z}, G) \to \mathcal{U}(G)$, $\alpha \mapsto \alpha(1)$ natural + iso since α is characterized by $\alpha(1)$)

Question How to find $\alpha : \mathrm{Hom}_{\mathcal{E}}(c,-) \Rightarrow F$ in general?

Thm2.10 [Yoneda lemma] Let $F : \mathcal{E} \to \mathrm{Set}$ a functor with $\mathcal{E}$ locsmall and $c \in \mathrm{Ob}(\mathcal{E})$. Then

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{E}}(c,-), F) \cong F(c)$$
$$\alpha \longmapsto \alpha_c(\mathrm{Id}_c)$$

Moreover this isomorphism is natural in c and in F.

Proof (•) $\alpha : \mathrm{Hom}_{\mathcal{E}}(c,-) \Rightarrow F$

we have $\alpha_x : \mathrm{Hom}_{\mathcal{E}}(c,x) \longrightarrow F(x)$

$$f : c \to x \longmapsto \alpha_x(f)$$

but $f : c \to x$ induces a commutative diagram (nat of α!)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(c,c) \arrow[r, "\alpha_c"] \arrow[d, "\mathrm{Hom}_{\mathcal{E}}(c,f)"'] & F(c) \arrow[d, "F(f)"] \\
\mathrm{Hom}(c,x) \arrow[r, "\alpha_x"'] & F(x)
\end{tikzcd}$$

at Id_c we have

$\hookrightarrow \alpha_x(f) \in F(x) = F(f)(\alpha_c(\mathrm{Id}_c))$

So α is completely determined by $\alpha_c(\mathrm{Id}_c)$. $\alpha \longmapsto \alpha_c(\mathrm{Id}_c)$ is injective.

(•) Conversely if $e \in F(c) \rightsquigarrow \alpha : \mathrm{Hom}(c,-) \Rightarrow F$ defined by

$$\alpha_x : \mathrm{Hom}_{\mathcal{E}}(c,x) \longrightarrow F(x)$$
$$f \longmapsto F(f)(e)$$

Check (1) α is a natural transf

(2) $e \longmapsto \alpha_e$ is the inverse bijection

(3) Naturality. $\square$

Universal II

Rem (1) $F : \mathcal{C} \to \mathrm{Ens}$ then (c,x) is universal element for F iff the natural transf $\alpha_x : \mathrm{Hom}(c,-) \Rightarrow F$ induced by x is an isomorphism

Indeed α_x iso iff $\forall$ ~~[illegible]~~ $c' \in \mathcal{C}$ $(\alpha_x)_{c'} : \mathrm{Hom}(c,c') \to F(c')$

$f \longmapsto F(f)(x)$

| is bijective iff $\forall c' \in \mathcal{C}$ $\forall y \in F(c')$ $\exists!\, f : c \to c'$; $F(f)(x) = y$.

(2) for universal arrow: use $\mathcal{D}(X, F(-))$ as before.

III Examples of object defined by universal properties

1) Product, Coproduct

Let $\mathcal{C}$ be a small category and $X, Y \in \mathrm{ob}(\mathcal{C})$

$\mathcal{C}_{X,Y}$ category objects tuples (Z,f,g) ; $Z \in \mathrm{ob}(\mathcal{C})$ $f \in \mathrm{Hom}(Z,X)$
$g \in \mathrm{Hom}(Z,Y)$

Morphism

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\
X \arrow[r, phantom, "\circ"] & {} & Y \arrow[dl, "g'"] \\
 & Z' \arrow[ul, "f'"] &
\end{tikzcd}$$

$\alpha : Z \to Z'$

making diag commute.

Def 2.11 A product of X and Y is a final object in $\mathcal{C}_{X,Y}$

↳ unwrapped: It is an object $X \times Y$ with two maps $\pi_X : X \times Y \to X$
$\pi_Y : X \times Y \to Y$

1.L ⊳

$$\begin{tikzcd}
 & Z \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] \arrow[d, "\exists!\alpha"] & \\
 \arrow[r, phantom, "\circ"] & X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] \arrow[r, phantom, "\circ"] & {} \\
X & & Y
\end{tikzcd}$$

Coro if it exists it is unique up to unique isomorphism in $\mathcal{C}_{X,Y}$.

Example In Set the cartesian product $X \times Y = \{(x,y);\ \begin{matrix} x \in X \\ y \in Y \end{matrix}\}$ together with the canonical projections is a product.

Dually

Def 2.11 A coproduct of X and Y is a product in $\mathcal{E}^{op}$

Unwrapped ~~#~~ satisfy: universal property

$$\begin{tikzcd}
 & X \arrow[dr, "i_X"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, bend left, "g"] \\
\forall \{ & & X \sqcup Y \arrow[d, dashed, "\exists ! \alpha"] & \\
 & & Z &
\end{tikzcd}$$

Example In Set the coproduct of X and Y is the disjoint union together with its canonical inclusion

$$X \sqcup Y := X \times \{0\} \cup Y \times \{1\} \qquad \begin{array}{llllll} i_X : & X & \to & X \sqcup Y & \leftarrow & Y . \\ & x & \mapsto & (x,0) \quad (y,1) & \leftarrow\!\shortmid & y \end{array}$$

(2) Equalizer and coequalizer

Def 2.12 Let $\mathcal{E}$ be a category and $X, Y \in Ob(\mathcal{E})$ $f, g : X \to Y \in Mor(\mathcal{E})$

Consider the contravariant functor

$$\begin{tikzcd}
F : \mathcal{E} \arrow[r] & \mathrm{Set} \\
c \arrow[r, mapsto] \arrow[d, "\partial"'] & \{\alpha : c \to X \in Mor(\mathcal{E});\ f\alpha = g\alpha\} \\
d \arrow[r, mapsto] & \{\beta : \underset{\overset{\uparrow \partial}{c}}{d} \to \cancel{X} ;\ f\beta = g\beta\} \arrow[u, "\beta \circ \partial \atop \overset{\uparrow}{\beta}"']
\end{tikzcd}$$

An equalizer in $\mathcal{E}$ is a representation of the contravariant functor F

$\rightsquigarrow$ Unwrapped Yoneda $Hom(-,c) \cong F \overset{1:1}{\longleftrightarrow}$ element of $F(c)$

so a representation is a pair (c, e) $c \in Ob(\mathcal{E})$ $e \in F(c)$ s.t. Yoneda nat trans is an iso!

concretely $\forall d \in ob(\mathcal{C})$ $\mathrm{Hom}_{\mathcal{C}}(d,c) \xrightarrow{\circ e} F(d) \ni \theta$ is an iso

$h \longmapsto F(h)(e)$

$$\begin{tikzcd}
 & d \arrow[dl, dashed, "\exists! \alpha"'] \arrow[d, "\forall \theta"] & \\
c \arrow[r, "e"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

__Example__ In Set $E = \{x \in X \,;\, f(x) = f(y)\} \hookrightarrow X$ is an equalizer.
$X \underset{g}{\overset{f}{\rightrightarrows}} Y$

Dually:

__Def 2.13__ dual notion of coequilizer: $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\pi} Z$; $\pi f = \pi g$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "\forall \theta"] & Z \arrow[dl, dashed, "\exists! \alpha"] \\
 & Z' \arrow[ur, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

__Example__ In Set $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ Consider $\sim$ the equivalence relation on Y generated by $f(x) \sim g(x)$ (smallest equiv relation with this property). Then $Y \xrightarrow{\pi} Y/_\sim$ is a coequalizer

__sketch__

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "h"] & Y/_\sim \arrow[dl, dashed] \\
 & Z &
\end{tikzcd}$$

by construction get $\pi f = \pi g$

$hf = hg \Rightarrow h$ compatible with $\sim$ so done by univ prop of quotient.

__III - Adjoint functors__

__Kan 1958__

__Def 2.14__ An adjunction (G, D) is a pair of functors $G : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{C}$ together with an isomorphism $\mathrm{Hom}_{\mathcal{D}}(G(c), d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, D(d))$ which is natural in both variables

__Not__ $G \dashv D$ and we say that G is a left-adjoint for D and D is a right adjoint for G