Example In Set the cartesian product $X \times Y = \{ (x,y) ;\ \substack{x \in X \\ y \in Y} \}$ together with the canonical projections is a product.

Dually

Def 2.11 A coproduct of X and Y is a product in $\mathcal{E}^{op}$

Unwrapped ~~#~~ satisfy: universal property

$$\begin{tikzcd} X \arrow[dr, "i_X"] \arrow[ddr, bend right, "\forall f"'] \arrow[ddr, phantom, bend right=20, "\circlearrowleft"] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, bend left, "g"] \arrow[ddl, phantom, bend left=20, "\circlearrowleft"] \\ & X \sqcup Y \arrow[d, dashed, "\exists! \alpha"] & \\ & Z & \end{tikzcd}$$

Example In Set the coproduct of X and Y is the disjoint union together with its canonical inclusion

$$X \sqcup Y := X \times \{0\} \cup Y \times \{1\} \qquad \begin{tikzcd}[row sep=tiny] i_X : X \arrow[r] & X \cup Y & Y \arrow[l] . \\ x \arrow[r, mapsto] & (x,0) \quad (y,1) & y \arrow[l, mapsto] \end{tikzcd}$$

(2) Equalizer and coequalizer

Def 2.12 Let $\mathcal{E}$ be a category and $X, Y \in Ob(\mathcal{E})$ $f, g : X \to Y \in Mor(\mathcal{E})$

Consider the contravariant functor

$$\begin{tikzcd} F : \mathcal{E} \arrow[r] & Set \\ c \arrow[r, mapsto] \arrow[d, "\partial"'] & \{ \alpha : c \to X \in Mor(\mathcal{E}) ;\ f\alpha = g\alpha \} \\ d \arrow[r, mapsto] & \{ \beta : \underset{\overset{\uparrow \partial}{c}}{d} \to X ;\ f\beta = g\beta \} \arrow[u, "\beta \circ \partial \ \overset{\uparrow}{\beta}"'] \end{tikzcd}$$

An equalizer in $\mathcal{E}$ is a representation of the contravariant functor F

⇝ Unwrapped Yoneda $Hom(-,c) \cong F \overset{1:1}{\longleftrightarrow}$ element of $F(c)$

so a representation is a pair (c, e) $c \in Ob(\mathcal{E})$ $e \in F(c)$ s.t. Yoneda nat trans is an iso!

concretely $\forall d \in ob(\mathcal{E})$ $Hom_{\mathcal{E}}(d,c) \xRightarrow{?e} F(d) \ni \partial$ is an iso

$$h \longmapsto F(h)(e)$$

$$\begin{tikzcd} & d \arrow[dl, dashed, "\exists ! \alpha"'] \arrow[d, "\forall \partial"] & \\ c \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

Example In Set $E = \{x \in X ; f(x) = f(y)\} \hookrightarrow X$ is an equalizer.

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

Dually:

Def 2.13 dual notion of coequilizer: $X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{\pi} Z$, $\pi f = \pi g$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "\forall \partial"'] & Z \arrow[dl, dashed, "\exists ! \alpha"] \\ & Z & \end{tikzcd}$$

Example In Set $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ Consider $\sim$ the equivalence relation on Y generated by $f(x) \sim g(x)$ (smallest equiv relation with this property). Then $Y \xrightarrow{\pi} Y/\sim$ is a coequalizer

sketch

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "h"] & Y/\sim \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

by construction get $\pi f = \pi g$

$hf = hg \Rightarrow h$ compatible with $\sim$ so done by univ prop of quotient.

III - Adjoint functors

Kan 1958

Def 2.14 An adjunction (G, D) is a pair of functors $G: \mathcal{E} \to \mathcal{D}$ and $D: \mathcal{D} \to \mathcal{E}$ together with an isomorphism $Hom_{\mathcal{D}}(G(c), d) \simeq Hom_{\mathcal{E}}(c, D(d))$ which is natural in both variables

Not $G \dashv D$ and we say that G is a left-adjoint for D
and D is a right adjoint for G

⑨

* $f: c \to c'$ look at $\eta_{c'}$

$$\begin{tikzcd}
(F(c), F(c)) \arrow[r, phantom, "\simeq"] \arrow[d, "F(f)\circ -"] & (c, GF(c)) \\
(F(c), F(c')) \arrow[r, phantom, "\simeq"] & (c, GF(c')) \arrow[u, "GF(f)\circ -"'] \\
(F(c'), F(c')) \arrow[r, phantom, "\simeq"] \arrow[u, "-\circ F(f)"] & (c', GF(c')) \arrow[u, "-\circ f"']
\end{tikzcd}$$

$Id \longmapsto \eta_c$

$Id \longmapsto \eta_{c'}$

If $G \dashv D$ we have $\forall\, c, d \in Ob(\mathcal{E}) \times Ob(\mathcal{D})$

$$Hom_{\mathcal{D}}(Gc, d) \xrightarrow[\alpha_{c,d}]{\sim} Hom_{\mathcal{E}}(c, D(d))$$

in particular when $d = Gc$ get $Hom_{\mathcal{D}}(Gc, Gc) \xrightarrow[\alpha_{c,G(c)}]{} Hom_{\mathcal{E}}(c, DGc)$

let $\eta_c : c \to DG(c)$ the image of Id_{Gc}

by naturality(*) of the isomorphism $\eta : Id_{\mathcal{E}} \Rightarrow DG$ is a natural transformation called the <u>unit</u> of the adjunction

Similarly when $c = D(d)$ we get $\varepsilon : GD \Rightarrow Id_{\mathcal{D}}$ a natural transformation called the <u>counit</u> of the adjunction.

Prop 2.15 Let $\mathcal{E} \underset{D}{\overset{G}{\rightleftarrows}} \mathcal{D}$ be two functors. Then $G \dashv D$ iff $\exists\, \eta : 1_{\mathcal{E}} \Rightarrow DG$

and $\varepsilon : GD \Rightarrow 1_{\mathcal{D}}$ s.t. the following diag commutes

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[rd, equal, "Id_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\
& F
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta G"] \arrow[rd, equal, "Id_D"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\
& G
\end{tikzcd}$$

(*) : $Hom(F(x), y) \to Hom(x, G(y))$

$F(x) \xrightarrow{\alpha} y \rightsquigarrow x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y)$

$F(y) \xrightarrow{F(\beta)} FG(y) \xrightarrow{\varepsilon_y} y \longleftarrow\!\mid\; y \xrightarrow{\beta} G(y)$

do the details.

Proof $\left\{ \begin{array}{l} \text{Riehl 4.2.6} \\ \text{Assem 3.7} \end{array} \right.$ give details for what it means.

$\Rightarrow$ did almost everything

$\Leftarrow$ use η et ε to construct the natural iso

Examples (1) $\mathcal{U} : Ab \to Set$ forget

$L : Set \to Ab$ free abelian gp

Then $L \dashv \mathcal{U}$

(2) $\mathcal{U} : Ab \to Gp$ forgetful describe left-adjoint

(3) $\mathcal{U} : Top \to Set$ f— describe left / Right adjoint.

(4) $Rep_k(G) \xrightarrow[Res^G_H]{} Rep_k(H)$ has Ind^G_H for left adjoint.
$H \leq G$

Thm 2.16 Let $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the following are equivalent

(1) F admits a left adjoint

(2) $\forall x \in \mathcal{D}$ the functor $\mathrm{Hom}_{\mathcal{D}}(x, F(-))$ is representable

(3) $\forall x \in \mathcal{D}$ $\exists$ a universal arrow $x \to F$.

"**Coro**" If they exist adjoints are unique up to isomorphisms $\underset{G \dashv F}{}$ ($F, F' \dashv G$ $\left(\mathrm{Hom}(-, G) \simeq \mathrm{Hom}(F, -) \simeq \mathrm{Hom}(F', -) \Rightarrow F \simeq F' \right)$

(2) $\Leftrightarrow$ (3) was already explained

(1) $\Leftrightarrow$ (2) We have by definition of adjunction $\mathrm{Hom}_{\mathcal{D}}(x, F(C)) \simeq \mathrm{Hom}_{\mathcal{C}}(Gx, C)$ natural in C so $\mathrm{Hom}_{\mathcal{D}}(x, F(-)) \simeq \mathrm{Hom}_{\mathcal{C}}(Gx, -)$. Conversely if $\mathrm{Hom}_{\mathcal{D}}(x, F(-)) \simeq \mathrm{Hom}_{\mathcal{D}}(Gx, -)$ this defines a functor $G : \mathcal{D} \to \mathcal{C}$ on objects. If $f : x \to y$ then f induces natural trans

$$\begin{tikzcd}
\mathrm{Hom}(Gx, -) \arrow[r, phantom, "\simeq"] & \mathrm{Hom}(x, F(-)) \\
\mathrm{Hom}(Gy, -) \arrow[r, phantom, "\simeq"] \arrow[u, dashed, "\exists ! \gamma"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Hom}(y, F(-)) \arrow[u, "- \circ f"']
\end{tikzcd}$$

and via Yoneda γ is determined by an element in $\mathrm{Hom}(Gx, Gy)$. Call it $G(f)$. $\square$

Morality: universal properties = adjoint functors!

IV - (Co)limits** can skip at first reading.

Def 2.17 (1) Let $\mathcal{C}, \mathcal{D}$ be two categories. Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \mathcal{D}^{\mathcal{C}}$ is the category with - objects functors from $\mathcal{C}$ to $\mathcal{D}$

- Morphisms natural transformations
- Compo vertical composition of nat trans.

It is called the functor category

(2) When $\mathcal{J}$ is a small category we also say that $\mathrm{Fun}(\mathcal{J}, \mathcal{D})$ is the category of diagrams of shape $\mathcal{J}$ in $\mathcal{C}$.

Example (1) $\mathbf{2}$ (it is a "two") be the category $\underset{1}{\bullet} \to \underset{2}{\bullet}$ with two objects 3 arrows : two identities and an arrow from $1 \to 2$.

Then 2×2 is

$$\begin{tikzcd}\bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet\end{tikzcd}$$

4 objects $5 + \underset{\uparrow \text{ identity}}{4}$ arrows

Then a functor from 2×2 to $\mathcal{E}$ is a commutative diagram in $\mathcal{E}$.

(2) $\mathcal{J}$ small category $\mathcal{E} \xrightarrow{\Delta} \operatorname{Fun}(\mathcal{J}, \mathcal{E})$ where $\Delta(c)$ is the
$c \longmapsto \Delta(c)$
constant functor at $\mathcal{E}$: $\Delta(c) : \mathcal{J} \to \mathcal{E}$

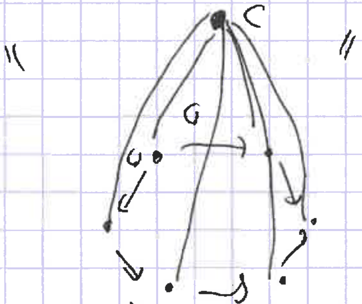
$$\begin{tikzcd}j \arrow[r, mapsto] \arrow[d, "f"'] & c \arrow[d, "Id_c"] \\ k \arrow[r, mapsto] & c\end{tikzcd}$$

Def 2.18 A cone above a diagram $F : \mathcal{J} \to \mathcal{E}$ with summit $c \in \mathcal{E}$ is a natural transformation $\lambda : \Delta(c) \Rightarrow F$

$\rightsquigarrow$ unwrapped $\forall j \in \mathcal{J}$ $\lambda_j : c \to F(j) \in \operatorname{Mor}(\mathcal{E})$ s.t. $\forall f : i \to j \in \mathcal{J}$

get

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[d, "\lambda_j"] \\ F(i) \arrow[r, "F(f)"'] \arrow[ur, phantom, "\circlearrowleft"] & F(j)\end{tikzcd}$$



dually: a cone under F (or cocone) with summit c is a natural transformation $\lambda : F \Rightarrow c$

Def 2.19 Let $F : \mathcal{J} \to \mathcal{E}$ a limit of F (= projective limit, or inverse limit) is a universal (final) cone above F. A colimit (= inductive limit or direct limit) is a universal (initial) cone under F

$\rightsquigarrow$ unwrapped :

$F : \mathcal{J} \to \mathcal{E}$ diagram then a limit of F is a pair $(\lim F, \phi)$ where

- $\lim F \in \mathcal{E}$
- $\phi : \Delta_{\lim F} \Rightarrow F$ "best"

s.t. $\forall \lambda : \Delta(c) \Rightarrow F$ $\exists!$ $f : x \to \lim F \in \mathcal{E}$;

$$\begin{tikzcd} & \Delta k \arrow[dl, dashed, "\Delta(f)"'] \arrow[d, Rightarrow, "\lambda"] \\ \Delta\lim F \arrow[r, Rightarrow, "\phi"'] \arrow[ur, phantom, "\circlearrowleft"] & F \end{tikzcd}$$

Concretely: $\forall j \in \mathcal{J}$

$$\begin{tikzcd} & x \arrow[dl, dashed, "\exists! f"'] \arrow[d, "\lambda_j"] \\ \lim F \arrow[r, "\phi_j"'] \arrow[ur, phantom, "\circlearrowleft"] & F(j) \end{tikzcd}$$

In compact form $\mathrm{Hom}_{\mathcal{E}}(-, \lim F) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{E})}(\Delta -, F)$

Exercise do the same for colimits

Rem (1) If a limit exists it is unique up to iso (unique iso that commutes with the legs!)

(2) If all limits exists, then lim can be promoted as a functor $\lim : \mathrm{Fun}(\mathcal{J},\mathcal{E}) \to \mathcal{E}$ using thm 2.16 ↳

$\eta : F \Rightarrow G$ get

$$\begin{tikzcd} \lim F \arrow[r, dotted, "\exists!"] \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowleft"] & \lim G \arrow[d, Rightarrow] \\ F \arrow[r, Rightarrow, "\eta"'] & G \end{tikzcd}$$

Coro 2.20 (1) If $\mathcal{E}$ has all $\mathcal{J}$-limits, then $\lim : \mathrm{Fun}(\mathcal{J}, \mathcal{E}) \to \mathcal{E}$ is a left adjoint to Δ

(2) If $\mathcal{E}$ has all $\mathcal{J}$-colimits then $\mathrm{colim} : \mathrm{Fun}(\mathcal{J}, \mathcal{E}) \to \mathcal{E}$ is a left adjoint to $\Delta : \mathcal{E} \to \mathrm{Fun}(\mathcal{J}, \mathcal{E})$

Examples / Definitions 2.21

(1) $\mathcal{J}$ discrete : only morphism are identities

$F : \mathcal{J} \to \mathcal{E}$ is the same as a collection $(X_i)_{i \in \mathcal{J}}$ of objects of $\mathcal{E}$

Then a limit of F is :

An object $\lim F \in \mathcal{E}$ with $\forall i \in \mathcal{J}$ $f_i : \lim F \to X_i$

s.t. $\forall$ $X \in ob\,\mathcal{E}$, $p_i : X \to X_i$ $\exists!\ \alpha : X \to \lim F$ s.t.

$\forall i, j$
$$\begin{tikzcd}
& X \arrow[ddl, bend right, "p_i"'] \arrow[d, dashed, "!\alpha"] \arrow[ddr, bend left, "p_j"] & \\
& \lim F \arrow[dl, "f_i"] \arrow[dr, "f_j"'] & \\
X_i & & X_j
\end{tikzcd}$$

$\rightsquigarrow$ $\lim F = \prod_{j \in \mathcal{J}} F(j)$ is the product of the $F(j)$s and $f_i =: \pi_i$ are the canonical projections.

(o) Dual $\operatorname{Colim} F = \coprod_{j \in \mathcal{J}} F(j)$ coproduct

(•) $\mathcal{J} = \bullet \rightrightarrows \bullet$ $F : \mathcal{J} \to \mathcal{E}$ is the data of two parallel morphisms

(Co)lim = (Co) equalizer.

• $\mathcal{J} =$
$$\begin{tikzcd}
& \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$
Then $F : \mathcal{J} \to \mathcal{E}$ is the data of
$$\begin{tikzcd}
& A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

$\lim F$ is called a pullback of f and g.

Univ prop
$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \times_C B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

• $\mathcal{J} = \omega^{op}$ $\cdots \to 2 \to 1 \to 0$ Then $\lim F$ is often called the "inverse limit" of F

Concretely F is the data of

Cone

$$\begin{tikzcd} & & C \arrow[dl, "\lambda_2"'] \arrow[d, "\lambda_1"] \arrow[dr, "\lambda_0"] & \\ \cdots \arrow[r] & F(2) \arrow[r, "\alpha_2"'] & F(1) \arrow[r, "\alpha_1"'] & F(0) \end{tikzcd}$$

Concretely

$a \in \varprojlim \mathbb{Z}/p^n\mathbb{Z}$

iff $a = (a_i)_{i \in n}$

s.t. $a_i \equiv a_j \mod p^i \ \forall i < j$

prop $(\alpha_i \circ \dots \circ \alpha_n) \circ \lambda_n = \lambda_i$

Typical example $F(n) = \mathbb{Z}/p^n\mathbb{Z} \in \mathrm{Ring}$ $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ ring of p adic integers

$J = 0 \to 1 \to 2 \to \cdots$

colim = direct lim

typical example $\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty)$ Prüfer gp

Def 2.22 Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ is $^{(co)}$complete if $\forall J$ small category $F: J \to \mathcal{E}$ diagram, F has a $^{(co)}$limit in $\mathcal{E}$

Thm 2.23 $\mathcal{E}$ is (co)complete iff $\mathcal{E}$ has all (co)product and (co)eq

Def 2.24 $F: \mathcal{E} \to \mathcal{D}$ preserves (co)limits if for every diagram $D: J \to \mathcal{E}$ and any (co)limit cone ~~of~~ (C, ϕ) of D, the image $(F(C), F\phi)$ is a (co)limit cone over $FD: J \to \mathcal{D}$

Moral $F(\lim D) \simeq \lim FD$ but stronger:

$$\begin{tikzcd} \lim D \arrow[d, "\phi_i"] & \rightsquigarrow & F(\lim D) \arrow[r, dashed, "\exists!\alpha"] \arrow[d, "F(\phi_i)"] & \lim FD \arrow[dl, "\lambda_i"] \\ D(i) & & FD(i) & \end{tikzcd}$$

and α is an iso!

Thm 2.25 (1) Right adjoint preserves limits

(2) Left adjoint preserves colimits

Proof We only need to prove (1), for two use the opposite category

11 bis

Def 2.22 $\mathcal{E}$ be a category. Then $\mathcal{E}$ is (co)complete if $\forall \mathbb{J}$ small category $F : \mathbb{J} \to \mathcal{E}$ diagram, then F has a (co)limit in $\mathcal{E}$

Thm 2.23 $\mathcal{E}$ is (co)complete iff $\mathcal{E}$ has all (co) products and (co)equalize

Idea Any small limit can be expressed as an equalizer

$$\lim_{\mathbb{J}} F \longrightarrow \prod_{j \in ob(\mathbb{J})} F_j \overset{c}{\underset{d}{\rightrightarrows}} \prod_{f \in Mor(\mathbb{J})} F(cod(f))$$

<u>c, d</u> :

c : $\alpha : i \to j$

$$\begin{tikzcd}
\prod_{j \in ob(\mathbb{J})} F_j \arrow[d, "\pi_i"] \arrow[rr, dashed, "\exists ! d"] & & \prod_{\substack{\alpha \in Mor(\mathbb{J}) \\ \alpha : i \to j}} F(cod(\alpha)) \arrow[ddll, "p_\alpha"] \\
F(i) \arrow[d, "F(\alpha)"] & & \\
F(j) & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\prod_{j \in ob(\mathbb{J})} F_j \arrow[d, "\pi_j"'] \arrow[r, dashed, "\exists ! c"] & \prod_{\alpha : i \to j} F(j) \arrow[dl, "p_\alpha"] \\
F(j) &
\end{tikzcd}$$

then has to check that $\lim_{\mathbb{J}} F$ is given by an equalizer of c and d.

illustration $\mathbb{J} =$

$$\begin{tikzcd}
& 1 \arrow[d] \\
2 \arrow[r] & 3
\end{tikzcd}$$

$F : \mathbb{J} \to \mathcal{E}$ is

$$\begin{tikzcd}
& X_1 \arrow[d, "\alpha"] \\
X_2 \arrow[r, "\beta"'] & X_3
\end{tikzcd}$$

get $X_1 \times X_2 \times X_3 \overset{c}{\underset{d}{\rightrightarrows}} X_3 \times X_3$ where

$$\begin{tikzcd}
& X_1 \times X_2 \times X_3 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dashed, "\exists ! c"] & \\
X_1 \arrow[d, "\alpha"'] & & X_2 \arrow[d, "\beta"] \\
X_3 & X_3 \times X_3 \arrow[l, "p_\alpha"] \arrow[r, "p_\beta"'] & X_3
\end{tikzcd}$$

$$\begin{tikzcd}
& X_1 \times X_2 \times X_3 \arrow[dl, "\pi_3"'] \arrow[dr, "\pi_3"] \arrow[d, dashed, "\exists ! d"] & \\
X_3 & X_3 \times X_3 \arrow[l, "p_\alpha"] \arrow[r, "p_\beta"'] & X_3
\end{tikzcd}$$

. If $E \xrightarrow[\gamma]{} X_1 \times X_2 \times X_3 \overset{c}{\underset{d}{\rightrightarrows}} X_3 \times X_3$

is s.t. $c\gamma = d\gamma$ we have

$\gamma_i := \pi_i \circ \gamma$

$$\begin{tikzcd}
& E \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] & \\
X_1 \arrow[r, "\alpha"'] & X_3 & X_2 \arrow[l, "\beta"]
\end{tikzcd}$$

and ~~$\forall$~~. $\alpha \circ \gamma_1 = \alpha \circ \pi_1 \circ \gamma$

$= p^\alpha \circ d \circ \gamma$

$= p^\alpha \circ e \circ \gamma = \pi_3 \circ \gamma = \gamma_3$

and similarly $\beta \circ \gamma_2 = \beta \circ \pi_2 \circ \gamma$

$= p_\beta \circ d \circ \gamma$

$= p_\beta \circ e \circ \gamma$

$= \pi_3 \circ \gamma = \gamma_3$

So an equalizer is a ~~lim~~ terminal object for

$$\begin{tikzcd} E \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & X_1 \arrow[d, "\alpha"] \\ X_2 \arrow[r, "\beta"'] & X_3 \end{tikzcd}$$

s.t. $\alpha \gamma_1 = \beta \gamma_2$, so a pullback.

Def 2.24 $F: \mathcal{C} \to \mathcal{D}$ preserves J-(co)limits if for every diagram $D: J \to \mathcal{C}$ and any ~~[illegible]~~ (co)limit cone (C, ϕ) of D the image $(F(C), F\phi)$ is a ~~[illegible]~~ (co)limit cone for $FD: J \to \mathcal{D}$

Moral $F(\lim D) \simeq \lim FD$ but much stronger

$$\begin{tikzcd} F \lim D \arrow[r, dashed, "\exists! \alpha"] \arrow[d, "F(\phi_i)"'] & \lim FD \arrow[dl, "\lambda_i"] \\ FD(i) & \end{tikzcd}$$

and saying that $(F \lim D, F\phi)$ is a lim cone implies that α is an isomorphism!

Prop 2.25 $\mathcal{C}$ locally small category. Then

(1) $\forall X \in ob(\mathcal{C})$ $Hom_{\mathcal{C}}(X, -)$ preserves all limits that exist in $\mathcal{C}$

(2) The contravariant functor $Hom_{\mathcal{C}}(-, X)$ carry colimits in $\mathcal{C}$ to limit in Set.

Proof proof 1 $\mathcal{J} \xrightarrow{D} \mathcal{E} \xrightarrow{\overset{=:F}{\mathrm{Hom}_{\mathcal{E}}(x,-)}} \mathrm{Set}$

- (L, λ) limit cone for D
- $(F(L), F(\lambda))$ cone in Set over FD

$$\begin{tikzcd} i \arrow[d, "\alpha"'] & & F(L) \arrow[dl, "F(\lambda)"'] \arrow[dr, "F(\lambda_j)"] \arrow[d, phantom, "\circlearrowleft"] & \\ j & \mathrm{Hom}_{\mathcal{E}}(x, F(i)) \arrow[rr, "\mathrm{Hom}_{\mathcal{E}}(x, F\alpha)"'] & & \mathrm{Hom}_{\mathcal{E}}(x, F(j)) \end{tikzcd}$$

Let us show that $(F(L), F(\lambda))$ is a limit cone for FD

$$\begin{tikzcd} (*) & S \arrow[d, Rightarrow, "f"'] & F(L) \arrow[dl, Rightarrow, "F(\lambda)"] \\ & FD & \end{tikzcd}$$

$f(i) : S \longrightarrow \mathrm{Hom}(x, F(i))$
$s \longmapsto f(s) =: f_i(s)$

then fixing s we get:

$$\begin{tikzcd} & x \arrow[dl, "f_i(s)"'] \arrow[dr, "f_j(s)"] & \\ F(i) \arrow[rr, "F(\alpha)"'] & & F(j) \end{tikzcd}$$

$F(\alpha) \circ f_i = f_j$
so $F(\alpha) \circ f_i(s) = f_j(s)$

so $(x, f_i(s))$ is a cone over F hence $\exists !\, u_s$:

$$\begin{tikzcd} x \arrow[r, "u_s"] \arrow[dr, "f_i(s)"'] & L \arrow[d, "\lambda_i"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F(i) \end{tikzcd}$$

Now set $u : S \longrightarrow \mathrm{Hom}(x, L)$ and we have
$s \longmapsto u_s$

$F(\lambda) \circ u(s) = F(\lambda) \circ u_s = f$ so gives the existence of our $u : S \to F(L)$ making $(*)$ a commutative diagram.

If v is another one then $v(s)$ makes

$$\begin{tikzcd} x \arrow[r, "v_s"] \arrow[dr, "f_i(s)"'] & L \arrow[d, "\lambda_i"] \\ & F(i) \end{tikzcd}$$

commutes

so $v = u$ □

Proof 2 $\mathrm{Hom}_{\mathcal{E}}(x, \lim D) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}, \mathcal{E})}(\Delta x, D) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}, \mathrm{Set})}(\Delta 1, \mathrm{Hom}_{\mathcal{E}}(x, D-))$

$\simeq \mathrm{Hom}_{Set}(1, \lim \mathrm{Hom}_{\mathcal{E}}(x, D-)) \simeq \lim \mathrm{Hom}_{\mathcal{E}}(x, D-)$ □

<u>Thm 2.26</u> (1) Right adjoint preserves limits

(2) left adjoint preserves colimits

<u>Proof</u>: we only do (1).
$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
$D : \mathcal{J} \to \mathcal{D}$ a diagram

<u>Goal</u> show that $(G \lim D, G\eta)$ is a limit cone for $G \circ D$

(•) <u>cone</u> <u>clear</u>

(•)

(*)
$$\begin{tikzcd} G \lim D \arrow[d, "G(\eta_i)"] & c \arrow[dl, "H_j"] \\ GD(j) & \end{tikzcd}$$
another cone $H_j \in \mathrm{Hom}(c, GD(j)) \simeq \mathrm{Hom}(F(c), D($

$H_j \longmapsto H_j^*$

<u>Fact</u> $(F(c), (H_c^*))$ is a cone over D $\exists! \, \tau : Fc \to \lim D$ s.t

$$\begin{tikzcd} F(c) \arrow[dr, dashed, "\exists! \tau"] \arrow[dd, Rightarrow, "H"'] & \\ & \lim D \arrow[dl, Rightarrow, "\eta"] \\ D & \end{tikzcd}$$

$\tau \in \mathrm{Hom}(Fc, \lim D) \simeq \mathrm{Hom}(c, G \lim D)$

$\tau \rightsquigarrow \tau^*$

Show that τ^* is a factorization for (*) moreover another such factor would transpose to τ, hence unique.

↳ <u>in compact form</u>

$\mathrm{Hom}_{\mathcal{E}}(C, \lim D) \simeq \mathrm{Hom}_{Fun(\mathcal{J}, \mathcal{C})}(\Delta C, GD) \simeq \mathrm{Hom}_{Fun(\mathcal{J}, \mathcal{D})}(F\Delta C, D) \simeq \mathrm{Hom}_{Fun(\mathcal{J}, \mathcal{D})}(\Delta Fc, D)$

$\simeq \mathrm{Hom}_{\mathcal{D}}(Fc, \lim D)$

$\simeq \mathrm{Hom}_{\mathcal{E}}(c, G \lim D)$ □

Chapter III Tensor product

Rings = associative with 1.

- A right A-module is an abelian group $(M,+)$ with $M \times A \to M$, $(m,a) \mapsto m \cdot a$

s.t. (1) $(m+n)a = am+an$ (3) $m(ab) = (ma)b$

(2) $m(a+b) = ma+mb$ (4) $m1_A = m$

by symmetry get notion of left-A-module

(•) If A, B are two rings an A-B-bimodule is an abelian gp M with a left A-module structure and a right A-module structure s.t. $\forall (a,b) \in A \times B$ $\forall m \in M$ $a \cdot (m \cdot b) = (a \cdot m) \cdot b$

Goal If M_A, ${}_AN$ are a right and a left A-module: Construct an abelian gp $M \otimes_A N$ "universal" with respect to "bilinear maps".

Def 3.1 M_A, ${}_AN$ as above. A bilinear map $f : M \times N \to G$ where G is an abelian group is a map f such that

$$\left[\begin{array}{l} (1)\ f(m_1+m_2, n) = f(m_1,n)+f(m_2,n) \\ (2)\ f(m, n_1+n_2) = f(m,n_1)+f(m,n_2) \\ (3)\ f(ma,n) = f(m,an) \end{array}\right.$$

(Also called "balanced" maps)

Thm 3.2 There exists an abelian group $M \otimes_A N$ together with $t : L \times M \to M \otimes_A N$ a bilinear map such that

$$\forall \quad \begin{tikzcd} L \times M \arrow[r, "g"', "\text{bilinear}"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & X \\ M \otimes_A N \arrow[ur, dashed, "\tilde{g}"', "\exists!\ \text{gp homo}"] & \end{tikzcd}$$

Proof

$L = \mathbb{Z}[M\times N]$ free abelian gp on $M\times N$

basis (m,n); $m\in M, n\in N$

$I = \langle (m_1+m_2,n)-(m_1,n)-(m_2,n),\ (m,n_1+n_2)-(m,n_1)-(m_2,n_2),$
$(ma,n)-(m,an)\rangle$ the subgroup generated by the relations that we want.

Set $M\underset{A}{\otimes}N := L/I$ $\qquad t: M\times N \hookrightarrow \mathbb{Z}[M\times N] \xrightarrow{\pi} L/I$
$\qquad (m,n) \longmapsto [(m,n)]$

by construction (•) $M\underset{A}{\otimes}N$ is an abelian group.

(•) t is bilinear

Universal property:

$$\begin{tikzcd} M\times N \arrow[r, "b"] \arrow[d, "t"] & G \\ M\underset{A}{\otimes}N & \end{tikzcd}$$

b bilinear

Then look at the universal properties

$$\begin{tikzcd} M\times N \arrow[r, "b"] \arrow[d, "i"] & G \\ \mathbb{Z}[M\times N] \arrow[ru, dashed, "\exists!\,\tilde{b}"'] \arrow[d, "\pi"] & \\ M\underset{A}{\otimes}N \arrow[ruu, dashed, "\exists!\,\bar{b}"'] & \end{tikzcd}$$

↺ ↺ linear

since $\tilde{b}$ respects the quotient

$(I \subseteq \ker(\tilde{b}))$

uniqueness

$$\begin{tikzcd} M\times N \arrow[r, "b"] \arrow[d, "i"] & G \\ \mathbb{Z}[M\times N] \arrow[ru, "\pi\circ f"] \arrow[d, "\pi"] & \\ M\underset{A}{\otimes}N \arrow[ruu, bend right, "f"'] & \end{tikzcd}$$

↺

then $\pi\circ f$ makes first triangle commute $\Rightarrow \pi\circ f = \tilde{b}$

then f makes second triangle $\Rightarrow f = \bar{b}$. $\square$

Rem ① $M\underset{A}{\otimes}N$ is unique up to unique iso

② $[(m,n)] =: m\otimes n$ "pure tensor". They generalize the tensor product

$x \in M \underset{A}{\otimes} N \iff x = \sum_{i=1}^{n} m_i \otimes n_i$.

↳ We want a <u>tensor</u> <u>functor</u>! Actually a bifunctor $- \underset{A}{\otimes} - \quad \mathrm{Mod}A \times A\mathrm{Mod} \to Ab$

"Morale".

$f : M \to N'$

$g : N \to N'$

$$\begin{tikzcd}
M \underset{A}{\otimes} N \arrow[r, "Id_M \otimes g"] \arrow[d, "f \otimes Id_N"] \arrow[dr, dashed, "f \otimes g"] & M \underset{A}{\otimes} N' \arrow[d, "f \otimes Id_{N'}"] \\
M' \underset{A}{\otimes} N \arrow[r, "Id_{M'} \otimes g"'] & M' \underset{A}{\otimes} N'
\end{tikzcd}$$

with $f \otimes g (m \otimes n) = f(m) \otimes g(n)$

of course need to be careful how $M \underset{A}{\otimes} N$ is defined ↗ <u>as</u> <u>univ</u> <u>prop</u> ↘ <u>as</u> <u>a</u> <u>quotient</u>

<u>Easier</u> with universal property

$$\begin{tikzcd}
L \times M \arrow[r, "(f{,}g)"] \arrow[d, "t"] & L' \times M' \arrow[d, "t'"] \\
L \underset{A}{\otimes} M & L' \otimes M'
\end{tikzcd}$$

$t' \circ (f,g)(la, m) = t'(f(la), g(m))$

$= t'(f(l)a, g(m))$

f A.lin ↗ $= t'(f(l), ag(m))$

t' balance ↗ $= t'(f(l), g(am))$

g A lin ↗

$= t' \circ (f,g)(l, am)$

similarly on can see that $t' \circ (f,g)$ is additive, so it is balinear

hence by universal property $\exists!\ f \otimes g : L \underset{A}{\otimes} M \to L' \underset{A}{\otimes} M'$

$l \otimes m \mapsto f(l) \otimes g(m)$

<u>Lemma</u> 3.3 $- \underset{A}{\otimes} -$ is a bifunctor

<u>Coro</u> 3.4

① If M is an B-A-bimodule, then $M \underset{A}{\otimes} N$ is a B-Module

② If N is an A-C bimodule, then $M \underset{A}{\otimes} N$ is a right Cmodule

③ If ${}_B M_A$ and ${}_A N_C$ are two bimodules then $M \underset{A}{\otimes} N$ is a B-C-bimodule.

<u>Proof</u> ① $b \cdot (m \otimes n) = bm \otimes n$ but need to check that this is well defined...

better $b \in B$ $\ell_b : M \to M$, $m \mapsto bm$ then $\ell_b \in \mathrm{End}_A(M)$ by functoriality

we have $\ell_b \otimes N : M \otimes N \to M \otimes N$, $m \otimes n \mapsto bm \otimes n$ so our action is well defined

and this is a B-module structure on the tensor product

(2) similar

(3) $\ell_b \otimes N$ and $M \otimes R_c$ commute. □ □

Examples ① $A \otimes_A N \simeq_A N$ as left A-modules

$$a \otimes n \overset{*}{\mapsto} a \cdot n$$
$$1 \otimes n \leftarrow n$$

* well defined as use the universal property

$$\begin{tikzcd} (a,n) \arrow[r, mapsto] & a\cdot n \\ A \times N \arrow[r] \arrow[d, "t"'] & N \\ A \otimes N \arrow[ur, dashed, "\exists!"'] \arrow[u, phantom, "\circlearrowleft", shift left=6] & \end{tikzcd}$$

(2) If R is a commutative ring (eg a field...) then an R-module is an R-R-bimodule $R \times M \times R \to M$, $(x, m, y) \mapsto mxy = myx$

↳ $M \otimes_R N$ is always an R-module

Moreover we recover the usual definition of tensor product as you know it.

$$\begin{tikzcd} V \times W \arrow[r, "t'"] \arrow[d, "t"'] & V \otimes W \\ V \otimes_R W & \end{tikzcd}$$

↑ usual

↑ non commutative

- t is balanced ⇒ bilinear with the action on $V \otimes W$.
- t' is R-bilinear ⇒ balanced so have factors

↳ so ~~[illegible]~~ by the usual argument they are isomorphic □

⚠ For a field - $\dim(V \otimes_k W) = \dim V \dim W$

- Wrong in general for a ring

ex if $\gcd(m, n) = 1$ then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \{0\}$.

Indeed: for $c \in \mathbb{Z}$ we have $c.a\otimes b = ca \otimes b = a \otimes cb$

$1 = mu + nv$ gives $1.(a\otimes b) = (mu)a\otimes b + (nu)a\otimes b$

$= mua \otimes b + \cancel{nva} a \otimes nub$

$= 0 + 0 = 0$. □

Thm 3.5 [Isomorphisme dûe à Cartan / Schapiro Lemma, Hom-tensor adj, Nakayama, Frobenius ...]

Let A, B be two rings ${}_AM_B$ be an A-B-bimodule.

We have a functor $-\otimes_A {}_AM_B : \mathrm{Mod}A \to \mathrm{Mod}B$

$X \mapsto X \otimes_A M$

and a functor $\mathrm{Hom}_B({}_AM_B, -) : \mathrm{Mod}B \to \mathrm{Mod}A$

$Y \mapsto \mathrm{Hom}_B({}_AM_B, Y)$

Then $-\otimes_A M$ is a left-adjoint to $\mathrm{Hom}_B({}_AM_B, -)$

Concretely we have natural isomorphisms

$$\mathrm{Hom}_B(X\otimes_A M_B, Y_B) \simeq \mathrm{Hom}_A(X_A, \mathrm{Hom}_B({}_AM_B, Y)_A)$$

before proving it we need an A-module structure on $\mathrm{Hom}_B({}_AM_B, Y_B)$ for every Y

$$\mathrm{Hom}_B({}_AM_B, Y_B) \times A \longrightarrow \mathrm{Hom}_B(M, Y)$$
$$(f, a) \longmapsto f\bullet a : M \to Y, \quad m \mapsto f(am)$$

Proof ① (•) $\mathrm{Hom}_B(X\otimes_A M_B, Y_B) \simeq \mathrm{Hom}_A(X_A, \mathrm{Hom}_B({}_AM_B, Y_B))$

$$\phi \longmapsto \left[\begin{array}{l} X \to \mathrm{Hom}(M, Y) \\ x \mapsto \left[\begin{array}{l} M \to Y \\ m \mapsto \phi(x\otimes m)\end{array}\right]\end{array}\right]$$

(•) $\mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y)) \longrightarrow \mathrm{Hom}_B(X\otimes M, Y)$

$$\psi \longmapsto \left[\begin{array}{l} X\otimes M \to Y \\ x\otimes m \mapsto \psi(x)(y)\end{array}\right.$$

Check two inverse bijection + binaturality

Proof 2 using calculus of adjunction

$\eta : \mathrm{Id} \Rightarrow \mathrm{Hom}(M, - \otimes M)$

$\eta_X : X \to \mathrm{Hom}(M, x \otimes M)$ $\qquad$ $\varepsilon_Y : \mathrm{Hom}(M, S) \otimes M \to S$

$x \longmapsto \left[\begin{array}{l} M \to x \otimes M \\ m \mapsto x \otimes m \end{array}\right.$ $\qquad$ $\phi \otimes m \longmapsto \phi(m)$

Check naturality + triangular identities. □